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PART 2

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Volume 29, No. 4 April 1965 Reviews 3329–4647

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GENERAL

See also 3702, 4638.

Meschkowski, Herbert 3329

★Einführung in die moderne Mathematik.
B·I-Hochschultaschenbücher, 75.
Bibliographisches Institut, Mannheim, 1964. 189 *pp*.

This book is part of an extensive series of "Hochschultaschenbücher" and provides a brief introduction to the ideas surrounding the elementary mathematical structures. In his foreword, the author reflects that some years ago an introduction to higher mathematics was principally limit calculus, while now it tends to be the science of formal systems. The book contains some problems and their solutions.

Table of Contents: Axioms, Sets, Rational Numbers, Groups, Rings and Fields, Lattices, Spaces, Real Numbers.

P. J. Davis (Providence, R.I.)

Burlak, J. 3330

★Russian-English mathematical vocabulary.
With a short grammatical sketch by K. Brooke. University Mathematical Texts.
Oliver and Boyd, Edinburgh-London; Interscience Publishers, Inc., New York, 1963. viii + 305 *pp*. 21 *s*.

This is (at least) the third Russian-English mathematical dictionary-with-grammar since 1961 (others are Lohwater [*Russian-English dictionary of the mathematical sciences*, Amer. Math. Soc., Providence, R.I., 1961; MR **23** #A3044] and Milne-Thomson [*Russian-English mathematical dictionary*, Univ. Wisconsin Press, Madison, Wis., 1962; MR **24** #A2520]). Although Brooke's grammar has some features that the others lack, notably an alphabetical table of final letters with indications of what they might signify, Lohwater's remains the most helpful because of its large number of illustrative examples from a variety of mathematical fields. All three dictionaries include (apparently random) selections of Russian versions of mathematicians' names; Burlak is particularly strong on those of interest in the history of mathematics, and on British names. Unfortunately, Burlak records the MR transliteration of Russian only as it was before 1962. The reviewer has made a number of spot checks of words from papers in several fields; whereas in most cases any one of the three dictionaries will lead to the correct meaning, at least after a little thought, Lohwater's seems considerably more likely to evoke the correct mathematical idiom. In the reviewer's opinion, a mathematician wanting to buy only one dictionary should still buy Lohwater's.

R. P. Boas, Jr. (Evanston, Ill.)

★Swedish-Russian technical dictionary 3331
[Шведско-русский политехнический словарь].
Compiled by V. F. Maksimov; Edited by L. Ja. Popilov and I. V. Èlhval'd.
Izdat. "Sovetskaja Ènciklopedija", Moscow, 1964. 1257 *pp*. 3.07 *r*.

This dictionary contains approximately 60,000 words and terms; the mathematical vocabulary will be of interest principally to applied mathematicians.

Mikhalszewska, N. E. [Михальшевская, Н. Е.]; Mikhalszewski, R. I. [Михальшевский, Р. И.] 3332

★Polish-Russian mathematical dictionary [Польско-русский математический словарь].
Glavnaja Redakcija Inostran. Naučno-Tehn. Slovarei Fizmatgiza, Moscow, 1963. 213 *pp*. 0.79 *r*.

This dictionary, which has 12,000 entries, will prove useful in reading general mathematics in Polish. The number of entries in the dictionary could have been substantially reduced by the omission of an excessive number of listings of the type $A + B$, where A and B have their customary meanings; for example, there are 146 such entries under "grupa", extending from "grupa abelowa" (abelian group) to "grupa zwarta" (compact group).

★Philosophy of mathematics: Selected readings. 3333
Edited and with an introduction by Paul Benacerraf and Hilary Putnam.
Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964. vii + 536 *pp*. $8.95.

This volume comprises a collection of some of the more important papers on the philosophy of mathematics that have appeared during the last 75 years or so, the majority of them belonging to the more recent period. The papers have been selected in such a way as to present the thread of the argument; the principle of selection is explained in a brief introductory chapter by the editors. There are four main parts, respectively on "The Foundations of Mathematics", "The Existence of Mathematical Objects", "Mathematical Truth", the last section being reserved for passages from Wittgenstein's *Remarks on the foundations of mathematics* [Macmillan, New York, 1956; MR **19**, 1], together with a number of comments and reviews of Wittgenstein's book. Within the four parts, the selection of papers (not in chronological order) is again carried out in such a way that the later papers frequently illuminate those that have preceded, and a particularly pleasing result is that philosophy of mathematics is exhibited in the form of a constant "disputation" (to use the title of Heyting's second paper in Part One). Thus the selection from Russell's *Introduction to mathematical philosophy*

[Allen & Unwin, London, 1919] follows the Frege extract, "The Concept of Number"; Castañeda's paper on "Arithmetic and Reality" blasts its predecessor, D. A. T. Gasking's "Mathematics and the World", etc. And the bibliography supplies additional controversial material where space has limited its reproduction in full.

It would be impossible to single out in detail every paper; some of the names from the list of contributors may give the reader some idea of its breadth and penetration: Carnap, Heyting, von Neumann, Brouwer, Frege, Russell, Hilbert, Quine, Goodman, Gödel, Bernays, Ayer, Nagel, Hempel, Poincaré, Dummett. There is no doubt that this selection will be a classical reference work for many years to come.

G. Buchdahl (Cambridge, England)

HISTORY AND BIOGRAHPY

See also 3359, 4231–4233.

Eves, Howard 3334

★**An introduction to the history of mathematics.**
Revised edition.
Holt, Rinehart and Winston, New York, 1964. xvi + 439 *pp*. $7.95.

The present book, a minor revision of the first edition [New York, 1953], attests the current popularity of the history of mathematics in the schools and universities (see, for example, the recent volume by A. Aaboe [*Episodes from the early history of mathematics*, Random House, New York, 1964; MR **28** #2956] sponsored by the School Mathematics Study Group).

The author has divided his book into two parts. Part I ("Before the Seventeenth Century") deals with numeral systems, Babylonian and Egyptian mathematics, Pythagorean mathematics, the problems of duplication, trisection and quadrature, Euclid's *Elements* and Greek mathematics after Euclid, Hindu and Arabian mathematics, and European mathematics up to 1600. Part II ("Seventeenth Century and Later European Mathematics") deals rapidly with later developments.

Bubnov, Nicolaus (Editor) 3335

★**Gerberti postea Silvestri II papae Opera Mathematica (972–1003).**
Accedunt aliorum opera ad Gerberti libellos aestimandos intelligendosque necessaria per septem appendices distributa.
Georg Olms Verlagsbuchhandlung, Hildesheim, 1963. cxix + 620 *pp*. (4 *plates*) *DM* 98.00.

Der Nachdruck der Originalausgabe [Friedländer, Berlin, 1899] ist sehr zu begrüßen. An ergänzendem Material ist zu nennen: P. Tannery und L'Abbé Clerval [Notices Extraits Manuscrits Bibliothèque Nat. **36** (1901), 487–543] und J. E. Hofmann [Abh. Preuss Akad. Wiss. Math.-Natur. Kl. **1942**, no. 8; MR **8**, 189].

J. E. Hofmann (Ichenhausen)

Kepler, Johannes 3336

★**Gesammelte Werke. Band VIII: Mysterium cosmographicum. Editio altera cum notis *De Cometis*, *Hyperaspistes*.**
Bearbeitet von Franz Hammer.
C. H. Beck'sche Verlagsbuchhandlung, Munich, 1963. 517 *pp*. (3 *inserts*) *DM* 65.00.

The present book is Volume 8 (Volume 7 appeared in 1953) in the planned series of Kepler's works begun by von Dyck and Max Caspar. In it Franz Hammer has presented three of Kepler's late works: Kepler's second and annotated edition of *Mysterium Cosmographicum* of 1621, his *De Cometis* of 1619, and his *Hyperaspistes* of 1625.

In the first edition (1596) of *Mysterium Cosmographicum* Kepler had related planetary orbits to nested regular polyhedra; in the second edition he tries to reconcile these results with his *Harmonice Mundi* (1619), where he used harmonic ratios to produce planetary parameters. In *De Cometis* he treats Halley's Comet as it appeared in September 1607, as well as the three comets of 1618; among other things, he endeavors to determine their motion, assuming that their paths are rectilinear. *Hyperaspistes* is a vehement polemic defending Brahe (d. 1601) against Chiaramonti, who in *Anti-Tycho*, a contribution to the controversy about the comets of 1618, had attacked Brahe's contention that comets are supralunar.

The Latin texts of these works are printed without translation, but followed by summaries and scanty notes and comments in German.

A. Aaboe (New Haven, Conn.)

Vekerdi, László 3337

The discovery of pre-Euclidean mathematics. (Hungarian)
Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **13** (1963), 133–150.

A brief survey of the discovery of the Greek mathematics of the 6th and 5th centuries B.C., centered on the problem of the so-called "geometrical algebra" of the second book of the *Elements*. It is suggested that the well-known interpretations of O. Neugebauer and B. L. van der Waerden are inspired partly by modern patterns of thought, and that the Greek "geometrical algebra" is, as H. G. Zeuthen already considered it, a simple tool of the Greek "synthetic" geometry.

Á. Szabó (Budapest)

Vekerdi, László 3338

Infinitesimal methods in Pascal's mathematics. (Hungarian)
Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **13** (1963), 269–285.

Pascal's infinitesimal method in the *Traité des sinus du quart de cercle* is discussed, and it is suggested that it may be an application of the method of indivisibles on infinitesimal concepts of the antique exhaustion method. Analogies to Descartes' method for the calculation of the area of the cycloid are discussed.

Á. Szabó (Budapest)

Boyer, Carl B. 3339

Pascal: The man and the mathematician.
Scripta Math. **26**, 283–307 (1963).

A critical account of Pascal's contributions to mathematics and their relations to previous results and the work of his contemporaries.

O. Ore (New Haven, Conn.)

upont, Pascal 3340
[illegible]same storico-critico del contributo di d'Alembert, Eulero, Poinsot, Poncelet ed altri al concetto dell'asse istantaneo di rotazione nei moti rigidi con un punto fisso.
Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **98** (1963/64), 537–572.
n analysis, in terms of vectors, of instantaneous axes of otation, beginning with D'Alembert's work on the utation of the Earth's axis. Besides the authors mentioned n the title, the studies of Burali-Forti, Marcolongo, and thers in rigid rotating systems are also briefly considered.
G. Huxley (Belfast)

upont, Pascal 3341
Il centro istantaneo di rotazione, nei moti rigidi piani, dal punto di vista storico, critico, applicativo.
Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **98** (1963/64), 331–354.
succinct study of centres of rotation in moving rigid lane systems, beginning with Descartes. The work of ernoulli, Cauchy, Chasles, and others on the subject is onsidered, and there is a neat treatment of stationary oints in rotating squares. The author ends by claiming or instantaneous centres of rotation and for centres of elocity a prime position in kinematics, statics, and nechanics.
G. Huxley (Belfast)

ryde, Matthew M. 3342
Waclaw Sierpiński—mathematician.
Scripta Math. **27**, 105–111 (1964).
n account of the professional life of Sierpiński in elebration of his eightieth birthday. Although there is no ibliography, references to his most important contribuions are mentioned in the text.

hevalley, Claude 3343
Emil Artin [1898–1962].
Bull. Soc. Math. France **92** (1964), 1–10.
n account of Artin's professional career, together with a escription of his most significant contributions. A bibliography, including unpublished lecture notes, is appended.

uraux, Werner 3344
Wilhelm Blaschkes Leben und Werk.
Mitt. Math. Ges. Hamburg **9**, no. 2, 24–40 (1963).
n account of Blaschke's career and principal contribuions. No bibliography is given.

ulliksen, Harold 3345
Samuel Stanley Wilks, 1906–1964.
Psychometrika **29** (1964), 103–104. (1 *plate*)
brief account of the professional career of Wilks. There no bibliography.

ibliography of papers of A. N. Kolmogorov 3346
published in [illegible]–1962. (Russian)
Uspehi Mat. Nauk **18** (1963), no. 5 (113), 121–123.

LOGIC AND FOUNDATIONS

See also 3333, 3381, 3383, 3385, 3386, 3388, 3405, 3406, 3408, 4224, 4645–4647.

Nagel, Ernest; Suppes, Patrick; Tarski, Alfred 3347
(Editors)
★**Logic, methodology and philosophy of science.**
Proceedings of the 1960 International Congress.
Stanford University Press, Stanford, Calif., 1962. ix + 661 *pp.* $12.50.
The papers of mathematical interest will be reviewed individually.

Novikov, P. S. 3348
★**Introduction à la logique mathématique.**
Traduit par Ch. Sarthou. Collection Universitaire de Mathématiques, XIV.
Dunod, Paris, 1964. viii + 332 *pp.* 48 *F.*
This is a French translation of a Russian work [*Elements of mathematical logic* (Russian), Fizmatgiz, Moscow, 1959] which was reviewed in MR **22** #5565. There is also an English translation by L. F. Boron [Oliver and Boyd, Edinburgh, 1964; MR **29** #2159].
H. B. Curry (University Park, Pa.)

Geymonat, Ludovico 3349
Riflessioni sul metodo assiomatico.
Math. Notae **19** (1964), 153–159.

Marcus, Solomon 3350
Sur un modèle logique de la partie du discours. (Romanian. Russian and French summaries)
Acad. R. P. Romîne Stud. Cerc. Mat. **13** (1962), 37–62.

Peklo, Bohuslav 3351
Einige Bemerkungen zu den deontischen Systemen, welche Sanktionen und mehrere Funktoren enthalten.
Logique et Analyse (*N.S.*) **5** (1962), 98–121.

Denjoy, Arnaud 3352
Rapports logiques associés, pour l'inclusion ou l'exclusion, vis-à-vis de p classes d'éléments dans un même espace.
C. R. Acad. Sci. Paris **257** (1963), 2594–2596.
Author's summary: "Les catégories définies par les systèmes de rapports logiques sont figurées, avec les notations de l'algèbre élémentaire, par des formules où chaque variable désigne concurremment un ensemble et sa fonction caractéristique."

Denjoy, Arnaud 3353
Catégories définies par l'association de rapports logiques d'inclusion ou d'exclusion, vis-à-vis de p classes d'éléments dans un même espace.
C. R. Acad. Sci. Paris **258** (1964), 765–767.
Author's summary: "Compléments à la Note précédente [#3352] pour en rectifier certains points et mener l'étude à son terme."

Rose, Alan 3354

Note sur la formalisation de calculs propositionnels polyvalents à foncteurs variables.

C. R. Acad. Sci. Paris **259** (1964), 967–968.

The author considers the problem of deriving, from a given axiomatisation of an M-valued propositional calculus without variable functors (variables for one-place functions from propositions to propositions), an axiomatisation of the corresponding calculus with variable functors. He has already obtained a result of this kind [same C. R. **258** (1964), 1951–1953; MR **28** #2967] which generalises a result of Rosser and Turquette [J. B. Rosser and A. R. Turquette, *Many-valued logics*, North-Holland, Amsterdam, 1951; MR **14**, 526]. It is assumed here that the (original) calculus is functionally complete [ibid., p. 13] and that its axiomatisation includes both a rule of substitution and the analogue of modus ponens. An axiomatisation of the extended calculus is obtained by modifying the rule of substitution (so as to allow substitution for variable functors) and adding a single new axiom. That a provable formula is assertable follows directly from an examination of the new axiom and the modified rule. Functional completeness ensures the availability of certain constant propositional functions, previously used by the author and Rosser and Turquette. Then, being given an assertable formula P of the extended calculus, there is indicated a method of obtaining a finite sequence of assertable formulae such that: (i) the first of them is P, (ii) the last, Q, contains no occurrences of variable functors, and (iii) each can be deduced from its predecessor in the extended axiomatisation. Now, since U is assertable, it can be deduced in the original axiomatisation, and thus one obtains, in a straightforward way, a proof of P in the new axiomatisation. *M. J. Wicks* (Singapore)

Saito, Setsuo 3355

Truth value assignment in predicate calculus of first order.

Notre Dame J. Formal Logic 4 (1963), 216–223.

A decision procedure is proposed for testing validity and consistency of formulas of the restricted predicate calculus. The procedure appears to be an extension of a well-known decision method applied to sentential calculus and is closely related to the use of semantic tableaux developed by E. W. Beth.

E. J. Cogan (Bronxville, N.Y.)

Maehara, Shoji 3356

On the interpolation theorem of Craig. (Japanese)

Sûgaku **12** (1960/61), 235–237.

This paper is an explanatory remark on the interpolation theorem of Craig in connection with a theorem of Beth [Beth, Nederl. Akad. Wetensch. Proc. Ser. A **56** (1953), 330–339; MR **15**, 385] and Gentzen's principal theorem for LK. As an example of possible applications of the interpolation theorem of Craig, the author gives a proof of Beth's theorem by it. An outline of a proof of the interpolation theorem by Gentzen's principal theorem for LK is also added. *K. Ono* (Nagoya)

Ackermann, Wilhelm 3357

Grundgedanken einer typenfreien Logik.

Essays on the foundations of mathematics, pp. 143–155. *Magnes Press, Hebrew Univ., Jerusalem*, 1961.

Es werden typenfreie formale Systeme besprochen, in denen das Komprehensionsaxiom uneingeschränkt gilt, aber das tertium non datur nicht allgemein herleitbar ist. Ein typenfreies System Σ_1, in dem sich die klassische Zahlentheorie und die verzweigte Analysis entwickeln lassen, ist als widerspruchsfrei bekannt [der Verfasser, Math. Z. **57** (1953), 155–166; MR **14**, 834; der Referent, Math. Ann. **125** (1953), 394–400; MR **15**, 386]. Der Nachteil dieses Systems, keine eigentliche Implikationsbeziehung zu enthalten und kein Deduktionstheorem zu besitzen, wird durch Einführung einer fiktiven Beweisbarkeit in einer nachweisbar widerspruchsfreien Erweiterung Σ_2 überwunden [der Referent, Math. Z. **61** (1954), 160–179; MR **16**, 662]. Um gewisse Härten in der Interpretation zu vermeiden, wie sie für Σ_2 bestehen, führt der Verfasser ein im wesentlichen äquivalentes System Σ_3 ein, das eine formal festgelegte Folgerungsbeziehung $U \vdash B$ (ist U beweisbar, so auch B) enthält. Schließlich werden die Grundgedanken für ein weiteres typenfreies System entwickelt, dem interpretationsmäßig ein schärferer Standpunkt zugrundegelegt ist. Es wird nämlich von einer sinnvollen Formel verlangt, daß sie keine sinnlosen Bestandteile enthält, wobei eine Formel U als sinnlos anzusehen ist, wenn sowohl U als auch $\neg U$ sich widersprechende Konsequenzen hat. In diesem System, das vermutlich als widerspruchsfrei nachgewiesen werden kann, läßt sich eine Art Zahlentheorie entwickeln, die jedoch nicht die klassische ist. In einer Erweiterung dieses Systems kann man die klassische Zahlentheorie und weitere Teile der klassischen Mathematik darstellen [der Verfasser, Arch. Math. Logik Grundlagenforsch. **5** (1960/1961), 96–111]. Ein relativer Widerspruchsfreiheitsbeweis des erweiterten Systems unter Bezugnahme auf die axiomatische Mengenlehre wird als wünschenswert angesehen. *K. Schütte* (Kiel)

Rasiowa, H. 3358

A generalization of a formalized theory of fields of sets on non-classical logics.

Rozprawy Mat. **42** (1964), 30 pp.

Let $\mathscr{L}$ be a propositional calculus containing among its binary and unary propositional connectives the signs for disjunction, conjunction and implication, and containing among its theorems all theorems of positive logic. ($\mathscr{L}$ can be, e.g., the classical, intuitionistic, minimal, positive or modal S4 propositional calculus.) With $\mathscr{L}$ a corresponding class of abstract algebras $\mathscr{A}$ ("$\mathscr{L}$-algebras") is associated. Also, with $\mathscr{L}$ a corresponding predicate calculus $\mathscr{L}^*$ of the second order is associated (containing as primitive signs the propositional connectives of $\mathscr{L}$, individual and set variables, quantifiers binding such variables, signs for the identity of individuals and for the equality and inclusion of sets, signs for binary and unary operations on sets—corresponding to the propositional connectives of $\mathscr{L}$—and signs for the universal set and, in case $\mathscr{L}$ contains a negation sign, for the empty set), with an axiom system characterizing the operations and relations, and with the usual rules of inference.

In case $\mathscr{L}$ is the classical propositional calculus, $\mathscr{L}^*$ can be interpreted as a formalized theory of fields of sets. A formula of $\mathscr{L}^*$ containing no individual variables and no quantifiers binding such variables is a theorem of $\mathscr{L}^*$ if and only if it is a theorem in the elementary formalized theory of Boolean algebras.

In the general case, under some additional suppositions concerning $\mathscr{L}$, an interpretation of the system $\mathscr{L}^*$ is given in terms of a model $\mathfrak{M}$ such that for any formula α of $\mathscr{L}^*$, α is a theorem of $\mathscr{L}^*$ if and only if α is valid in $\mathfrak{M}$. Furthermore, a formula of $\mathscr{L}^*$ of the form $A=B$ or $A \subset B$ is a theorem of $\mathscr{L}^*$ if and only if it is a theorem of the elementary formalized theory of $\mathscr{L}$-algebras based on $\mathscr{L}$-logic, whereas the question remains open whether the same holds for arbitrary formulas of $\mathscr{L}^*$ containing no individual variables and no quantifiers binding such variables. *V. Devidé* (Zagreb)

Moreno, Alberto 3359

Propositional logic in Juan de Santo Tomás. (Spanish)

Notre Dame J. Formal Logic 4 (1963), 113–134.

This is an analysis of some of the work of John of St. Thomas (1589–1644) from the point of view of symbolic logic. His view of simple and compound propositions is explained, as are his notions of consequence and argumentation. Negation, disjunction, and conjunction emerge from his work as connectives, while implications of various sorts are, for him, relations among statements. Among his rules of argumentation a substantial number of inference rules available in classical sentence calculi are to be found. Relevant passages from the original Latin text are supplied in notes. *E. J. Cogan* (Bronxville, N.Y.)

Massey, Gerald J. 3360

Note on Copi's system.

Notre Dame J. Formal Logic 4 (1963), 140–141.

The author argues that a proof in the first edition of the reviewer's *Symbolic logic* [Macmillan, New York, 1954; MR 17, 223] is fallacious. The proof in question is intended to establish that the nineteen elementary valid argument forms (of Chapter 3) supplemented by the principles of conditional proof and indirect proof (of Chapter 3) provide a complete system of truth functional logic. The author agrees that the nineteen forms augmented by the strengthened rule of conditional proof (of Chapter 4) is proved complete by the argument of the text, but objects that in Chapter 3 the conditional proof of a formula nowhere contains the formula itself as a whole line, thus making the argument of the text inapplicable. The author is careful to state that his objection is not to the conclusion drawn in the text, which "remains an open question", but to the proof offered in the text. His objection seems to be correct to the present reviewer. (A simpler objection would turn on the fact that the argument in the text makes no reference to the rule of indirect proof, although the text admits that that rule serves to strengthen the proof apparatus consisting of the nineteen forms plus the "weak" conditional proof rule.)

I. M. Copi (Ann Arbor, Mich.)

Canty, John Thomas 3361

Completeness of Copi's method of deduction.

Notre Dame J. Formal Logic 4 (1963), 142–144.

The author settles the open question pointed out by Massey in the note reviewed above [#3360]. The author's proof turns not at all on conditional proof, but entirely on the nineteen forms, plus the rule of indirect proof (IP). Using only the nineteen forms the author proves the validity of a lemma: $q \vee (p \cdot -p) \cdot r .\therefore. q$. Then, using the rule of indirect proof, he proves his Theorem 1: Corresponding to every derived rule which can be demonstrated in RS there is an argument which can be proved valid by Copi's method of deduction (CMD). His proof can be described as follows. For any derived rule $P_1, \cdots, P_n \vdash Q$ of RS, by the deduction theorem there is a demonstration in RS that $\vdash P \supset Q$, where P is the conjunction of $P_1, \cdots, P_n$. Now it is obvious that by CMD a formal proof of validity can be constructed for the argument $P_1, \cdots, P_n, -Q \therefore N$, where N is the disjunctive normal form of the wff $P \cdot -Q$. By the analyticity of RS, $P \supset Q$ is a tautology, hence $P \cdot -Q$ is a contradiction, and every disjunct of N contains a contradiction. Hence by iterated uses of the lemma we obtain by CMD a formal proof of a contradiction from $P_1, \cdots, P_n, -Q$. Now by the rule IP the validity of the argument $P_1, \cdots, P_n \therefore Q$ is proved valid by CMD. Now the completeness of CMD follows from the deductive completeness of RS which contains a derived rule $P_1, \cdots, P_n \vdash Q$ corresponding to any argument $P_1, \cdots, P_n \therefore Q$ that can be proved valid by the use of truth tables. *I. M. Copi* (Ann Arbor, Mich.)

Nagornyĭ, N. M. 3362

On realizable and fulfillable logical-arithmetical formulae. (Russian)

Dokl. Akad. Nauk SSSR 157 (1964), 529–531.

The author proposes a definition of realizability which differs from the original one of Kleene [J. Symbolic Logic 10 (1945), 109–124; MR 7, 406] in some details; it is based on that of Šanin [Trudy Mat. Inst. Steklov. 43 (1955); MR 19, 4]. The object, if any, which realizes a formula P is a partial recursive function or an ordered pair; with it is associated an index (*zapis*, also translatable as "record", "transcription", etc.) which is its Gödel number in some fixed Gödel numeration. If P is a prime formula $T=S$, where T and S are constant terms, the realization, if any, of P is a binary general recursive function R such that $R(m,n)=0$ if and only if $m=n$; and this realizes P if and only if T and S have the same numerical value. If P is compound, the realization, if any, of P is determined in terms of the indices of the major components of P. The author then defines a second notion of fulfillability (*osupolnenie*) by an inductive definition of the same structure except that "realization" is replaced by "fulfillment" and the index of a fulfillment is always (if it exists at all) the number 0. The author states without proof the following two theorems: (1) If constructive logic is used, then a formula P has a realization if and only if it has a fulfillment; (2) If classical logic is used, then the class H of those formulas for which realizability and fulfillability are equivalent cannot be recognized by any algorithm. *H. B. Curry* (University Park, Pa.)

Fuhrken, G. 3363

Skolem-type normal forms for first-order languages with a generalized quantifier.

Fund. Math. 54 (1964), 291–302.

The author considers a language (with denumerably many non-logical constants) obtained from the first-order language by adjoining to it a new quantifier Q. Each ordinal α determines an interpretation L_α of the language

under which the usual connectives and the quantifiers $\wedge$, $\vee$ have the ordinary meaning, whereas $QxFx$ means: there are at least $\aleph_\alpha$ elements satisfying Fx.

Let V_α be the set of formulas which are identically true in the interpretation L_α. The main results of the paper are: (1) If $\aleph_\alpha$ is regular, then $V_1 \subseteq V_\alpha$; (2) $V_\alpha \subseteq V_0$ for every α.

In order to obtain these results the author introduces two "normal forms" for sets of formulas. One of them is used when $\alpha = \beta + 1$ and the other when α is a limit number. The first normal form for a set Σ of formulas is a set Σ^* of first-order formulas (without the quantifier Q) containing two additional predicates, a unary predicate U and a ternary predicate F. The set Σ^* has the property that a relational system $\mathfrak{A}$ of power $\aleph_{\beta+1}$ is a model of Σ if and only if there are a set X (an interpretation of U) and a ternary relation Y (an interpretation of F) such that the relational system $(\mathfrak{A}, X, Y)$ is a model of $\Sigma^* \cup \{QxUx\}$. The word "model" refers, of course, to the interpretation L_α of the logical constants. The idea behind the construction of Σ^* is that U is an auxiliary set of power $\aleph_\beta$, and sentences of the form $QxHx$ are replaced everywhere by sentences stating that there is a one-one mapping of the universe into the set $\{x : Hx\}$; furthermore, one assures (by adding suitable sentences to Σ^*) that for each sentence of the form $QxHx$ occurring as a part of a formula in Σ either the universe can be mapped in one-one way into $\{x : Hx\}$ or the set $\{x : Hx\}$ can be so mapped into $\{x : Ux\}$.

The second normal form is based on a similar principle.

Theorems on normal forms jointly with results of Vaught on Skolem-Löwenheim forms for two cardinals [cf. R. L. Vaught, Bull. Amer. Math. Soc. **69** (1963), 299–313; MR **26** #4912] yield results (1) and (2) stated at the beginning. Another important corollary is the compactness theorem for L_1 which states that a set Σ of formulas is satisfiable (under the interpretation L_1 of logical constants) if and only if every finite subset of Σ is satisfiable.

In addition to these results, the paper lists a couple of theorems on ordinals α, β for which the inequality $V_\alpha - V_\beta \neq 0$ holds. Finally, the author formulates some open problems concerning conditions under which $V_\alpha = V_\beta$, respectively, $V_\alpha \subseteq V_\beta$. *A. Mostowski* (Warsaw)

Vaught, R. L. 3364

The completeness of logic with the added quantifier "there are uncountably many".

Fund. Math. **54** (1964), 303–304.

Using results of G. Fuhrken [#3363], the author shows that the set V_1 is recursively enumerable. More generally, if S is a recursively enumerable set of formulas, then so is the set of (semantically defined) consequences of S, logical constants being interpreted according to L_1.

The author notes that the set V_1 is defined in set theory. If one limits the notion of "sets" used in the definition of V_1 to constructible sets, one obtains a definition of another possible set V_1^c. The author shows, however, that the equation $V_1 = V_1^c$ is provable in the Zermelo-Fraenkel set theory. Thus one can always eliminate the generalized continuum hypothesis and more generally, the axiom of constructibility from a proof of a theorem saying that a particular formula F belongs to V_1. *A. Mostowski* (Warsaw)

Pi Calleja, Pedro 33[illegible]

Formalization of the Russell counter-paradox. (Spanish)

Math. Notae **19** (1964), 147–151.

The author makes a few remarks about Russell's parado[x] emphasizing the analogous problem of a catalogue listi[ng] all those catalogues which do not list themselves. He al[so] refers to the resolution of the paradox in axiomatic s[et] theory. No new slant on the paradoxes is offered. *E. Mendelson* (Flushing, N.Y.)

Hintikka, Jaakko 336[6]

The modes of modality.

Acta Philos. Fenn. Fasc. **16** (1963), 65–81.

A non-technical exposition of the author's theory o[f] model systems as a unifying concept in the study o[f] modal logics; somewhat more extensive than the author's previous sketch [Theoria (Lund) **27** (1961), 119–128; MR **25** #2953]. *E. Mendelson* (Flushing, N.Y.)

Sacks, Gerald E. 336[7]

The recursively enumerable degrees are dense.

Ann. of Math. (2) **80** (1964), 300–312.

The author shows that, given recursively enumerable (RE) degrees **b** and **c** with **b** < **c**, there is an RE degree **d** with **b** < **d** < **c**. (The case **b** = **0** had been previously proved by Muchnik and Friedberg.) Given RE sets B and C of degrees **b** and **c**, respectively, the author constructs an RE set D of degree **d**. The set B is "copied" in D. In addition, a partial copy of C is made in D for each i to insure that $D \neq \{i\}^B$. A complete copy would insure this, since C is not recursive in B; but only a partial copy must be made, since we want $C \neq \{e\}^D$. As usual, a priority scheme is set up between these requirements. A difficulty is that the requirement $C \neq \{e\}^D$ may interfere infinitely often with a lower priority requirement $D \neq \{i\}^B$. However, this can happen only when $\{e\}^C$ is not total; the author then says that e is unstable. The construction insures that for given i, there will be infinitely many stages at which no unstable $e < i$ is preventing us from making the ith copy of C in D; so we can achieve $D \neq \{i\}^B$. A tedious but straightforward argument then shows that D is recursive in C. *J. R. Shoenfield* (Durham, N.C.)

Sacks, Gerald E. 3368

A maximal set which is not complete.

Michigan Math. J. **11** (1964), 193–205.

The author shows that if **c** is a degree such that $\mathbf{0} < \mathbf{c} \leq \mathbf{0}'$, then there is a degree **b** containing a maximal recursively enumerable set such that $\mathbf{c} \not\leq \mathbf{b}$ (so that $\mathbf{b} \neq \mathbf{0}'$). Given a set C of degree **c**, the author constructs a maximal recursively enumerable set M of degree **b** by combining Friedberg's construction of a maximal set with standard priority methods for insuring that M is not recursive in C. Difficulties arise because a requirement for making M maximal may necessitate putting infinitely many numbers in M, thus interfering with a lower priority requirement infinitely often. This difficulty is surmounted by methods previously developed by the author [#3367 above]. *J. R. Shoenfield* (Durham, N.C.)

Mendelson, Elliott 3369

On some recent criticism of Church's Thesis.

Notre Dame J. Formal Logic **4** (1963), 201–205.

The author rejects three recent arguments directed against Church's thesis. The first, by J. Porte, is found to confuse "effective computability" with "human computability". The second, by R. Péter, interprets Kleene's definition of recursive function more narrowly than is intended in the statement of Church's thesis. And the third argument by L. Kalmar is found to contain the tacit assumption that the class of proofs in arbitrary consistent extensions of a given first-order theory p is effectively enumerable. Church's thesis does not lead to a contradiction, but merely implies the falsity of this assumption.

D. L. Kreider (Hanover, N.H.)

Hermes, Hans 3370

Unentscheidbarkeit der Arithmetik.

Math.-Phys. Semesterber. 11 (1964), 20–34.

This is a brief but very clear exposition of matters connected with the title. It contains a condensed explanation of a Turing machine and a sketch of the proof by way of the undecidability of semi-Thue systems. For details the author refers to his book [*Aufzählbarkeit, Entscheidbarkeit, Berechenbarkeit*, Springer, Berlin, 1961; MR 26 #1252]. *H. B. Curry* (University Park, Pa.)

Vučković, Vladeta 3371

On some possibilities in the foundations of recursive arithmetics of words. (Serbo-Croatian summary)

Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hrvatske Ser. II 17 (1962), 145–157 (1963).

The author generalises his previous work on multiply successor arithmetics to systems whose "numerals" are finite strings of letters of an alphabet a_ξ, $\xi \in I$. It is shown that even when no conditions are imposed upon the alphabet (other than that the letters are distinct and one of them is distinguished), the characteristic function for word equality is primitive recursive and the analogue of induction is provable. If the index set I is a semi-group, the associative law for addition is provable, but the general associative law for linear operations requires the semi-group to be commutative. When I is a group (with neutral element 0, and $-\xi$ the inverse of ξ), if a_0 now denotes the empty word, so that the alphabet is a_ξ, $\xi \in I-\{0\}$, and if we impose the axioms

$$a_{-\xi}a_\xi = a_\xi a_{-\xi} = a_0$$

for all $\xi \in I-\{0\}$, and define $-X$ by

$$-a_0 = a_0, \qquad -(a_\xi X) = a_{-\xi}+(-X), \qquad \xi \in I-\{0\},$$

then $X = -A$ is the solution of each of the equations

$$X+A = a_0, \qquad A+X = a_0.$$

R. L. Goodstein (Leicester)

Péter, Rózsa 3372

Über die Primitiv-Rekursivität einiger den Aufbau von Formeln charakterisierenden Wortfunktionen.

Acta Math. Acad. Sci. Hungar. 14 (1963), 149–172.

Formulas of the propositional calculus are treated as words on an appropriate alphabet, and the notion of primitive recursive word-function is used. The step-by-step inward decomposition of a formula into successive components is systematized by means of a Kantorovich graph, the nodes of which are indexed in a simple fashion by words on two fixed letters. The graph of a formula is shown to depend primitive recursively on the formula. Let $\mathrm{Pos}(x, y) \equiv [y$ is the index of a node of the graph of a formula x, and y represents a positive component of $x]$, and let $\mathrm{Neg}(x, y)$ be defined similarly for negative components. Then Pos and Neg are primitive recursive. There is a primitive recursive function with primitive recursive inverse which for each formula maps its Łukasiewicz parenthesis-free notation onto its Pawlak parenthesis-free notation. *P. Axt* (E. Lansing, Mich.)

Dijkman, J. G. 3373

Markov chains and intuitionism. III. Note on continuous functions with an application to Markov chains.

Nederl. Akad. Wetensch. Proc. Ser. A 67 = *Indag. Math.* 26 (1964), 256–261.

Part II appeared in same Proc. 66 (1963), 275–281 [MR 27 #6304]. The limit properties of continuous functions defined in an interval at the end points of the interval are considered. The following intuitionistic result has no classical counterpart: If $f(t)$ is a function defined for all real numbers t such that $(t \nless a) \vee (t<a)$ and if $\lim_{t\downarrow x} f(t)$ exists for every x, then we have $\lim_{t\uparrow a} f(t) = \lim_{t\downarrow a} f(t)$. These results are motivated by a theorem of Doob's on Markov chains.

K. L. Chung (Stanford, Calif.)

Žuravlev, Ju. I. 3374

An estimate of complexity of local algorithms for certain extremal properties on finite sets. (Russian)

Dokl. Akad. Nauk SSSR 158 (1964), 1018–1021.

This paper concerns algorithms for deciding for finite sets of a family $\mathfrak{M}$ whether or not predicates of a given class P are true. Such an algorithm is said to be local if its procedure in making a decision for a set $\mathfrak{A}$ is conditioned by its decision for sets $\mathfrak{B}$ in $\mathfrak{M}$ in a neighborhood of $\mathfrak{A}$, where neighborhoods are interpreted distinctly in special situations. Among special local algorithms are those for obtaining minimal disjunctive normal forms and those for constructing minimal paths in lattice graphs. The results of the paper give estimates of complexity of local algorithms in terms of the sizes of neighborhoods and of the cardinal number of the class P of predicates.

E. J. Cogan (Bronxville, N.Y.)

Rutledge, J. D. 3375

On Ianov's program schemata.

J. Assoc. Comput. Mach. 11 (1964), 1–9.

Ju. I. Janov's program schemata [Problemy Kibernet. 1 (1958), 75–127; MR 24 #B1735; for a leisurely English exposition see the reviewer's article in *Logik und Logikkalkül*, pp. 159–178, Verlag Karl Alber, Freiburg, 1962] are strings of symbols that abstractly characterize the sequential and control properties of computer programs (without algebraically inessential details). Alternatively, Janov gave such characterizations in matrix form. In terms of and for such schemata, Janov devised constructive equivalence criteria—"equivalence" in what the author rightly calls a "very strong" sense.

The author aims at expounding Janov's results in the context of a judiciously revamped formal apparatus of his

own, and he presents inter alia an algorithm for the equivalence problem as conceived and solved by Janov. Pointing out the proximity and relevance of such considerations to the equivalence problem for finite automata, he also specifies a procedure for generating all members of an equivalence class of schemata, given one member.

{It appears, from a review, that some of the author's results, in particular, the tie-up with finite-automata theory, were independently obtained by S. Igarashi [J. Information Processing Soc. Japan 3 (1962), 66–72; MR 28 #4979].} *E. M. Fels* (Munich)

Elgot, C. C.; Rutledge, J. D. 3376

RS-machines with almost blank tape.

J. Assoc. Comput. Mach. **11** (1964), 313–337.

Authors' summary: "Finite automata which communicate with counters or with tapes on a single-letter alphabet with end mark are studied. A typical machine system studied here consists of a family of machines; the finite automaton part of each of the machines is identical; each machine has one reset counter (almost blank loop tape) and one non-reset counter (almost blank straight tape); the first counter counts up to a, say, and the second counts running up to a, b, respectively. The system "accepts" those pairs (a, b) such that the (a, b)-machine eventually halts when started in standard position. Thus the system defines a binary relation on natural numbers. Some solvability and unsolvability results are obtained concerning the emptiness of the set accepted by a system or the emptiness of the intersection of the sets accepted by two or more systems. Some of the theorems strengthen results of Rabin and Scott [IBM J. Res. Develop. 3 (1959), 114–125; MR 21 #2559]. It is shown for a certain class of systems that the relations defined by them are exactly the same as those definable in the elementary theory of addition of natural numbers. For another class of systems it is shown that an intersection problem is equivalent to Hilbert's tenth problem."

Frey, T. 3377

Über die Konstruktion nichtvollständiger Automaten.

Acta Math. Acad. Sci. Hungar. **15** (1964), 375–381.

Let F^* be a set of words (over a finite alphabet) which contains u if it contains uv. Two incomplete Moore sequential machines A and B, with start states, are called F^* equivalent if, for each word in F^* as input, the outputs from A and B coincide. Using standard techniques, the author constructs for each machine defined over F^* a smallest state F^* equivalent machine.

S. Ginsburg (Van Nuys, Calif.)

Frey, T. 3378

Über die Konstruktion endlicher Automaten.

Acta Math. Acad. Sci. Hungar. **15** (1964), 383–398.

Regular sets are here given by automata. The author considers functions f of regular sets which yield regular sets. In addition, he seeks constructions realizing f which yield reduced (i.e., minimal state) automata if the original automata are reduced. All the functions discussed, which include union, intersection, star, and initial subwords among others, are already known to preserve regular sets (many, in fact, with the same constructions in the paper). The only new results are those relating to when the constructed automaton is reduced.

S. Ginsburg (Van Nuys, Calif.)

Kratko, M. I. 3379

The algorithmic unsolvability of a problem in the theory of finite automata. (Russian)

Diskret. Analiz. *No.* 2 (1964), 37–41.

A finite automaton is said to be initial if it has an inner state from which its operation always begins, and the basis of a logical net is said to be initial if each finite automaton in the basis is initial. A basis is called τ-complete if any bounded determinate operator may be realized by a τ-translation of the basis. This paper is devoted to a proof that the problem of discriminating τ-completeness of initial bases is algorithmically unsolvable. *E. J. Cogan* (Bronxville, N.Y.)

SET THEORY

See also 4022.

La Menza, Francisco 3380

On the foundations of arithmetic. (Spanish)

Math. Notae **19** (1964), 171–177.

A few deductions are carried out, based on Dedekind's definition of finite set; for example, the union of two finite sets is shown to be finite. However, in trying to prove the trichotomy law for finite sets, the author takes as obvious a theorem which is actually equivalent to the trichotomy law for arbitrary sets.

E. Mendelson (Flushing, N.Y.)

Ono, Katuzi 3381

New formulation of the axiom of choice by making use of the comprehension operator.

Nagoya Math. J. **23** (1963), 53–71.

Es handelt sich um Formalisierungen der Mengenlehre im Rahmen der gewöhnlichen Prädikatenlogik mit der binären Relation $\in$ als einziger Grundrelation. Die Gleichheit $=$ und die Identität $\Box$ sind folgendermaßen definiert: $x = y \equiv \forall z\,(z \in x \leftrightarrow z \in y)$, $x \Box y \equiv \forall t\,(x \in t \leftrightarrow y \in t)$. Jede Menge r bestimmt in natürlicher Weise zwei binäre Relationen r (bezüglich der Gleichheit) und $\underset{\Box}{r}$ (bezüglich der Identität). Durch den Komprehensionsoperator wird aus einer binärer Relation Γ eine binäre Relation (Γ) gebildet: $x(\Gamma)y \equiv \forall s\,(s \in x \leftrightarrow s\Gamma y)$. Für das Produkt $\Gamma\Delta$ aus zwei binären Relationen Γ und Δ gilt: $x\Gamma\Delta y \equiv \exists z\,(x\Gamma z \wedge z\Delta y)$. In dieser Symbolik läßt sich eine strenge Fassung des Auswahlaxioms folgendermaßen formulieren: (A) $x \in y \to y \ni (\eta)y$. Dabei ist η eine spezielle binäre Relation. Zwei verschiedene schwache Fassungen des Auswahlaxioms lauten: (B) $\exists r \forall xy\,(x \in y \in m \to y \ni (\underset{\Box}{r})y)$, (C) $\exists r \forall xy\,(x \in y \in m \to y \ni (r)y)$. Aus dem Axiom (C) und der Voraussetzung, daß es zu jeder Menge s eine Einheitsmenge $\{s\}$ gibt, folgt das Extensionalitätsaxiom. (Hiermit fällt die Gleichheit mit der Identität zusammen.) Der Verfasser untersucht die Beziehungen zwischen den verschiedenen Auswahlaxiomen (A)–(C) und gewissen Verallgemeinerungen dieser Axiome im Rahmen seiner Objekttheorien OF [dasselbe J. 22 (1963), 119–167; MR

27 #5669] und OZ [ibid. 29 (1962), 105–168; MR 26 #36], die nur je ein Axiom haben, das jeweils unter Bezugnahme auf eine Satellitenrelation formuliert ist. Fügt man zu dem Axiom von OF das Auswahlaxiom (C) hinzu (wobei die Satellitenrelation von OF durch die einfachere Satellitenrelation von OZ ersetzt werden kann), so ergibt sich eine Theorie, die nach Angabe des Verfassers mit der Mengentheorie von Fraenkel (unter Anschluß des Fundierungsaxioms) äquivalent ist. *K. Schütte* (Kiel)

Kurepa, G. [Kurepa, Đuro] 3382

On rank-decreasing functions.

Essays on the foundations of mathematics, pp. 248–253. *Magnes Press, Hebrew Univ., Jerusalem*, 1961.

The author considers any partially ordered set B in which every chain is well-ordered and extends the theory of regressive functions for such sets [cf. W. Neumer, Math. Z. 54 (1951), 254–261; MR 13, 331; the reviewer, Acta Sci. Math. (Szeged) 17 (1956), 139–142; MR 18, 551; G. Kurepa, Z. Math. Logik Grundlagen Math. 4 (1958), 148–156; MR 20 #4499; R. Ricabarra, Rev. Mat. Cuyana 2 (1956), 1–27; MR 22 #1518]. Any B determines a well-defined sequence of mutually disjoint sets R_0B, R_1B, $\cdots$, $R_\alpha B$, $\cdots$, where R_0X denotes the set of the initial elements of X and where $R_\alpha B=R_0(B-\bigcup_{\xi<\alpha}R_\xi B)$. For each $x\in B$ one has a well-determined ordinal number γx such that $R_{\gamma x}$ contains x. Let $\gamma M=(\gamma x : x\in M)$ for each $M\subseteq B$. The set γB is an initial section of ordinal numbers. The ordinal type of γB is supposed to be an initial number ω_α such that $\mathrm{cf}(\alpha)>0$. A set $M\subseteq B$ is said to be a stationary set in B, provided the set γM has the corresponding property in γB, i.e., γM intersects every closed set $\subseteq B$ cofinal to γB. Each function f whose domain D and antidomain fD are parts of B and which satisfies $\gamma fx=\gamma x$ for $x\in D\cap R_0B$ and $\gamma fx<\gamma x$ for $x\in D-R_0B$ is said to be a γ-regressive (or rank-decreasing) function. For a rank-increasing function one has $\gamma x<\gamma fx$ $(x\in D)$. The main results are as follows. If ω_α is a regular initial number and if for every $\beta\in\gamma B$ the cardinal number of $R_\beta B$ is $<\aleph_\alpha$, then for every many-valued rank-increasing function f from B into B such that, for each $x\in B$, the set (fx) of values fx at x is a non-stationary set, the set fB is a non-stationary set (Theorem 3.5). Let, for every $\beta\in\gamma B$, the cardinal number of $R_\beta B<\aleph_{\mathrm{cf}(\alpha)}$. If M is any stationary set in B and if f is any γ-regressive function from M into B, then there exists a stationary set $M_0\subseteq M$ such that $\sup\gamma fM_0<\inf\gamma M_0$ (Theorem 4.2).

G. Fodor (Szeged)

Keisler, H. Jerome 3383

Good ideals in fields of sets.

Ann. of Math. (2) 79 (1964), 338–359.

This paper deals with certain Boolean-algebraic properties of ideals in fields of sets, somewhat related to infinite additivity. The results have important applications in model theory, but are also of independent interest.

Let D be a set of sets directed by inclusion (i.e., $(\forall x, y\in D)(\exists z\in D)(z\supseteq x\cup y)$), and let f and g be functions on D with values in the set $S(X)$ of all subsets of X. We write $g\geq f$ if for all $x\in D$, $g(x)\supseteq f(x)$. We say that an ideal I in the field $S(X)$ is D-good if for every monotonic function f on D into I there exists an additive function $g\geq f$ on D into I.

For each ideal I let $G(I)$ be the least cardinal $|D|$ of a directed set D such that I is not D-good, if there is such a D. By Theorem 3.2 (credited to C. C. Chang), if $|Y|=G(I)$, then I is not $S_\omega(Y)$-good, where $S_\omega(Y)$ is the set of all non-void finite subsets of Y. The author's main results are the following. Let $|X|=\alpha$ and let α^+ be the least cardinal greater than α. (A) Assuming $2^\alpha=\alpha^+$, there are 2^{2^α} prime ideals I in $S(X)$ such that $G(I)=\alpha^+$; (B) There are 2^{2^α} prime ideals I in $S(X)$ such that $G(I)=\omega^+$. The paper is concluded with a list of interesting problems, mostly concerned with the function G. *A. Lévy* (Jerusalem)

Keisler, H. Jerome 3384

On cardinalities of ultraproducts.

Bull. Amer. Math. Soc. 70 (1964), 644–647.

This paper answers some unsolved problems of five or six years standing concerning the cardinalities of ultraproducts. The more interesting consequences of these results concern the cardinalities of ultrapowers, and they are stated by the author in Theorem A as follows (D is a nonprincipal ultrafilter over a set I of infinite power λ). (i) If α is infinite and D is not countably complete, then $\alpha^I/D=(\alpha^I/D)^\omega$. (ii) For any α, γ, and D, $(\alpha^\gamma)^I/D\geq(\alpha^I/D)^\gamma$. (iii) If D is uniform, then $(\alpha^{(\lambda)})^I/D=(\alpha^I/D)^\lambda=\alpha^\lambda$, where $\alpha^{(\lambda)}=\sum_{\beta<\lambda}\alpha^\beta$. Each part of Theorem A has a corresponding generalization (given in Theorem 2.2) to ultraproducts $\prod_{i\in I}\alpha_i/D$. The author observes that the proofs of these theorems can be made uniform by introducing the notion of (β,γ)-regular ultrafilters and proving a result (Theorem 2.1) about them. Essentially, Theorems A and 2.2 follow from Theorem 2.1 by noticing that (Lemma 1.3) every countably incomplete ultrafilter is (ω,ω)-regular and every uniform ultrafilter is (λ,λ)-regular. Another consequence of Theorem 2.1 concerns the cardinal $\alpha=\prod_{i\in I}n_i^{q(n_i)}/D$, where $\lim_{m\to\infty}q(m)=\infty$. It turns out (Theorem 2.3) that if $\beta=\prod_{i\in I}n_i/D$ is infinite, then $\alpha\geq\beta^\omega$.

C.-C. Chang (Los Angeles, Calif.)

Keisler, H. J.; Tarski, A. 3385

From accessible to inaccessible cardinals. Results holding for all accessible cardinal numbers and the problem of their extension to inaccessible ones.

Fund. Math. 53 (1963/64), 225–308.

The authors investigate the classes C_0, C_1, C_2 of all infinite cardinals satisfying the following conditions (a), (b), and (c), respectively: (a) There exists an α-complete field of sets with at most α generators in which some α-complete proper ideal cannot be extended to an α-complete prime ideal; (b) In the field of all subsets of a set of power α every α-complete prime ideal is principal; (c) There exists an α-complete field of sets in which some α-complete proper ideal cannot be extended to an α-complete prime ideal.

It has been known for some time that $\omega\notin C_0\subseteq C_1\subseteq C_2$, and that all accessible cardinals belong to C_0 [P. Erdős and A. Tarski, *Essays on the foundations of mathematics*, pp. 50–82, Magnes Press, Hebrew Univ., Jerusalem, 1961; MR 29 #4695]. It has turned out, contrary to expectations, that a very large class of inaccessible cardinals is a part of C_0. This was first proved by Tarski [Logic, Methodology and Philosophy of Science (Proc. 1960 Internat. Congr.), pp. 125–135, Stanford Univ. Press, Stanford, Calif., 1962; MR 27 #1382], using a metamathematical result of W. P. Hanf [Fund. Math. 53 (1963/64), 309–324; MR 26 #3943].

H. J. Keisler gave a different model-theoretical proof of a similar result concerning the class C_1 [Logic, Methodology and Philosophy of Science (Proc. 1960 Internat. Congr.), pp. 80–86, Stanford Univ. Press, Stanford, Calif., 1962].

The authors give a purely mathematical discussion of the problems involved in (a), (b), and (c) and prove that a very large class of infinite cardinals α possesses the properties (a), (b), and (c).

"Actually we do not know at present any non-denumerable cardinal which is capable of a 'constructive characterization' (in some very general and rather loose sense of the term) and for which we could not prove that it possesses the properties discussed. Nevertheless, the straightforward question whether all cardinals larger than ω possess these properties is still open, and it does not seem very likely that the methods now available will bring forth an answer to this question" (from the introduction of the authors).

The authors define a family of subclasses of C_1 called normal classes and define some "constructively characterised" operations on normal classes (or on sequences of normal classes) such that these operations lead from a class X to a new class Y which includes X. The results concerning the extent of class C_1 consist in establishing induction principles which say that the above operations yield normal classes when applied to normal classes and are based on the theorem that the class of all accessible cardinals is normal. The proofs of these results are given in full in § 1. In § 3, Theorem 3.4, it is proved that C_0 itself is normal and, as a corollary of this, C_0 is a proper part of C_1 unless C_1 contains all non-denumerable cardinals. On the other hand, as an improvement of the results of § 1 it is proved that various "constructively defined" classes of cardinals are included in C_0. Some of the proofs given in § 3 are of sketchy character. The authors mention that they are going to publish a paper considering metamathematical problems exhibiting the same pattern. There they will give much simpler metamathematical proofs for results which imply all the results of § 3.

The authors discuss several mathematical problems exhibiting the same pattern as the problems involved in (a), (b), and (c). Most of these problems can be shown to be equivalent to one of the problems discussed in detail, and so the authors get important corollaries of their results in various branches of mathematics: general set theory, theory of Boolean algebras, theory of measure, group theory, topology and functional analysis. These results can be considered as characterizations of the classes C_0, C_1, C_2 and are proved in §§ 4, 2 and 5, respectively.

The following corollaries of the results give a general survey of the problems and results of the respective sections.

Theorem 4.35: The hypothesis $C = C_0$ implies (and is implied by) each of several statements, e.g.,

(i) For every $\alpha > \omega$ there is an α-complete field of subsets of α which includes $S_\omega(\alpha)$, is α-generated by a set of power α, and in which every α-complete prime ideal (more generally, every α-complete and δ-saturated ideal with $2^\delta < \alpha$) is principal.

Theorem 2.50: The hypothesis $C = C_1$ implies (and is implied by) each of several statements, e.g.,

(i) In every β-complete Boolean algebra every countably complete prime ideal is β-complete.

Theorem 5.12: The hypothesis $C = C_2$ implies (and is implied by) each of several statements, e.g.,

(i) For every α there is a β such that, for every $\gamma \geq \beta$, some α-complete proper ideal in $S(\gamma)$ cannot be extended to any α-complete prime ideal (more generally, to any α-complete and δ-saturated proper ideal with $2^\delta < \alpha$).

These theorems have further corollaries under the assumption that the generalised continuum hypothesis holds. Theorem 2.51 (iii) states that if the generalised continuum hypothesis holds, or at least $[\omega_1, 2^\omega]$ does not contain regular limit cardinals, the hypothesis that every non-denumerable α belongs to C_1 is equivalent to the statement that there is no (countably additive non-trivial) measure on any set.

It is to be mentioned that in some places the authors point out interesting and important unsolved problems; see, e.g., pages 230, 267 and 305. The paper contains complete references on the known results on the topic and restates the most important of them.

A. Hajnal (Berkeley, Calif.)

Monk, D.; Scott, D. 3386

Additions to some results of Erdős and Tarski.

Fund. Math. 53 (1963/64), 335–343.

The authors prove some results on the topic treated by H. J. Keisler and A. Tarski [#3385].

The relations $T(\alpha, \beta)$, $\bar{R}(\alpha, \beta)$ are defined for infinite α, β. $T(\alpha, \beta)$ holds if and only if the α-product space of a β-termed sequence of discrete two-point topological spaces is α-compact. $\bar{R}(\alpha, \beta)$ holds if and only if in every α-complete field α-generated by at most β elements every α-complete proper ideal can be extended to an α-complete prime ideal. Theorem 1.8 states that $\bar{R}(\alpha, \beta)$ and $T(\alpha, \beta)$ are equivalent unless α is singular and $\alpha = |2^\alpha|$. $C(\alpha)$ is said to hold if the α-product space of an arbitrary sequence of α-compact topological spaces is α-compact. Theorem 1.9 states that $C(\alpha)$ is equivalent to the statement that $T(\alpha, \beta)$ holds for every β. A ramification system is said to be an α-ramification system if it is a system of order α and for every $\xi < \alpha$ the set of all elements of order ξ has power less than α. $Q(\alpha)$ is said to hold if every α-ramification system has a simply ordered subset of type α. Theorem 2.1 states that for every strongly inaccessible α, $Q(\alpha)$ and $T(\alpha, \alpha)$ are equivalent.

In Theorems 3.1 and 3.2 equivalents of $T(\alpha, \beta)$ are given in terms of general set theory. They are called first and second separation principles, respectively.

A. Hajnal (Berkeley, Calif.)

Klaua, Dieter 3387

Über einen nichtarchimedischen Elementebereich.

Z. Math. Logik Grundlagen Math. 10 (1964), 221–260.

The author considers the system Z consisting of all eventually zero $(\omega^* + \omega)$-sequences of real numbers. Under natural definitions, Z is a non-archimedean ordered field. Questions studied are powers, roots, convergence, covering theorems, etc.

L. Gillman (Rochester, N.Y.)

Doss, Raouf 3388

On Gödel's proof that $V = L$ implies the generalized continuum hypothesis.

Notre Dame J. Formal Logic 4 (1963), 283–287.

The proof mentioned in the title is modified as follows: For a set m of ordinals closed with respect to the operations J_i $(0 \le i < 9)$, K_1, K_2, O known from Gödel's monograph [*The consistency of the continuum hypothesis*, Princeton Univ. Press, Princeton, N.J., 1940; MR 2, 66] and with respect to still one more auxiliary operation O_1, the author defines by transfinite induction a function H on ordinals such that $H(J_i(\langle\alpha, \beta\rangle)) = \mathfrak{F}_i(H(\alpha), H(\beta))$ for $0 < i < 9$ and $H(J_0(\langle\alpha, \beta\rangle)) = H``m \cap J_0(\langle\alpha, \beta\rangle)$. He shows that the relations $F(\alpha) \in F(\beta)$ and $H(\alpha) \in H(\beta)$ are equivalent for $\alpha, \beta \in m$, and similarly for the relations $F(\alpha) = F(\beta)$ and $H(\alpha) = H(\beta)$. If G is an isomorphism from m to an ordinal, then $H(\alpha) = F(G(\alpha))$ for $\alpha \in m$. Proofs of these theorems are brief and quite simple. The author shows that they can replace the rather cumbersome proof given on pp. 54–60 of Gödel's monograph. *A. Mostowski* (Warsaw)

COMBINATORIAL ANALYSIS

See also 3419, 4046, 4162.

Bizley, M. T. L. 3389

A problem in permutations.

Amer. Math. Monthly **70** (1963), 722–730.

The problem in question is the enumeration of the arrangements of m letters and a_i letters A_i, $i = 1, 2, \cdots, n$, such that apart from the M's, no two like letters are together. The problem is solved in two ways and generalized to the case where there are a_i letters A_i, $i = 1, 2, \cdots, n$, and at most b_i of the letters A_i are together. The (multivariable) generating function for the numbers of the latter case is shown to be $[1 - \sum_{i=1}^{n} x_i(1 - x_i^{b_i}) \times (1 - x_i^{b_i+1})^{-1}]^{-1}$. Recurrence relations, multiple sum expressions, and the relation of the symmetrical case (when all of the various kinds of letters are subject to the same restrictions) to the two-pack matching problem are also given. *J. Riordan* (Murray Hill, N.J.)

Dembowski, P. 3390

Classes of geometric designs.

Rend. Mat. e Appl. (5) **23** (1964), 14–21.

Author's introduction: "The purpose of this purely expository paper is to outline some geometric aspects of the theory of (balanced incomplete block) designs. Our discussion falls into two parts which are only loosely connected. In Section 1 we give the basic definitions and discuss the question how the finite projective and affine spaces can be characterised among the more general projective, resolvable, and affine designs. Section 2 deals with finite inversive planes; these form a special class of 3-designs. We shall be mainly interested in the characterisation of the miquelian finite inversive planes, and some results in this direction will be given."

Moon, John W. 3391

A note on "Pattern variants on a square field".

Psychometrika **28** (1963), 93–95.

A k-pattern on an m-by-m square of m^2 like cells is a selection of k cells. Two patterns are equivalent if one can be obtained from the other by a rotation or reflection of the square. The number g_{mk} of inequivalent k-patterns has been found by S. J. Prokhovnik [Psychometrika 24 (1959), 329–341]; here it is formulated in terms of George Pólya's fundamental theorem of enumerative combinatorial analysis. The cycle index is

$$(f_1^{m^2} + 2f_1^m f_2^{m(m-1)/2} + 3f_2^{m^2/2} + 2f_4^{m^2/4})/8 \quad \text{for } m \text{ even},$$

$$(f_1^{m^2} + 4f_1^m f_2^{m(m-1)/2} + f_1 f_2^{(m^2-1)/2} + 2f_1 f_4^{(m^2-1)/4})/8 \quad \text{for } m \text{ odd},$$

and the store enumerator is $1 + x$. It is also noted that the theorem is adapted to other specifications of patterns and equivalence relations; in particular, the patterns on a rectangle whose cells may be marked in two ways, and whose equivalences are specified by the group of reflections of the rectangle, are enumerated. No attention is given to the development of the theorem for numerical results, nor to more complicated equivalences such as that in F. Harary [Pacific J. Math. 8 (1958), 743–755; MR 21 #2598]. *J. Riordan* (Murray Hill, N.J.)

Raney, George N. 3392

A formal solution of $\sum_{i=1}^{\infty} A_i e^{B_i X} = X$.

Canad. J. Math. **16** (1964), 755–762.

Recursive formulas are obtained for the enumeration problem involving certain collections of labeled rooted trees having coloured nodes and edges. A direct solution in terms of multinomial coefficients and power products is also given using a modification of a method of Prüfer. These results are reformulated to obtain a formal solution of the equation

$$X = \sum_{i=1}^{\infty} A_i \exp(B_i X)$$

by expressing X as a multiple power series in A_i and B_i. *M. S. Cheema* (Tucson, Ariz.)

Johnsen, E. C. 3393

The inverse multiplier for abelian group difference sets.

Canad. J. Math. **16** (1964), 787–796.

The author proves (I) If -1 is a multiplier of a difference set with parameters v, k, λ of an Abelian group G, then v and λ are even and $v = (m/\alpha)[(m+\alpha)^2 - 1]$, $k = m(m+\alpha)$, $\lambda = m\alpha$, where m and α are integers and $m \equiv \alpha + 1$ (2). (II) If 2^{s_1} is the order of the subgroup of element of order 2 of G and if $2^{s_1} < (k/\lambda) + 1$, then -1 is not a multiplier. *H. B. Mann* (Madison, Wis.)

Erdős, P.; Hanani, H. 3394

On a limit theorem in combinatorial analysis.

Publ. Math. Debrecen **10** (1963), 10–13.

Given a set E of positive integers and positive integers k, such that $l \le k \le n$, let $M(k, l, n)$ denote a minimal system of k-tuples such that every l-tuple is contained in at least one k-tuple of the system. Let $m(k, l, n)$ denote a maximal system of the k-tuples such that every l-tuple is contained in at most one set of the system. The number of k-tuples

in these systems is denoted by $\bar{M}(k, l, n)$ and $\bar{m}(k, l, n)$, respectively. Put

$$\mu(k, l, n) = \binom{k}{l}\binom{n}{l}^{-1}\bar{M}(k, l, n),$$

$$\nu(k, l, n) = \binom{k}{l}\binom{n}{l}^{-1}\bar{m}(k, l, n),$$

so that $\nu(k, l, n) \leq 1 \leq \mu(k, l, n)$ with equality only if

$$\binom{k-h}{l-h}\bigg|\binom{n-h}{l-h} \qquad (h = 0, 1, \cdots, l-1).$$

Hanani [see #4161 below] has proved that the equalities hold for $l=2$, $k=3, 4, 5$ and for $l=3$, $k=4$.

Erdős and Rényi [Publ. Math. Debrecen 4 (1956), 398–405; MR 18, 3] have proved that for every k

$$\lim_{n=\infty} \mu(k, 2, n) = \gamma_k \quad \text{and} \quad \lim_{k=\infty} \gamma_k = 1.$$

In the present paper it is proved that

$$\lim_{n=\infty} \mu(k, 2, n) = \lim_{n=\infty} \nu(k, 2, n) = 1 \qquad (k = 2, 3, \cdots),$$

$$\lim_{n=\infty} \mu(p+1, 3, n) = \lim_{n=\infty} \nu(p+1, 3, n) = 1,$$

where p is a power of a prime.

L. Carlitz (Durham, N.C.)

Foata, Dominique 3395

Un théorème combinatoire non commutatif.

C. R. Acad. Sci. Paris **258** (1964), 5128–5130.

An identity is established which generalizes the master theorem of MacMahon in the theory of permutations. The first part of the paper is devoted to the introduction of notation and to statements of previous results.

P. S. Dwyer (Ann Arbor, Mich.)

ORDER, LATTICES

See also 3382, 3411, 3525, 3535, 3556, 3567, 4023.

Szász, Gábor 3396

★Introduction to lattice theory.

Third revised and enlarged edition. MS revised by R. Wiegandt; translated by B. Balkay and G. Tóth.

Academic Press, New York-London; Akadémiai Kiadó, Budapest, 1963. 229 *pp*. $8.50.

This English version of Szász's text differs somewhat from the German edition [*Einführung in die Verbandstheorie*, Akadémiai Kiadó, Budapest, 1962; MR **25** #2011]. It has a new tenth chapter on direct and subdirect decompositions not included in the German edition. The bibliography, which is good but not exhaustive, has now 232 entries instead of 198. The added references are mostly to recent papers, some of which are referred to in the revised text.

Some of the new sections included in this edition are on length, the maximum and minimum conditions, irreducible and prime elements of a lattice, complete sublattices of a complete lattice, compact elements, and compactly generated lattices. Some of the problems at the ends of the chapters are new. The titles of the chapters are: (1) Partly ordered sets, (2) Lattices in general, (3) Complete lattices, (4) Distributive and modular lattices, (5) Special subclasses of the class of modular lattices, (6) Boolean algebras, (7) Semimodular lattices, (8) Ideals of lattices, (9) Congruence relations, (10) Direct and subdirect decompositions.

This is a clearly written introductory text from which one can learn painlessly most of the fundamental notions and methods of lattice theory. It is not intended for the specialist or research worker. For the most difficult topics the author refers the reader to the literature. It includes little or nothing about ordered linear spaces or lattice ordered rings, groups, and semigroups.

O. Frink (University Park, Pa.)

Szász, G. [Szász, Gábor] 3397

On independent systems of axioms for lattices.

Publ. Math. Debrecen **10** (1963), 108–115.

Let $L=(L; \cap, \cup)$ be an algebra with two binary operations denoted by $\cap$ and $\cup$, and consider the equations: (1) $(x \cap y) \cap z = x \cap (y \cap z)$; (2) $(x \cup y) \cup z = x \cup (y \cup z)$; (3) $x \cap y = y \cap x$; (4) $x \cup y = y \cup x$; (5) $x \cap (x \cup y) = x$; (6) $x \cup (x \cap y) = x$; (7) $x \cap x = x$; (8) $x \cup x = x$; (9) $x \cap (y \cap z) = (x \cap y) \cap (x \cap z)$; (10) $x \cup (y \cup z) = (x \cup y) \cup (x \cup z)$; (11) $x \cap (y \cap z) = (y \cap x) \cap (z \cap x)$; (12) $x \cup (y \cup z) = (y \cup x) \cup (z \cup x)$ and the relation (13) $x \cap y = y$ if and only if $y \cup x = x$. Each of the systems I [(1)–(6)], II [(1)–(4), (7), (13)], III [(3)–(6), (9), (10)] is known to be an independent system of axioms for lattices, and the author points out that IV [(5), (6), (11), (12)] is another such system.

The purpose of this report is to determine independence examples for these systems consisting of the least possible number of elements. For instance, the independence of (3)–(6) in I can be shown with algebras consisting of two elements, whereas the independence of (1) and (2) in I can be shown with algebras consisting of five elements. The author shows that these examples are best possible in the following sense: If $(L; \cap, \cup)$ is an algebra consisting of at most four elements in which the axioms (3)–(6) hold, then either both or more of (1) and (2) hold in L. Similar results are obtained for the systems II and III, and the author exhibits independence examples for the system IV, each containing but two elements.

J. Kist (University Park, Pa.)

Birkhoff, Garrett 3398

A new interval topology for dually directed sets.

Univ. Nac. Tucumán Rev. Ser. A **14** (1962), 325–331.

A dually directed set is a partially ordered set in which every pair of elements has an upper bound and a lower bound. Every lattice is dually directed. For such sets the author defines a new interval topology in terms of the order as follows. Let the family C of "closed bounded sets" consist of arbitrary intersections of finite unions of closed intervals $[a, b] = \{x \in D: a \leq x \leq b\}$. Then the closed sets of the dually directed set D in the new interval topology are those sets all of whose intersections with members of the family C are members of C. Any family formed in this way from a family of sets C is called the residual C^* of C. For example, in a k-space the family of all closed sets is the residual of the family of compact sets.

The author shows that for certain important dually directed sets the new interval topology is more useful

and better behaved than the "old" interval topology defined by the reviewer, in which the semi-infinite closed intervals form a sub-basis for the closed sets. For example, the new interval topology of the Euclidean space E^n considered as a direct product of n chains coincides with its usual topology, while the old interval topology does not. He shows also that a closed set in the new topology is bounded if and only if it is a member of the family C, and that a subset of a conditionally complete lattice which is closed in the new topology is compact if and only if it is bounded.

The author states incorrectly (Theorem 1) that the old and new interval topologies coincide if and only if D has universal bounds O and I. Actually the two topologies coincide for all chains, whether they have universal bounds or not. The question of when the two topologies coincide remains open, as does the question of when the new interval topology of a direct product of ordered sets coincides with the cartesian product of the new topologies of the factors. *O. Frink* (University Park, Pa.)

Nash-Williams, C. St J. A. 3399

On well-quasi-ordering lower sets of finite trees.

Proc. Cambridge Philos. Soc. **60** (1964), 369–384.

A lower set P of a quasi-ordered set Q is a subset such that $x \in P$ and $y \leq x$ imply $y \in P$, and LQ is the set of lower sets of Q ordered by inclusion. It is here shown that if T is the set of finite trees with the usual quasi-order, L^nT is well-quasi-ordered for all n. *Graham Higman* (Oxford)

Kosmák, Ladislav 3400

Sur la notion de borne supérieure. (Russian summary)

Spisy Přírod. Fak. Univ. Brno **1963**, 227–228.

The author constructs a lattice which contains a set M which has a least upper bound but which is such that, for every $a \in M$, the set $M-(a)$ does not have a least upper bound. *R. J. Crittenden* (Providence, R.I.)

Papert, Dona 3401

Congruence relations in semi-lattices.

J. London Math. Soc. **39** (1964), 723–729.

Let L be any semi-lattice and let C be the lattice of congruence relations on L. The author shows that C has the following two properties: (i) every element ($\neq 1$) of C is an intersection of dual atoms; (ii) every $s \in C$ has a pseudo-complement $s^* \in C$. L can be embedded in C in the following way. For every $x \in L$ we define $\bar{x} \in C$ by defining $u \equiv v\ (\bar{x})$ to mean $u \vee x = v \vee x$. It is proved that this element $\bar{x}$ is of the form s^*. The author proves that every completely distributive Boolean algebra can be represented as the lattice of congruence relations of a semi-lattice. *E. T. Schmidt* (Budapest)

Avann, S. P. 3402

Increases in the join-excess function in a lattice.

Math. Ann. **154** (1964), 420–426.

In the present paper the author continues his investigations of the join-excess function $\nu[a]$ of a finite lattice L [same Ann. **142** (1960/61), 345–354; MR **23** #A81]; he is mainly concerned with the various ways in which $\nu[a]$ increases in an ascending chain. Several sufficient conditions are given in order that $\nu[a, b] \geq 0$ for $a, b \in L$. Another result is the following. If, in a Jordan-Dedekind lattice L, $\nu[c] = \nu[d]$ for $c \geq d$, then the quotient lattice c/d is lower semi-distributive. *Ph. Dwinger* (Delft)

Jakubík, J.; Kolibiar, M. 3403

Über euklidische Verbände.

Math. Ann. **155** (1964), 334–342.

The authors generalize the notion of euclidean lattice introduced by F. Klein-Barmen [same Ann. **140** (1960), 263–277; MR **22** #6739; J. Reine Angew. Math. **195** (1955), 121–126; MR **18**, 6] to that of α-euclidean lattice, and prove some theorems about the generalization. Let L be a lattice with 0-element, and let α be a cardinal number. An element x of L is said to be primary if the closed interval $[0, x]$ is a chain. L is said to be α-euclidean if it has the following three properties: (1) L contains the join of any set of pairwise disjoint primary elements of cardinal number less than α; (2) every element of L is the join of such a set of pairwise disjoint primary elements; and (3) if z is in L and $z \leq \bigcup x_i$, where the set $\{x_i\}$ has cardinal number less than α, and $z \cap x_i = 0$ for $i \neq k$, then $z \leq x_k$. If α is $\aleph_0$, then α-euclidean reduces to euclidean in the sense of Klein-Barmen.

By the α-product of a family of chains with 0-element is meant the set of all elements x of the direct product of the chains such that the cardinal number of those components of x which are different from 0 is less than α. The authors show that a lattice is α-euclidean if and only if it is an α-product of chains with 0-element. They show also that a lattice S is isomorphic to a dual ideal of an α-product of chains if and only if every principal dual ideal of S is an α-euclidean lattice. Finally, they show that S is isomorphic to an α-product of chains if and only if every principal dual ideal of S and the dual of every principal ideal of S is α-euclidean. *O. Frink* (University Park, Pa.)

Skornjakov, L. A. [Скорняков, Л. А.] 3404

★Complemented modular lattices and regular rings [Дедекиндовы структуры с дополнениями и регулярные кольца].

Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1961. 198 *pp.* 0.75 *r.*

It has been almost thirty years since von Neumann's early work on complemented modular lattices, continuous geometries, and regular rings. During that period, numerous contributions have been made to these subjects. von Neumann's results have been polished and generalized, and several important new facts about complemented modular lattices have been discovered. The author has done an excellent job of assembling the main theorems of von Neumann and his followers into a short but comprehensive book. The first four sections deal with the representation theorems for complemented modular lattices. The culmination is von Neumann's theorem that every complemented modular lattice with a homogeneous basis of rank four or greater is isomorphic to the lattice of all principal left ideals of a regular ring. The proof of this result is based on ideas introduced by Amemiya [J. Math. Soc. Japan **9** (1957), 263–279; MR **19**, 1154]. Section 5 deals with complete, complemented modular lattices. The results obtained are specialized in Section 6 to continuous

geometries. In particular, the theorem that every continuous geometry admits a unique dimension function is proved. These sections are based on Chapter IV of Maeda's book [*Kontinuierliche Geometrien*, Springer, Berlin, 1958; MR 19, 833]. Section 7 is devoted to uniqueness questions, the main results being those of von Neumann and the author. Section 8 treats *-regular rings, and the application which Kaplansky made of them to prove that every complete, orthocomplemented modular lattice is continuous. It is essentially Kaplansky's proof [Ann. of Math. (2) **61** (1955), 524–541; MR 19, 524] which is given for this theorem, rather than the more direct lattice-theoretic argument of Amemiya and Halperin [Canad. J. Math. **11** (1959), 481–520; MR **22** #1530]. The final section contains a list of 37 interesting problems, along with a discussion of work related to these problems (up to about 1960). It is the reviewer's opinion that the author's book fills very well a definite gap in the mathematical literature. It offers students direct access to some of von Neumann's most beautiful discoveries. The original notes of von Neumann, remarkable as they are, show signs of having been written in the heat of discovery. The only other book on continuous geometries, Maeda's work cited above, is more a monograph than a textbook. The author has taken advantage of numerous advances which have been made in the field of continuous geometries since von Neumann's and Maeda's books were written. The result is a well-organized, lucid book which can serve either as a textbook or a research tool. *R. S. Pierce* (Seattle, Wash.)

Glagolev, V. V. 3405

An estimate of the complexity of the contracted normal form for almost all functions of the logic of algebra. (Russian)

Dokl. Akad. Nauk SSSR **158** (1964), 770–773.

The author is here concerned with complexities introduced by algorithms for computing minimal disjunctive normal forms (dnf's) of functions of the algebra of logic. Such algorithms begin by searching out reduced dnf's, and choosing those that are minimal from the irreducible dnf's among them. The upper estimate obtained of irreducible dnf's of almost all functions of n variables of algebraic logic is $2^{2^n c_3 \log n \cdot \log \log n}$. A property $Q(f)$ is said to hold for almost all functions of the algebra of logic if $\psi_Q(n)/2^{2^n}$ approaches 1 as n becomes infinite, where $\psi_Q(n)$ is the number of functions of n variables having the property $Q(f)$.

E. J. Cogan (Bronxville, N.Y.)

Glagolev, V. V. 3406

A function of a Boolean algebra the number of whose units is equal to the number of monotone functions of n variables. (Russian)

Diskret. Analiz. No. 2 (1964), 10–11.

A Boolean function of 2^n arguments is constructed and shown to be the characteristic function of the set M_n of monotonic Boolean functions of n arguments.

E. J. Cogan (Bronxville, N.Y.)

Dwinger, Ph. 3407

The dual space of the inverse limit of an inverse limit system of Boolean algebras.

Nederl. Akad. Wetensch. Proc. Ser. A **67** = *Indag. Math.* **26** (1964), 164–172.

Consider an inverse limit system of Boolean algebras $\{B_\alpha, f_{\alpha\beta}\}$ with inverse limit B_∞ [see the reviewer's paper, Proc. Amer. Math. Soc. **2** (1951), 566–576; MR **12**, 524]. Let C be an arbitrary Boolean algebra with homomorphic maps g_α into the B_α. Suppose, further, that $f_{\alpha\beta}g_\beta = g_\alpha$ for each pair $\alpha \leq \beta$. Then (Theorem 1) there exists a unique $g \in \mathrm{Hom}(C, B_\infty)$ such that $\pi_\alpha g = g_\alpha$, where π_α is a certain projection of B_∞ into B_α, and dually (Theorem 2). The direct limit X^∞ of the dual spaces can be made (Theorem 5) into a 0-dimensional Hausdorff-space, and the dual space of B_∞ is the largest 0-dimensional Hausdorff compactification of the space obtained from X^∞ by identifying each two points which cannot be separated from each other by open sets. Theorem 6: If the projections π_α of B_∞ into B_α are onto, then the largest 0-dimensional Hausdorff compactification of X^∞ is the dual space X of B_∞, and X^∞ is dense in X (this last correcting an error of the reviewer's) and can be a proper subset of X. {The author has communicated that there are minor misprints on page 165, line 12, and on page 170, line 1, and that, on page 171, the definitions for x_n and x_n^* have been omitted from the proof of Theorem 7.} *F. Haimo* (Belmont, Mass.)

Daigneault, Aubert 3408

Freedom in polyadic algebras and two theorems of Beth and Craig.

Michigan Math. J. **11** (1964), 129–135.

The author proves two theorems in polyadic algebra which are equivalent to the well-known model-theoretic results of Beth [Nederl. Akad. Wetensch. Proc. Ser. A **56** (1953), 330–339; MR **15**, 385] and of Craig [J. Symbolic Logic **22** (1957), 269–285; MR **21** #3318]. By a polyadic algebra is meant a locally finite polyadic algebra of infinite degree. The two theorems are as follows. (1) Let B^* be a family of polyadic algebras, let M^* be a family of monomorphisms of a polyadic algebra M into members of B^*, and let B be the free product of B^* with amalgamation of M^*. Then the intersection of any two factors in B is precisely the amalgamated part. (2) Let V and W be respectively a set of predicates and a set of constants freely generating an equality polyadic algebra A. Let $V = V_1 \cup V_2$ and $V_0 = V_1 \cap V_2$, and for $i = 0, 1, 2$, let A_i be the subalgebra of A generated by V_i and W. Then, whenever $a_1 \leq a_2$ with $a_1 \in A_1$ and $a_2 \in A_2$, there exists $a_0 \in A_0$ such that $a_1 \leq a_0 \leq a_2$. Theorem 1 also has a version for equality polyadic algebras. The proofs are carried out in polyadic algebra, but are in part similar to the proof of Craig's theorem due to A. Robinson [Nederl. Akad. Wetensch. Proc. Ser. A **59** (1956), 47–58; MR **17**, 1178].

H. J. Keisler (Madison, Wis.)

Benzaken, Claude 3409

Définition et propriétés de certaines familles de fonctions booléennes croissantes.

C. R. Acad. Sci. Paris **259** (1964), 1369–1371.

This note is an announcement of results which form a continuation of work as yet unpublished. A set of finite Boolean functions forms a family provided it is closed under reduction; the act of obtaining a new function $\bar{f}$ from f by identification of p ($p \leq n$) of its n variables, and composition; replacing one or more variables by other functions. MS is the family of increasing Boolean functions which are non-constant and majorize their respective

duals. $M\Sigma$ is the family generated by $U(x, y, z) = x + yz$. The author considers MS_p $(p \geq 3)$, the set of $f \in MS$ such that each reduction by at least p variables belongs to $M\Sigma$. It turns out that MS_p is a family and $MS_3 = MS \supset MS_4 \supset \cdots \supset M\Sigma$. When $p \geq 4$, a generator for MS_p is found; also the maximal families in MS_p $(p \geq 3)$ are found.

B. Brainerd (Toronto, Ont.)

Rosenblatt, David 3410

On the graphs of finite idempotent Boolean relation matrices.

J. Res. Nat. Bur. Standards Sect. B **67B** (1963), 249–256.

This paper presents a graph-theoretic characterization of those binary relations over a finite set which are idempotent under the operation of composition. The set of all binary relations over a finite set of n elements forms a relation algebra under the operations of union, intersection, converse and composition [cf. G. Birkhoff, *Lattice theory*, Amer. Math. Soc., Providence, R.I., 1961; MR **23** #A815]. This algebra may be represented geometrically by the set of all directed graphs with n nodes, and algebraically by the set of all Boolean matrices with n^2 entries. The author proves that a relation is idempotent under composition if and only if its associated graph is a "permanent" graph, in a sense defined at length in the paper. He then derives from this result assorted corollaries for special cases, and discusses applications to ordering and flow problems in large-scale information systems.

{The author calls attention to the fact that line 1, column 1, on page 256 is missing, and should read: "where $\hat{R}$, $\hat{S}$ denote the unions $\bigcup_{n=1}^{\infty} \hat{R}^n$, $\bigcup_{n=1}^{\infty} S^n$. Moreover,".}

R. T. Prosser (Lexington, Mass.)

GENERAL MATHEMATICAL SYSTEMS

See also 3520.

Schmidt, E. T. 3411

Universale Algebren mit gegebenen Automorphismengruppen und Kongruenzverbänden.

Acta Math. Acad. Sci. Hungar. **15** (1964), 37–45.

Es wird der folgende Satz bewiesen: Zu einer beliebig vorgegebenen Gruppe G und zu einem beliebig vorgegebenen kompakt erzeugten Verband V existiert eine universale Algebra $\mathbf{A}$, sodass die Automorphismengruppe von $\mathbf{A}$ mit G und der Kongruenzverband von $\mathbf{A}$ mit V isomorph ist. (Ein vollständiger Verband V heisst kompakt erzeugt, wenn sich jedes Element als Vereinigung kompakter Elemente darstellen lässt; dabei heisst das Element $a \in V$ kompakt, wenn stets zu $a \in \bigcup_{\lambda \in \Lambda} x_\lambda$ eine endliche Teilmenge Λ' von Λ existiert mit $a \in \bigcup_{\lambda \in \Lambda'} x_\lambda$.)

Das angegebene Resultat verallgemeinert einen Satz von G. Birkhoff [Rev. Un. Mat. Argentina **11** (1946), 155–157; MR **7**, 411], nach dem zu einer beliebig vorgegebenen Gruppe G eine universale Algebra existiert, deren Automorphismengruppe zu G isomorph ist. Weiterhin liegt mit dem angegebenen Resultat eine Verallgemeinerung eines Satzes von G. Grätzer und E. T. Schmidt [Acta Sci. Math. (Szeged) **24** (1963), 34–59; MR **27** #1391] vor, nach dem zu einem beliebig vorgegebenen kompakt erzeugten Verband V eine universale Algebra existiert, deren Kongruenzverband zu V isomorph ist. (Nach G. Birkhoff und O. Frink [Trans. Amer. Math. Soc. **64** (1948), 299–316; MR **10**, 279] sind die Kongruenzverbände universaler Algebren kompakt erzeugt.)

Der Verfasser konstruiert weiterhin eine einfache universale Algebra zu vorgegebener Automorphismengruppe.

W. Benz (Frankfurt a.M.)

Nöbauer, Wilfried 3412

Transformation von Teilalgebren und Kongruenzrelationen in allgemeinen Algebren.

J. Reine Angew. Math. **214/215** (1964), 412–418.

Let T be an ideal of a polynomial ring $R = r[x_1, \cdots, x_s]$, where r is a commutative ring with unit; further, let $U(x) = (u_1, \cdots, u_s)$ be an s-tuple of polynomials from R; then T^U, the set of all polynomials $f(x_1, \cdots, x_s)$ with $f(u_1, \cdots, u_s) \in T$, is again an ideal of R. The present paper generalizes this type of transformation by defining a generalized distributive and associative law in a universal algebra.

If θ is an n-ary and ψ an r_ν-ary operation in a universal algebra A, θ is distributive in the kth place if

$$\theta(a_1, \cdots, a_{k-1}, \psi(b_1, \cdots, b_r), a_{k+1}, \cdots, a_n) = \psi(\theta(a_1, \cdots, a_{k-1}, b_1, a_{k+1}, \cdots, a_n), \cdots, \theta(a_1, \cdots, b_r, \cdots, a_n)).$$

Now suppose A has r-ary operations ψ_ν and an n-ary operation θ, distributive at the kth place for every ψ_ν. Designate by $(n-1)A$ the set of $(n-1)$-tuples from A. Let T be a subalgebra of $A[\psi_\nu]$ and ρ a congruence relation in it, U a subset of $(n-1)A$. Define T^U by $a \in T^U$ if $\theta(u_1, \cdots, u_{k-1}, a, u_{k+1}, \cdots, u_n) \in T$ for every $(n-1)$-tuple $(u_1, \cdots, u_{k-1}, u_{k+1}, \cdots, u_n)$ in U. Define a relation ρ^U in an analogous manner. Then T^U is a subalgebra of $A[\psi_\nu]$ and ρ^U is a congruence relation in it. Finally, denote by T_U the subalgebra generated by all elements $\theta(u_1, \cdots, a, \cdots, u_n)$ with $a \in T$, the $(n-1)$-tuple in U, and by ρ_U the congruence relation generated by pairs of such θ-operands from $a \equiv b \pmod{\rho}$.

From these general definitions one may prove such theorems as the preservation of all inclusions if $U_1 \subseteq U_2$. One may also consider special cases for A. The author proceeds in both directions, but the bulk of his consideration is given to general formulae rather than to applications.

J. D. Swift (Los Angeles, Calif.)

THEORY OF NUMBERS

See also 3466, 3473, 3807, 4075, 4640.

Sierpiński, W. 3413

Sur une conséquence d'une hypothèse sur les polynômes.

Rend. Circ. Mat. Palermo (2) **11** (1962), 283–284.

L'auteur déduit d'une conjecture de Bouniakovsky [cf. L. E. Dickson, *History of the theory of numbers*, Vol. I, p. 333, Stechert, New York, 1934] la proposition suivante: Si $f(x)$ est un polynôme en x aux coefficients entiers et s'il existe une infinité de nombres naturels x tels que $f(x)$ est un nombre premier, alors, quelque soit le nombre naturel s, il existe une infinité de nombres naturels x tels que $f(x)$ est un produit de s nombres premiers distincts.

L'auteur démontre cette proposition pour le cas d'un polynôme linéaire.

A. Schinzel (Columbus, Ohio)

Rényi, A. 3414
On the distribution of values of additive number-theoretical functions.
Publ. Math. Debrecen **10** (1963), 264–273.
Let $f(n)$ be an additive number-theoretic function. Put

$$f^*(p) = f(p) \quad \text{if} \quad |(fp)| \leq 1,$$
$$f^*(p) = 1 \quad \text{if} \quad |f(p)| > 1.$$

The reviewer proved that if

$$(1) \qquad \sum_p f^*(p)/p \quad \text{and} \quad \sum_p (f^*(p))^2/p$$

both converge, then $f(n)$ has a distribution function [J. London Math. Soc. **13** (1938), 119–127].

The author gives a new analytic proof of this theorem. He uses a method developed by the author and Turán [Acta Arith. 4 (1958), 71–84; MR **20** #3112]. (The reviewer and Wintner proved [Amer. J. Math. **61** (1939), 713–721; MR **1**, 40] that the existence of the distribution function implies the convergence of the two series (1).)
P. Erdős (Budapest)

Rotkiewicz, A. 3415
Sur les nombres premiers p et q tels que $pq|2^{pq}-2$.
Rend. Circ. Mat. Palermo (2) **11** (1962), 280–282.
L'auteur démontre trois théorèmes sur les nombres pseudopremiers dont le plus intéressant est le suivant: Pour tout nombre premier p il existe une infinité de nombres composés n, tels que $n\,|\,2^n-2$ et $p\,|\,n$.
A. Schinzel (Columbus, Ohio)

Rotkiewicz, A. 3416
Sur les formules donnant des nombres pseudopremiers.
Colloq. Math. **12** (1964), 69–72.
The positive integer n is pseudoprime if n is composite and n divides 2^n-2. The author presents several formulas which give directly an infinite number of pseudoprimes. Thus, if p is a prime, then $2^{2p-1}/3$ is pseudoprime for $p>3$ and $2^{2p+1}/5$ is pseudoprime for $p>5$. If $F_n = 2^{2^n}+1$, then the integer $M=(2^{F_{n_1}}-1)(2^{F_{n_2}}-1)\cdots(2^{F_{n_k}}-1)$ is pseudoprime if $n_1<n_2<\cdots<n_k$ and $n_k<2^{n_1}$. Using these results, it is shown that each of the progressions $8k+1$, $8k+3$, $8k+5$ and $8k+7$ contains an infinite number of pseudoprimes. The author has proved, in a less elementary fashion, the corresponding theorem for progressions $ak+b$, where $(a,b)=1$ [C. R. Acad. Sci. Paris **257** (1963), 2601–2604; MR **29** #61].
J. B. Kelly (Tempe, Ariz.)

Erdős, P.; Gordon, B.; Rubel, L. A.; Straus, E. G. 3417
Tauberian theorems for sum sets.
Acta Arith. **9** (1964), 177–189.
Let $K=\{k_0,k_1,k_2,\cdots\}$ be any sequence of real numbers, $0<k_0\leq k_1\leq k_2\leq\cdots$. Let, for $x>0$, $S(x)$ be the number of solutions of $\varepsilon_0k_0+\varepsilon_1k_1+\cdots\leq x$, with $\varepsilon_i=0$ or 1 for each i. Put

$$A = \liminf S(x)/x, \qquad B = \limsup S(x)/x,$$
$$\alpha = \liminf 2^n/k_n, \qquad \beta = \limsup 2^n/k_n.$$

The authors show that $A=\alpha$ and $B\geq 2\alpha-\alpha^2/\beta$. Moreover, they show that for any α, β with $0<\alpha<\beta<2\alpha$, it is possible to find a sequence K such that $B=2\alpha^2-\alpha^2/\beta$.

A direct consequence is that if $A=\alpha>0$, then $B=\beta=A=\alpha$; this was previously obtained by Gordon and Rubel [Illinois J. Math. **4** (1960), 367–369; MR **22** #12084]. It is also shown that $A=\alpha=0$ need not imply $B=\beta=0$.

A further theorem states that if $S(x)-x$ is bounded, then $\sum|k_n-2^n|<\infty$; if the k_n are integers, this means that $k_n=2^n$ for all large n. An extension is given to the case that $S(x)-x$ lies between $-f_1(x)$ and $f_2(x)$, where f_1 and f_2 are positive and non-decreasing, and such that $x-f_1(x)$ and $x+f_2(x)$ are increasing.

A final theorem states that $S(x)$ cannot be asymptotically equivalent to $x^a f(x)$, where $0<a<1$, f is continuous, positive, slowly oscillating, and $x^a f(x)$ is strictly increasing.
N. G. de Bruijn (Eindhoven)

Sastry, K. P. R. 3418
On the generalised type Möbius functions.
Math. Student **31** (1963), 85–88 (1964).
Let n and k denote rational integers, and let $\prod_{i=1}^r p_i^{a_i}$ denote the standard form of n. The author defines a function $\mu_k(n)$ as follows: $\mu_k(1)=1$, $\mu_k(n)=0$ if n has a kth power divisor, and otherwise $\mu_k(n)=(-1)^{\sum a_i}$. When $k=2$, this function coincides with the Möbius function $\mu(n)$. The author proves six theorems concerning $\mu_k(n)$, all of which are immediate generalizations of theorems valid for $\mu(n)$.
W. Ljunggren (Oslo)

Salmeri, Antonio 3419
Introduzione alla teoria dei coefficienti fattoriali. (English summary)
Giorn. Mat. Battaglini (5) **10** (90) (1962), 44–54.
If n and k are positive integers, and if $n>k$, the factorial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}$ is defined as the sum of all products of k different integers chosen from $1, 2, \cdots, n$. Thus in particular $\begin{bmatrix} n \\ 1 \end{bmatrix}=\frac12 n(n+1)$, $\begin{bmatrix} n \\ n \end{bmatrix}=n!$; also conventionally $\begin{bmatrix} n \\ 0 \end{bmatrix}=1$, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}=1$, $\begin{bmatrix} n \\ k \end{bmatrix}=0$ $(n<k)$. There is some analogy with binomial coefficients. Properties obtained include the following:

$$\prod_{k=1}^n (a+kb) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} a^{n-k}b^k;$$

$$\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} = (n+1)!, \qquad \sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} = 0;$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = n\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ k \end{bmatrix} = \sum_{s=k}^n s\begin{bmatrix} s-1 \\ k-1 \end{bmatrix}$$
$$= \frac{1}{k}\sum_{s=0}^{k-1} \binom{n+1-s}{k+1-s}\begin{bmatrix} n \\ s \end{bmatrix}.$$

(The last formula is not proved.) $\begin{bmatrix} n \\ k \end{bmatrix}$ is a polynomial in n of degree $2k$, which from examples seems always to contain the factors $(n+1)n(n-1)\cdots(n-k+1)$.
I. M. H. Etherington (Edinburgh)

Togashi, Akiyo; Uchiyama, Saburô 3420
On the representation of large even integers as sums of two almost primes. I.
J. Fac. Sci. Hokkaido Univ. Ser. I **18** (1964), 60–68.

In connection with Goldbach's famous conjecture the theorem that every sufficiently large even integer is representable as a sum of two integers, each of which has at most three prime factors, was given by A. I. Vinogradov. Here the authors, combining the sieve methods of Viggo Brun and A. Selberg, offer another proof of the theorem.

H. Gupta (Chandigarh)

Uchiyama, Saburô 3421

On the representation of large even integers as sums of two almost primes. II.

J. Fac. Sci. Hokkaido Univ. Ser. I 18 (1964), 69–77.

In this paper, the author, refining the method used in the paper reviewed above [#3420], proves that every sufficiently large even integer N can be represented as a sum of two integers n_1 and n_2 such that $(n_1, n_2) = 1$ and $n_1 n_2$ is a product of not more than five prime factors.

H. Gupta (Chandigarh)

Karst, Edgar 3422

Search limits on divisors of Mersenne Numbers.

Nordisk Tidskr. Informations-Behandling 2 (1962), 224–227.

The author has searched for, and in 227 cases has found, relatively small factors q of Mersenne numbers $2^p - 1$, p a prime > 3000 (mostly p in regions just beyond 3000, 10000, 55000000 and 2^{19}). In this note he gives the upper limit (mostly 10^8) to which the search for q was made, in order to prevent unnecessary work by other researchers.

D. H. Lehmer (Berkeley, Calif.)

Shanks, Daniel 3423

An analytic criterion for the existence of infinitely many primes of the form $\frac{1}{2}(n^2+1)$.

Illinois J. Math. 8 (1964), 377–379.

Let $f(z) = \sum_{n=0}^{\infty} z^{n(n+1)/2}$, and write $g(z) = f^3(z) - 3f(z) = -2 + \{f(z)-1\}^3 - \{f(z)-1\} = \sum_{m=0}^{\infty} a_m z^m$, say. The author points out that, for $m > 0$, $4m+1$ is a prime of the form $\frac{1}{2}(n^2+1)$ if and only if $a_m = (1/m!)g^{(m)}(0) < 0$. Consequently there exist infinitely many primes of the form $\frac{1}{2}(n^2+1)$ if and only if g possesses infinitely many negative coefficients in its Taylor expansion near $z = 0$.

H. Halberstam (Nottingham)

Cohen, Eckford 3424

Arithmetical notes. X. A class of totients.

Proc. Amer. Math. Soc. 15 (1964), 534–539.

Part IX appeared in Acta Arith. 7 (1961/62), 417–420 [MR 26 #2406].

A divisor d of a positive integer n is called unitary if $(d, n/d) = 1$. An integer n is called k-free if it has no prime factor of multiplicity $\geq k$ and k-full if it has no prime factor of multiplicity $< k$. The author defines the following functions: $(m, n)_*$ is the largest common divisor of m and n which is a unitary divisor of n; $\alpha_k(n)$ is the maximal k-free unitary divisor n; $\bar{\phi}_k{}^*(n)$ is the number of integers m in a residue system mod n such that $(m, \alpha_k(n))_* = 1$. $\phi^*(n) = \lim_{k\to\infty} \bar{\phi}_k{}^*(n)$; $\mu_k{}^*(n)$ is a multiplicative function such that for p a prime,

$$\mu_k{}^*(p^e) = -1 \quad \text{if} \quad 1 \leq e < k,$$
$$= 0 \quad \text{if} \quad e \geq k;$$

and $\mu^*(n) = \lim_{k\to\infty} \mu_k{}^*(n)$. It will be observed that $\mu_2{}^*$ is the ordinary Möbius function. Also, if $\delta = n/d$, the notation $\sum_{d\cdot\delta=n}$ is used to indicate summation over unitary divisors of n.

The following results, among others, are proved. (1) If $q_k(n)$ denotes the characteristic function of the k-free integers, then $g(n) = \sum_{d\cdot\delta=n} f(d) q_k(\delta)$ if and only if $f(n) = \sum_{d\cdot\delta=n} g(d)\mu_k{}^*(\delta)$; (2) $\bar{\phi}_k{}^*(n) = \sum_{d\cdot\delta=n} d\mu_k{}^*(\delta)$;

$$(3) \qquad \sum_{n\leq x} \bar{\phi}_k{}^*(n) = \tfrac{1}{2}c_k x^2 + O(x \log^2 x)$$

uniformly in k, where

$$c_k = \prod_p \left\{1 - \frac{1}{p^{2k-1}}\left(\frac{p^{2k-2}-1}{p+1}\right)\right\}.$$

Letting $k \to \infty$ in each of the above produces "unitary analogues" of classical theorems, e.g., (1) becomes $g(n) = \sum_{d\cdot\delta=n} f(d)$ if and only if $f(n) = \sum_{d\cdot\delta=n} g(d)\mu^*(\delta)$.

There are some minor misprints.

S. L. Segal (Rochester, N.Y.)

Carlitz, L. 3425

A sequence of integers related to the Bessel functions.

Proc. Amer. Math. Soc. 14 (1963), 1–9.

Let j_i be the zeros of the Bessel function $J_0(z)$. Define a_r for $r \geq 1$ by

$$a_r = 4^r r!(r-1)! \sum_i j_i^{-2r}.$$

Then a_r is an integer.

The author proves some congruence properties of these integers. For example, if p is a prime, $a_{mp} \equiv a_m \pmod{p}$. If $n > p$, then p divides na_n. If $n > 1$, then $n-1$ divides a_n.

Some of the congruence properties are expressed in terms of the coefficients in the power series for $[J_0(z)]^{-2}$.

The numbers a_n are tabulated in factored form for $n = 1(1)18$ and small factors $< 10^4$ are given for $n = 19(1)30$.

D. H. Lehmer (Berkeley, Calif.)

Carlitz, Leonard 3426

Some q-identities related to the theta functions.

Boll. Un. Mat. Ital. (3) 17 (1962), 172–178.

Es sei $|q| < 1$. Verfasser setzt $(n)! = (1-q^2)\cdots(1-q^{2n})$, $(0)! = 1$ und $1/(-n)! = 0$ für natürliche Zahlen n. Er beweist die Formel

$$\sum_{s=0}^{\infty} \frac{q^{2s}}{(s)!(n+s)!} \prod_{r=1}^{\infty} (1-q^{2r})^2 = \sum_{r=0}^{\infty} (-1)^r q^{r(r+1+2n)}$$

für alle ganzen n. Zunächst findet er sie unter Benützung einer bekannten Formel, die θ_4 enthält, und einer Reihenentwicklung von $F(x)^{-1}$, wobei

$$F(x) = \prod_1^{\infty} (1-q^{2n-1}x)(1-q^{2n-1}x^{-1})$$

ist. Dann leitet er sie noch unter Benützung einer von F. H. Jackson [Proc. London Math. Soc. (2) 2 (1904/05), 192–220; ibid. (2) 3 (1905), 1–23] eingeführten und von W. Hahn [Math. Nachr. 2 (1949), 340–379; MR 11, 720] diskutierten Funktion her. Schließlich leitet er noch zwei Verallgemeinerungen der obigen Formel her. Bemerkte Druckfehler: In dem Produkt für $F(x)$ sollte n erst bei 1 beginnen. In der Reihe (4) ist das von x unabhängige Glied

nicht 1, sondern lautet so wie der Koeffizient von x^n+x^{-n} für $n=1$. In der ersten Formel unter Punkt 4. fehlt im Nenner ein Index s. *G. Lochs* (Zbl 163, 302)

Carlitz, L. 3427

Generalized Dedekind sums.

Math. Z. **85** (1964), 83–90.

The author derives a reciprocity formula for a generalized Dedekind sum defined by

$$s_p(h,k;x,y) = \sum_{\mu \,(\mathrm{mod}\, k)} \bar{B}_p\left(h\frac{\mu+y}{k}+x\right)\bar{B}_1\left(\frac{\mu+y}{k}\right),$$

where x, y are real, p, h, k are integers, $p \geq 0$, $(h,k)=1$, and $\bar{B}_p$, $\bar{B}_1$ are periodic Bernoulli functions. Special cases have been treated by Rademacher [1963 Number Theory Institute in Boulder, Colorado] and by the reviewer [Duke Math. J. **17** (1950), 147–157; MR **11**, 641]. *T. M. Apostol* (Pasadena, Calif.)

Karanicoloff, Chr. 3428

Sur une équation diophantienne considérée par Goormaghtigh.

Ann. Polon. Math. **14** (1963), 69–76.

Let $g_n(x)=1+x+\cdots+x^n$. The author is concerned with integral solutions of equations of the type $g_m(x)=g_n(y)$, where $m>n>1$, $y>x>1$. Goormaghtigh [Intermed. Math. **24** (1917), 88; errata. p. 153] proved that there were exactly two 4-tuples (m,n,x,y) such that $g_m(x)=g_n(y)<10^6$. Kanold [Math. Ann. **132** (1956), 246–255; MR **18**, 718] proved that for fixed x and y there are finitely many m and n such that $g_m(x)=g_n(y)$. Davenport, Lewis and Schinzel [Quart. J. Math. Oxford Ser. (2) **12** (1961), 304–312; MR **25** #1152] proved that for fixed m and n there are finitely many x and y such that $g_m(x)=g_n(y)$, and in case $(n,m)=d>1$ or $m=n+1$, the solutions could be enumerated by use of Runge's method. In this paper the author shows that the only solution of $g_{2n}(x)=g_n(y)$ is $n=2$, $y=5$, $x=2$. The proof is elementary. *D. J. Lewis* (Ann Arbor, Mich.)

Brudno, Simcha 3429

A further example of $A^4+B^4+C^4+D^4=E^4$.

Proc. Cambridge Philos. Soc. **60** (1964), 1027–1028.

The example is $A=955$, $B=1770$, $C=2634$, $D=5400$, $E=5491$. The author outlines the search method on an IBM 709 by which this solution was found. See also J. Leech [same Proc. **54** (1958), 554–555; MR **20** #2301]. *O. Taussky-Todd* (Pasadena, Calif.)

Chowla, S. 3430

A remark on a conjecture of C. L. Siegel.

Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 774–775.

The author states that Siegel [Abh. Preuss. Akad. Wiss. Phys.-Math. Kl. **1929**, 1–70] made the conjecture: Let $f(x,y)=0$ be the equation of an algebraic curve of genus $p>0$ over an algebraic number field; then, if the diophantine equation $f(x,y)=0$ has only finitely many (integral) solutions, a bound can be found for the number of solutions which depends only on the number of coefficients of f. The author remarks that this conjecture is false since, for example, he has proved [J. Indian Math. Soc. **20** (1934), 121–128] that, for some $c>0$ and infinitely many n, the equation $x^2+y^3=n$ has at least $c\log\log n$ solutions in positive integers. *E. S. Barnes* (Adelaide)

Froda, Alexandre 3431

Sur l'irrationalité du nombre 2^e.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **35** (1963), 472–478.

Put

$$\varepsilon_r = \sum_{s=1}^{r}\frac{1}{s!},\quad x_r=1,\quad y_r=2^{\varepsilon_r},\quad q_r=\frac{2^{\varepsilon_r}}{r},\quad p_r=\frac{q_{r+1}}{q_r},$$

$$\alpha_r=\frac{y_r}{x_r},\quad \alpha=\lim \alpha_r = 2^{e-1},\quad \xi_r = 1-\left[\frac{x_r}{q_r}\right]+\frac{x_r}{q_r},$$

$$\eta_r=\frac{y_r}{q_r}-\left[\frac{y_r}{q_r}\right].$$

Further, assume that $\alpha=u/v$, where $u\geq 1$, $v\geq 1$, $(u,v)=1$, is rational. The following four properties are shown to hold.

$$\frac{y_{r+1}}{q_{r+1}}-\frac{y_r}{q_r}\geq\frac{y_{r+2}}{q_{r+2}}-\frac{y_{r+1}}{q_{r+1}};$$

$$0<\frac{x_{r+1}}{q_{r+1}}-\frac{x_r}{q_r}<\frac{x_{r+2}}{q_{r+2}}-\frac{x_{r+1}}{q_{r+1}}\quad\text{for sufficiently large } r;$$

$\eta_r=0$ for all r; $\xi_{r_s}\geq\xi_{r_{s+1}}$ for an infinite sequence of suffixes r_s. In a previous paper [Math. Scand. **12** (1963), 199–208; MR **29** #2222] it was shown that these properties imply the irrationality of α, hence of 2^e, if the sequence $\{\alpha_r\}$ consists of rational elements. This is not so in the present case, but, according to the author, this condition is superfluous. *K. Mahler* (Canberra)

Davenport, H. 3432

A note on Diophantine approximation. II.

Mathematika **11** (1964), 50–58.

Let $\|\alpha\|$ denote the distance from the real number α to the nearest integer. Cassels [Mathematika **3** (1956), 109–110; MR **18**, 875] proved the following. Let $\lambda_1,\cdots,\lambda_r$ be given real numbers, then there exists a real number α such that all numbers $\alpha+\lambda_1,\cdots,\alpha+\lambda_r$ are "badly approximable"; more precisely, they satisfy $\|(\alpha+\lambda_q)u\|>C/u$ $(q=1,\cdots,r)$ for all positive integers u, where $C=C(r)>0$, i.e., the partial quotients of their continued fractions are bounded.

In the paper reviewed here this result is generalized in different directions. One of the simplest is the following: "Suppose $f_1(\alpha),\cdots,f_r(\alpha)$ are functions of a real variable α such that their first derivatives are continuous in some interval containing α_0, and suppose that none of these derivatives is zero for $\alpha=\alpha_0$. Then there exists α such that $\|f_q(\alpha)u\|>C/u$ $(q=1,\cdots,r)$ for all positive integers u, where $C=C(f_1,\cdots,f_r)>0$. The set of those α has the cardinal of the continuum".

This theorem can be extended to simultaneous Diophantine approximation of k-tuples of real numbers. In the two-dimensional case this result reads: "Let $f_1(\alpha,\beta),\cdots,f_r(\alpha,\beta)$ and $g_1(\alpha,\beta),\cdots,g_r(\alpha,\beta)$ be real functions with continuous partial derivatives of the first order in a neighbourhood of some point (α_0,β_0). Suppose that the Jacobian of each pair (f_q,g_q) does not vanish for (α_0,β_0)

($q=1, \cdots, r$). Then there exists a set of pairs (α, β) with the cardinal of the continuum, for which

$$\max(\|f_q(\alpha, \beta)u\|^2, \|g_q(\alpha, \beta)u\|^2) > C/u$$

for all integers $u>0$ and for $q=1, \cdots, r$, where

$$C = C(f_1, \cdots, f_r, g_1, \cdots, g_r) > 0".$$

Finally, the author shows that one can go even further and find values of functions which possess properties of being badly approximable of different dimensions at the same time. The exact wording of these results is omitted in this review.

The author uses a geometrical method of proof already used by himself in a special case in Part I [*Studies in mathematical analysis and related topics*, pp. 77–81, Stanford Univ. Press, Stanford, Calif., 1962; MR 26 #3671].

J. Popken (Amstelveen)

Davenport, H. 3433

Corrigendum: "On a principle of Lipschitz".

J. London Math. Soc. **39** (1964), 580.

A correction to an earlier paper [same J. **26** (1951), 179–183; MR **13**, 323].

Bernstein, Leon 3434

Periodicity of Jacobi's algorithm for a special type of cubic irrationals.

J. Reine Angew. Math. **213** (1963/64), 137–146.

In two previous papers [same J. **213** (1963), 31–38; MR **27** #5727; Math. Nachr. (to appear)], the author studied periodic continued fractions for special nth roots. Proceeding further in this direction here, the author proves the following theorem: In the case $\omega=\sqrt[3]{(D^3+3D)}$, with an integer $D>1$, Jacobi's algorithm for the set $1, \omega, \omega^2$, is periodic with the four-lined pre-period given here, and the four-lined period given here.

E. Frank (Chicago, Ill.)

Roberts, J. B. 3435

Relations between the digits of numbers and equal sums of like powers.

Canad. J. Math. **16** (1964), 626–636.

The author derives a general identity involving the digits of integers in an arbitrary system of notation and deduces a few solutions of the Tarry-Escott problem [Lehmer, Scripta Math. **13** (1947), 37–41; MR **9**, 78]. For instance, we have Theorem 4: The integers from 0 to db^m-1, inclusive, whose base d expansions do not have a units digit equal to any one of a fixed, but arbitrary, set of $d-b$ of the integers $0, 1, \cdots, d-1$, may be split into b classes such that the sum of the tth powers of the elements in a given class is the same for all classes for all t from 0 to $m-1$, inclusive, and the splitting may be accomplished in $(b-1)!^{m-1}$ ways. Theorem 7: For each t, $0\leq t<m$, the sum of the tth powers of the integers from 0 to 3^m-1, inclusive, having no 1's and an odd number of 2's in their base 3 expansions equals the sum of the tth powers of those integers in the same range having no 1's and an even number of 2's. A numerical example for $m=3$ is

$$0^t+8^t+20^t+24^t = 2^t+6^t+18^t+26^t, \qquad t=0, 1, 2.$$

J. W. Andrushkiw (S. Orange, N.J.)

Callahan, Francis P. 3436

Density and uniform density.

Proc. Amer. Math. Soc. **15** (1964), 841–843.

An elementary proof is given that the multiples modulo 1 of any irrational number θ are uniformly dense in the unit interval. Whereas the standard elementary proof uses rational approximations of θ, the device here is to prove that any subinterval I of length $1/k$ gets a proportional share of the multiples of θ in the limiting sense, k being any positive integer. This is done by choosing an integer t so that $t\theta$ is close to $1/k$ (mod 1) and then observing that I and its translates $I+rt\theta$ taken modulo 1 with $1\leq r\leq k-1$ approximately cover the unit interval. The proof generalizes to the n-dimensional case of uniform density, if density in the n-cube is assumed.

I. Niven (Eugene, Ore.)

Christilles, William Edward 3437

A result concerning integral binary quadratic forms.

Pacific J. Math. **14** (1964), 795–796.

Let M be properly represented by the integral positive definite form $ax^2+bxy+cy^2$ of discriminant $-D=b^2-4ac$. It is well known that any factor of M is properly represented by some form of discriminant $-D$. Let $a_ix^2+b_ixy+c_iy^2$ ($i=1, \cdots, k$) be the reduced primitive forms of discriminant $-D$. The author shows, by a simple elementary argument, that if $(M, D)=1$ and $M\leq 3D/16$, then, in any factorization M_1M_2 of M, one of the factors is an a_i for some i.

{Reviewer's remark: The bound $3D/16$ may easily be improved to $D/4$, and the result is then best possible.}

E. S. Barnes (Adelaide)

Danicic, I. 3438

On the fractional parts of θx^2 and ϕx^2.

J. London Math. Soc. **34** (1959), 353–357.

Nach Heilbronn [Quart. J. Math. Oxford Ser. **19** (1948), 249–256; MR **10**, 284] gibt es zu jedem $\varepsilon>0$ ein $C(\varepsilon)$, so daß für jedes reelle θ und jedes $N\geq 1$ eine ganze Zahl x existiert, für die $1\leq x\leq N$, $\|\theta x^2\|<CN^{-1/2+\varepsilon}$ gilt. Für die simultane Approximation zweier beliebiger reeller Zahlen θ, ϕ gilt: Zu jedem $\varepsilon>0$ gibt es ein $C(\varepsilon)$ und ein ganzes x, so daß $1\leq x\leq N$, $\|\theta x^2\|<CN^{-1/8+\varepsilon}$, $\|\phi x^2\|<CN^{-1/8+\varepsilon}$ gilt, wobei wieder $N\geq 1$ gegeben ist. Der Beweis erfolgt mit Hilfe von trigonometrischen Summen. Man vergleiche die neue Arbeit des Verfassers [Mathematika **5** (1958), 30–37; MR **20** #3103], in der eine Verallgemeinerung auf beliebige quadratische Formen durchgeführt wurde.

N. Hofreiter (Zbl 88, 257)

Pall, G. 3439

Simultaneous representation by adjoint quadratic forms.

Acta Arith. **9** (1964), 271–284.

Let φ be an n-ary quadratic form with real coefficients and φ' its adjoint form, with matrices $A=(a_{ij})$ and $A'=(a_{ij}')$, respectively. Two real numbers m and m' are said to be simultaneously represented by φ and φ' if there exist integers x_i, z_i' ($i=1, 2, \cdots, n$) such that

$$m = \sum a_{ij}x_ix_j, \qquad m' = \sum a_{ij}'z_i'z_j', \qquad 0 = \sum x_iz_i',$$

where the sums are over all i and j. The pair of column

vectors $x=(x_i)$ and $z'=(z_i')$ is called a simultaneous representation, and the representation is primitive if each vector is.

An algorithm is given which produces all the simultaneous representations of given m and m' by φ and φ', each set of primitive representations (a set being an aggregate Wx, and $W'z'$, W running over the unimodular automorphs of φ) being associated with a unique class of quadratic forms in $n-2$ variables and a certain set of solutions of certain quadratic congruences modulo m and m'. In particular there is Theorem 6. Every set of simultaneous and primitive representations of nonzero numbers m and m' by the real nonsingular n-ary quadratic form φ and its adjoint φ' is associated with a unique class of matrices G of order $n-2$ and, if we select a particular matrix G in this class, with a unique G-set. One such set of representations obtains for every matrix G and accompanying G-set for which a certain matrix B is equivalent to the matrix of φ. *Burton W. Jones* (San Jose)

Watson, G. L. 3440

A problem of Dade on quadratic forms.

Mathematika **10** (1963), 101–106.

Let $f(x_1, \cdots, x_n)$, $n \geq 3$, be a non-singular quadratic form with rational integer coefficients whose greatest common divisor is 1. The problem is that of finding an algebraic extension E of $\mathbf{Q}$ such that the equation $f(x_1, \cdots, x_n)=1$ has a solution in integers $x_1, \cdots, x_n$ of E. Using theorems from the theory of integral quadratic forms, the author proves the following theorems, which give constructions for E as quadratic extensions of $\mathbf{Q}$. Theorem 1: Suppose $n \geq 4$, then there exists a rational integer q such that $f(y_1+z_1\sqrt{q}, \cdots, y_n+z_n\sqrt{q})=1$ has a solution in rational integers y_i, z_i $(1 \leq i \leq n)$. Theorem 2: Suppose that $n=3$, then there exist rational integers q, r such that the equation $f(y_1+z_1\sqrt{q}+w_1\sqrt{r}, \cdots, y_3+z_3\sqrt{q}+w_3\sqrt{r})=1$ has a solution in rational integers y_i, z_i, w_i $(1 \leq i \leq 3)$.

J. V. Armitage (Durham)

Dade, E. C. 3441a

Algebraic integral representations by arbitrary forms.

Mathematika **10** (1963), 96–100.

Dade, E. C. 3441b

A correction.

Mathematika **11** (1964), 89–90.

Let $\mathfrak{O}$ be the integral closure of the rational integers in some algebraic closure K of the rationals, and let $f(X_1, \cdots, X_n)$ be a polynomial with relatively prime coefficients in $\mathfrak{O}$. The first of these papers generalizes the result of Watson [see #3440 above] as follows: For some $x_1, \cdots, x_n \in \mathfrak{O}$, $f(x_1, \cdots, x_n)$ is a unit. An immediate deduction from this is that if $f(X_1, \cdots, X_n)$ is homogeneous, then $f(x_1, \cdots, x_n)=1$ for suitable $x_1, \cdots, x_n \in \mathfrak{O}$.

The proof depends on a theorem of Steinitz, which in turn requires that a certain determinant shall be divisible by ideals $\mathfrak{q}_1, \cdots, \mathfrak{q}_s$ of $\mathfrak{O}$. Siegel has pointed out that the proof of this last is incomplete at one point, and the author supplies a correct version in the second of these papers.

It should be noted that, whereas Watson's result [#3440] essentially gives the extension in which $x_1, \cdots, x_n$ lie in terms of the coefficients of the form, the present generalization is just an existence theorem.

J. V. Armitage (Durham)

Ehrhart, Eugène 3442

Calcul de la mesure d'un polyèdre entier par un décompte de points.

C. R. Acad. Sci. Paris **258** (1964), 5131–5133.

In Euclidean k-space, let G_1 denote the lattice of points with integral coordinates, let $G_n=G_1/n$ and let Π be any polyhedron whose vertices are points of G_1. The volume and surface area (les mesures réticulaires) of Π are expressed in terms of the numbers of points in the sets $G_n \cap \Pi$ $(n=1, 2, \cdots, [\frac{1}{2}(k+1)])$.

J. H. H. Chalk (Toronto, Ont.)

Von Wolff, M. R. 3443

Application of the domain of action method to $|xy| \leq 1$.

Illinois J. Math. **8** (1964), 500–522.

Let $\mathscr{S}$ be a star domain in the affine plane, symmetric about O. A set of points $\mathscr{P}$ is said to provide a packing for $\mathscr{S}$ if the domains $\{\mathscr{S}+P\}$, where $P \in \mathscr{P}$, have the property that no domain $(\mathscr{S}+P_0)$ contains the center of another in its interior; such a point set $\mathscr{P}$ is called an admissible point set for $\mathscr{S}$. Let $A(t)$ denote the number of points of $\mathscr{P}$ in the square $|x|<t$, $|y|<t$. Then the density $\mathscr{D}(\mathscr{P})$ of $\mathscr{P}$ is defined as $\limsup_{t\to\infty} A(t)/4t^2$. A norm distance $N(X)=N(OX)$ is a real-valued function which is non-negative, continuous and homogeneous. A convex distance function or Minkowski distance, where M is a norm distance with the additional properties that $M(PQ)=0$ implies $P=Q$, $M(PQ) \leq M(PR)+M(RQ)$.

Let $\mathscr{P}$ be a point set in the plane, and let M be a Minkowski distance. The domain of action $D(P)=D(P, M, \mathscr{P})$ of a point P relative to M and $\mathscr{P}$ is the set of all points X in the plane for which $M(PX) \leq M(QX)$, $Q \in \mathscr{P}$, $Q \neq P$.

Consider the domain $\mathscr{S}: |xy| \leq 1$. Let

$$M(P_1, P_2) = \tfrac{1}{2}(|x_2-x_1|+|y_2-y_1|).$$

N. E. Smith [Ph.D. Dissertation, McGill Univ., Montreal, Que., 1951] proved that a critical lattice gives the closest packing of $\mathscr{S}: |xy| \leq 1$; in other words, if $\mathscr{P}$ is any $\mathscr{S}$-admissible point set, $\mathscr{D}(\mathscr{P}) \leq 1/\sqrt{5}$. M. Rahman [Ph.D. Dissertation, McGill Univ., Montreal, Que., 1957] indicated that one might get as sharp a result by using domain of action method; this is indeed the case as the author proves that if O is any point of an admissible point set for the region $\mathscr{S}: |xy| \leq 1$, then $|D(O)| \geq \sqrt{5}$.

M. S. Cheema (Tucson, Ariz.)

Chalk, J. H. H. 3444

A local criterion for the covering of space by convex bodies.

Acta Arith. **9** (1964), 237–243.

Let K be a convex body in n-dimensional Euclidean space R^n, and Λ_0 the integer lattice. Let $\mu=\inf\{t: tK+\Lambda_0=R^n\}$ be the non-homogeneous minimum, and let $\lambda_K{}^*$ be the smallest value of t such that for some $d \in R^n$ the convex

body $tK+d$ contains "a K-dimensional set of $K+1$ points of Λ_0". The following inequality is proved:

$$\mu \leq \tfrac{1}{2}(n+1)\lambda_n^* \qquad \text{if } n \text{ is odd,}$$

$$\leq \tfrac{1}{2}n\left(1+\frac{1}{n+1}\right)\lambda_n^* \quad \text{if } n \text{ is even.}$$

The value of μ is calculated for the modified octahedron.

A. M. Macbeath (Birmingham)

Keller, H. B.; Swenson, J. R. 3445

Experiments on the lattice problem of Gauss.

Math. Comp. **17** (1963), 223–230.

Where $A(r)$ is the number of lattice points, i.e., points with integer coordinates, in the circle $x^2+y^2 \leq r^2$, it has been shown by Hua that

$$(1) \qquad E(r) = A(r) - \prod r^2 = O(r^\theta)$$

holds for $\theta = 13/20$ and by Hardy that it does not hold for $\theta = \frac{1}{2}$. It has been conjectured that (1) holds for $\theta > \frac{1}{2}$.

The authors describe a method of calculating $E(r)$ and tabulate sample values of it for r up to 250,002. For some values of $r > 10^5$ there are values of $\ln |E(r)|/\ln r$ greater than Hua's upper bound of 0.65, while for all calculations performed, it was found that $|E(r)|/r^{1/2} < 7$. Thus it appears that the range of values of r considered, though large, is still not large enough to support any conjecture [cf. the reviewer and Gotlieb, Math. Comp. **16** (1962), 282–290; MR **27** #5722].

W. Fraser (Toronto, Ont.)

Maak, Wilhelm 3446

Gitterpunktsummen.

Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II **1964**, 59–66.

Bekanntlich ist $\Gamma(2)$ (die Gruppe der ganzzahligen Matrizen $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ mit $a \equiv d \equiv 1$ (4), $b \equiv c \equiv 0$ (2)) eine freie Gruppe von zwei Erzeugenden $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ und $B = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$. Die Anzahl der Faktoren A bzw. B, aus denen ein Element $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2)$ aufgebaut ist, ist eine Funktion der ganzrationalen Zahlen a, b, c, d. Es wird eine Methode angegeben, die es gestattet, diese Anzahle durch Abzählen von Gitterpunkten, also ohne eigentliche Rechnung zu bestimmen. Ganz ähnlich Abzählungen können dazu dienen, die Werte der Rademacherschen Φ-Funktion [Rademacher, J. Reine Angew. Math. **167** (1932), 312–336] und damit die der Dedekindsummen zu bestimmen. Es wird die Transformationstheorie der Weierstraßschen σ-Funktion benutzt. Für die angegebenen und weitere Sätze wird demnächst Herr Burde elementare Beweise an anderer Stelle veröffentlichen. Die angegebenen Methoden führen zu einer Fülle weiterer Sätze, die mehr oder weniger interessant erscheinen, je nach der Einstellung, die man gegenüber den Fragestellungen einnimmt, die sie berühren.

J. W. S. Cassels (Cambridge, England)

Gyires, B. 3447

On a generalization of Wilson's theorem.

Acta Arith. **6** (1960), 185–191.

$A_0, A_1, \cdots, A_{r-1}$ seien $(n \times n)$-Matrizen aus ganzen rationalen Zahlen, E die zugehörige Einheitsmatrix. Ersetzt man die Elemente 1 der Einheitsmatrix r-ter Ordnung der Reihe nach durch $A_0, A_1, \cdots, A_{r-1}$ und besetzt man alle übrigen Felder durch Nullen, so entsteht eine Matrix der Ordnung nr, die mit $\langle A_0, A_1, \cdots, A_{r-1}\rangle$ bezeichnet sei. Ferner seien die Koeffizienten von $f(z) = \sum a_{\rho-1}z^{\rho-1} + z^r$ $(\rho = 1, 2, \cdots, r)$ ganz rational, die Wurzeln α_ρ von $f(z) = 0$ paarweise verschieden und die Lösungen β_ρ von $f(z) \equiv 0$ (mod p), wo p Primzahl ist, paarweise inkongruent; außerdem $M(z) = \sum A_{\rho-1}z^{\rho-1}$, $\mathfrak{A} = \langle M(\alpha_1), \cdots, M(\alpha_r)\rangle$, $\mathfrak{B} = \langle M(\beta_1), \cdots, M(\beta_r)\rangle$. Schließlich seien $\mathfrak{V}$ und $\mathfrak{W}$ die Matrizen, die aus den Vandermondeschen Matrizen r-ter Ordnung $(\alpha_{ij}) = (\alpha_j^{i-1})$ bzw. $(\beta_{ij}) = (\beta_j^{i-1})$ entstehen, wenn man jedes Element durch sein Produkt mit E ersetzt. Dann gilt, wie Verfasser zeigt, I. $\mathfrak{V}\mathfrak{A}\mathfrak{V}^{-1} \equiv \mathfrak{W}\mathfrak{B}\mathfrak{W}^{-1}$ (mod p), woraus durch Übergang zu den Determinanten II. $|M(\alpha_1)| \cdots |M(\alpha_r)| \equiv |M(\beta_1)| \cdots |M(\beta_r)|$ folgt. Durch Spezialisierung ergibt sich aus II. unter anderem: $M(1) \cdots M(p-1) \equiv M(\omega_1) \cdots M(\omega_{p-1})$, wo $\omega_1, \cdots, \omega_{p-1}$ die $(p-1)$-ten Einheitswurzeln sind. Diese Kongruenz zu beweisen, war vom Verfasser schon früher als Aufgabe gestellt worden. Mit $M(z) = 1$ folgt aus ihr der Wilsonsche Satz. Der Beweis von I. wird mit Hilfe einer leichten Verallgemeinerung eines Satzes von J. Wellstein [Jber. Deutsch. Math.-Verein. **42** (1932), Abt. 2, 7–9] geführt. Zum Schluß liefert Verfasser noch einen direkten Beweis für ein Nebenresultat von L. Rédei und P. Turán [Acta Arith. **5** (1959), 223–225; MR **21** #4148], das er so formuliert: Notwendig und hinreichend dafür, daß eine zyklische Matrix $(p-1)$-ter Ordnung mod p den Rang k hat, ist $s_{p-1} \equiv \cdots \equiv s_{k+1} \equiv 0$ (mod p); $s_k \not\equiv 0$, wobei s_i die Summe der Hauptminoren i-ter Ordnung ist.

E. Schönhardt (Zbl **104**, 267)

Kirin, Vladimir G. 3448

A note on Wilson's theorem. (Serbo-Croatian summary)

Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hrvatske Ser. II **17** (1962), 181–182 (1963).

For $n \neq 4$, Wilson's theorem is expressed in the form $(n-1)! \equiv -1 \bmod n$ [$\equiv 0 \bmod n$] if and only if n is prime [composite], and it is shown, by induction on k $(0 \leq k \leq n-1)$, that $(n-k-1)!k! \equiv (-1)^{k+1} \bmod n$ [$\equiv 0 \bmod n$] if and only if n is prime [composite].

J. H. H. Chalk (Toronto, Ont.)

Glazkov, V. V. 3449

On a class of finite homomorphisms. (Russian)

Dokl. Akad. Nauk SSSR **158** (1964), 33–36.

Let $h(n)$, $n = 1, 2, \cdots$, be a generalized principal character, i.e., a completely multiplicative function of n, not identically zero, taking only a finite number of (complex) values and satisfying $\sum_{n \leq x} h(n) = ax + O(1)$, $a \neq 0$. The author shows in a basically elementary way that every such character is a Dirichlet character.

S. Knapowski (Marburg)

Katai, I. [Kátai, I.] 3450

An asymptotic formula in the theory of numbers. (Russian)

Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **6** (1963), 83–87.

The following conditional result is proved. If for all σ_0,

$\frac{1}{2} \leq \sigma_0 \leq 1$, $T > 2$, the number of zeros of Riemann's zeta-function $\zeta(\sigma+it)$ in the rectangle $\sigma_0 \leq \sigma \leq 1$, $|t| \leq T$, is equal to $O(T^{2(1-\sigma_0)} \log^2 T)$, then

$$\sum_{n \leq u} \left(\sum_{p<n} \frac{1}{n-p} \right)^2 = u + o(u),$$

where the inner sum is taken over all primes $p < n$.

J. Kubilius (Vilnius)

Kátai, I. 3451

On certain sets of integers.

Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **6** (1963), 43–48.

Let $c_1, \cdots, c_{n+1}$ $(n \geq 2)$ be integers different from zero. A sequence of positive integers $u_1, u_2, \cdots$ is called an A-set if the equality $c_1 u_{i_1} + \cdots + c_n u_{i_n} = c_{n+1} u_{i_{n+1}}$ implies $i_1 = \cdots = i_n = i_{n+1}$. Denote by $A(x)$ the largest number of integers which can be selected from 1, 2, $\cdots$, $[x]$ to form an A-set, $a(x) = A(x)/x$. Let m be a positive integer which does not divide the numbers c_i $(i = 1, \cdots, n+1)$. A set $a_1, \cdots, a_k$ of incongruent residues modulo m will be called a $\mathscr{D}$-set if the congruence $c_1 a_{i_1} + c_n a_{i_n} \equiv c_{n+1} a_{i_{n+1}} \bmod m$ implies $i_1 = \cdots = i_n = i_{n+1}$. Denote by $\mathscr{D}(m)$ the largest number of residues modulo m to form a $\mathscr{D}$-set, $d(m) = \mathscr{D}(m)/m$.

The author obtains the following theorems. (1) If $c_1 + \cdots + c_n \neq c_{n+1}$, then $\lim \inf_{x\to\infty} a(x) > 0$, $\lim \inf_{m\to\infty} d(m) > 0$. (2) If $c_1 + \cdots + c_n = c_{n+1}$, then $\lim_{x\to\infty} a(x) = 0$. (3) If $c_1 + \cdots + c_n = c_{n+1}$, $c_l \not\equiv 0 \bmod m$ $(l = 1, \cdots, n+1)$, then $\lim_{m\to\infty} d(m) = 0$.

The assertion (2) in the case $n = 2$, $c_1 = c_2 = 1$ was conjectured by P. Turán and P. Erdős and solved by K. F. Roth [J. London Math. Soc. **28** (1953), 104–109; MR **14**, 536].

J. Kubilius (Vilnius)

Rémond, Paul 3452

Évaluations asymptotiques dans certains semi-groupes.

C. R. Acad. Sci. Paris **258** (1964), 4179–4181.

Let $\mathscr{E}(E^*)$ be a commutative semigroup having a subset of primes E^* such that if $A \in \mathscr{E}$, then A has a unique expression in the form

$$A = P_1^{\alpha_1} \cdots P_n^{\alpha_n} \qquad (P_i \in E^*, \alpha_i \in Z).$$

Assume that there is a norm N from $\mathscr{E}$ to the natural numbers with $N(A) \geq 2$ and $N(e) = 1$. The purpose of this note is to extend certain results of H. Delange [Ann. Sci. École Norm. Sup. (3) 78 (1961), 1–29]. A normed semigroup $\mathscr{E}(E^*)$ is called a D semigroup if

$$\sum_{A \in \mathscr{E}} N(A)^{-s} = u(s)(s-1)^{-\mu},$$

$s = \sigma + it$, $\sigma > 1$, $\mu > 0$, $u(s)$ holomorphic for $\sigma \geq 1$, $u(s) \neq 0$ for $\sigma \geq 1$. The set E^* is said to be regular if, for $\sigma > 1$,

$$\sum_{P \in E^*} N(P)^{-s} = -\mu \log(s-1) + r(s),$$

$\mu \geq 0$ and $r(s)$ holomorphic for $\sigma \geq 1$. μ is called the density of E^*. A normed semigroup $\mathscr{E}$ is a D semigroup if and only if the set E^* of primes is regular with density $\mu > 0$. In addition, the author announces certain asymptotic relations for D semigroups and deduces the following corollary. If $k = 2, 4, p^a, 2p^a$, where p is a rational prime > 2,

$$\sum_{\substack{n \leq x \\ n \equiv 1 \pmod k}} 1 \sim \left(\frac{\phi(\phi(k))}{[\phi(k)]^{\phi(k)}[\phi(k)-1]!} \right) \left(\frac{x(\log\log x)^{\phi(k)-1}}{\log x} \right),$$

where the summation extends over those n which are not the product of two integers congruent to 1 mod k.

R. Ayoub (University Park, Pa.)

Wright, E. M. 3453

Partition of multipartite numbers into k parts.

J. Reine Angew. Math. **216** (1964), 101–112.

Let $q(k; n_1, \cdots, n_j)$ be the number of partitions of the vector $(n_1, \cdots, n_j)$ into k parts $(x_1, \cdots, x_j)$, where the x_i are non-negative integers. The author estimates q for large $n_1, \cdots, n_j$. The problem is first reduced to studying the coefficients c_n of $J(x) = \sum_{n=0}^\infty c_n x^n = \prod_{i=1}^p (1-x^{a_i})^{-1}$, where $a_1 + \cdots + a_p$ is a partition of k. If $u \geq 1$, let $s(u)$ be the number of $a_i \equiv 0 \pmod u$. By partial fractions one gets $J(x) = \sum_{s(u)>0} J_u(x)$, where the poles of $J_u(x)$ are at the primitive uth roots of unity. If $J_u(x) = \sum_{n=0}^\infty C(u, n) x^n$, then $C(u, n)$ is of the form $\sum_{l=0}^{s(u)-1} \varepsilon_l(u, n) n^l$, where $\varepsilon_l(u, n+u) = \varepsilon_l(u, n)$. Using this to define $C(u, n)$ for $n < 0$, one gets $C(u, n) = (-1)^{p-1} C(u, -n-k)$. Putting $N = n + \frac{1}{2}k$, this leads for $u = 1$ and 2 to an expansion $C(u, n) = \sum_{0 \leq 2\nu \leq s(u)-1} \chi_u(\nu) N^{s(u)-1-2\nu}$, where $\chi_1(\nu)$ and $(-1)^n \chi_2(\nu)$ are independent of n. Explicit formulas are obtained for χ_1 and χ_2 in terms of Bernoulli numbers. Then $q(k; n_1, \cdots, n_j)$ is expanded in powers of $N_i = n_i + \frac{1}{2}k$, and these formulas are used to evaluate the leading terms. For $k \leq 4$ this gives explicit formulas for q.

B. Gordon (Los Angeles, Calif.)

Hagis, Peter, Jr. 3454

On a class of partitions with distinct summands.

Trans. Amer. Math. Soc. **112** (1964), 401–415.

In an earlier paper [same Trans. **102** (1962), 30–62; MR **26** #3688] the author obtained a convergent series representation and asymptotic formulas for the number $p_a(n)$ of partitions of n into parts each congruent mod p (for a given fixed prime p) to one of a fixed set a of r residues mod p. In the present paper he deals by means of similar analytic techniques with the corresponding problem for partitions into distinct (i.e., unequal) parts.

H. Halberstam (Nottingham)

Hagis, Peter, Jr. 3455

Partitions into odd and unequal parts.

Amer. J. Math. **86** (1964), 317–324.

In an earlier paper [same J. **85** (1963), 213–222; MR **27** #3613] the author obtained a convergent series and asymptotic formulas for the number of partitions of n into positive odd summands. In the present paper he uses similar methods to deal with the corresponding problem for partitions into distinct odd parts and, in a concluding section, interprets his results in terms of modular forms.

H. Halberstam (Nottingham)

Chowla, S. 3456

On Gaussian sums.

Proc. Nat. Acad. Sci. U.S.A. **48** (1962), 1127–1128.

The author proves the following theorem. Let

$$\sum_{n=1}^{p-1} \chi(n) e^{2\pi i n/p} = s(z)\sqrt{p};$$

then $s(z)$ is a root of unity only for the case of a quadratic character $\chi^2 = \chi_0$.

A. Vinogradov (RŽMat 1963 #3 A98)

Wiesing, Eduard 3457

Elementare Beweise des Primzahlsatzes mit Restglied. II.

J. Reine Angew. Math. **214/215** (1964), 1–18.

The author continues his researches concerning the elementary proof of the prime number theorem; the formula

$$(1)\qquad \psi(x) = x + O(x/\log^{\alpha} x)$$

is now proved for every α as announced in a previous paper [same J. **211** (1962), 205–214; MR **27** #119]. The author also sketches the proof of the corresponding result concerning primes in an arithmetical progression.

The main tool in the proof is an analogue of the celebrated Selberg formula. If we set

$$(2)\qquad \rho(x) = \sum_{n \leq \exp x} \left(\frac{1}{n} - \frac{\Lambda(n)}{n}\right) - 2C,$$

the fundamental result in the paper is the proof that

$$(3)\qquad \rho(x) = O(x^{-\alpha}) \quad \text{for every } \alpha,$$

from which (1) can be easily deduced.

The author follows in the proof a cyclic method. From (3) he deduces the relations:

$$(4)\qquad \begin{aligned} \rho(x) &= \frac{1}{x}\int_{0-}^{x} \rho(x-y)\,d\rho(y) + O(x^{-\beta}),\\ |\rho(x)-\rho(y)| &\leq |x-y| + O(x^{-\beta}) \end{aligned}$$

for $\beta < \alpha + 1$, and from (4) he can also deduce (3) with β instead of α. The deduction of (3) from (4) is given only for a certain $\alpha' > \beta - 1$. The proof for $\alpha = \beta$ is only briefly sketched.

The proof of (1) for every α, as is pointed out by the author, was obtained at the same time by E. Bombieri following a different method founded on an important extension of the Selberg formula [Riv. Mat. Univ. Parma (2) **3** (1962), 393–440; MR **27** #4804]. *M. Cugiani* (Milan)

Cohen, Eckford 3458

Some asymptotic formulas in the theory of numbers.

Trans. Amer. Math. Soc. **112** (1964), 214–227.

Denote by $Q^2(n)$ the largest square divisor of n. Let α be an arbitrary non-negative real number and S any non-empty set of positive integers. Then for $x \to \infty$,

$$\sum_{\substack{n \leq x\\ Q(n) \in S}} \left(\frac{n}{Q^2(n)}\right)^{\alpha} = \frac{6x^{\alpha+1}}{\pi^2(\alpha+1)} \sum_{\substack{n=1\\ n \in S}}^{\infty} n^{-2\alpha-2} + O(x^{\alpha+1/2}R_{\alpha}(x,S))$$

uniformly in S, where

$$R_{\alpha}(x,S) = \sum_{\substack{n \leq \sqrt{x}\\ n \in S}} n^{-2\alpha-1},$$

if this sum is non-empty; otherwise, $R_{\alpha}(x,S) = 1$.

The author proves two more asymptotic formulas: (1) for an analogous sum with the additional condition $(n, Q^2(n)) = 1$, and (2) for an analogous sum with summands $(n/Q^*(n))^{\alpha}$, where $Q^*(n)$ is the largest divisor of n with no simple prime factors.

He deduces from these results 42 corollaries, which yield estimates due to Cesàro [Ann. Mat. Pura Appl. (2) **13** (1885), 251–268], Feller and Tornier [Math. Ann. **107** (1932), 188–232], Kanold [J. Reine Angew. Math. **193** (1954), 250–252; MR **16**, 569], Rényi [Acad. Serbe Sci. Publ. Inst. Math. **8** (1955), 157–162; MR **17**, 944] and the author [Duke Math. J. **28** (1961), 183–192; MR **25** #3916]. *J. Kubilius* (Vilnius)

Linnik, Ju. V.; Skubenko, B. F. 3459

Asymptotic distribution of integral matrices of third order. (Russian. English summary)

Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. **19** (1964), no. 3, 25–36.

The authors give an asymptotic formula for the number of integral matrices $X = (x_{ij})$ of third order with determinant $\det(X) = N$, $N \to \infty$, where the X's are subjected to the following additional restrictions: (1) $X \equiv A \pmod p$ ($p \nmid N$ a given prime, A a given integral matrix of third order with determinant $N \pmod p$); (2) $\bar{X} = (N^{-1/3}x_{ij})$ belongs to a prescribed sufficiently regular domain Ω.

S. Knapowski (Marburg)

Dade, E. C.; Taussky, O. 3460

On the different in orders in an algebraic number field and special units connected with it.

Acta Arith. **9** (1964), 47–51.

Let $\mathfrak{C} = \mathbf{Z}[\theta]$, $\mathfrak{C}' = \mathbf{Z}[\lambda]$ be two principal orders in an algebraic number field F of degree n over the rationals $\mathbf{Q}$, $\mathfrak{C}' \subset \mathfrak{C}$. Each element of the ring $\mathfrak{C}[\bar{X}_0, \cdots, \bar{X}_{n-1}]$, where $\bar{X}_0, \cdots, \bar{X}_{n-1}$ are independent over F, has a unique representation in the form $B_0 + B_1\theta + \cdots + B_{n-1}\theta^{n-1}$, $B_i \in \mathbf{Z}[\bar{X}_0, \cdots, \bar{X}_{n-1}]$. In particular, the powers Λ^i of $\Lambda = \bar{X}_0 + \bar{X}_1\theta + \cdots + \bar{X}_{n-1}\theta^{n-1}$ may be expressed as $\Lambda^i = B_{i0} + B_{i1}\theta + \cdots + B_{i,n-1}\theta^{n-1}$, where $B_{ik} \in \mathbf{Z}[\bar{X}_0, \cdots, \bar{X}_{n-1}]$ and B_{ik} is homogeneous of degree i in $\bar{X}_0, \cdots, \bar{X}_{n-1}$. The determinant of the matrix (B_{ik}), $i, k = 0, 1, \cdots, n-1$, is a homogeneous polynomial $z(\bar{X}_0, \cdots, \bar{X}_{n-1})$ of degree $n(n-1)/2$. A second polynomial $y(\bar{X}_0, \cdots, \bar{X}_{n-1}; \theta)$ is defined, which depends on the ratios of the differents of θ and λ (the definition requires too many preliminaries to be given here), and it is shown that the norm from $F(\bar{X}_1, \cdots, \bar{X}_{n-1})$ to $\mathbf{Q}(\bar{X}_1, \cdots, \bar{X}_{n-1})$ of $y(\bar{X}_1, \cdots, \bar{X}_{n-1}; \theta)$ is $z(\bar{X}_0, \cdots, \bar{X}_{n-1})^2$. It is an immediate deduction that the following three statements are equivalent: (1) $\mathfrak{C} = \mathfrak{C}'$; (2) $z(X_0, \cdots, X_{n-1}) = \pm 1$; (3) $y(X_1, \cdots, X_{n-1}; \theta)$ is a unit in $\mathbf{Z}[\theta]$ with norm $+1$, where the $X_0, \cdots, X_{n-1}$ are rational integers. An alternative proof of these three is outlined. The results generalize and underlie observations on orders in cubic fields with which the authors introduce the paper. *J. V. Armitage* (Durham)

Fröhlich, A. 3461

On the l-classgroup of the field $P(\sqrt[l]{m})$.

J. London Math. Soc. **37** (1962), 189–192.

Let P be the field of rational numbers, l an odd prime, m is an l-power free integer ($\neq \pm 1$). Let G be the absolute class group of $K = P(\sqrt[l]{m})$ and h its order. Let e be the number of l cyclic factors of G.

Let $t = 1 +$ number of distinct prime factors of m if $(m, l) = 1$ and $X^l \equiv m \pmod{l^2}$ has no solution and $t = r$ otherwise. Let $s =$ number of distinct prime factors of m which are $\equiv 1 \pmod l$. The author proves the following

theorems: (I) $e \geq t - \frac{1}{2}(l+1)$; (II) $e \geq s$; (III) If for $i = 1, 2, \cdots, s$, and $j = 1, 2, \cdots, t$, p_j is an lth power residue mod p_i $(i \neq j)$, then $l^{s+t-(l+1)/2} | h$.

R. Ayoub (University Park, Pa.)

Birch, B. J. 3462

Waring's problem for $\mathfrak{p}$-adic number fields.
Acta Arith. **9** (1964), 169–176.

Let K be an algebraic number field and let $J(K, d)$ be the order generated by the dth powers of integers of K. Siegel conjectured that there is a number $G(d)$ such that every large enough element of $J(K, d)$ is a sum of at most G dth powers. The author proved [Proc. Cambridge Philos. Soc. **57** (1961), 449–459; MR **26** #1306] that if M is a large enough totally positive integer of K, which can be expressed as a sum of at most $s \geq 2^d + 1$ dth powers in every $\mathfrak{p}$-adic completion of K, then M is a sum of at most s totally positive dth powers of integers of K. [Cf. O. Körner, Math. Ann. **147** (1962), 205–239; MR **25** #3022; G. Rieger, ibid. **148** (1962), 83–88; MR **26** #106.] This settled the analytic part of the argument required to prove Siegel's conjecture; the arithmetic part remained open. In this paper the author proves that if $K_\mathfrak{p}$ is the $\mathfrak{p}$-adic completion of K at $\mathfrak{p}$, then every element of $K_\mathfrak{p}$ which is a sum of dth powers of integers of $K_\mathfrak{p}$ is a sum of at most d^{16d^2} such dth powers. Thus any sufficiently large totally positive element of $J(K, d)$ is a sum of at most $\max(2^d + 1, d^{16d^2})$ such dth powers.

The author proves the more general result: given any set $\mathfrak{x}$ of integers of $K_\mathfrak{p}$, then there is a set $\mathfrak{y}$ consisting of at most d^{16d^2} integers such that $s_j(\mathfrak{x}) = s_j(\mathfrak{y})$, for $1 \leq j \leq d$, where s_j denotes the elementary symmetric function of weight j. The foregoing result follows from this on expressing sums of dth powers in terms of $s_1, \cdots, s_d$. The main advantage of using symmetric functions is that one is then able to use Hensel's lemma and so pass from congruences mod $\mathfrak{p}^r$ to equalities.

For an alternate approach to the $\mathfrak{p}$-adic problem [see C. P. Ramanujam, Mathematika **10** (1963), 137–146; MR **29** #5811]. For the problem of the identification of the order $J(K, d)$, see Ramanujam [loc. cit.] and P. T. Bateman and R. M. Stemmler [Illinois J. Math. **6** (1962), 142–156; MR **25** #2059].

J. V. Armitage (Durham)

Erdős, P.; Heilbronn, H. 3463

On the addition of residue classes mod p.
Acta Arith. **9** (1964), 149–159.

Let p be a prime, $a_1, \cdots, a_k$ distinct nonzero residue classes modulo p, N a residue class modulo p. Let $F(N)$ denote the number of solutions of the congruence $e_1 a_1 + \cdots + e_k a_k \equiv N \pmod{p}$, where $e_\nu = 0$ or 1. The authors prove: (I) $F(N) > 0$ provided $k \geq 3\sqrt{(6p)}$; (II) $F(N) = 2^k p^{-1}(1 + o(1))$ provided $k^3 p^{-2} \to \infty$ as $p \to \infty$. The second result is shown to be best possible, while the first result is shown to be nearly best possible. It is conjectured that $F(N) > 0$ provided $k \geq 2\sqrt{p}$. This first result follows from elementary combinatorial arguments including the Cauchy-Davenport theorem. The proof of the second result makes use of finite Fourier series and results from diophantine approximations.

D. J. Lewis (Ann Arbor, Mich.)

FIELDS AND POLYNOMIALS

See also 3387, 3463, 3473, 3482, 3525, 3577.

Kesava Menon, P. 3464

A class of quasi-fields having isomorphic additive & multiplicative groups.
J. Indian Math. Soc. (*N.S.*) **27** (1963), 71–90.

By a "quasi-field" the author means a set Q in which "addition" and "multiplication" are defined such that (1) Q together with addition is an abelian group; (2) Q together with multiplication is an abelian group (the "0" of addition is not excluded); (3) the quasi-distributive law $a(b+c) + a = ab + ac$ holds. The author constructs such quasi-fields by considering the set F of functions defined on the set $\mathscr{S}$ of finite subsets of a set S with values in a commutative ring R with unity. He then sets out to construct a logarithmic function by extending the range of F to include values in rings $R[x, y, \cdots]$. To do this he considers the function f^n and shows that for a given $S_i \in \mathscr{S}$, $f^n(S_i)$ is a polynomial of the form

$$\sum n(n-1)\cdots(n-r+1) f_r(S_i), \quad \text{where} \quad f_r(S_i) \in R.$$

He now introduces indeterminates x, y, etc., and proposes to define f^x, f^y, etc., in $R[x, y, \cdots]$ by treating $n \in \mathbf{N}$ as an indeterminate. The reviewer was unable to follow the argument at this point. The remainder of the paper is a straightforward investigation of the properties of log and exp and concludes with an account of finite quasi-fields (when S has a finite number of elements).

J. V. Armitage (Durham)

Mahler, K. 3465

An inequality for the discriminant of a polynomial.
Michigan Math. J. **11** (1964), 257–262.

Let $f(x) = a_0 x^m + a_1 x^{m-1} + \cdots + a_m = a_0(x - \alpha_1)\cdots(x - \alpha_m)$ be a polynomial with complex coefficients of degree $m \geq 2$, and put

$$M(f) = |a_0| \prod_{k=1}^{m} \max(1, |\alpha_k|),$$

$$P = \prod_{1 \leq h < k \leq m} (\alpha_h - \alpha_k),$$

$$Q = P \prod_{|\alpha_k| > 1} \alpha_k^{-(m-1)}.$$

The principal estimates obtained by the author are as follows. (i) Let $\alpha_r \neq \alpha_s$. Then $|Q/(\alpha_r - \alpha_s)| \leq 3^{-1/2} m^{(m+2)/2}$. This inequality is best possible except, perhaps, for the value of the constant $3^{-1/2}$. (ii) Let $D(f) = a_0^{2m-2} P^2$ denote the discriminant of f. Then

$$|D(f)| \leq m^m (M(f))^{2m-2},$$

with equality if and only if f has the form $f(x) = a_0 x^m + a_m$, where $|a_0| = |a_m| > 0$. (iii) Suppose that the zeros of f are distinct and let $\Delta(f)$ be the minimal distance between any two zeros:

$$\Delta(f) = \min_{1 \leq h < k \leq m} |\alpha_h - \alpha_k|.$$

Then

$$\Delta(f) > 3^{1/2} m^{-(m+2)/2} |D(f)|^{1/2} (M(f))^{-(m-1)}.$$

Now since, as was pointed out in an earlier paper

[Mathematika 7 (1960), 98–100; MR 23 #A1779], $M(f) \le |a_0| + \cdots + |a_n|$, the last two estimates lead to further inequalities for $D(f)$ and $\Delta(f)$, which are sometimes more useful in practice. As an application of (iii), it is shown that the imaginary part of every non-real zero of f exceeds

$$(3/4)^{1/2} n^{-(n+2)/2} \{|a_0| + \cdots + |a_n|\}^{-(n-1)}$$

in modulus. *L. Mirsky* (Sheffield)

Honda, Taira 3466

On the absolute ideal class groups of relatively metacyclic number fields of a certain type.

Nagoya Math. J. 17 (1960), 171–179.

Let $\mathfrak{G}$ be a finite group containing a subgroup $\mathfrak{H}$ with the following property: $\mathfrak{H} \cap \rho\mathfrak{H}\rho^{-1} = \{1\}$ for an arbitrary element ρ in $\mathfrak{G}$ not belonging to $\mathfrak{H}$. By the theorem of Frobenius the elements of $\mathfrak{G}$ not belonging to any conjugate subgroup of $\mathfrak{H}$, together with the identity, form a normal subgroup $\mathfrak{N}$ in $\mathfrak{G}$. The author considers the case when the groups $\mathfrak{H}$ and $\mathfrak{N}$ are both cyclic (in this case $\mathfrak{G}$ is called metacyclic of type F). Let the normal field L/P (L and P are fields of algebraic numbers of finite degree) have as Galois group a metacyclic group of type F, and let K, Ω be intermediate fields of L/P belonging to $\mathfrak{N}$ and $\mathfrak{H}$, respectively. Let $a_{L/K}$ denote the number of two-sided classes of L/K, $\mathfrak{K}_L$ the group of ideal classes of L, and $\mathfrak{K}_{L/\Omega}$ the subgroup of $\mathfrak{K}_L$ consisting of two-sided classes of the field L/Ω. The basic result of the paper is the following. If $a_{L/K} = 1$, then $\mathfrak{K}_L \cong \mathfrak{K}_{L/\Omega}^{(t)}$, the direct product of the group $\mathfrak{K}_{L/\Omega}$ taken t times, where t is the order of the group $\mathfrak{H}$. This result is used to prove the following theorem. If the class number of the cyclic field $Q(\zeta_l)$ (Q is the field of rational numbers and l is a prime) is equal to 1 and if the prime q has order $l-1$ in the group of residue classes modulo l^2, then the group of classes of the field $Q(\zeta_l, \sqrt[l]{q})$ is isomorphic to the direct product taken $l-1$ times of the group of classes of the field $Q(\sqrt[l]{q})$.

S. P. Demuškin (RŽMat 1962 #6 A193)

Inaba, Eizi 3467

Normal form of generalized Artin-Schreier equations.

Natur. Sci. Rep. Ochanomizu Univ. 14 (1963), 1–15.

Let k be a field of characteristic p, and M a non-singular matrix in k_n. A matrix equation (1) $X^q = MX$, where $X = (x_{ik})$ and $X^q = (x_{ik}{}^q)$ ($q = p^r$), is called a generalized Artin-Schreier equation. The author continues his study of a correspondence between the minimal Galois extension K of k which contains a solution of (1) and a representation of its Galois group $G(K/k)$ in matrices over the finite field P of q elements [same Rep. 13 (1962), no. 2, 1–13; MR 27 #5751]. In the present paper a normal form of the equation (1) is obtained by replacing M by a q-semilinear transformation T of an n-dimensional vector space over k. A general Jordan form for M is found and its characteristic polynomial is studied [see also Ore, Trans. Amer. Math. Soc. 35 (1933), 559–584; Jacobson, Ann. of Math. (2) 38 (1937), 484–507]. The reducibility of T and of the corresponding representation of $G(K/k)$ are shown to be related. This is applied to show that if k is a field containing the $(q-1)$st root of unity and satisfying the Hilbert irreducibility theorem, then k has Galois extensions with groups isomorphic to a general type of group of almost triangular non-singular matrices in P_n.

S. A. Amitsur (Jerusalem)

Yakabe, Iwao 3468

An extension theorem for semi-valuations of the first kind.

Mem. Fac. Sci. Kyushu Univ. Ser. A 18 (1964), 50–55.

A one-valued function $s(x)$ on the multiplicative group $\{x\} = K^*$ of a field into an additive abelian l-group Γ is called a semi-valuation if $s(xy) = s(x) + s(y)$ and $s(x+y) \ge \inf\{s(x), s(y)\}$. The semi-valuation is said to have order n if Γ has n maximal l-closed upper classes (which are sublattices of Γ); such a semi-valuation is of the "first kind". The author proves the following extension theorem. "If k is a subfield of a field K, then any semi-valuation s of the first kind of k, of order n, can be extended to a semi-valuation of the first kind of K, of the same order n. In particular, if K is a finite algebraic extension of k, then the number of distinct extensions of s in K, of order n, is finite and is not greater than d^n, where d is the degree of separability of K over k." For the proof, reference is made to the extension theorem for ordinary valuations and a reduction of semi-valuations to sets of valuations [cf. the author's papers, same Mem. 16 (1962), 1–8; MR 26 #115; ibid. 17 (1963), 10–28; MR 28 #87].

O. F. G. Schilling (Lafayette, Ind.)

ABSTRACT ALGEBRAIC GEOMETRY

See also 3574, 3582, 3586, 4066.

Lascu, Alexandru T. 3469

Two intersection formulas in algebraic geometry.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) 35 (1963), 435–442.

Let V be an algebraic variety, U a simple subvariety of V, and $t: V' \to V$ the monoidal transformation of center U. The author gives a formula for $X' \cdot t^{-1}(x_0)$, where $x_0 \in U$ and X' is a V'-divisor linearly equivalent to the subvariety $t^{-1}(U)$. Now suppose that U is a proper component of $A \cdot B$, where A and B are subvarieties of V, with $\mathrm{codim}_V(B) = 1$; let A', B' be the proper transforms of A and B in V'; then the coefficient of U in $t(A' \cdot B')$ is $i(A \cdot B, U; V) - m(U, A)m(U, B)$.

P. Samuel (Waltham, Mass.)

Mizuno, Hirobumi 3470

Sulle equivalenze e corrispondenze algebriche. II.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) 35 (1963), 486–495.

Part I appeared in Rend. Mat. e Appl. (5) 20 (1961), 338–354 [MR 25 #2064]. Let V be a complex non-singular projective variety of dimension n. For every integer h, $0 \le h < n$, the author defines a canonical cycle-class (for rational equivalence) $\Theta_h(V)$ of dimension h. For $h = 0$ one gets the Severi class (projection of the self-intersection of the diagonal in $V \times V$); for $h = n-1$ one gets the classical canonical divisor class. Relations between the cycle-classes $\Theta_h(V_i)$, $i = 0, \cdots, n$, where (V_i) is an

increasing sequence of subvarieties of V with $\dim(V_i) = i$. Application to ramified coverings $p: V \to U$.

P. Samuel (Waltham, Mass.)

Zariski, Oscar 3471

On the superabundance of the complete linear systems $|nD|$ (n large) for an arbitrary divisor D on an algebraic surface.

Univ. e Politec. Torino Rend. Sem. Mat. **20** (1960/61), 157–173.

The contents of this article were published with complete proofs in Ann. of Math. (2) **76** (1962), 560–615 [MR **25** #5065]. This is the same as that in Atti Convegno Internaz. Geometria Algebrica (Torino, 1961), pp. 105–121; Rattero, Turin, 1962 [MR **26** #2440].

Y. Nakai (Hiroshima)

Lluis, Emilio [Lluis Riera, Emilio] 3472

Algebraic varieties with certain tangent conditions. (Spanish)

An. Inst. Mat. Univ. Nac. Autónoma México **2** (1962), 9–10.

This is a statement of the results of another paper of the same title [Bol. Soc. Mat. Mexicana (2) **7** (1962), 47–56; MR **26** #4995].

Siegel, Carl Ludwig 3473

Moduln Abelscher Funktionen.

Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II **1963**, 365–427.

Let $\Omega = (Z, T)$ be a $2n \times n$ period matrix, where $T = \operatorname{diag}(t_1, \cdots, t_n)$ with natural integers t_i and $Z = X + iY$ symmetric, $Y > 0$. If $t_1 \geq 3$, the Jacobian theta functions yield a singularity-free model of the corresponding Abelian manifold in the projective space of dimension $t = |T|$. Let $\vartheta_i(w) = \vartheta_i(w_1, \cdots, w_n)$ be these theta functions and consider as many indeterminates $\xi_1, \cdots, \xi_t$. With $(n+1)t$ further indeterminates λ_{ik} form $\eta_i = \sum \lambda_{ik}\xi_k$ and $j_i(w) = \sum \lambda_{ik}\vartheta_k(w)$. Now let $\Phi(\lambda, \eta)$ be a homogeneous polynomial in the λ and η of minimal degree in the latter which vanishes for $\eta_i = j_i(w)$. It has the following properties. (1) The degree is $m = n!t$. (2) If $\Phi(\lambda, \eta) = \sum \lambda_{\alpha}\Phi_{ik}(\xi)$, all algebraic equations between the $\vartheta_i(w)$ are consequences of $\Phi_{ik}(\vartheta(w)) = 0$. (3) The coefficients of Φ depend naturally on the period matrix. (4) Φ can explicitly be described as follows: let $\varphi_l(\xi)$, $l = 1, \cdots, h$, all be power products of the ξ_t in some order and $\alpha_l = \alpha_l(\lambda)$ the $(h-1)$-rowed subdeterminants of the matrix $(\varphi_l(\vartheta_k(w)))$, then

$$(1) \qquad \Phi = \sum_{l=1}^{h} \alpha_l\varphi_l(\xi).$$

The chief object of the paper is the study of the behavior of the coefficients of Φ under modular transformation of the period matrix Z (T being left fixed throughout). A tool for this is the following Hermitian scalar product of theta functions. Put $w = Zu + Tv$ with real vectors u and v and for two Jacobian functions $f(w)$, $g(w)$

$$(f, g) = \int_W f(w)\overline{g(w)}e^{(\pi/2)T(w-\bar{w})}\,du_1 \cdots dv_1 \cdots,$$

where W is the period torus. It is easily shown that the integrand is a function defined on W_1 and that two different theta functions are orthogonal while $(\vartheta_i, \vartheta_i) = 2|Y|^{-1/2}$.

Now Jacobian functions of characteristic xy are considered which are linear combinations of these theta series

$$(2) \quad \vartheta(x, y; Z, w) = \sum_m \exp \pi i(Z[m + T^{-1}x] + 2(m + T^{-1}x)^t(w + y)).$$

To these functions the transformations

$$Z_1 = (TAT^{-1}Z + TB)(CT^{-1}Z + D)^{-1},$$
$$w_1 = ((CT^{-1}Z + D)^t)^{-1}w$$

are applied, where A, B, C, D are rational integral matrices with

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^t \begin{pmatrix} 0 & T \\ -T & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = s\begin{pmatrix} 0 & T \\ -T & 0 \end{pmatrix}$$

with a further rational integer s. If $s = 1$, these matrices form the modular group $\Gamma(T)$; subsequently the following normal subgroup $\Delta(T)$ will be used:

$$(A - E)T^{-1},\ BT^{-1},\ CT^{-1},\ (D - E)T^{-1} \text{ integral.}$$

With a substitution described above and a Jacobian function $f(w_1)$ of characteristic xy and period matrix (Z_1T)

$$f_1(w) = \exp(-\pi i(s^{-1}w^tT^{-1}C^tw_1))f(w_1)$$

is a Jacobian function of characteristic

$$x_1 = A^tx + C^ty + \tfrac{1}{2}\{A^tTC\}, \qquad y_1 = B^tx + D^ty + \tfrac{1}{2}\{B^tTD\}$$

and period matrix $s(ZT)$, where $\{\cdots\}$ denotes the vector whose components are the diagonal elements of the matrix in parentheses. Furthermore, the scalar product remains invariant: $(f, g) = (f_1, g_1)$.

This transformation is specially applied to theta functions of the more general type (2) and under the restriction to $w = 0$. It yields a unitary representation of the group $\Gamma(T)$ whose coefficients can of course be written in terms of Gaussian sums. The kernel of this representation turns out to be the subgroup $\Theta(T) \subset \Delta(T)$ defined by the conditions that $AT^{-1}B^t$, $CT^{-1}D$ have even coefficients in the diagonal. A similar result which is less easy to formulate concerns the transformation of the functions (2) with $w \neq 0$; it is a decisive tool in the following problem.

Under the transformations $Z, w \to Z_1, w_1$ belonging to the modular group $\Gamma(T)$, the coefficients $\alpha_l = \alpha_l(\lambda)$ undergo corresponding transformations. The ratios $\alpha_{l_1} : \alpha_{l_2}$, however, remain invariant if and only if the transformation belongs to the congruence subgroup $\Delta(T)$. It is now very easy to form invariants for the whole modular group $\Gamma(T)$ by symmetrization. Now the α_l are polynomials in the indeterminates λ_{ik} the coefficients of which are functions of Z and the w_i. Let $q_0, q_1, \cdots, q_r$ be all these coefficients, then the quotients $f_i = q_i/q_0$ are functions of Z only. The author even proves that to every $Z = Z_0$ there is such an ordering of the q_i that the f_i are regular in the neighborhood of this point. (The statement has been somewhat simplified; in fact, this is only true if the $\alpha_l(\lambda)$ are relatively prime polynomials in the λ, the formulation of the theorem in the general case being more complicated.)

The last fact is necessary to prove the following deep theorem: All modular functions of $\Delta(T)$ are rational functions of the f_i. The proof is of course rather involved. Among other things it uses the fact that all modular functions can be represented as quotients of Eisenstein series [the author, Math. Ann. 116 (1939), 617–657; MR 1, 203]. In the case $T=tE$ it is eventually shown that all modular functions can be represented as quotients of Fourier series with integral rational coefficients.

Taking into account the contents of the author's paper quoted before [loc. cit.] (which has to be generalised, however, since it assumes $T=E$) and furthermore the result of Igusa [Amer. J. Math. 84 (1962), 175–200; MR 25 #5040], we may expect the story to end here. But the author continues, attempting to determine all algebraic equations between the functions f_i or, what is the same, all homogeneous algebraic equations between the q_i. The latter do not only depend on Z but also on the w_i. The principle of determining these equations consists of two theorems. (1) Let $Q(Z,w)$ be a homogeneous polynomial in the q_i. If sufficiently many of the initial coefficients of the development of Q in a power series in the w_i vanish, then Q vanishes identically. (2) If sufficiently many of the initial coefficients in the Fourier expansion of Q with respect to Z vanish, then Q vanishes identically. The latter theorem has been proved and used by the author on previous occasions. From these theorems we can easily derive a "constructive procedure" to obtain all algebraic relations between the f_i since we know them to be generated by those of a bounded degree. However, the conditions seem rather involved, and it may be premature to expect explicit information. *M. Eichler* (Basel)

Serre, Jean-Pierre 3474

Exemples de variétés projectives conjuguées non homéomorphes.

C. R. Acad. Sci. Paris 258 (1964), 4194–4196.

Author's summary: "Soit V une variété projective non singulière, définie sur un corps de nombres algébriques K; si φ est un plongement de K dans C, soit V_φ la variété complexe déduite de V par extension des scalaires au moyen de φ. On sait que les nombres de Betti de V_φ sont indépendants de φ. Cette propriété d'invariance ne s'étend pas au groupe fondamental: nous construisons ci-dessous une variété V et deux plongements φ et ψ tels que $\pi_1(V_\varphi)$ ne soit pas isomorphe à $\pi_1(V_\psi)$; en particulier, V_φ et V_ψ ne sont pas homéomorphes." *R. C. Hartshorne* (Cambridge, Mass.)

Ramanujam, C. P. 3475

A note on automorphism groups of algebraic varieties.

Math. Ann. 156 (1964), 25–33.

Let X be an irreducible algebraic variety and G a group of automorphisms of X. Assume that G is "connected" [i.e., any two elements of G belong to some algebraic irreducible family $S\to G$ of automorphisms) and "finite-dimensional" (i.e., for every such injective family $S\to G$, dim (S) is bounded). Then there exists on G a unique structure of algebraic groups such that the operation law $G\times X\to X$ is a morphism and is infinitesimally injective. Furthermore, for every algebraic family $S\to G$ of automorphisms of X, $S\to G$ is a morphism.

Suppose now that X is complete (more generally, that $H^0(X,F)$ is finite-dimensional for every torsion-free algebraic coherent sheaf F). Then the connected component G of Aut(X) is "finite-dimensional", and therefore admits a unique algebraic group structure, with the properties above. The group G is an extension of a subgroup of Pic (X) by a linear group.

Finally, let P be a locally isotrivial principal fiber space over a complete variety X_0. Then the connected component G of Aut (P) is an algebraic group, and the elements g of G which preserve the base X_0 form a closed subgroup of G. *P. Samuel* (Waltham, Mass.)

Baldassarri, Mario 3476

Osservazioni sulla struttura dei fasci lisci.

Univ. e Politec. Torino Rend. Sem. Mat. 20 (1960/61), 101–108.

Soit V une variété algébrique irréductible et localement normale au sens de Serre; l'auteur dit qu'un faisceau algébrique cohérent F sur V a la propriété d'extension si toute section définie en dehors d'un fermé dont toutes les composantes sont de codimension >1 se prolonge en une section sur tout V. Si F a cette propriété et n'a pas de torsion, tout faisceau de la forme Hom (G, F) avec G cohérent a la propriété d'extension; il en est de même d'un faisceau localement libre. Ces propriétés et quelques autres analogues sont données sans démonstration. *P. Cartier* (Strasbourg)

LINEAR ALGEBRA

See also 3459, 3830, 3892, 3927, 3941, 3988, 3989, 4162, 4183, 4184, 4186.

Mal'cev, A. I. 3477

★Foundations of linear algebra.

Translated from the Russian by Thomas Craig Brown; edited by J. B. Roberts.

W. H. Freeman & Co., San Francisco, Calif.-London, 1963. xi+304 pp. $7.50.

This is a translation from the second Russian edition [Gostehizdat, Moscow, 1956].

Toscano, Letterio 3478

Su gli operatori permutabili di secondo ordine.

Giorn. Mat. Battaglini (5) 10 (90) (1962), 55–71.

An operator X on a linear space S is said to commute to the second order with an operator A on the same space if $AX-XA=P$ is an operator which commutes with A, i.e., $AP-PA=0$. In particular, A and X are called associated operators if $AX-XA=I$, the identity operator. Numerous identities for associated operators are derived by elementary methods. The author does not state, but seems to require, that all operators considered must have inverses. *O. Wyler* (Albuquerque, N.M.)

Wonenburger, Maria J. 3479

A decomposition of orthogonal transformations.

Canad. Math. Bull. 7 (1964), 379–383.

Let M be a finite-dimensional vector space over a field of characteristic not 2 having a nondegenerate symmetric

bilinear form of index 0. The author proves that an orthogonal transformation on M is the product of at most two involutions. The proof utilizes the fact that the product of symmetries relative to orthogonal hyperplanes is an involution. *W. E. Deskins* (E. Lansing, Mich.)

Calabi, Eugenio 3480

Linear systems of real quadratic forms.

Proc. Amer. Math. Soc. **15** (1964), 844–846.

The author gives a simple topological proof for the following theorem. Let $P(x)$, $Q(x)$ be two real quadratic forms over R^n $(x \in R^n)$, with $3 \leq n < \infty$. If the only $x \in R^n$ satisfying $P(x) = Q(x) = 0$ is $x = 0$, then there exists a linear combination of P and Q with real coefficients, that is, a positive definite quadratic form over R^n.

He remarks that the theorem is false for $n = 2$ and is partially correct for $n = \infty$. *W. Klingenberg* (Mainz)

Lynn, M. S. 3481

On the Schur product of H-matrices and non-negative matrices, and related inequalities.

Proc. Cambridge Philos. Soc. **60** (1964), 425–431.

If $A = (a_{ij})$, $B = (b_{ij})$ are $n \times n$ matrices, their "Schur product" $A \circ B$ is the $n \times n$ matrix whose (i, j)-th element is $a_{ij}b_{ij}$. Further, $A\begin{bmatrix} i_1, \cdots, i_p \\ j_1, \cdots, j_p \end{bmatrix}$ denotes the determinant of the submatrix of A specified by the rows $i_1, \cdots, i_p$ and columns $j_1, \cdots, j_p$. If $A = (a_{ij})$ is real, we define α_{ij} as $|a_{ij}|$ or $-|a_{ij}|$ according as $i = j$ or $i \neq j$, and we denote by A' the matrix (α_{ij}). The real $n \times n$ matrix A is said to be of class $\mathscr{H}_n$, a class first studied by A. Ostrowski [Comment. Math. Helv. **10** (1937), 69–96], if all principal minors of A' are positive. The main results established in the present paper are as follows. (i) If $A, B \in \mathscr{H}_n$, then $A \circ B \in \mathscr{H}_n$. (ii) If $A \in \mathscr{H}_n$, then $\det A' \leq |a_{11} \cdots a_{nn}|$. (iii) If $A \in \mathscr{H}_n$ and $1 \leq p \leq n$, then

$$\det A' \leq A'\begin{bmatrix} 1, \cdots, p \\ 1, \cdots, p \end{bmatrix} \cdot A'\begin{bmatrix} p+1, \cdots, n \\ p+1, \cdots, n \end{bmatrix}.$$

(iv) If $A, B \in \mathscr{H}_n$, then

$$\det(A \circ B)' + \det A' \cdot \det B' \geq |b_{11} \cdots b_{nn}| \det A' + |a_{11} \cdots a_{nn}| \det B'.$$

These results receive an added interest from the fact that similar conclusions are known to be valid for the class of real, symmetric, positive definite matrices. In the last section of the paper a different type of theorem is established. Let λ_A denote the spectral radius of A. Then $\lambda_{A \circ B} \leq \lambda_A \lambda_B$ for any non-negative matrices A, B; and if A, B are irreducible and at least one of them has only positive diagonal elements, then this inequality is strict. *L. Mirsky* (Sheffield)

Kazimirskiĭ, P. S. 3482

On the factorization of a polynomial matrix into linear factors. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 446–448.

Let $A(x) = \sum_{j=0}^{s} A_{s-j}x^j$, $s \geq 2$, A_j square matrices of order n $(j = 0, \cdots, s)$ with elements from an algebraically closed field P of characteristic 0, $|A_0| \neq 0$, $A_*(x)A(x) = A(x)A_*(x) = E|A(x)|$, E the unit matrix, $a_1, \cdots, a_1$ (k_1 terms), $\cdots$, $a_l, \cdots, a_l$ (k_l terms) n roots of the equation $|A(x)| = 0$, $k_1 + \cdots + k_l = n$, k_i the multiplicity of the root,

$$M_{A(x_n)} = \left\| \begin{matrix} A_*(a_1) \\ \cdot \quad \cdot \quad \cdot \\ A_*^{(k_1 - 1)}(a_1) \\ \cdot \quad \cdot \quad \cdot \\ A_*(a_l) \\ \cdot \quad \cdot \quad \cdot \\ A_*^{(k_l - 1)}(a_l) \end{matrix} \right\|.$$

Three lemmas lead to the following theorem: There exists a matrix $B(x) = B_0x + B_1$, $|B_0| \neq 0$, with the given roots and which divides $A(x)$ from the left if and only if the rank of $M_{A(x_n)}$ is n. *J. Reichbach* (Tel Aviv)

Fiedler, Miroslav 3483

Relations between the diagonal elements of two mutually inverse positive definite matrices. (Russian summary)

Czechoslovak Math. J. **14** (89) (1964), 39–51.

The results of this paper have to do with an n-by-n Hermitian positive definite matrix $A = (a_{ij})$ and its inverse $A^{-1} = (\alpha_{ij})$ and necessary and sufficient conditions on the diagonal elements. Two theorems are the following. Theorem 3.1. Let λ_1 be the least, λ_n the greatest proper value (eigenvalue) of A, $q = \lambda_n/\lambda_1$. Then

$$(\operatorname{tr} A \operatorname{tr} A^{-1})^{1/2} \geq q^{1/2} + q^{-1/2} + n - 2 \geq 2 \max \sqrt{(a_{ii}\alpha_{ii})} + n - 2,$$

where the maximum value is over all i. In the first inequality, equality is attained if and only if all remaining proper values of A are equal to $(\lambda_1\lambda_n)^{1/2}$. (Necessary and sufficient conditions for the second equality are also given.)

Theorem 3.2. For A and A^{-1} as given, it follows that

$$a_{ii} > 0, \qquad \alpha_{ii} > 0, \qquad a_{ii}\alpha_{ii} \geq 1,$$
$$\sqrt{(a_{ii}\alpha_{ii})} - 1 \leq \sum [\sqrt{(a_{jj}\alpha_{jj})} - 1]$$

for all i, where the sum is over all $j \neq i$. Conversely, let a_{ii}, α_{ii} $(i = 1, \cdots, n)$ be $2n$ real numbers which satisfy the above inequalities. Then there exists a positive definite (even real) matrix $A = (a_{ik})$ such that its diagonal elements coincide with the given numbers a_{ii} and the diagonal elements of its inverse matrix with α_{ii}.

The author also gives conditions for the equalities to hold and applications of his results.

Burton W. Jones (San Jose)

Pimenov, R. I. 3484

Flagtensor algebra. (Russian. English summary)

Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. **19** (1964), no. 3, 150–152.

A flagspace is a linear space S with a sequence of subspaces $L_1 \subset L_2 \subset \cdots \subset L_n$. A flagtensor algebra is a tensor algebra for linear transformations of S which have the L_i as invariant subspaces. These can be considered relative to any structure defined in S (euclidean, non-euclidean, affine, etc.). Different definitions of flagtensors are given and their equivalence established. The invariant euclidean flagtensors are discussed.

H. W. Guggenheimer (Minneapolis, Minn.)

Kozin, F. [Kozin, Frank] 3485

On functions of three vectors.

Publ. Math. Debrecen **10** (1963), 120–122.

The author proves the following theorem. Let X, Y, Z be three vector spaces, and let f be a real-valued function on $X \times Y \times Z$. The inequalities

$$\text{(I)} \qquad f(x+x_1, y+y_1, z) \geq f(x, y, z)+f(x_1, y_1, z),$$

$$\text{(I')} \qquad f(x+x_1, y, z+z_1) \leq f(x, y, z)+f(x_1, y, z_1)$$

for all vectors x, x_1, y, y_1, z, z_1 jointly imply the two opposite inequalities. The proof uses only the fact that X, Y, Z are additive abelian groups.

O. Wyler (Albuquerque, N.M.)

Kozin, Frank; Menger, Karl 3486

A self-dual theory of real determinants.

Publ. Math. Debrecen **10** (1963), 123–127.

The determinant of a square matrix is linear and alternating in rows and columns, and $\det I = 1$ if I is a unit matrix. The authors replace these postulates, for real matrices, by the following conditions: (1) det A is a sublinear and subalternating function of the columns of A, and a superlinear and superalternating function of the rows of A; (2) $\det I = 1$ is kept. {By "subalternating in columns" the following is meant. If A has two equal columns, then $\det A \leq 0$. "Superalternating in rows" means that $\det A \geq 0$ if A has two equal rows.}

O. Wyler (Albuquerque, N.M.)

Tutschke, W. 3487

Eine hinreichende Bedingung für die Existenz positiver Lösungen von linearen Gleichungssystemen.

Monatsb. Deutsch. Akad. Wiss. Berlin 5 (1963), 663–667.

The author proves a sufficient condition regarding the existence of positive solutions x_j of the linear system

$$\sum_{j=1}^{q} A_{ij}x_j = 0, \qquad i = 1, \cdots, q.$$

The result is the following. Suppose $A_{ij} > 0$ for all $i \neq j$ and let all principal minors of the matrix $\|A_{ij}\|$, $i, j = 1, 2, \cdots, n-1$, be positive numbers. From $B_i = -A_{iq} \geq 0$ it follows that $x_i > 0$ for all $i = 1, \cdots, q-1$ and $\sum_{i=1}^{q-1} B_i > 0$. The result is important in the case of applications in the theory of periodic solutions of linear elliptic partial differential equations which are not self-adjoint.

M. Pinl (Cologne)

ASSOCIATIVE RINGS AND ALGEBRAS

See also 3464, 3522, 3525, 3550, 3551.

Fuchs, L.; Halperin, I. 3488

On the imbedding of a regular ring in a regular ring with identity.

Fund. Math. **54** (1964), 285–290.

When imbedding a ring A in a ring with identity by the standard method, one can replace the ring of integers by any commutative ring S with identity over which A is an algebra. Denote the resulting ring by $(A; S)$ (reviewer's notation); then A is isomorphic to an ideal in $(A; S)$. The authors show that there exists a commutative ring M with identity such that every regular ring is an algebra over M. (Regularity is used in the sense of von Neumann.) From this they deduce that any regular ring R is isomorphic to an ideal in a regular ring with identity, namely, $(R; M)$. {The proof that $(R; M)$ is regular can be shortened considerably. In fact, if E is any ring with an ideal R such that R and E/R are regular, then E is regular; see the proof of Theorem 2 in Brown and McCoy [Proc. Amer. Math. Soc. 1 (1950), 165–171; MR 11, 628].}

C. W. Kohls (Syracuse, N.Y.)

Gilmer, Robert W., Jr. 3489

Integral domains which are almost Dedekind.

Proc. Amer. Math. Soc. **15** (1964), 813–818.

An integral domain J with unit is said to be almost Dedekind if and only if J_P is a Dedekind domain for every maximal ideal P of J. Let D be an integral domain with unit. Theorem: D is almost Dedekind if and only if (i) nonzero proper ideals of D are maximal, and (ii) primary ideals of D are prime powers. Theorem: Let D be almost Dedekind. D is a Dedekind domain if and only if every nonzero proper ideal of D is contained in only finitely many maximal ideals. In particular, an almost Dedekind domain with only a finite number of maximal ideals is a principal ideal domain.

In terms of valuations, D is Dedekind if and only if D is a Krull domain [cf. Zariski and Samuel, *Commutative algebra*, Vol. II, p. 82, Van Nostrand, Princeton, 1960; MR **22** #11006]. Relations between the ideal structure of D and D' with $D \subseteq D' \subseteq K$ (D almost Dedekind, K its quotient field) are investigated. In particular, D' is proved to be almost Dedekind.

H. Gross (Bozeman, Mont.)

Samuel, Pierre 3490

Classes de diviseurs et dérivées logarithmiques.

Topology **3** (1964), *suppl.* 1, 81–96.

Let A be a Krull ring (i.e., a commutative ring A such that the totality $D(A)$ of divisors of A forms an ordered group whose positive elements satisfy the minimum condition), and let K be its quotient field. Let G be a cyclic group of automorphisms of A. G operates also on K. Let K' be the fixed subfield, and let $A' = A \cap K'$. Also A' is a Krull ring, and its quotient field is K'. Let $C(A)$ $[C(A')]$ be the group of divisor classes of A $[A']$. The author first studies the kernel of the canonical homomorphism $\bar{j}$ of $C(A')$ into $C(A)$ induced by the canonical map j of the divisor group $D(A')$ into $D(A)$. With a fixed generator s of G, set $h(x) = sx/x$ $(x \in K)$, and for a divisor $\mathfrak{b}$ of A' mapped by j to a principal divisor (a) of A, let $\varphi(\mathfrak{b})$ be $h(a) \bmod h(U)$, U denoting the group of units in A. φ is shown to induce a monomorphism $\bar{\varphi}$: $\operatorname{Ker}(\bar{j}) \to (U \cap \operatorname{Im}(h))/h(U)$. If every prime divisor of A' is unramified in A, then $\bar{\varphi}$ is onto. If A is factorial (i.e., $C(A) = 1$), then $\operatorname{Ker}(\bar{j}) = C(A')$. If, for instance, A is a polynomial ring $k[x_1, \cdots, x_d]$ $(d \geq 2)$ over a field k possessing a root w of 1 with exponent n prime to the characteristic of k, and if $sx_i = wx_i$, $s\kappa = \kappa$ $(\kappa \in K)$, then $\ker(\bar{j}) = C(A')$ is shown to be cyclic of order n. The same is the case if $A = k[x, y]$ and $sx = wx$, $sy = w^{-1}y$, $s\kappa = \kappa$ $(\kappa \in K)$. On the other hand, with a field k of prime characteristic p, if $A = k[x, y]$ and $s\lambda = \lambda$ $(\lambda \in k[x])$, $sy = y + x$, then A' is also factorial. Leaving these Galois descent cases, the

author turns to his main object, p-radical descent, where he considers, on assuming K to be of prime characteristic p, a derivation D of K with $D(A)\subset A$, and sets $K'=\operatorname{Ker}(D)$, $A'=K'\cap A$ (which is again Krull). As before, $j: C(A')\to C(A)$ is considered, and a monomorphism ϕ of $K(j)$ into $\mathfrak{D}/\mathfrak{D}'$ is obtained, where $\mathfrak{D}$ is the set of elements of A which are logarithmic derivatives in K and $\mathfrak{D}'$ is the set of logarithmic derivatives of units. If $[K:K']=p$ and no prime ideal of height 1 in A contains $D(A)$, then ϕ is onto. By means of the formulas of Barsotti and Cartier [Barsotti, Illinois J. Math. **2** (1958), 43–70; MR **21** #2654; Cartier, Bull. Soc. Math. France **86** (1958), 177–251; MR **21** #4957] and Hochschild [Trans. Amer. Math. Soc. **79** (1955), 477–489; MR **17**, 61], it is proved that $t\ (\in K)$ is a logarithmic derivative if and only if $D^{p-1}(t)-at+t^p=0$ with $a\in K'$, provided $[K:K']=p$. For a polynomial ring $A=k[x,y]$ (of characteristic p), cases are studied in which A' turns out to be $k[x^p, y^p, xy]$, respectively, $k[x^p, y^p, x^i+y^j]$ (i,j prime to p). In other cases, $C(A')$ is shown to be cyclic of order p, respectively, of type $(p,\cdots,p)$ and order p^f with $f<\gcd(i,j)$. For the ring $A=k[[x,y]]$ of power series, similar considerations are made to determine $C(A')$ for $A'=k[[X,Y,Z]]$ with $Z^p=XY$, respectively, $Z^2=X^{2l+1}+Y^{2l+1}$ $(p=2)$. For the last case, $C(A'[[T]])$ is also determined. The examples recover some results about factoriality in the author's paper [Bull. Soc. Math. France **89** (1961), 155–173; MR **25** #3046]. Open questions are given, including: Is $\mathfrak{D}\cap AD(A)=\mathfrak{D}'$? In case A is either a polynomial ring or a ring of power series, is $\mathfrak{D}/\mathfrak{D}'$ an algebraic group over k?

T. Nakayama (Nagoya)

Hoechsmann, Klaus 3491

Algebras split by a given purely inseparable field.

Proc. Amer. Math. Soc. **14** (1963), 768–776.

The construction of central simple algebras (of finite dimension) over a field K as a cross product (N,Γ,α) is known to hold also in the following situation [Teichmüller, Deutsche Math. 1 (1936), 92–102]: Γ is a finite group of automorphisms of a commutative ring $N=N_1\oplus\cdots\oplus N_n$ which is a direct sum of finite field extensions N_i of K, and K is the field of all elements fixed by Γ; $\alpha\in H^2(\Gamma,N^*)$ and Γ is assumed to be transitive on the set of fields $\{N_i\}$, and the identity is the only automorphism of Γ leaving all the elements of any of the fields N_i fixed. Let K be a field of characteristic p, and let E be an inseparable extension of K. Assume that E^* contains a maximal multiplicative set $X\supseteq K^*$ such that the cosets of X mod K^* form a K-linear basis of E, then the group $G=X/K^*$ (which is independent of X) is a finite group. The author proves that a K-central simple algebra A such that $(A:K)=(E:K)^2$ is split by E if and only if $A\cong(N,\Gamma,\xi)$ for some N, $\xi\in H^2(\Gamma,K^*)$ and $\Gamma\cong G$. This result is useful for all p-algebras since every p-algebra A over K has a splitting field which is inseparable over K and satisfies the preceding requirements.

S. A. Amitsur (Jerusalem)

Roquette, Peter 3492

Isomorphisms of generic splitting fields of simple algebras.

J. Reine Angew. Math. **214/215** (1964), 207–226.

Let k be a commutative field, and A a k-central simple algebra of finite dimension and of Schur exponent S. The author has defined a generic splitting field $F_m(A)$ (a Brauer field) for every multiple m of s [Math. Ann. **150** (1963), 411–439; MR **27** #4832]. The question arises under what conditions are two Brauer fields $F_m(A)$ and $F_n(B)$ isomorphic, and the following is proved. Theorem 1: If $m\geq n$ and $A\sim B^l$ for $l\geq 0$, then $F_n(A)$ is isomorphic with a subset of $F_m(B)$. The proof depends on the following interesting lemma: Let F, K be two field extensions of an infinite field k and of finite transcendence degree (tr.d.); if F is isomorphic with a subfield of a purely transcendental extension of K and $\mathrm{tr.d.}(F/k)\leq\mathrm{tr.d.}(K/k)$, then F is isomorphic with a subfield of K itself. The second result is Theorem 2: If l is prime to the exponent e of A, then $F_m(A)\cong F_m(A^l)$ if one of the following holds: (a) $m>s$, (b) $l\equiv -1 \pmod e$, (c) A has a Galois splitting field F which is cyclic over k and of degree s over k. The latter result is extended to the case that F is solvable over k. This last result covers the case of all algebras A over algebraic number fields.

{Reviewer's remark: A more general construction of $F_m(A)$ was introduced by the reviewer for the case $m^2=(A:k)$ [Ann. of Math. (2) **62** (1955), 8–43; MR **17**, 9], and there the cyclic part of case (c) of Theorem 2 was proved. This proof is asserted by the author to depend on a corollary which he claims contains an error; but in fact this proof is correct and does not depend on the quoted corollary.}

S. A. Amitsur (Jerusalem)

Bostock, F. A.; Patterson, E. M. 3493

A generalisation of Divinsky's radical.

Proc. Glasgow Math. Assoc. **6**, 75–87 (1963).

Let A be an associative ring, M a right A-module and F a set of A-endomorphisms of M closed under the circle operation: $f\circ g=f+g-fg$. An F-permissible module N is a submodule of M with the property that for every $m\in N$ there exists $f\in F$ such that $f(m)=m$. It is shown that M contains a maximal F-permissible submodule $\Delta(M,F)$ which satisfies the basic properties of a radical. This is a generalization of a radical introduced by Divinsky [Proc. Amer. Math. Soc. **9** (1958), 62–71; MR **20** #52], which is obtained for $M=A$ and F the ring of all left multiplications by the elements of the ring A. It is shown that the ascending chain condition for submodules of M, or the descending chain condition for left ideals in F, implies the existence of a $g\in F$ which leaves every element of $\Delta(M,F)$ invariant. Let $M_s(\cdot)$ denote the set of all finite-rowed matrices over $(\cdot)$ and let $P(\cdot)$ be the set of all polynomials in $(\cdot)$; then $\Delta(M_s(M), M_s(F))=M_s[\Delta(M,F)]$ and $\Delta(P(M),P(F))=P[\Delta(M,F)]$.

S. A. Amitsur (Jerusalem)

Kegel, Otto H. 3494

A remark on maximal subrings.

Michigan Math. J. **11** (1964), 251–255.

The following theorem is proved. If the maximal subring M of the ring R is solvable, then M is an ideal. The set of all nilpotent elements of R is a solvable ideal; it is weakly nilpotent if M is weakly nilpotent.

Z. Janko (Canberra)

Kegel, Otto H. 3495

On rings that are sums of two subrings.

J. Algebra **1** (1964), 103–109.

Let R be a ring, A and B subrings of R, and let $R = A + B$. If A and B are nilpotent, the author showed that R is also [Math. Ann. **149** (1962/63), 258–260; MR **28** #3049]. This note generalizes this result in several directions. The ideal I of R is solvably [nilpotently] embedded in R if for every epimorphism f of R such that $f(I) \neq 0$, there is an ideal J of $f(R)$ contained in $f(I)$ such that $J^2 = 0$ $[f(R)J = Jf(R) = 0]$. $S(R)$ denotes the sum of all solvably embedded ideals of R and $N(R)$ the sum of all nilpotently embedded ideals of R. These two ideals are the radicals of the two properties and behave in the usual manner. If $R = A + B$, then $N(A)N(B) = N(B)N(A) = 0 \bmod S(R)$. The author also studies an analogous situation involving descending chains of ideals, but the details are too complicated to be reproduced here. He then raises the following: (1) If $R = A + B$, is the ring generated by the Levitzki radicals of A and B locally nilpotent or at least nil? (2) If $R = A + B$, where A and B are proper subrings with nonzero Levitzki radicals, is R not simple? As a partial answer to (1) the author shows that if $R = A + B$, with A locally nilpotent, then the subring generated by A and $N(B)$ is locally nilpotent and $N(B)$ is contained in the Levitzki radical of R. *A. Rosenberg* (Ithaca, N.Y.)

Herstein, Israel N. 3496

Sul teorema di Goldie.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **35** (1963), 23–26.

Goldie bewies, dass ein Primring, der gewisse Maximalbedingungen für Rechtsideale erfüllt, immer einen Quotientenring besitzt, der ein einfacher Ring mit Minimalbedingung ist [Proc. London Math. Soc. (3) 8 (1958), 589–608; MR **21** #1988]. Der Beweis ist von, u.a., C. Procesi vereinfacht worden [erscheint in Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **34** (1963), 372–377; MR **28** #2130]. In einer späteren Arbeit [Proc. London Math. Soc. (3) **10** (1960), 201–220; MR **22** #2627], konnte dann Goldie beweisen, dass auch ein Ring, der keine nilpotenten Ideale enthält und dieselben Maximalbedingungen erfüllt, einen Quotientenring besitzt, der ein halbeinfacher Ring mit Minimalbedingung ist. Der Verfasser gibt hier einen verhältnismässig einfachen Beweis dieser letzten Tatsache unter Benutzung des ersten obigen Resultates. Der Verfasser hofft, dass die Beweisführung von Procesi sowie die Vorliegende, die Sätze von Goldie den Studierenden der höheren Semester, mit etwaigen italienischen Kenntnissen, leicht zugänglich machen werden. Die Sätze können nämlich jetzt gleich nach den Wedderburnschen Sätzen gebracht werden.

A. Rosenberg (Ithaca, N.Y.)

Herstein, I. N.; Small, Lance 3497

Nil rings satisfying certain chain conditions.

Canad. J. Math. **16** (1964), 771–776.

If every element of the ring R is nilpotent, then R need not be nilpotent (not even locally, as a recent result of Golod shows), but in the presence of suitable finiteness conditions, nilpotency can be inferred. The authors conjecture that any nil ring satisfying the ascending chain condition for left annihilators is nilpotent. Without any further conditions they prove (Lemma 1) that any homomorphic image $H \neq 0$ of such a ring has an ideal $I \neq 0$ with $I^2 = 0$. This result is strengthened by aid of further conditions. Theorem 1: If the ring R satisfies the ascending chain condition on left and on right annihilators, then any nil subring of R is nilpotent. Similarly, Theorem 3: If the nil ring R satisfies the ascending chain condition on left annihilators and on direct sums of left ideals, then R is nilpotent.

In a slightly different direction they establish (Theorem 2): If the ring R satisfies a polynomial identity, R is locally nilpotent exactly if there are two nil subrings A and B of R such that $R = A + B$.

{Remark: In connection with Theorem 1, compare also J. Levitzki [Israel J. Math. **1** (1963), 215–216; MR **29** #1230].} *O. H. Kegel* (Frankfurt a.M.)

Kertész, Andor 3498

Über artinsche Ringe.

Wiss. Z. Humboldt-Univ. Berlin Math.-Natur. Reihe **12** (1962/63), 823–826.

An Artin ring is defined as an associative ring in which the left ideals satisfy the descending chain condition. This expository paper surveys what is known of the structure of Artin rings, and then gives a list of six open problems.

R. C. Hartshorne (Cambridge, Mass.)

Pareigis, Bodo 3499

Einige Bemerkungen über Frobenius-Erweiterungen.

Math. Ann. **153** (1964), 1–13.

For a Frobenius extension R/S [Kasch, S.-B. Heidelberger Akad. Wiss. Math.-Natur. Kl. **1960/61**, 87–109; MR **24** #A1932; Nakayama and Tsuzuku, Nagoya Math. J. **17** (1960), 89–110; MR **23** #A1687; ibid. **19** (1961), 127–148; MR **25** #3959a; ibid. **20** (1962), 205; MR **25** #3959b], Kasch [Math. Z. **77** (1961), 219–227; MR **28** #1227] has introduced its so-called Nakayama automorphism (N.A.) as an automorphism of the centralizer $P = Z_R(S)$ of S in R. If R/S is β-Frobenius and β can be extended to an automorphism α of R, then the ring $\mathrm{Hom}(R_S, R_S)$ is α^{-1}-Frobenius over (the left-multiplication ring of) R, and the author proves first that (the centralizer of R in $\mathrm{Hom}(R_S, R_S)$ is the right-multiplication of P on R and) if ψ^* is an N.A. of R/S, then the N.A. of $\mathrm{Hom}(R_S, R_S)/R$ is given by what corresponds to $\psi^*\alpha^{-1}$ on P. Then, for $R \supset S \supset T$, he asks: Assuming that two of the extensions R/S, S/T, R/T are Frobenius, is the third one also Frobenius? For instance, it is shown that if S/T is 1-Frobenius, $T \subseteq Z_S(S)$ and R/T is β-Frobenius with the N.A. mapping S $(\subseteq Z_R(T))$ onto S, and if R_S is projective, then R/S is α-Frobenius. Finally, after Gruenberg and Kasch [cf. MR **28** #1227 cited] the notion of Spur is defined for general (non-free) Frobenius extensions, considering certain dual systems (which are not bases); and by means of the Spur the Maschke-Ikeda-Kasch characterization of relative projective and injective modules is proved for such a general case.

T. Nakayama (Nagoya)

van Leeuwen, L. C. A. 3500

On ring extensions of Szép.

Nederl. Akad. Wetensch. Proc. Ser. A **67** = *Indag. Math.* **26** (1964), 40–47.

Let the additive group of a ring $\mathfrak{R}$ be a direct sum of the additive groups of two of its subrings R, P. The ring $\mathfrak{R}$

is then isomorphic to the set $\{(a, \alpha) | a \in R, \alpha \in P\}$, where addition is defined componentwise, and multiplication is given by $(a, \alpha)(b, \beta) = (ab + \alpha_l b + a\beta_r + a_l\beta + \alpha b_r + \alpha\beta)$, where α_l, β_r, a_l, b_r are homomorphisms satisfying obvious but complicated conditions. Additional restrictions on these maps are given to obtain some decomposition theorems and to provide a unit in $\mathscr{R}$. *S. A. Amitsur* (Jerusalem)

Raynaud, Michèle 3501

Solution d'un problème universel relatif aux modules projectifs de classe zéro.

C. R. Acad. Sci. Paris **258** (1964), 2457–2460.

A projective module P over a commutative ring A is of class zero if $P \oplus A^q = A^n$ for integers $q \leq n$. In this note the authors shows that the problem of finding such P can be reduced to the study of a special class of rings of the form $A_0 = Z[X_{ij}, Y_{jl}]/I$, where Z is the ring of integers and I is the ideal generated by elements of the form $\sum_i X_{ji} Y_{ik} - \delta_{jk}$ in the polynomial ring over Z. It turns out that modules P of class zero over A are obtained as $P = P_0 \otimes_{A_0} A$, where P_0 is of class zero over A_0.

Finally, the author investigates the properties of the rings A_0; they are regular, unique factorization domains. *J. P. Jans* (Seattle, Wash.)

Faith, Carl 3502

A structure theory for semialgebraic extensions of division algebras.

J. Reine Angew. Math. **209** (1962), 144–162.

Let Φ be a commutative field, $A \supset B$ $(A \neq B)$ two Φ-algebras and S a set of polynomials in $\Phi[x]$ each with a zero-free coefficient. The algebra A is an (S-) semi-algebraic (s.a.) extension of B if for every $a \in A$ there is a polynomial $f_a[x] \in S$ such that $f_a[a] \in B$. The paper deals with the problem of determining the proper s.a. extensions of an algebra B. If $A = D_n$, D a division algebra, and A is a proper s.a. extension of a subalgebra B, then D is an algebraic algebra and A itself is algebraic in each of the following cases: (1) D is commutative, (2) Φ is not countably finite, (3) B is a division ring and $\Phi \neq \mathrm{GF}[2]$. The s.a. extensions of a division algebra B are of the form $Q \oplus P$, where P is an algebraic algebra and Q is a directly irreducible semi-algebraic extension of B. In case $\Phi \neq \mathrm{GF}[2]$, then the only possible case in which $Q \neq 0$ is that Q be of characteristic p and Q is a p-radical extension of B, i.e., for every $q \in Q$, some power $q^{p^n} \in B$. Additional results are obtained from s.a. extensions of bounded degree, and some of the previous results are extended to Φ-algebras, where Φ is an arbitrary commutative ring. *S. A. Amitsur* (Jerusalem)

Faith, Carl; Utumi, Yuzo 3503

Quasi-injective modules and their endomorphism rings.

Arch. Math. **15** (1964), 166–174.

Some results on injective modules are extended to quasi-injective modules. For example, let M_R be a quasi-injective module over a ring R, S its endomorphism ring, and J the Jacobson radical of S. Then it is shown that $J = \{b \in S | M_R$ is an essential extension of ker $b\}$, S/J is a regular ring, S_S is injective if $J = 0$, and every finite set of orthogonal idempotents of S/J can be lifted to S. It is also shown that if a ring R with unity is regular or left self-injective and I is a right ideal of R, then every set $\{u_1 + I, \cdots, u_n + I\}$ of orthogonal idempotents of $R - I$ (i.e., $u_i - u_i^2, u_i u_j \in I$ for all i, j with $i \neq j$) can be lifted to R. *R. E. Johnson* (Rochester, N.Y.)

Osofsky, B. L. 3504

On ring properties of injective hulls.

Canad. Math. Bull. **7** (1964), 405–413.

Let R be a ring with unity and $\hat{R}$ the injective hull of the module R_R. It is shown by one example that $\hat{R}$ need not be a ring extension of R and by another example that $\hat{R}$ might be a ring extension of R greater than the Utumi ring of quotients of R. If $\hat{R}$ is a ring extension of R, then $\mathrm{Hom}_R(\hat{R}, \hat{R})$ is shown to be an injective $\hat{R}$-module. *R. E. Johnson* (Rochester, N.Y.)

Ioffe, L. S. 3505

The radical of a module. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 820–826.

The author introduces the radical associated with the following definition of semisimplicity for unital left Λ-modules A: $0 \notin \lambda(\Lambda - \{0\})$ implies $0 \notin \lambda(A - \{0\})$. After some preliminaries, he is concerned with conditions on Λ for the radical (in every A) to consist just of the elements of A annihilated (each) by some non-left zero divisor. His condition (sufficient and, under side conditions, necessary) is that for every λ and μ, λ not a left zero divisor, there exist ν and π, ν not a left zero divisor, such that $\nu\mu = \pi\lambda$. *J. R. Isbell* (New Orleans, La.)

Budach, Lothar; Kerstan, Johannes 3506

Über eine Charakterisierung der Grellschen Schemata.

Math. Nachr. **27** (1963/64), 253–264.

The authors give a characterization of Grell rings (noetherian integral domains of rank one) in the language of scheme theory. A subset M of the affine scheme $X = \mathrm{Spec}(A)$ is called an "absteigend" subset if any generalization of a point in M is also in M, and a subset N is called β-closed if $(N, \tilde{A}|N)$ is an affine scheme. Budach showed in an earlier paper [same Nachr. **25** (1963), 339–380; MR **28** #89] that any "absteigend" subset is β-closed if A is a Grell ring. In this paper it is shown that this property is also sufficient for a noetherian integral domain A to be a Grell ring provided A satisfies a certain condition. (This condition is shown to be superfluous by Budach [#3507 below].) *Y. Nakai* (Hiroshima)

Budach, L. 3507

Zum Begriff des Modulquotienten.

Monatsb. Deutsch. Akad. Wiss. Berlin **6** (1964), 81–85.

Let A be a commutative ring with unit and consider the category of all A-modules. For each finitely generated ideal $\mathfrak{a}$ of A, the notion of quotient functor is defined, and the most important example is the functor: $M \to \mathrm{Hom}(\mathfrak{a}, M/t(\langle\mathfrak{a}\rangle, M))$, where $t(\langle\mathfrak{a}\rangle, M)$ is the set of all m in M which are annihilated by certain powers of $\mathfrak{a}$. The author discusses also some other functors related to this. *M. Nagata* (Kyoto)

Sehgal, S. K. 3508

Jacobson theory of ringoids.

Notre Dame J. Formal Logic 4 (1963), 206–215.

A ringoid is a set R together with two partially defined binary operations that, insofar as they are defined, satisfy the axioms for a ring. For example, if $a+b$ is defined, then so is $b+a$, and $a+b=b+a$. Also, for each $a \in R$ there is $0_a \in R$ such that $a+0_a=a$ and $0_a+x=x$ whenever 0_a+x is defined. The Jacobson theory of rings is extended to ringoids, with definitions and proofs modeled on the theory for rings. In conclusion, Barratt ringoids are examined. The question is raised whether "in a general ringoid, if ab and bc are defined, then $a(bc)$ is defined".

S. Stein (Davis, Calif.)

Betsch, Gerhard 3509

★Struktursätze für Fastringe.

Inaugural-Dissertation zur Erlangung des Grades eines Doktors der Naturwissenschaften der Mathematisch-Naturwissenschaftlichen Fakultät der Eberhard-Karls-Universität zu Tübingen.

Paul Jllg, Photo-Offsetdruck, Stuttgart, 1963. 62 *pp.*

An additively written group Γ is an N-group for the near-ring N if and only if γn is defined in N so that $\gamma(n+m)=\gamma n+\gamma m$ and $\gamma(nm)=(\gamma n)m$ for all $\gamma \in \Gamma$ and $n, m \in N$. $\Gamma \neq \{0\}$ is (1) essentially minimal if and only if $\Gamma N \neq \{0\}$ and Γ contains only the trivial N-subgroups; (2) monogenous if and only if $\Gamma=\gamma N$ for some $\gamma \in \Gamma$; (3) strongly monogenous if and only if $\Gamma N \neq \{0\}$ and $\gamma N=\Gamma$ or $\{0\}$ for each $\gamma \in \Gamma$. The author defines four radical-like ideals in N: J_0 is the intersection of $A(\Gamma)$ of irreducible, monogenous N-groups Γ; J_1 is the intersection of $A(\Gamma)$ of irreducible, strongly monogenous N-groups Γ; J_2 is the intersection of $A(\Gamma)$ of essentially minimal N-groups Γ; D is the intersection of maximal modular right ideals of N. (Here $A(\Gamma)$ is the set of all n in N such that $\Gamma n=\{0\}$.) If the required N-groups or right ideals do not exist, then the corresponding radical is defined to be N. The author proves that $J_0 \subseteq D \subseteq J_1 \subseteq J_2$ and gives examples where $D \subset J_1=J_2$, $D=J_1 \subset J_2$, and $J_0 \subset D$. He also obtains various structure theorems for the different "semisimple" near-rings, including and generalizing results of D. W. Blackett, A. Fröhlich, R. R. Laxton, H. Wielandt, and the reviewer. Finally, he extends some results of Arens and Kaplansky on biregular rings to certain near-rings, paralleling the development in Chapter 9 of N. Jacobson's *Structure of rings* [Amer. Math. Soc., Providence, R.I., 1956; MR 18, 373].

W. E. Deskins (E. Lansing, Mich.)

NON-ASSOCIATIVE ALGEBRA

Sagle, Arthur A. 3510

On anti-commutative algebras with an invariant form.

Canad. J. Math. 16 (1964), 370–378.

Algebras which are anti-commutative and possess a symmetric bilinear form $f(x, y)$ such that $f(xy, z)=f(x, yz)$ for all x, y, z in the algebra are considered. Lie and Mal'cev algebras are examples of such systems. Generalizations of Lie and Mal'cev algebras are obtained by taking the non-commutative Jordan algebras $\mathfrak{A}$ of L. J. Paige [Portugal. Math. 16 (1957), 15–18; MR 20 #5795] and considering the algebras $\mathfrak{A}^-$. The anti-commutative algebra $\mathfrak{A}^0=\mathfrak{A}^-/\mathfrak{C}$, where $\mathfrak{C}$ is the set of all x in $\mathfrak{A}^-$ such that $xy-yx=0$ for each y in $\mathfrak{A}^-$, is used to construct other simple non-commutative Jordan and anti-commutative algebras. It is proved that $\mathfrak{A}^0$ is simple if and only if $f(x, y)$ is non-degenerate on $\mathfrak{A}$. It is also proved that if $\mathfrak{A}$ is any finite-dimensional, anti-commutative algebra with invariant form $f(x, y)$ over F of characteristic not 2, then there exists a non-commutative Jordan algebra $\mathfrak{B}$ with identity element 1 such that $\mathfrak{B}^-/1F$ is isomorphic to $\mathfrak{A}$. If $f(x, y)$ is non-degenerate and the mapping $x \to R_x$, where R_x is right multiplication by x, is injective, then $\mathfrak{B}$ is simple.

L. A. Kokoris (Chicago, Ill.)

Martindale, Wallace S., 3rd 3511

Lie derivations of primitive rings.

Michigan Math. J. 11 (1964), 183–187.

Let R be a ring with associated Lie ring R_L. Some information has accumulated about the relation of the structure and mappings of R to those of R_L. In particular, derivations of R are R_L derivations and under severe restrictions on R may exhaust or almost exhaust the R_L derivations. The author adds to what is known in this matter as follows: If R is primitive, is not of characteristic 2, and contains a non-trivial idempotent, then R can be imbedded in a primitive ring $\bar{R}$ such that every derivation of R_L is induced by the sum of a derivation of R into $\bar{R}$ and a trace-like additive mapping sending R into the center of $\bar{R}$ and $[R, R]$ into 0.

W. G. Lister (Stony Brook, N.Y.)

Romberg, Walter 3512

Pseudobewertungen und Arithmetik in nichtassoziativen Algebren. I.

Math. Nachr. 26 (1963/64), 287–306.

Suppose that A/k is a non-associative algebra with multiplicative unit of finite rank over the field k. The author considers real-valued functions w on A, called pseudo-valuations, for which (I) $w(a)$ is finite for $a \neq 0$, $w(0)=\infty$; (II) $w(a-b) \geq \min(w(a), w(b))$; (III) $w(ab) \geq w(a)+w(b)$; (IV) given $\alpha=w(a)$ for some $a \in A$, then there exists at least one "α-w-element" $a_\alpha \in A$ such that $w(a_\alpha)=\alpha$ and $w(a_\alpha b)=\alpha+w(b)$ for all $b \in A$; (V) the restriction of w to k is a valuation, $\mathfrak{k}$ the associated residue class field. As in the associative theory of H. Benz [J. Reine Angew. Math. 209 (1962), 72–81; MR 26 #139] the valuation ring, residue class algebra $\mathfrak{R}$, ramification degree e and residue class degree f and completion are defined. If the restriction of w to k is discrete, then the customary product formula $ef=[A:k]$ for ramification and residue class degrees for complete A holds. Next, denote by $L(a)$ the left transformation $x \to ax$, $x \in A$. Suppose that $\gamma, \delta \in \{w(z), z \in A\}=\Gamma$ and for γ, δ, $\gamma+\delta$ pick w-elements π_γ, π_δ, $\pi_{\gamma+\delta}$. Then

$$w((\pi_\gamma a \cdot \pi_\delta b)L(\pi_{\gamma+\delta})^{-1}) \geq w(a)+w(b),$$

whence $\vartheta_{\gamma,\delta}(a \bmod w, b \bmod w)=(\pi_\gamma a \cdot \pi_\delta b)L(\pi_{\gamma+\delta})^{-1} \bmod w$, $w=\{z \in A,\ w(a)>0\}$ defines a two-sided distributive multiplication on the additive group $\mathfrak{R}$ considered as a $\mathfrak{k}$-module, thus giving rise to an algebra $\mathfrak{R}(\gamma, \delta)$. Normalization of the discrete value group to Γ gives rise to e^2 algebras, $0 \leq \gamma, \delta < e$, $\mathfrak{R}(\gamma, \delta)$ which make up a "w-$\mathfrak{R}$-system" $\mathfrak{S}$. This system is uniquely determined to within isotopy, i.e., different choices of the elements

π_γ, π_δ, $\pi_{\gamma+\delta}$ lead to isotopic algebras. A system $\mathfrak{S}$ is termed irreducible if a $\mathfrak{k}$-submodule of $\mathfrak{R}$ which is a two-sided ideal in all $\mathfrak{R}(\gamma, \delta)$ is necessarily equal to 0 or to $\mathfrak{R}$. The author proves that an algebra A/k is two-sided simple and each element of its center is a w-element if the associated w-$\mathfrak{R}$-system is irreducible. Furthermore, the existence of algebras A/k with prescribed w-$\mathfrak{R}$-systems $\mathfrak{S}$ is established. Finally, the general results are applied to the study of arithmetic in Cayley algebras, Jordan algebras and the lifting of idempotents (Hensel's Lemma) for complete algebras. *O. F. G. Schilling* (Lafayette, Ind.)

Oehmke, Robert H.; Sandler, Reuben 3513

The collineation groups of division ring planes. I. Jordan division algebras.

J. Reine Angew. Math. **216** (1964), 67–87.

Given a projective plane π and four points in π, no three of which are collinear, a coordinate ternary ring R may be defined. Starting with another (ordered) set of four points, the ternary ring S obtained is isomorphic with R if and only if π admits a collineation mapping one ordered set of points onto the other. Earlier results are summarized relating properties of R, each of which is a weaker condition than that R be an associative division ring, to transitivity properties of central collineations in π. The relation of isotopy between (non-associative) division rings R, S associated with planes π, π' is equivalent with isomorphism of the planes when neither R nor S is alternative. When R is a non-alternative division ring coordinatizing π, the authors study the collineation group $G(\pi)$. Among the collineations are the shears and translations; the group generated by these is the group $E(\pi)$ of elementary collineations. Those collineations which fix both of two specially chosen points form a group $H(\pi)$, and $G(\pi)=H(\pi)E(\pi)$, $E(\pi)$ being a normal subgroup. The group $H(\pi)$ may be identified with the group $A(R)$ of autotopisms of R; in this identification, the group $B(R)$ of automorphisms of R is identified with the subgroup $H_1(\pi)$ of $H(\pi)$ fixing a third point whose coordinates are $(1, 1)$. Thus the cosets of $H_1(\pi)$ in $H(\pi)$ correspond to the points in the orbit of $(1, 1)$ under $H(\pi)$. Such points (a, b) are called admissible pairs; associated with each pair is a coordinatization of π, and the pair is admissible if and only if the new coordinate ring $S_{a,b}$ is isomorphic with R. When R is a finite-dimensional (commutative) Jordan division algebra over a field of characteristic $\neq 2, 3$, the admissible pairs (a, b) are shown to be those with both a and b in Z, the center of R, i.e., all products of three elements of R involving at least one of a, b are associative. This result yields a normal subgroup $H_2(\pi)$ of $H(\pi)$, $H_2(\pi)$ being isomorphic to the direct sum $Z+Z$, with $H_1(\pi)=H(\pi)/H_2(\pi)$. A normal series for $G(\pi)$ is thus obtained, whose factors are either abelian or (in the case of one factor) the automorphisms of R.

G. B. Seligman (New Haven, Conn.)

Thedy, A. 3514

Note zu einer Arbeit von R. H. Oehmke und R. Sandler.

J. Reine Angew. Math. **216** (1964), 88–90.

The determination of admissible pairs in the paper above [#3513] is made to depend ultimately on the following: If R is a finite-dimensional simple Jordan algebra over an algebraically closed field of characteristic $\neq 2, 3$ with center Z, and if $(zm)y=z(my)$ for all $z, y \in R$, then $m \in Z$. Oehmke and Sandler prove this result by using the classification of simple Jordan algebras. The author shows that the conclusion holds for R a commutative algebra over a ring K admitting an associative linear form λ whose associated bilinear form is non-singular (thus $\lambda((ab)c)=\lambda(a(bc))$ and $\langle a, b\rangle=\lambda(ab)$ is non-singular). Unpublished work of Artin, Hel Braun and Koecher is known to the author to have established the existence of such a linear form in cases which include those needed by Oehmke and Sandler.

G. B. Seligman (New Haven, Conn.)

Kosier, Frank 3515

A generalization of alternative rings.

Trans. Amer. Math. Soc. **112** (1964), 32–42.

Rings satisfying

$$(x^2, y, z) = x \circ (x, y, z), \qquad (z, y, x^2) = x \circ (z, y, x),$$

and $(x, x, x)=0$ are studied, where $x \circ y = xy+yx$. It is known that alternative rings satisfy these identities, so that the rings studied here are generalizations of alternative rings. Rings satisfying the above three identities are shown to be power-associative. Although the decomposition of a power-associative algebra is known, a decomposition $A=A_1+A_{1/2}+A_0$ relative to an idempotent e is obtained directly, where x is in A_i if and only if $e \circ x = 2ix$. It is also proved that A_1 and A_0 are orthogonal subrings, $A_iA_{1/2}+A_{1/2}A_i \subseteq A_{1/2}$ for $i=0, 1$, $(x_i, y_{1/2}, e)=(e, y_{1/2}, x_i)$ and $x_{1/2} \circ y_{1/2}$ is in A_0+A_1 for all x_i in A_i, $i=0, 1$, and all $x_{1/2}$, $y_{1/2}$ in $A_{1/2}$. Using these results, it is proved that if A has no non-zero ideal such that $x^2=0$ for every x in the ideal, then, for some idempotent e, $A=A_{11}+A_{10}+A_{01}+A_{00}$, where x is in A_{ij} if and only if $ex=ix$, $xe=jx$. Also, $A_{ij}A_{km}=\delta_{jk}A_{im}$, except that $A_{ij}^2 \subseteq A_{ji}$ for $i \neq j$. These results and further consequences of this decomposition are used to prove that a simple ring with an idempotent $e \neq 1$ satisfying the defining identities is either an associative ring or a Cayley-Dickson algebra over its center. The fact that a semi-simple algebra has a unity element and is a direct sum of simple algebras follows from the proof of the corresponding results in Kleinfeld, Kosier, Osborn, and Rodabaugh [same Trans. **110** (1964), 473–483; MR **28** #1221]. Some examples of algebras satisfying the identities of this paper are given.

L. A. Kokoris (Chicago, Ill.)

HOMOLOGICAL ALGEBRA

See also 3501, 3524, 3536, 3866.

Eilenberg, Samuel 3516

Homological algebra. (Spanish)

An. Inst. Mat. Univ. Nac. Autónoma México **1** (1961), 117–145.

Expository lecture.

Freyd, Peter 3517

★Abelian categories. An introduction to the theory of functors.

Harper's Series in Modern Mathematics.

Harper & Row, Publishers, New York, 1964. xi + 164 pp. \$7.00.

The author of this remarkable little book evidently believes, with the reviewer, that the theory of categories and functors now exists as an autonomous part of mathematics, separate from homological algebra, and that no more excuse is necessary for the publication of a book (as opposed to a research paper) on the subject than is required for a book on point-set topology or group theory.

The book he has chosen to write has the form of a textbook and indeed devotes a considerable amount of its space to exercises. The formal exposition takes, as the author observes, a "geodesic course" towards the Mitchell embedding theorem [Mitchell, Amer. J. Math. **86** (1964), 619–637; MR **29** #4783] which asserts that a small abelian category (i.e., one with a set of objects; the index reference is to "Kittygory") may be embedded as a full exact subcategory of a category of modules.

This geodesic course leads through a chapter on fundamentals of categories, e.g., subobjects, difference kernels and completeness, and a chapter on fundamental notions for abelian categories, such as additivity, pullback and pushout diagrams (a diagram which is both is a "Doolittle" diagram) and the nine-lemma. These are followed (in Chapter 3) by considerations at the same level for functors. The next chapter gives the precise statement of the embedding theorem, proves it for complete abelian categories with projective generators, and adds some metatheorems to indicate its importance.

Chapter 5 introduces the heavy machinery, viz., the theory of functor-categories and the representation functor. Chapter 6 treats injective envelopes. Chapter 7 develops the properties of the category of left exact functors from a small abelian category into the category of abelian groups: this category is complete, abelian, has injective envelopes and an injective cogenerator, and the representation functor is a full exact embedding of the domain category into it. From these results the Mitchell theorem follows immediately.

The selection of the Mitchell theorem as the goal, and its proof as the organizing principle, of the book is perhaps somewhat idiosyncratic. On the one hand, the economy of the "geodesic course" means that the many notions introduced are very little explored. On the other, new embedding theorems, e.g., that of Gabriel and Popesco [#3518 below] are rapidly being added to the literature of the field, which is, of course, in considerable flux. Indeed, the author himself has shown [Proc. Nat. Acad. Sci. U.S.A. **49** (1963), 19–20; MR **26** #3756] that the representation embedding into the category of left exact functors has virtues not touched on in the book. But it is, no doubt, a defense against this accusation of idiosyncracy that the adoption of this organization may have been the only way of producing in finite time a book on a rapidly changing field.

Moreover, to describe this book solely in terms of its formal expository content would be grossly to misdescribe it. The exercises occupy an amount of text roughly comparable to the exposition and, indeed, are perhaps not properly labelled as exercises. They consist rather of outline treatments of large parts of category theory, which the reader is invited to fill in for himself. Thus the exercises in Chapter 2 treat the fundamentals of the theory of additive categories and the notion of group in a category. Those for Chapter 3 contain the notions of adjoint functors and the author's important existence theorem for adjoint functors, as well as an interesting application to (logical) model theory. The pace is maintained throughout: an exercise in Chapter 7 is to prove that the category of sheaves on a topological space is abelian, using the purely functor-theoretic methods developed there.

This is a "textbook" of course only for the reader of considerable mathematical maturity. For such a reader it can provide a rapid induction into this important new field.

A. Heller (Urbana, Ill.)

Popesco, Nicolae; Gabriel, Pierre 3518

Caractérisation des catégories abéliennes avec générateurs et limites inductives exactes.

C. R. Acad. Sci. Paris **258** (1964), 4188–4190.

This is an important addition to the growing family of embedding theorems for abelian categories [cf. #3517 above].

If C is a Grothendieck category, i.e., is abelian with exact direct limits, and U is a generator, then $M \to \operatorname{Hom}(U, M)$ embeds C in the category of right $\operatorname{Hom}(U, U)$ modules. The image is identified as a localization in the sense of Gabriel [Bull. Soc. Math. France **90** (1962), 323–448] with respect to a complete Serre subcategory. Thus C is equivalent to a quotient-category of the category of modules. The methods are those of the latter reference.

A. Heller (Urbana, Ill.)

You, Cheng-ye 3519

A proof of Pitcher inequalities of the critical point theory. (Chinese. English summary)

Acta Sci. Natur. Univ. Pekinensis **10** (1964), 207–210.

Author's summary: "Pitcher's proof of his inequalities

$$\bar{M}_p \geq R_p+\eta_p+\eta_{p-1},$$

$$\sum_{p=0}^{k}(-1)^{k-p}\bar{M}_p \geq \sum_{p=0}^{k}(-1)^{k-p}R_p+\eta_k \quad (k=0,\cdots,n)$$

of the critical point theory [Bull. Amer. Math. Soc. **64** (1958), 1–30; MR **20** #2648] is based on the concept of 'skeleton'. Our proof given in the present paper is simpler and closer to the usual proof of Morse's inequalities. An essential step is the following lemma. Let H_i be finitely generated modules over a principal ideal domain, $i=1, 2, 3$. If $H_1 \xrightarrow{\lambda} H_2 \xrightarrow{\mu} H_3$ is exact, then

$$R(\operatorname{Ker}\lambda)+E(H_1)+E(H_3) \geq E(H_2),$$

where $R(\operatorname{Ker}\lambda)$ is the rank of the kernel of the homomorphism λ, and $E(H_i)$ is the number of torsion coefficients of H_i. We give also an outline of another simple proof of Pitcher's inequalities by means of the universal coefficient theorem."

Drbohlav, Karel 3520

Concerning representations of small categories.

Comment. Math. Univ. Carolinae **4** (1963), 147–151.

The author obtains rather sharp results on the problem: For cardinals m, n, p, if a category has m objects and no coterminal set of n maps, is it isomorphic with a category of sets of power $<p$? The methods used are the familiar ones, which are apparently not powerful enough to solve (even) this problem completely.

J. R. Isbell (New Orleans, La.)

Yang, Kung-wei 3521

On some finite groups and their cohomology.

Pacific J. Math. 14 (1964), 735–740.

A finite group G is said to have periodic cohomology of period k if k is the least positive integer such that $\hat{H}^k(G, Z)$ contains a maximal generator. The following proposition is proved. Let G be a finite group whose 2-Sylow subgroups are not isomorphic to a generalized quaternion group. Then G has periodic cohomology of period 4 if and only if G has a representation

$$G = \{\sigma, \tau : \sigma^s = 1, \tau^t = 1, \tau\sigma\tau^{-1} = \sigma^{-1}\}$$

with the conditions (i) s is an odd integer > 1; (ii) t is a positive even integer prime to s [Cartan and Eilenberg, *Homological algebra*, pp. 260–265, Princeton Univ. Press, Princeton, N.J., 1956; MR 17, 1040]. Further, it is shown that all possible cohomologies of such a group G can be realized by direct sums of G-modules which belong to a specific finite family of G-modules. *K. Kondo* (Tokyo)

Ozeki, Hideki 3522

Chern classes of projective modules.

Nagoya Math. J. 23 (1963), 121–152.

L'auteur définit les classes de Chern d'un module projectif M de type fini et de rang constant sur une algèbre commutative R. Ces classes sont des éléments des modules de cohomologie de R donnés par le complexe des formes multilinéaires alternées sur le module des dérivations de R. La définition utilise une connexion linéaire dans M et repose sur la formule qui exprime les classes de Chern d'un espace fibré vectoriel complexe à partir de la forme de courbure associée à une connexion. L'auteur démontre que les classes ainsi définies ne dépendent pas de la connexion choisie et vérifient la formule du produit pour la somme directe de deux modules.

J. L. Koszul (Grenoble)

Cartier, P. 3523

Sur l'acyclicité du complexe des formes différentielles.

Ann. Scuola Norm. Sup. Pisa (3) 16 (1962), 45–74.

The author proves the Poincaré and Grothendieck lemmas, and their analogues for flows, on the basis of the following two simple algebraic theorems. On the Abelian group M let there be given a system of operators D_i $(1 \le i \le n)$ which commute pairwise, and introduce the differential $d = \sum_{i=1}^n d_i$ into the Z-module $C = M \otimes E$ (where E denotes the exterior algebra over the free Abelian group Z^n with generators $e_1, \cdots, e_n$), where

$$d_i(m \otimes u) = (D_i m) \otimes (e_i \wedge u), \qquad m \in M,\ u \in E.$$

Let us set

$$M_0 = (0), \quad M_i = \sum_{j=1}^{i} D_j(M), \qquad i = 1, \cdots, n.$$

If D_{i+1} induces an injective mapping on M/M_i for every i, $0 \le i < n$, then $H^p(C) = 0$ for $p \ne n$ and $H^n(C) = M/M_n$.

The second theorem is a duality theorem. Let

$$M_0' = M, \quad M_i' = \bigcap_{j=1}^{i} \operatorname{Ker} D_j, \qquad i = 1, \cdots, n.$$

If the operator D_{i+1} is surjective on M_i' for every i, $0 \le i < n$, then $H^p(C) = 0$ for $p \ne 0$, and $H^0(C) = M_n'$.

E. Golod (RŽMat 1963 #6 A307)

GROUP THEORY AND GENERALIZATIONS

See also 3393, 3411, 3446, 3452, 3466, 3573, 3870, 3987, 4040–4042.

Fuchs, L.; Schmidt, E. T. 3524

★Proceedings of the Colloquium on Abelian Groups.

Tihany (Hungary), September 1963.

Akadémiai Kiadó (Publishing House of the Hungarian Academy of Sciences), Budapest, 1964. 162 *pp*. $6.50.

A collection of 15 papers from the colloquium held at Tihany, 2–7 September 1963. The papers will be reviewed individually.

Dubreil, P.; Dubreil-Jacotin, M. L. 3525

★Leçons d'algèbre moderne.

Deuxième édition. Collection Universitaire de Mathématiques.

Dunod, Paris, 1964. vii+401 *pp*. 36.00 *F*.

Contents: (I) Lois de composition; (II) Demi-groupes, groupes; (III) Génération des groupes; (IV) Anneaux, corps; (V) Ensembles ordonnés, treillis; (VI) Axiome de Zorn; (VII) Anneaux noethériens; (VIII) Compléments de théorie des groupes; (IX) Espaces vectoriels; (X) Corps, équations algébriques.

This work was written as a textbook for an already existing course, the content of which was apparently preordained. Possibly for this reason the book is somewhat conservative; indeed, it really could have been written twenty years ago. Nevertheless, it is good. As one would expect from the authors, unusual attention is given to the study of general multiplicatively closed systems. This helps give the book its unique character and effectively illustrates the use of the basic axioms. The style is severe, light on motivation, and with some tendency to prove things in settings considerably more general than their main realm of applicability would justify to many people. Criticism of content would be unfair without knowledge of the specific course for which the book is intended. Suffice it to say that what the book does, it does well and with style.

W. E. Jenner (Chapel Hill, N.C.)

Grindlinger, M. D. [Greendlinger, Martin D.] 3526

On Magnus's generalized word problem. (Russian)

Sibirsk. Mat. Ž. 5 (1964), 955–957.

Let $G = \mathrm{Gp}\{X; \Phi\}$ be a finitely presented group, and for any subset X' of X, denote by H the subgroup of G generated by X'. By Magnus's generalized word problem the author understands the problem of finding an algorithm for deciding when a given word w in X belongs to H (this was solved by Magnus [Math. Ann. 106 (1932), 295–307] when Φ consists of a single relation). Assume that (i) $\Phi = \{R_1, \cdots, R_m\}$ is closed under taking inverses and cyclic conjugates; (ii) each R_i is reduced as a free word; (iii) if $R_i \ne R_j^{-1}$, then cancellation in R_iR_j absorbs less than one-sixth the length of R_i. When these conditions hold and when the left-hand half of each R_i (including

the middle letter, if any) includes a letter not in X', the author gives such an algorithm in explicit form.
P. M. Cohn (Chicago, Ill.)

Grindlinger, E. I. 3527
Solution of the isomorphism problem for a class of semigroups. (Russian)
Sibirsk. Mat. Ž. **5** (1964), 768–792.
The isomorphism problem is solved for a special class of semigroups given by a finite number of generators and relations. Calling a "defining word" the left or right side of one of the defining relations, we can state the condition as follows: no defining word is a proper part of another defining word.

This class includes the class for which the author had solved the word problem [same Ž. **5** (1964), 77–85; MR **28** #3079]. He remarks that he does not known whether the word problem is solvable in the whole class under consideration. *S. Stein* (Davis, Calif.)

Hulanicki, A.; Newman, M. F. 3528
Corrigendum: "Existence of unrestricted direct products with one amalgamated subgroup".
J. London Math. Soc. **39** (1964), 672.
A correction to a recent paper [same J. **38** (1963), 169–175; MR **26** #6232] in which the proof of part (b) of Theorem 2.3 is replaced.

Oyama, Tuyosi 3529
On the groups with the same table of characters as alternating groups.
Osaka J. Math. **1** (1964), no. 1, 91–101.
The author proves the following: A finite group G which has the same table of characters as an alternating group A_n is isomorphic to A_n. By an elaborate examination of the multiplication table of conjugate classes in G, he shows the existence of elements $a_1, \cdots, a_{n-2}$ in G which satisfy the relations $a_1^3=1$, $a_2^2=\cdots=a_{n-2}^2=1$, $(a_ia_{i+1})^3=1$, $(a_ia_j)^2=1$ for $j>i+1$. The argument is more or less similar to that used by the reviewer in proving the similar theorem for a symmetric group [J. Inst. Polytech. Osaka City Univ. Ser. A **8** (1957), 1–8; MR **19**, 387]. The first two lemmas in the paper are used effectively to determine the orders of elements in some conjugate classes.
H. Nagao (Osaka)

Vaida, D. 3530
Un problème de G. Birkhoff. (Russian summary)
C. R. Acad. Bulgare Sci. **15** (1962), 801–803.
Answering a question of the reviewer, the author shows that if $0^+=0$, $x=x^+-(-x)^+$, and $x+y^+-x=(x+y-x)^+$ in a group with unary operation $f(x)=x^+$, then one gets an l-group by postulating either (i) $(x^+-y)^++y=(y^+-x)^++x$, or (ii) $(x^++y^+)^+=x^++y^+$ and (ii') $(x-y)^+=x-y$ if and only if $(x^+-y^+)^+=x^+-y^+$.
G. Birkhoff (Cambridge, Mass.)

Ito, Noboru 3531
On transitive permutation groups of prime degree. (Japanese)
Sûgaku **15** (1963/64), 129–141.
In the first half of the paper, the author gives an excellent exposition of the main problems in the investigation of transitive groups of degree p (p a prime) and several methods which have been used or will be useful for them. Then the following two theorems concerning the solvability of transitive groups of degree p are proved. (1) Let G be a transitive group of degree p and let H be the subgroup of G fixing one letter. If the center of H is not trivial, then G is solvable. (2) Let G be as above. If each irreducible representation of G has degree prime to p, then G is solvable. ((1) has also been obtained by Wielandt independently.) Finally, the author sketches the proof of the following main theorem. A non-solvable transitive group of degree $p=2q+1$, where p and q are both prime, is triply transitive if $p>11$. *H. Nagao* (Osaka)

Learner, A. 3532
Residual properties of polycyclic groups.
Illinois J. Math. **8** (1964), 536–542.
It is proved that if G is polycyclic, there exists a finite set π of primes such that G is residually a finite π-group. In fact, let $G=G_0>G_1>\cdots>G_k=1$ be a normal series for G with factors finite or free abelian, and let τ be a finite non-empty set of primes including all primes dividing the orders of finite factors of this series. Let ω be the product of the primes in τ. Then as π one may take τ, together with all primes dividing the index of the centraliser of G_{i-1}/G_i whenever G_{i-1}/G_i is finite, and all primes dividing the index of the centralizer of $G_{i-1}/G_iG_{i-1}^{\omega}$, whenever G_{i-1}/G_i is infinite. *Graham Higman* (Oxford)

Kohler, Joseph 3533
Finite groups with all maximal subgroups of prime or prime square index.
Canad. J. Math. **16** (1964), 435–442.
A theorem of Huppert [Math. Z. **60** (1954), 409–434; MR **16**, 332] says that if every maximal subgroup of a finite group G is of prime index, then every chief factor of G is of prime order. One of the main results of this paper is a generalization of the above theorem to finite groups with the property that every maximal subgroup is of prime or prime square index (property M). The author proves that if G is a finite group with property M and if the order $|G|$ of G is odd, then every chief factor of G is of prime or prime square order. The oddness of $|G|$ is shown to be necessary by constructing a group of even order with the property M that has a chief factor of order greater than a given number.

The following two theorems are also proved. (1) Let $\Delta(G)$ be the intersection of the non-normal subgroups of G. A finite group G has property M if and only if $G/\Delta(G)$ is a subdirect product of primitive solvable permutation groups on a prime or a prime square number of letters. (2) If $|G|$ is not divisible by 6 and if every maximal subgroup of G has the property M, then G is solvable.
B. Chang (Vancouver, B.C.)

Bauman, S. 3534
Nonsolvable IC groups.
Proc. Amer. Math. Soc. **15** (1964), 823–827.
Two subgroups H and K of a group G are said to be non-incident if $H\not\subset K$ and $K\not\subset H$. A finite group G is said to

be an IC group if for any two non-incident subgroups H and K, $H \cap K$ has only cyclic Sylow groups. The author proves that if G is a nonsolvable IC group then G is isomorphic to one of the following: (i) SL(2, 4), (ii) SL(2, 8), (iii) SL(2, 5).

D. L. Barnett (Dayton, Ohio)

Balcerzyk, S. 3535

On groups of functions defined on Boolean algebras.

Fund. Math. **50** (1961/62), 347–367.

Let G be an Abelian group of cardinality $\mathfrak{m}$ and $\mathscr{B}$ a Boolean $\mathfrak{m}$-additive algebra with maximal element e. The group $S(\mathscr{B}, G)$ is defined by the set of functions x defined on G and satisfying the following conditions. (1) $x(g) \in \mathscr{B}$ for all $g \in G$. (2) $\bigcup_{g \in G} x(g) = e$. (3) $x(g) \cap x(g') = 0$ for $g \neq g'$. The sum $z = x + y$ of two such functions x and y is defined by the equation $z(g) = \bigcup_{g' \in G} (x(g') \cap y(g - g'))$ for $g \in G$. For each element x of $S(\mathscr{B}, G)$, put $\nu(x) = \bigcup_{0 \neq g \in G} x(g)$. Let $\mathscr{J}$ be an arbitrary (finitely additive) ideal in the algebra $\mathscr{B}$ and $\mathscr{J}_\delta$ the least σ-ideal containing $\mathscr{J}$. The set $S(\mathscr{J})$ is defined by all elements $x \in S(\mathscr{B}, G)$ with $\nu(x) \in \mathscr{J}$. Then, it is a subgroup of $S(\mathscr{B}, G)$.

The author studies the structure of the factor groups $S(\mathscr{J}_\delta)/S(\mathscr{J})$ for the torsion-free group G. He proves that they are torsion-free algebraically compact groups (in the sense of Kaplansky). In a special case, he gives the cardinal invariants of these groups. The homomorphism groups of groups $S(\mathscr{B}, G)$ in slender groups are also studied. (The author communicated some of the results in an earlier paper [Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 7 (1959), 141–142; MR **21** #7245].)

K. Honda (Tokyo)

Yahya, S. M. 3536

Kernel of the homomorphism $A' \otimes B' \to A \otimes B$.

J. Natur. Sci. and Math. **3** (1963), 139–145.

Let $\phi: A' \to A$ and $\psi: B' \to B$ be monomorphisms of abelian groups. The author considers the question of whether the kernel K of the homomorphism

$$\phi \otimes \psi: A' \otimes B' \to A \otimes B$$

is generated by monomials $a' \otimes b'$, $a' \in A'$, $b' \in B'$. He gives two examples, one for mixed groups, and another for finite groups, to show that the answer is not always yes. He gives also some sufficient conditions for the answer to be affirmative. For example, if ϕ is an epimorphism, or if $\phi(A')$ contains the torsion subgroup of A, or if both A and B are divisible, then K is generated by monomials of the form $a' \otimes b'$, where either $\phi(a') = ha$, and $hb' = 0$, or $\psi(b') = hb$, and $ha' = 0$.

C. W. Curtis (Eugene, Ore.)

Šemetkov, L. A. 3537

Subgroups of π-supersoluble groups. (Russian)

Dokl. Akad. Nauk BSSR **8** (1964), 495–496.

A group is called π-supersoluble if it possesses a principal series with cyclic π-quotients and (arbitrary) π'-quotients. A group is of type S if it is not nilpotent while all the proper subgroups are. The author shows: If R is the maximal normal π'-subgroup of the π-supersoluble group G, then (i) G/R is the extension of its π-Sylow group by an abelian group; (ii) for all primes p of π' dividing $o(G/R)$ there are p-nilpotent pq-subgroups of type S in G for a certain prime q in π with $q \equiv 1 \bmod p$. Two further theorems are interesting corollaries.

H. Heineken (Frankfurt a.M.)

Feit, Walter; Thompson, John G. 3538

Solvability of groups of odd order.

Pacific J. Math. **13** (1963), 775–1029.

In this impressive paper, which occupies the entire issue, the authors present a complete proof of the solvability of finite groups of odd order; thus they give an affirmative answer to the famous Burnside conjecture that all finite non-abelian simple groups are of even order. Not only the result itself, which has many consequences of importance, but the method employed, is significant. It is difficult to state an indication of their long involved proof and explain the reason why the oddness of the group order implies the solvability. The oddness of the group order predominates the proof and is used in all sorts of conceivable ways. The reviewer makes no attempt here to indicate the places in the proof where the oddness of the order is essential.

If there were a non-solvable group of odd order, then a non-solvable group of the smallest possible odd order would be a non-abelian simple group, all of whose proper subgroups are solvable. Such a simple group is called a minimal simple group. The proof of this paper consists in deriving a contradiction from the assumed existence of a minimal simple group of odd order. The proof is an expansion of ideas contained in papers by the reviewer [Proc. Amer. Math. Soc. **8** (1957), 686–695; MR **19**, 248] and Feit, Hall and Thompson [Math. Z. **74** (1960), 1–17; MR **22** #9539], where special cases of Burnside's conjecture are solved. In these papers special assumptions are placed on the structure of the centralizers, and the minimal simple group to be disposed of satisfies the property that every maximal subgroup is a Frobenius group and is the normalizer of a nilpotent Hall subgroup. From this property one can apply the theory of group characters, the theory of so-called exceptional characters, which provides a certain limitation on the index of a nilpotent Hall subgroup in a maximal subgroup. One is then able to derive a contradiction if the order of the group involved is odd. Chapter V of this paper is an application of group characters. In the general case of this paper, maximal subgroups may not be Frobenius groups, so that the theory of exceptional characters has to be generalized to take care of situations which may occur in the present case. This generalization is given in Chapter III of this paper. The details are too technical to be presented here, but roughly speaking, given a subgroup H which is "nicely" imbedded in G and given a set S of generalized characters of H which satisfies "nice" properties, one can define a particular isometry τ of S into the ring of generalized characters of G. This makes it possible to study the ring of generalized characters of G by means of $\tau(S)$; thus one obtains a certain information on the structure of G from the structure of H. In Chapter III the authors discuss a tamely imbedded (i.e., "nicely" imbedded) subset, of which H is the normalizer, and a coherent set of characters of H, which is "nice". A particular mapping τ is defined for a coherent set. The main discussion is to find a sufficient condition for a union of several coherent sets to be coherent. Theorem 10.1 is a main result in this direction and is a generalization of a

result of Brauer and Feit on exceptional characters [cf. Feit, Illinois J. Math. 4 (1960), 170–186; MR 22 #4784]. The coherency of a certain set often comes from the structure of H. Chapter III contains several results of this nature. All results in this chapter are of a general character and do not refer to a minimal simple group of odd order. In Chapter V, the ring of generalized characters of G is studied by using $\tau(S)$ for certain coherent sets S of generalized characters of various maximal subgroups. A delicate discussion about a not necessarily coherent set comes in also. After about seventy pages of computation, the outcome is not quite a contradiction. But one obtains a contradiction unless there are two maximal subgroups of a special kind, which are interrelated in a particular fashion. These two groups generate G. Their structure and the way they intersect give rise to relations among the set of generators of G. The final chapter, Chapter VI, is a study of these generators and relations (about twenty pages). The authors are then able to show that the particular relations they obtain are not possible in finite groups. One of the distinguished maximal subgroups is isomorphic to an extension of the linear group $y = ax + b$, where a, b are elements taken from a finite field with a restriction that the norm of a over the prime field is 1. From this fact the authors are able to translate the relations among generators into equations in a finite field and then derive a desired contradiction.

Chapter IV is a study of maximal subgroups by means of group-theoretical methods, and the main object is to show that maximal subgroups satisfy nice properties so as to make possible an application of group characters. Here the method the authors employ is quite interesting. Many parts of their argument can be and have been generalized to handle minimal simple groups in general or simple groups in which every proper subgroup involves only a particular class of simple groups [see the work of D. Gorenstein and J. H. Walter in J. Algebra 1 (1964), 168–213].

Let p be a prime divisor of the order of G and S a Sylow p-group of G. Let A be a maximal abelian normal subgroup of S. Assume that the minimal number of generators of A is at least three. The study of subgroups U such that U is normalized by A and $U \cap A = \{1\}$, is important. The first major result is the following transitivity theorem (Theorem 17.1). Let q be a prime different from p. If Q_1 and Q_2 are two q-subgroups, normalized by A and maximal under this restriction, then there is an element x of $C_G(A)$ such that $Q_2 = x^{-1}Q_1x$. This transitivity theorem is true if G is a p-solvable group. Even in the solvable case, this is not quite trivial and is one of the many results the authors have to establish in their course of proof. The fact that this result holds for minimal simple groups is amazing. In this theorem, not only the conjugacy of Q_1 and Q_2, but also the location of the element x which conjugates Q_1 and Q_2, is important. The second step is to prove the existence of a Hall subgroup with respect to a particular set of primes. Let a set of primes π satisfy the property that if p and q are two primes in π, then G contains a proper (hence solvable) subgroup generated by elementary abelian subgroups of order p^3 and q^3, respectively. The existence theorem asserts that if π satisfies the above property, then G contains a Hall π-subgroup. The passage from the transitivity theorem to this existence theorem is not short (about fifty pages) and represents one of the highest points ever achieved in the theory of finite groups. The proof is quite technical and involved. The prime p such that a Sylow p-group does not normalize any p'-subgroup has to be treated separately, but at the end G has a Hall π-subgroup H such that $N_G(H)$ is a maximal subgroup and the intersection of H and its conjugate H^x is not too large.

If a Sylow p-group S does not contain an elementary abelian normal subgroup of order p^3, the structure of S is restricted (and in fact for $p > 2$ a complete classification is available [Blackburn, Acta Math. 100 (1958), 45–92; MR 21 #1349; Proc. London Math. Soc. (3) 11 (1961), 1–22; MR 23 #A208]). This and the above existence theorem are used to study maximal subgroups of G, which are proved to have a structure closely related to Frobenius groups. Another result is that two elements of a nilpotent Hall subgroup H of G are conjugate in G if and only if they are conjugate in the normalizer of H. Also, one has to know that certain sets are tamely imbedded before one is able to apply character theory. The required proof occupies the long last section of Chapter IV (about forty pages).

Chapters I and II are preliminaries, and contain many new properties of finite groups, the importance of which is recognized by the success of this work.

M. Suzuki (Urbana, Ill.)

Šmel'kin, A. L. 3539

Wreath products and varieties of groups. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 1063–1065.

The "base group" of a wreath product of two groups is a direct power of the first group; if this is replaced by a verbal power, the corresponding construction yields a "verbal wreath product". Theorem 1 is a technical result on verbal wreath products. It enables the author to re-prove some theorems of G. Baumslag [Math. Z. 81 (1963), 286–299; MR 27 #1503], e.g., (Theorem 2) that if N is a normal subgroup of a free group F and $V(N)$ is a verbal subgroup of N, and if both F/N and $N/V(N)$ are residually finite (residually finite p-groups), then so is $F/V(N)$. Further applications include Theorem 3. Let F, N, $V(N)$ be as above; let F have finite rank, let N be generated (qua normal subgroups of F) by a recursively enumerable set of elements, and let the verbal subgroup function V be defined by a recursively enumerable set of words. The word problem is solvable in $F/V(N)$ if it is solvable in F/N and in the countable rank free group of the variety associated with V. Theorem 4. The product of two non-trivial varieties can be generated by a finite group if and only if the first variety is nilpotent, the second is abelian, and their exponents are relatively prime positive numbers. This solves a problem for which a partial solution had been given by P. M. Neumann [Quart. J. Math. Oxford Ser. (2) 14 (1963), 46–50; MR 27 #1523]. *L. G. Kovács* (Canberra)

Kemhadze, Š. S. 3540

On outer nilpotent automorphism groups. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **34** (1964), 265–270.

The automorphism σ of the arbitrary group G is called stable or outer nilpotent if σ induces the identity in each factor of some finite normal series of G [see B. I. Plotkin, Dokl. Akad. Nauk SSSR **130** (1960), 977–980; MR 22

#A228]. The author proves that the set of elements of G inducing stable inner automorphisms in G coincides with the Baer nilradical $N(G)$ which is the maximal normal nilsubgroup of G. If σ is a stable automorphism of G and $g \in G$, then $[g, \sigma] = g^{-1}g^{\sigma} \in N(G)$.

H. Salzmann (Frankfurt a.M.)

Kemhadze, S. S. 3541

On stable groups of automorphisms. (Russian)

Dokl. Akad. Nauk SSSR **158** (1964), 510–512.

A set Φ of automorphisms of a group G is called stable if G has an ascending normal series of Φ-admissible subgroups such that every element of Φ induces the identity automorphism on every factor of this series. A subset of G is stable if the corresponding set of inner automorphisms is stable. A single automorphism (or element) of G is stable if the corresponding one-element set is stable. A group is quasi-nilpotent if every finitely generated subgroup of it is subnormal; the quasi-nilpotent radical of a group is the union of its quasi-nilpotent normal subgroups [the author, same Dokl. **155** (1964), 1003–1005; MR **28** #3098]. If G, Φ are as above, then $[G, \Phi]$ denotes the subgroup generated in G by the elements $g \cdot \sigma g$ where $g \in G$, $\sigma \in \Phi$, $\sigma: g \to \sigma g$.

Theorem 1: In any group G, the set of all stable elements coincides with the quasi-nilpotent radical. Theorem 2: The subgroup generated by a finite subset A of a group G is contained in the quasi-nilpotent radical of G if and only if A is stable. Theorem 3: If Φ is a stable group of automorphisms of G, then $[G, \Phi]$ is contained in the quasi-nilpotent radical of G. Theorem 4: Let Φ be any group of automorphisms of a group G. The set of those stable automorphisms of G which lie in the locally nilpotent radical of Φ is a normal subgroup of Φ. Seven corollaries are given.

L. G. Kovács (Canberra)

Baer, Reinhold 3542

The hypercenter of functorially defined subgroups.

Illinois J. Math. **8** (1964), 177–230.

In this paper a functor $\mathfrak{f}$ on a suitable class $\mathfrak{D}$ of groups is a rule assigning to each group G in $\mathfrak{D}$ a subgroup $\mathfrak{f}G$, such that if σ is a homomorphism of G into H, $(\mathfrak{f}G)^{\sigma} \subseteq \mathfrak{f}H$, with equality if σ is an epimorphism. Furthermore, $\mathfrak{f}$ is an n-functor, where n is a positive integer, if for all G, $\mathfrak{f}G$ is the join of the groups $\mathfrak{f}U$, where U runs over the n-generator subgroups of G. For instance, the nth term of the lower central series is an n-functor on the class of all groups; and (using the well-known but not yet available determination by J. G. Thompson of all minimal simple groups) the intersection of the terms of the derived series is a 2-functor on the class of finite groups.

Using his own characterisations of Engel elements in Noetherian groups [Math. Ann. **133** (1957), 256–270; MR **19**, 248], the author first proves that if $\mathfrak{f}$ is an n-functor and $\mathfrak{f}G$ is Noetherian, then (a) $\mathfrak{f}G$ is nilpotent of finite class if $\mathfrak{f}U$ is of finite class for every $(n+1)$-generator subgroup U of G, and (b) $\mathfrak{f}G$ is contained in the hypercenter of G if, for every $(n+1)$-generator subgroup U of G, $\mathfrak{f}U$ is contained in the hypercenter of U. He then discusses conditions under which, in the first of these statements, $n+1$ can be replaced by n. If $n>1$ and G is assumed finite, a painstaking analysis of the least criminal yields a list of conditions too complicated to be reproduced here. From these, it is deduced that if $\mathfrak{f}$ is an n-functor with $n>1$, and $\mathfrak{f}G$ is finite, then $\mathfrak{f}G$ is nilpotent if and only if (a) for all n-generator subgroups U of G, $\mathfrak{f}U$ is nilpotent and (b) for all subgroups S ($\neq G$) of $\mathfrak{f}G$, $S/\mathfrak{f}S$ is soluble. There is a similar result if $\mathfrak{f}G$ is only assumed to be Noetherian, (b) being replaced by the requirement that $\mathfrak{f}G$ is soluble.

The author turns next to the hypercenter of $\mathfrak{f}G$. The normal subgroup N of G is said to be $\mathfrak{f}$-hypercentralized if $N \cap \mathfrak{f}G$ is contained in the hypercenter of $\mathfrak{f}G$, and $\mathfrak{f}$-n-hypercentralized if $N \cap \mathfrak{f}U$ is contained in the hypercenter of $\mathfrak{f}U$ whenever U is a subgroup of G generated by $n+1$ elements, at least one of which is in U. An $\mathfrak{f}$-hypercentralized normal subgroup is $\mathfrak{f}$-n-hypercentralized for all n, and if $\mathfrak{f}$ is an n-functor, $n>1$, an $\mathfrak{f}$-$(n+1)$-hypercentralized normal subgroup is $\mathfrak{f}$-hypercentralized. Various additional hypotheses are given under which, in this last statement, $\mathfrak{f}$-$(n+1)$-hypercentralized can be replaced by $\mathfrak{f}$-n-hypercentralized.

The paper contains also a number of variations of the ideas here mentioned, and numerous counterexamples.

Graham Higman (Oxford)

Baer, Reinhold 3543

Erreichbare und engelsche Gruppenelemente.

Abh. Math. Sem. Univ. Hamburg **27** (1964), 44–74.

A normal chain in a group is a set of subgroups which is well-ordered by inclusion, which contains the union of each of its subsets, and each member of which, except the last, is normal in its successor. The subgroup U of the group G is accessible if there is a normal chain to which both U and G belong. If $\mathfrak{e}$ is any group property, and if G is any group, $\mathfrak{e}G$ denotes the join of the accessible $\mathfrak{e}$-subgroups of G. This paper is mainly concerned with the two questions (i) whether $\mathfrak{e}G$ is necessarily an $\mathfrak{e}$-group, and (ii) whether $\mathfrak{e}(G/\mathfrak{e}G)$ is necessarily 1.

First, the author obtains an abstract characterization of properties $\mathfrak{e}$ for which the answer to both questions is positive. This is so, for instance, if $\mathfrak{e}$ is the property of being locally nilpotent, or of possessing a normal chain containing 1 and G with cyclic factors.

If, however, $\mathfrak{e}$ is the property of being locally noetherian, the answer to (i) is positive but to (ii) is negative. This suggests the study of the property of possessing a normal chain, including 1 and G, with locally noetherian factors. Because a join of accessible locally noetherian subgroups is locally noetherian, this is equivalent to asking that non-trivial homomorphic images of G have a non-trivial locally noetherian normal subgroup. The new property proves useful in the study of classes of elements defined by variations on the Engel conditions.

Graham Higman (Oxford)

Baer, Reinhold 3544

Der reduzierte Rang einer Gruppe.

J. Reine Angew. Math. **214/215** (1964), 146–173.

The rank of a group is the smallest number of elements required to generate it, and the reduced rank is the smallest number of conjugate classes of elements required to generate it, with the convention that both the rank and the reduced rank of a trivial group are 1. If the group is finite, then its reduced rank is the rank of its factor group by the derived group, and the purpose of this paper

is to obtain conditions on an infinite group which ensure the same equality.

The normal subgroup N of the group G is said to be small if there is no proper normal subgroup X of G such that $G=NX$. The union of all small normal subgroups of G is also the intersection of all maximal normal subgroups of G, and is here called the radical of G. Then the result proved is that if the radical of G is itself a small normal subgroup, and if the minimal condition holds for the normal subgroups of G between the derived group and its intersection with the radical, then the reduced rank of G is the rank of G/G'.

The paper also contains interesting characterisations of the group properties of having rank or reduced rank less than a fixed cardinal; and of the groups which are direct products of a finite number of simple groups, or of simple groups and free cyclic groups. *Graham Higman* (Oxford)

Kirsch, Arnold 3545

Über die Endomorphismen der endlichen Bewegungsgruppen und ihre Veranschaulichung.

Math.-Phys. Semesterber. **11** (1964), 48–70.

The author observes that, whenever a finite rotation group $\mathfrak{G}$ has a normal subgroup $\mathfrak{N}$, it also has a subgroup isomorphic to the quotient group $\mathfrak{G}/\mathfrak{N}$. He illustrates this fact by drawing a figure which $\mathfrak{G}$ leaves invariant and decorating it with a pattern which is invariant for $\mathfrak{N}$ but is changed by every element (except the identity) of the other subgroup. For instance, if $\mathfrak{G}$ is the octahedral group and $\mathfrak{N}$ its tetrahedral subgroup, the figure is a cube and the pattern is an inscribed regular tetrahedron; the "other subgroup" (of order 2) is generated by the half-turn about the line joining the midpoints of two opposite edges of the cube. *H. S. M. Coxeter* (Hanover, N.H.)

White, George K. 3546

On generators and defining relations for the unimodular group $\mathfrak{M}_2$.

Amer. Math. Monthly **71** (1964), 743–748.

The unimodular group $\mathfrak{M}_2$ consists of all 2×2 matrices with rational integral entries and determinant ± 1. The author gives defining relations for $\mathfrak{M}_2$ as an abstract group on two generators, and briefly discusses various simple relations satisfied by these generators.

I. Reiner (Urbana, Ill.)

Wan, Zhe-xian [Wan, Cheh-hsian] 3547

On the structure of the quotient group of the unitary group with respect to its commutator subgroup.

Acta Math. Sinica **12** (1962), 352–368 (*Chinese*); *translated as Chinese Math.* **3** (1963), 382–398.

Let the [skew] field K have an involutary anti-automorphism, the action of which will be denoted by a bar. Let H be an n-by-n invertible hermitian [skew-hermitian] matrix over K. Such a matrix defines by $(x, y)=xH\bar{y}'$ a sesquilinear form on an n-dimensional vector space over K—with the vectors as row vectors, and with $'$ as transposition. Assume, furthermore, that this form is trace-valued and that H is not symmetric if char $K\neq 2$. Then the author establishes the following theorem: If $n=2\nu>2$, then the factor group $U_n(K, H)/TU_n(K, H)$ of the group of all H-unitary n-by-n matrices over K by the subgroup generated by all H-unitary transvections is isomorphic to the factor group K^*/SC, where K^* is the multiplicative group of K, S its subgroup of H-symmetric elements, and C its commutator subgroup. The number ν denotes the index of the sesquilinear form defined by H.

This result is contained in the more general Theorem 1 of G. E. Wall [Inst. Hautes Études Sci. Publ. Math. No. 1 (1959); MR **21** #3517], where a somewhat different description of the subgroup SC is given.

O. H. Kegel (Frankfurt a.M.)

Carter, Roger; Fong, Paul 3548

The Sylow 2-subgroups of the finite classical groups.

J. Algebra **1** (1964), 139–151.

The authors determine the Sylow 2-subgroups, and their normalizers, in the finite classical groups $\mathrm{GL}_n(q)$, $\mathrm{Sp}_{2n}(q)$, $\mathrm{U}_n(q)$, $\mathrm{O}_{2n+1}^{\pm}(q)$, $\mathrm{O}_{2n}(\pm 1, q)$, where q is an odd prime power. Let Z_k be the cyclic group of order k, T_m the wreath product $Z_2 \wr Z_2 \wr \cdots \wr Z_2$ (m terms). Let Q_n be a Sylow 2-subgroup of $\mathrm{GL}_n(q)$, $\mathrm{Sp}_n(q)$ or $\mathrm{U}_n(q)$, and let $n=2^{r_1}+\cdots+2^{r_t}$, where $r_1<r_2<\cdots<r_t$. Then $Q_n\cong Q_{2^{r_1}}\times\cdots\times Q_{2^{r_t}}$, $Q_{2^r}\cong Q_2 \wr T_{r-1}$. In the full linear and unitary cases, $N(Q_n)\cong Q_n\times Z_k\times\cdots\times Z_k$ (t Z_k's) for suitable k. In the symplectic case, $(N(Q_n):Q_n)=1$ or 3^t according as $q\equiv\pm 1$ or $\pm 3 \pmod 8$. The results for the orthogonal groups are similar. *G. E. Wall* (Sydney)

Foulser, David A. 3549

The flag-transitive collineation groups of the finite Desarguesian affine planes.

Canad. J. Math. **16** (1964), 443–472.

A collineation group of a plane is flag-transitive if it is transitive on incident point-line pairs, and the author solves completely the problem of determining the groups described in the title. Let π be a finite Desarguesian affine plane, let G be its full collineation group, and let E be the subgroup of G generated by the elations. If π is of order n, its points can be identified with the elements of a two-dimensional vector space over the field $\mathrm{GF}(n)$ of n elements, and hence with the elements of $\mathrm{GF}(n^2)$. The mappings $x\to ax^{p^b}+c$ (b an integer, $a\neq 0$, c in $\mathrm{GF}(n^2)$) of $\mathrm{GF}(n^2)$ into itself then form a subgroup U of G. The author's first main theorem is that, with a finite number of exceptions, a flag-transitive group of collineations either contains E or is contained in a conjugate under G of U. Moreover, the exceptions and the minimal flag-transitive subgroups of U are determined explicitly. Next, with one exception, again explicitly determined, flag-transitive subgroups of G are isomorphic only if they are conjugate.

The doubly-transitive collineation groups are, of course, flag-transitive, and the author goes on to determine which of his groups are doubly transitive. For these the isomorphism theorem takes a stronger form, for, again with one exception, non-isomorphic planes cannot have isomorphic doubly transitive collineation groups, and this remains true if one of them is non-Desarguesian. This enables the author to show that a finite non-Desarguesian plane cannot have a solvable doubly transitive collineation group, unless it is the near-field plane of order 9.

Graham Higman (Oxford)

Berman, S. D.; Gudivok, P. M. 3550
Indecomposable representations of finite groups over the ring of p-adic integers. (Russian)
Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 875–910.

Let ZG denote the group ring of a finite group G over the ring Z of rational integers. Consider representations of G by matrices with entries in Z, or equivalently, consider left ZG-modules having finite Z-bases. Although the present paper is couched in matrix terminology, for the purpose of this review we have rephrased its results in terms of the more natural module concepts. The main theorems in this paper have been independently obtained by Heller and the reviewer [Ann. of Math. (2) **76** (1962), 73–92; MR **25** #3993; ibid. (2) **77** (1963), 318–328; MR **26** #2520].

The problem of finding all ZG-modules, for a specific group G, was first attacked by Diederichsen [Abh. Math. Sem. Hansischen Univ. **13** (1940), 357–412; MR **2**, 4], using the ideas developed in a fundamental work of Zassenhaus [ibid. **12** (1938), 276–288]. Since every ZG-module is expressible as a direct sum of indecomposable modules, though not necessarily in a unique manner, the problem is at once reduced to determining a full set of indecomposable modules. Let $n(ZG)$ denote the number of modules in such a set.

For the case where G is cyclic of prime order, Diederichsen determined a full set of indecomposable ZG-modules. In addition, he gave an example to show that $n(ZG)$ is infinite for G cyclic of order 4. This example was fallacious, however, as was pointed out independently by Roiter [Vestnik Leningrad. Univ. **15** (1960), no. 19, 65–74; MR **23** #A1730] and Troy [Ph.D. Dissertation, Univ. Illinois, Urbana, Ill., 1961]. Both gave a full set of indecomposable ZG-modules for this case, in which $n(ZG)=9$.

This raised the question: For which groups G is the number $n(ZG)$ finite? It soon became evident that in order to answer this question, it was necessary to first solve the corresponding problem for p-adic integral representations, where p is prime. Let $Z_p{}^*$ denote the ring of all p-adic integers in the p-adic completion Q_p of the rational field Q.

For the moment, let H be any p-Sylow subgroup of G. By using a theorem of D. G. Higman [Duke Math. J. **21** (1954), 377–381; MR **16**, 794], together with the basic result that the Krull-Schmidt theorem holds for $Z_p{}^*G$-modules, it is trivial to show that $n(Z_p{}^*G)$ is finite if and only if $n(Z_p{}^*H)$ is finite. The problem is thereby reduced to the study of p-adic integral representations of p-groups, which explains the significance of the present work.

This paper gives a complete answer to the question: When is $n(Z_p{}^*G)$ finite? The same answer was independently given by Heller and the reviewer, and is as follows. Theorem: The number of indecomposable $Z_p{}^*G$-modules is finite if and only if the p-Sylow subgroups of G are cyclic, of order at most p^2.

By using this theorem, it is possible to decide for which groups the number $n(ZG)$ is finite. This was done by A. Jones [Michigan Math. J. **10** (1963), 257–261; MR **27** #3698], who proved that the number of indecomposable ZG-modules is finite if and only if for each prime p, the p-Sylow subgroups of G are cyclic of order at most p^2.

We turn now to a more detailed description of the present work. With each $Z_p{}^*G$-module M we may associate a Q_pG-module $Q_p\otimes M$ (tensor product over $Z_p{}^*$). Let us call the composition factors of $Q_p\otimes M$ the "Q_p-composition factors" of M; they are a collection of irreducible Q_pG-modules.

Suppose that G is cyclic of order p^m, with generator g, and for each n, $1\leq n\leq m$, let ε_n denote a primitive p^nth root of unity over Q_p. Form the Q_pG-module $K_n=Q_p(\varepsilon_n)$, on which g acts as multiplication by ε_n. Then $K_0, K_1, \cdots, K_m$ form a full set of irreducible Q_pG-modules. Set $J_n=Z_p{}^*[\varepsilon_n]$, a $Z_p{}^*G$-module in K_n.

Keeping the above notation, suppose that the $Z_p{}^*G$-module M has at most two types of Q_p-composition factors, say K_{n_1} and K_{n_2}, occurring with certain multiplicities. It is easily shown that M is an extension of a submodule M_1 by a factor module M_2, where for $i=1, 2$, M_i is a direct sum of copies of J_{n_i}. Since $\mathrm{Ext}^1_{Z_p{}^*G}(J_{n_2}, J_{n_1})$ can be computed explicitly, and turns out to be a local ring, the following theorem can be established. Let G be a cyclic p-group, and consider $Z_p{}^*G$-modules with at most two types of Q_p-composition factors. Then the number of such modules which are indecomposable is finite. Indeed, the only such indecomposables are J_{n_1}, J_{n_2}, and a finite set of non-split extensions of J_{n_1} by J_{n_2}.

The authors next construct various infinite families of indecomposable $Z_p{}^*G$-modules, for G cyclic of order p^m, $m\geq 3$. These modules, too complicated to list here, have three or more types of Q_p-composition factors, and depend on one or more matrix parameters. These matrix parameters have entries in some one of the rings J_n defined above. The existence of such families of modules implies that $n(Z_p{}^*G)$ is infinite for $m\geq 3$. Another consequence is that for $m\geq 5$, the problem of finding all indecomposable $Z_p{}^*G$-modules is at least as difficult as the classical unsolved problem of classifying pairs of matrices (over a field) under simultaneous similarity.

For H an abelian (p, p)-group, the authors give an infinite family of indecomposable $Z_p{}^*H$-modules. This easily implies that if G is any non-cyclic p-group, then $n(Z_p{}^*G)$ is infinite.

Several of the above constructions have been generalized by Dade [Ann. of Math. (2) **77** (1963), 406–412; MR **26** #2521] and Gudivok [Dokl. i Soobšč. Užgorodsk. Univ. Ser. Fiz.-Mat. Istor. Nauk **1962**, no. 5, 73; ibid. **1962**, no. 5, 81–82]. Let K be a finite extension of Q_p, let R be the valuation ring of K, and suppose that G is a p-group such that there exist at least 4 types of irreducible KG-modules. Dade and Gudivok both proved that under these assumptions, there exist infinitely many indecomposable RG-modules.

Turning next to the case where G is cyclic of order p^2, the authors of the present paper prove that $n(Z_p{}^*G)=4p+1$, and give a full set of indecomposable modules for this case. These coincide with those determined by Heller and the reviewer [loc. cit.]. The list originally given by the authors [Dokl. Akad. Nauk SSSR **145** (1962), 1199–1201; MR **25** #3095] is incorrect.

A somewhat more general theorem is established here, namely the following: *Let G be any cyclic p-group, and consider the set S of all $Z_p{}^*G$-modules having at most three types of Q_p-composition factors, one of which is the trivial Q_pG-module K_0. Let the other two types of Q_p-composition factors be K_r and K_s, where $0<r<s$. Then the number of indecomposable modules in the set S is precisely $5+4(p-1)p^{r-1}$.*

I. Reiner (Urbana, Ill.)

Gudivok, P. M. 3551

Representations of finite groups over certain local rings. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukrain. RSR **1964**, 173–176.

Let Z_p^* be the ring of p-adic integers, and Q_p its quotient field. Suppose that K is any finite extension field of Q_p with valuation ring R. For G a finite group, denote by $n(RG)$ the number of inequivalent indecomposable representations of G by matrices with entries in R. If H is any p-Sylow subgroup of G, a standard argument shows that $n(RG)$ is finite if and only if $n(RH)$ is finite. The author gives brief outlines of a number of interesting theorems about the finiteness of $n(RH)$. These generalize earlier results by various authors [Z. I. Borevič and D. K. Faddeev, Vestnik Leningrad. Univ. **14** (1959), no. 7, 72–87; MR **21** #4968; A. Heller and the reviewer, Ann. of Math. (2) **76** (1962), 73–92; MR **25** #3993; ibid. (2) **77** (1963), 318–328; MR **26** #2520; S. D. Berman and the author, #3550 above].

From these earlier results it follows that $n(RH)$ is infinite when H is non-cyclic. The present author proves that $n(RH)$ is infinite whenever H has at least four irreducible matrix representations over K. (This has also been shown by E. C. Dade [Ann. of Math. (2) **77** (1963), 406–412; MR **26** #2521].) This latter hypothesis is certainly fulfilled if H is cyclic of order p^m, $m \geq 3$, and also if H is cyclic of order at least 5, once we assume that R contains all pth roots of 1.

A rather surprising theorem is as follows: If R contains all p^2th roots of 1, and H is cyclic of order at least 3, then $n(RH)$ is infinite. This is the first example known in which there are only three irreducible K-representations of H and yet $n(RH)$ is infinite.

Another interesting result is the fact that $n(RH)$ is finite if H is cyclic of order p^2, where p is unramified in the extension K over Q_p.

Using a number of such results, the author obtains necessary and sufficient conditions for the finiteness of $n(RH)$, where K is a cyclotomic extension of R_p.

I. Reiner (Urbana, Ill.)

Isaacs, I. M.; Passman, D. S. 3552

Groups whose irreducible representations have degrees dividing p^e.

Illinois J. Math. **8** (1964), 446–457.

Let G be a finitely generated group and p a prime. Suppose that all irreducible representations of G over the complex numbers have degrees dividing p^e for a fixed exponent e. The authors' first main result is that G has a subinvariant series $G = A_e \supset A_{e-1} \supset \cdots \supset A_0$ such that A_0 is abelian, and A_i/A_{i-1} is an elementary abelian p-group with not more than $2i+1$ generators. In particular, G has an abelian subgroup A_0 whose index divides $p^{e(e+2)}$. The second main result is that a finitely generated group G all of whose irreducible complex representations have degree 1 or p, for a fixed prime p, is of one of the following types: (a) G is abelian; (b) G has a normal abelian subgroup of index p; or (c) G has a center Z with G/Z being a group of order p^3 and period p. Conversely, if G is a finite group of one of the above types, then its irreducible complex representations all have degree 1 or p. If G is finitely generated, of one of the above types, then G has a complete set of representations of degree 1 or p. Examples are given to show that all the types can occur. The first result is a sharpened version of Theorem III in an earlier paper of the authors [Canad. J. Math. **16** (1964), 299–309; MR **29** #4811]. For $p=2$ the second result reduces to a theorem of Amitsur [Illinois J. Math. **5** (1961), 198–205; MR **23** #A225]. The proofs are interesting but are too long to be described in this review.

C. W. Curtis (Eugene, Ore.)

Armstrong, J. W. 3553

A note on endomorphism groups.

Publ. Math. Debrecen **10** (1963), 116–119.

It is known that the structure of an abelian group G is determined by its ring of endomorphisms $\mathscr{E}(G)$, but not by its endomorphism group End(G) [D. K. Harrison, same Publ. 7 (1960), 316–319]. The author characterizes those groups G, among the class of direct sum of cyclic groups, that are determined by End(G). They are bounded groups with some restriction on ranks.

Ti Yen (E. Lansing, Mich.)

Fuchs, L. 3554

On group homomorphic images of partially ordered semigroups.

Acta Sci. Math. (Szeged) **25** (1964), 139–142.

This paper is an investigation of those partially ordered semigroups which can be mapped order-homomorphically onto a partially ordered group. The result obtained is a generalization of a theorem of M.-L. Dubreil-Jacotin concerning lattice-ordered semigroups [C. R. Acad. Sci. Paris **232** (1951), 287–289; MR **12**, 473]. Let S be a partially ordered semigroup with identity element e, and let $a, b \in S$. The generalized right residual of a by b is $\langle a\cdot b\rangle = \{x \in S | xb \leq a\}$, and similarly $\langle a.\cdot b\rangle = \{x \in S | bx \leq a\}$. S is generalized residuated if for all $a, b \in S$, neither $\langle a\cdot b\rangle$ nor $\langle a.\cdot b\rangle$ is empty. If $U \subseteq S$, an element $u \in U$ is left multiplicatively maximal if $v \in S$ and $vu \in U$ imply $v \leq e$. An equivalence relation θ on S is a Dubreil-Jacotin congruence if (1) $a \equiv b$ (θ) implies $ca \equiv cb$ (θ) and $ac \equiv bc$ (θ), (2) $a \leq c \leq b$ and $a \equiv b$ (θ) imply $a \equiv c$ (θ), and (3) $a \equiv a'$ (θ), $e \equiv e'$ (θ) and $a' \leq e'$ imply $a \leq e$. The author proves the following theorem. Let S be a partially ordered semigroup with identity element e. Then these are equivalent. (A) θ is a Dubreil-Jacotin congruence on S and S/θ is a group. (B) (i) S is generalized residuated, (ii) $\langle e\cdot .a\rangle = \langle e.\cdot a\rangle$ for all $a \in S$, (iii) $\langle e\cdot .a\rangle$ contains a left multiplicatively maximal element for all $a \in S$, and (iv) $\langle e\cdot .a\rangle = \langle e\cdot .b\rangle$ if and only if $a \equiv b$ (θ).

W. C. Holland (Madison, Wis.)

Domoščnickaja, N. E. 3555

On the axiomatics of almost monotone ordered semigroups. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 804–814.

Let $(G, +, <)$ be an additively written semigroup with irreflexive order $<$. Let $<$ be linear and positive (i.e., $a+b>a, b$ for every $a, b \in G$; $<$ need not be compatible with $+$). The author shows that her previous conditions for the existence of a non-trivial isotone homomorphism of $(G, +, <)$ in the semigroup of all real numbers ordered in the natural way are equivalent to the following conditions. There exist e_1, e_2, e_3, e_4 such that, for every a, b, c, (1) $a<b$ implies $a+c<b+c+e_1$ and $c+a<$

e_1+c+b; (2) $a<b$ implies $na<nb+e_2$ for every integer n; (3) $n(a+b)<na+nb+e_3$ and $na+nb<n(a+b)+e_3$ for every integer n; (4) for every a there exists an integer n such that $ne_4>a$. It is shown that conditions (1)–(4) are independent. If $(G, +, <)$ satisfies (1)–(3), it is called an almost monotone ordered semigroup.

B. M. Schein (*Šaĭn*) (Saratov)

Conrad, Paul 3556

The relationship between the radical of a lattice-ordered group and complete distributivity.

Pacific J. Math. **14** (1964), 493–499.

In a lattice-ordered group G, denote by R_g, for each element $g\neq 0$ in G, the join of the lattice ideals of G not containing g, and let $R(G)$ be the intersection of the ideals R_g. $R(G)$ is the radical of G, and this paper begins a study of groups with $R(G)=0$. In particular, if G is representable, that is, is lattice-isomorphic to a subdirect sum of a cardinal sum of totally ordered groups, $R(G)=0$ if and only if G is completely distributive.

Graham Higman (Oxford)

Chehata, C. G. 3557

On a relation on ordered groups.

Proc. Math. Phys. Soc. U.A.R. (*Egypt*) No. 25 (1961), 79–82 (1964).

Let $[a, b]=a^{-1}b^{-1}ab$, $[a, b, c]=[[a, b], c]$, etc. An example is constructed to show that for elements a, b in an ordered group, the relation $[a, b^m, a, a]=e$ for some integer $m\neq 0, 1$ need not imply $[a, b, a, a]=e$. The example G is generated by elements $\cdots, a_{-2}, a_{-1}, a_0, a_1, a_2, \cdots$ and g, with the relations $g^{-1}a_mg=a_{m+1}$ for all m, $[a_m, a_n]=e$ for all m and all $n>m+1$, $[a_m, a_{m+1}, a_n]=e$ for all m and all $n\neq m$, and $[a_m, a_{m+1}, a_m, a_n]=e$ for all m and all n.

If C is the subgroup of G generated by the elements $c_m=[a_m, a_{m+1}, a_m]$ $(m=0, \pm 1, \pm 2, \cdots)$, B is the subgroup generated by the elements c_m and $b_m=[a_m, a_{m+1}]$ $(m=0, \pm 1, \pm 2, \cdots)$ and A is the subgroup generated by the elements a_m $(m=0, \pm 1, \pm 2, \cdots)$, the series $G\rhd A\rhd B\rhd C\rhd\{e\}$ has free Abelian factors. Suitably chosen orders on these factors can be combined to construct an order on G. In G, $[a_0, g^m, a_0, a_0]=e$ for all $m\neq 1$, but $[a_0, g, a_0, a_0]\neq e$.

J. C. Ault (Leicester)

Budach, L. 3558

Struktur Noetherscher kommutativer Halbgruppen.

Monatsb. Deutsch. Akad. Wiss. Berlin **6** (1964), 85–88.

Let S be a commutative semigroup. Consider the conditions: (i) the maximum condition for congruence relations in S; (ii) the existence of a finite system of generators for S. The fact that (ii) implies (i) was proved by L. Rédei [*Theorie der endlich erzeugbaren kommutativen Halbgruppen*, p. 153, Physica-Verlag, Würzburg, 1963; MR **28** #5130] and by the reviewer [Comment. Math. Univ. Carolinae **4** (1963), 87–92; MR **29** #4821]. The author proves that (i) implies (ii) and for the fact that (ii) implies (i) he indicates a simple proof. If S is regular and with identity element, then the maximum condition for ideals implies the existence of a finite subset F such that S is generated by $F\cup G$, where G is the group of all invertible elements.

{The reviewer feels the reader should know that the meanings of some basic symbols which the author does not explain can be found in the reviewer's paper [#3559 below].}

K. Drbohlav (Prague)

Drbohlav, Karel 3559

Zur Theorie der Kongruenzrelationen auf kommutativen Halbgruppen.

Math. Nachr. **26** (1963/64), 233–245.

Following the classical ideal theory of commutative rings, the ideal theory of semigroups has been studied by many people. In this paper the author treats it from the standpoint of congruence relations. Let S be a commutative semigroup, C a congruence relation on S, and $\mathfrak{C}(S)$ the set of all congruence relations on S. The universal relation on S is denoted by 1. For $C\in\mathfrak{C}(S)$ and $x\in S$, a congruence relation $C(x)$ is defined by $C(x)=\{(u, v)|xuCxv\}$, and the ideals $[C]^0$ and $[C]$ are associated with C as follows: $[C]^0=\{x\in S|C(x)=1\}$, $[C]=\{x\in S|x^\rho\in[C]^0$ for some positive integer $\rho\}$. C is called irreducible if $C=D\cap E$, $D\in\mathfrak{C}(S)$, $E\in\mathfrak{C}(S)$, $C\neq D$ imply $C=E$. C is prime if $x\in S$, $C\neq C(x)$ imply $x\in[C]^0$; C is primary if $x\in S$, $C\neq C(x)$ imply $x\in[C]$. S is supposed to satisfy the maximal condition for congruence relations, that is, any non-empty subset of $\mathfrak{C}(S)$ contains at least one maximal congruence relation. Then the following results are obtained. (I) Any irreducible congruence C is primary. (II) Any $C\in\mathfrak{C}(S)$ has at least one canonical factorization: $C=C_1\cap\cdots\cap C_r$, $C_i\in\mathfrak{C}(S)$, $i=1, \cdots, r$, where C_i is primary, $[C_i]\neq[C_j]$, $i\neq j$, and this factorization is irredundant. (III) The canonical factorization is unique in the following sense: if C has two canonical factorizations $C=C_1\cap\cdots\cap C_k\cap C_{k+1}\cap\cdots\cap C_r=D_1\cap\cdots\cap D_k\cap D_{k+1}\cap\cdots\cap D_s$, then $r=s$ and $[C_i]=[D_i]$, $i=1, \cdots, r$, for some permutation of indices; and furthermore, if $[C_i]\not\supset[C_j]$ $(i=1, \cdots, k$; $j=k+1, \cdots, r(=s))$, then $C_1\cap\cdots\cap C_k=D_1\cap\cdots\cap D_k$. The author introduces the concept of congruence ideals: an ideal A of S is a congruence ideal of S if there is a congruence relation C such that $[C]^0=A$. Under the maximal condition for congruence relations, the statements (I), (II) and (III) hold if the congruence relation C is replaced by the congruence ideal A ("irreducible", "primary" and "canonical factorization" are also defined on congruence ideals in parallel) and if the radical $[A]$ of A is defined by $[A]=\{x|x^\rho\in A$ for some $\rho\}$. Furthermore, the conditions for congruence ideals to be prime or primary are given.

T. Tamura (Davis, Calif.)

Petrich, Mario 3560

Sur certaines classes de demi-groupes. I, II.

Acad. Roy. Belg. Bull. Cl. Sci. (5) **49** (1963), 785–798; *ibid.* (5) **49** (1963), 888–900.

A complex [a non-empty subset] H of a semigroup D is strong ("fort") if xv, uv, $uy\in H$ imply $xy\in H$ for every $u, v, x, y\in D$, and is bilaterally strong if uxv, uyv, $wxz\in H$ imply $wyz\in H$. D is [bilaterally] rectangular if each one-element complex is [bilaterally] strong. If $e\in D$ is an idempotent, $G^{(e)}$ denotes the maximum subgroup of D containing e.

There are found several characterizations of idempotent rectangular semigroups (e.g., $Dx\cap xD=\{x\}$ for every $x\in D$). D is called inversed ("inversé") if for every $x\in D$ there exists x' such that xx' is an idempotent. D is an inversed [bilaterally] rectangular semigroup if and only if

$D^2 \cong E \times G$ [$D^3 \cong E \times G$], where E is a rectangular band, G is a group, and $xey = xy$ [$xeyez = xyz$] for every idempotent $e \in D$. Each subsemigroup of D is [bilaterally] strong if and only if $D^2 \cong E \times G$ [$D^3 \cong E \times G$], where E is a rectangular band, G is a periodic group (and all subgroups of G are invariant), $abc \in G^{(e)}$ implies $ac \in G^{(e)}$ [$axbyc \in G^{(e)}$ implies $abc \in G^{(e)}$] for every idempotent e. All complexes of D are [bilaterally] strong if and only if D satisfies one of the following conditions: (a) $xy = x^2$ for all x, y; (b) $xy = y^2$; (c) D^2 is the group of order 2 [(a) $axb = ayb$ for all $a, b, x, y \in D$; (b) D^3 is the group of order 2]. (The bilateral case is treated in the second part of the paper.)

The end of the paper is devoted to the modifications of the results mentioned for special kinds of semigroups (globally idempotent and idempotent groups).

B. M. Schein (*Šain*) (Saratov)

Petrich, Mario 3561

On the structure of a class of commutative semigroups. (Russian summary)

Czechoslovak Math. J. **14** (89) (1964), 147–153.

Der Verfasser bezeichnet eine kommutative Halbgruppe S als N-Halbgruppe, wenn (1) $xz = yz \Rightarrow x = y$, (2) von je zwei Elementen aus S jedes eine Potenz des andern teilt und (3) S keine Idempotente enthält. Die Theorie der Zerlegungen der kommutativen Halbgruppen in archimedische, d.h. (2) genügende Unterhalbgruppen findet man in § 4.3 des Buches von Clifford und Preston [*The algebraic theory of semigroups*, Amer. Math. Soc., Providence, R.I., 1961; MR **24** #A2627]. Die Bedingung (1) kann man durch die Separabilität von S ersetzen, d.h. durch die Bedingung (4) $x^2 = y^2 = xy \Rightarrow x = y$. In einer endlich erzeugten N-Halbgruppe gibt es zu je zwei Elementen x und y zwei natürliche Zahlen p und q mit $x^p = y^q$.

Der Verfasser beweist, daß jede N-Halbgruppe mit zwei Erzeugenden a_1 und a_2 und natürlichen Zahlen m_1 und m_2, die bezüglich $a_2/a_1^{m_1}$ und $a_1/a_2^{m_2}$ minimal sind, zu folgender Halbgruppe S isomorph ist:

$$S = \{(k_1, k_2); k_1 = 0, 1, 2, \cdots; \quad k_2 = 0, 1, 2, \cdots, m_2 - 1 : k_1 + k_2 > 0\}$$

mit der Multiplikation

$$(k_1, k_2)(l_1, l_2) = (k_1 + l_1 + jm_1, k_2 + l_2 - jm_2),$$

wobei die ganze rationale Zahl j durch die Bedingung $0 \le k_2 + l_2 - jm_2 < m_2$ festgelegt ist. S kann man auch als diejenige Unterhalbgruppe der multiplikativen Gruppe der komplexen Zahlen beschreiben, die von $a_1 = 2^{1/m_1} e^{(2\pi i)/m_1}$ und $a_2 = 2^{1/m_2} e^{(4\pi i)/m_2}$ erzeugt wird.

E.-A. Behrens (Frankfurt a.M.)

Brown, Thomas C. 3562

On the finiteness of semigroups in which $x^r = x$.

Proc. Cambridge Philos. Soc. **60** (1964), 1028–1029.

Der Verfasser gibt einen neuen Beweis für den folgenden Satz von J. A. Green und D. Rees [dieselben Proc. **48** (1952), 35–40; MR **13**, 720] über die Äquivalenz der Vermutung von Burnside für Gruppen mit der für Halbgruppen: Für jede natürliche Zahl r, $r \ge 2$, ist die Endlichkeit aller endlich erzeugten und der Identität $x^r = x$ genügenden Halbgruppe S damit äquivalent, daß jede endlich erzeugte Gruppe von Exponenten $r - 1$ endlich ist. Der Beweis beruht auf dem Lemma, daß, falls die beiden Worte W und AXB in den Erzeugenden von S dasselbe Element von S repräsentieren, dies Element auch durch $W \cdot (XW)^{r-1}$ repräsentierbar ist. Daraus läßt sich nämlich schließen, daß für ein Element t von S, das durch ein alle Erzeugenden von S enthaltendes Wort repräsentierbar ist, die Unterhalbgruppe tSt von S eine Gruppe ist.

E.-A. Behrens (Frankfurt a.M.)

Ljapin, E. S. 3563

Conditions for complete embeddability in semigroups. (Russian)

Leningrad. Gos. Ped. Inst. Učen. Zap. **238** (1962), 3–20.

Several authors have considered embedding of semigroups as ideals (with various additional properties) in special semigroups; see, for example, Gluskin [Izv. Akad. Nauk SSSR Ser. Mat. **22** (1958), 439–488; MR **20** #2386; Mat. Sb. (N.S.) **47** (89) (1959), 111–130; MR **21** #4197; Dokl. Akad. Nauk SSSR **131** (1960), 1004–1006; MR **22** #6859] and the author [Mat. Sb. (N.S.) **52** (94) (1960), 589–596; MR **22** #12155; Leningrad. Gos. Ped. Inst. Učen. Zap. **103** (1955), 5–29]. In this paper an abstract version of these embedding theorems, and a number of sharpened results, are given.

Let A be a semigroup and $\varnothing \ne N \subset A$. If $a, a' \in A$ and $ax = a'x$ for $x \in N$, then a and a' are said to be equi-acting on N. Consider relations ρ between a semigroup B and a subsemigroup A of B such that if $A \subset B(\rho)$, then (1) A is a left ideal in B; (2) $\phi(A) \subset \phi(B)(\rho)$ for every homomorphism ϕ; (3) if $A \subset K \subset B$ and K is a subsemigroup, then $A \subset K(\rho)$. Particular such relations are the following: $A \subset B(\rho_0)$ if $A \subset B(\rho)$ and B contains no two elements equi-acting on A; $A \subset B(\rho_1)$ if $A \subset B(\rho)$ and every non-isomorphic homomorphism of B is non-isomorphic on A; $A \subset B(\rho_2)$ if $A \subset B(\rho)$ and for all subsemigroups B' such that $A \subset B'(\rho)$ and $B' \subset B$ ($B' \ne B$), there is a non-isomorphic homomorphism of B' that is an isomorphism on A. Connections among ρ_0, ρ_1, ρ_2 are explored, and a fourth relation ρ_3, too complicated to define here, is also introduced and studied. Sample theorems: $\rho_0 \subset \rho_1 \cup \rho_2$; if $A \subset B(\rho_0)$, then $A \subset B(\rho_1)$ if and only if B contains no two elements equi-acting on A.

Edwin Hewitt (Seattle, Wash.)

Makkai, M. 3564

Solution of a problem of G. Grätzer concerning endomorphism semigroups.

Acta Math. Acad. Sci. Hungar. **15** (1964), 297–307.

Let $\mathfrak{A}$ be a universal algebra, i.e., a non-empty set with a finite or infinite system of finitary operations on this set. $\mathfrak{E}(\mathfrak{A})$, $\mathfrak{M}(\mathfrak{A})$, $\mathfrak{H}(\mathfrak{A})$, denote, respectively, the semigroups of all endomorphisms, all monomorphisms (i.e., one-to-one endomorphisms) and all epimorphisms (i.e., endomorphisms onto) of $\mathfrak{A}$ under the usual operation of superposition. G. Grätzer ["Some Results on Universal Algebras", Mimeographed Notes, Budapest, 1962] proposed to find under what conditions a given triplet (E, M, H) of semigroups is isomorphic to some triplet $(\mathfrak{E}(\mathfrak{A}), \mathfrak{M}(\mathfrak{A}), \mathfrak{H}(\mathfrak{A}))$. He has found three necessary conditions: (C1) E has a unit element 1; (C2) M is a subsemigroup of E, all elements of M are right regular (i.e., right cancellative) in

E, $E-M$ is a right ideal of E and $1 \in M$; (C3) H satisfies the dual conditions: H is a subsemigroup of E, all elements of H are left regular in E, $E-H$ is a left ideal of E and $1 \in H$. E. Fried and the author have found the further necessary condition: (C4) If $m \in M$, $h \in H$, $x \in E$, $y \in E$ and $xm = hy$, then there exists $z \in E$ such that $x = hz$.

The author proves that conditions (C1)–(C4) are both necessary and sufficient. (C1) and (C2) form an abstract characterization of $(\mathfrak{E}(\mathfrak{A}), \mathfrak{M}(\mathfrak{A}))$, (C1) and (C3) form a characterization of $(\mathfrak{E}(\mathfrak{A}), \mathfrak{H}(\mathfrak{A}))$. It is noteworthy that for (E, M, H) satisfying (C1)–(C4), there exists $\mathfrak{A}$ with one-place operations only.

{Remark of the reviewer: Lemma 1, which is used for the proof of sufficiency, is very similar to the results of È. G. Šutov [Učen. Zap. Udmurtsk. Ped. Inst. Vyp. 12 (1958), 24–36].} *B. M. Schein* (*Šain*) (Saratov)

Lallement, Gérard 3565

Décompositions matricielles et homomorphismes d'un demi-groupe.

C. R. Acad. Sci. Paris **259** (1964), 273–276.

In this paper the author discusses the homomorphisms of arbitrary semigroups onto completely 0-simple semigroups by using the concept of matricial congruences.

A rectangular 0-band is a semigroup B with 0 such that, for any $a, b \in B$, $aba = a$ or 0, and for any $a, b \in B$, $a \neq 0$ and $b \neq 0$, there is $x \in B$ such that $axb \neq 0$. If a semigroup D has as an image a rectangular 0-band B, the congruence class W corresponding to the zero of B is a two-sided primary ideal of D. A primary ideal W of D is an ideal of D which satisfies: $aDb \subseteq W$ implies $a \in W$ or $b \in W$. A congruence $\mathfrak{M}$ on D, as well as the corresponding decomposition, is called W-matricial if (1.1) W is a class modulo $\mathfrak{M}$, (1.2) $aba \notin W$ implies $aba \equiv a \bmod \mathfrak{M}$. A semigroup D admits a W-matricial decomposition if and only if the primary ideal W satisfies: (2) $abc \in W$ implies $ab \in W$ or $bc \in W$. An equivalence $\mathscr{R}$ is called left [right] W-zero if it is left [right] compatible, if W is a class modulo $\mathscr{R}$, and if $x, y \in D$ and $xy \notin W$ imply $xy \equiv x$ [$xy \equiv y$] mod $\mathscr{R}$. Then a relation is a W-matricial congruence if and only if it is the intersection of a left W-zero equivalence and a right W-zero equivalence. (This expression of intersection is unique.) Consider a strong, regular family of weakly unitary subsemigroups [see the author, same C. R. **258** (1964), 3609–3612; MR **28** #4046]. Their union K satisfies: (3.1) it is reflexive ($ab \in K$ implies $ba \in K$) and its residue W is a primary ideal satisfying (2); (3.2) there exist a left W-zero equivalence $\mathscr{R}$, whose classes are R_i ($i \in I$) and W, and a right W-zero equivalence $\mathscr{L}$, whose classes are L_λ ($\lambda \in \Lambda$) and W, such that for $i \in I$, $K \cap R_i$ is a right unitary subsemigroup and for $\lambda \in \Lambda$, $K \cap L_\lambda$ is a left unitary subsemigroup [see Dubreil, *Algèbre*, Tome I, deuxième édition, Gauthier-Villars, Paris, 1954; MR **16**, 328]. Conversely, every complex K satisfying (3.1) and (3.2) is a union of subsemigroups satisfying (2). A W-matricial congruence $\mathfrak{M} = \mathscr{R} \cap \mathscr{L}$ satisfying (3.2) relative to a complex K which satisfies (3.1) is called "connected to K", similarly for the matrix of $D/\mathfrak{M}$. The homomorphisms of a semigroup onto a completely 0-simple semigroup are associated with the pairs formed by a complex K and a matrix connected with K contracted from that of $D/B(W)$, where $B(W)$ is the intersection of the smallest left W-zero equivalence and the smallest right W-zero equivalence. *T. Tamura* (Davis, Calif.)

Lajos, Sándor 3566

A criterion for Neumann regularity of normal semigroups.

Acta Sci. Math. (*Szeged*) **25** (1964), 172–173.

A semigroup S is said to be Neumann regular in case x belongs to xSx for every x in S; it is said to be normal in case $xS = Sx$ for every x in S. The author proves that a normal semigroup is regular if and only if every left ideal of it is idempotent. *D. J. Foulis* (Gainesville, Fla.)

Aizenštat, A. Ja. 3567

On homomorphisms of semigroups of endomorphisms of ordered sets. (Russian)

Leningrad. Gos. Ped. Inst. Učen. Zap. **238** (1962), 38–48.

Let Ω be an ordered (= partially ordered) set. The set Σ of all endomorphisms (i.e., isotonic transformations) of Ω is a semigroup under the natural multiplication of transformations.

It is well known that if S is a semigroup, then each of its ideals generates some congruence. If all congruences of S are generated by ideals, then S is called semi-simple. This term is due to E. S. Ljapin [Izv. Akad. Nauk SSSR Ser. Mat. **14** (1950), 367–380; MR **12**, 154]. The author shows that Ω is a finite linearly ordered set if and only if Σ is semi-simple. She also finds all ideals of Σ in this case: each of the ideals has the form $I_k = \{S \in \Sigma : S\Omega$ contains k elements at most$\}$. There exist semi-simple semigroups that are not isomorphic to the semigroups of all endomorphisms of finite linearly ordered sets.

B. M. Schein (*Šain*) (Saratov)

Hoehnke, Hans-Jürgen 3568

Zur Theorie der Gruppoide. VII.

Math. Nachr. **27** (1963/64), 289–298.

This paper is the continuation of the author's earlier paper [same Nachr. **25** (1963), 191–198; MR **27** #3735]. Let Γ be the direct product of a group G and a Brandt groupoid S_I, where $S_I = \{e_{ik} | i, k \in I\}$, and Λ is a subgroupoid of Γ. Further, Δ denotes an Ehresmann subgroupoid of Λ, and $\mathfrak{T}(\Gamma)$ the Loewy group [cf. the author's earlier paper, loc. cit.]. Assume that Δ satisfies the following condition: For $t \in \mathfrak{T}(\Gamma)$ with $t\Delta t^{-1} \subseteq \Lambda$, there is $s \in R(\Lambda)$ such that $s^{-1}\Delta^{(\mu)}s = t\Delta^{(\mu)}t^{-1}$, $\Delta = \bigcup_{\mu \in M} \Delta^{(\mu)}$, $\Delta^{(\mu)} \cap \Delta^{(\nu)} = \varnothing$, $\mu \neq \nu$. The main theorem of this paper gives the number of distinct conjugate subgroupoids of Λ with respect to $\mathfrak{T}(\Gamma)$, that is, $\Delta \subseteq \Lambda \cap (t^{-1}\Lambda t)$, under the above assumption on Δ.

T. Tamura (Davis, Calif.)

Cardoso, Jayme Machado 3569

Subtractive quasi-groups. (Portuguese)

Gaz. Mat. (*Lisboa*) **24** (1963), 7–10.

A multiplicative groupoid G is a quasi-group when $ax = b$ and $ya = b$ always has unique solutions. A subtractive groupoid (s.g.) has right unit and inverses with respect to it; in addition, it satisfies the special associative commutative laws $(ab)c = (ac)b$, $a(bc) = c(ba)$. The author proves various properties of such systems; among the principal results are: An s.g. is a quasi-group; it is commutative if and only if its identity is bilateral; for finite s.g. the theorem of Lagrange about the expansion in co-sets holds. *O. Ore* (New Haven, Conn.)

TOPOLOGICAL GROUPS AND LIE THEORY

See also 3524, 3679, 3680, 3879, 3894, 3927, 4066, 4106.

Varopoulos, N. Th. 3570

A theorem on cardinal numbers associated with a locally compact Abelian group.

Proc. Cambridge Philos. Soc. **60** (1964), 701–704.

The following conjecture was announced by E. Hewitt in his paper [Fund. Math. **53** (1963), 55–64; MR **27** #4889]. Let G be an abelian group and let τ_1 and τ_2 be two Hausdorff locally compact topologies on G so that G becomes a topological group in the topology τ_1 as well as in τ_2. Let τ_1 be strictly stronger than τ_2. (That is, there are more open sets in τ_1 than in τ_2.) Then there are at least 2^c continuous characters of G in the topology τ_1 which are discontinuous in the topology τ_2.

In this paper this conjecture is proved. The line of argument runs as indicated in a previous review of the reviewer [see MR **27** #4889]. {Reviewer's remark: The reviewer also obtained this result independently (to appear in Math. Z.). K. Ross also solved this problem independently (unpublished). The method is the same in all three cases.} *M. Rajagopalan* (Urbana, Ill.)

Freudenthal, H. 3571

Ein Zerlegungssatz für im Kleinen kompakte Gruppen.

Arch. Math. **15** (1964), 161–165.

Let G be a locally compact group, and A a connected abelian subgroup such that the left coset space G/A is compact. Then there is a normal vector subgroup N and a compact subgroup C in G such that G splits as a semidirect product of N and C. This was first proved by Hofmann and Mostert [Mem. Amer. Math. Soc. No. 43 (1963); MR **27** #1529]. Another proof was recently given by J. Tits [Topology **3** (1964), suppl. 1, 97–107; MR **28** #2170]; the statement of this result was incorrectly formulated in MR **28** #2170. The author gives another proof after proving the following Theorem II. Let G be a topological group whose identity component is a Lie group with Lie algebra $\mathfrak{G}$. Define $\mathfrak{F}=\{X\in\mathfrak{G}: X^G$ is bounded$\}$. Then $\mathfrak{F}$ is a Lie sub-algebra of $\mathfrak{G}$, and if F is the corresponding Lie subgroup of G, then F is a connected and maximally almost periodic normal subgroup of G. This gives a much simpler and more elegant proof than the original. (Tits's approach is different from either and proves more.) Another consequence of Theorem II is Theorem III: If G is a simple Lie group with a relatively compact set b^G of conjugates for some $b\in G$, then G is itself compact. His Theorem IV is the same as Theorem X of Hofmann and Mostert [loc. cit.].

P. S. Mostert (New Orleans, La.)

Flachsmeyer, Jürgen; Zieschang, Heiner 3572

Über die schwache Konvergenz der Haarschen Masse von Untergruppen.

Math. Ann. **156** (1964), 1–8.

Theorem 1: If $\{\mu_i\}$ is a set of (suitably normed) Haar measures of closed subgroups X_i of a Hausdorff locally compact group, then weak convergence of the μ_i is equivalent to Hausdorff convergence of the X_i. Theorem 2: The set of Haar measures of all closed subgroups, together with the zero measure, is compact.

J. G. Wendel (Ann Arbor, Mich.)

Gerstenhaber, Murray; Rothaus, Oscar S. 3573

The solution of sets of equations in groups.

Proc. Nat. Acad. Sci. U.S.A. **48** (1962), 1531–1533.

Let H be a compact connected Lie group of rank m, $W_1,\cdots,W_r$ words in elements of H ("constants") and indeterminates $x_1,\cdots,x_r$ and d_{ij} the exponent sum of x_j in W_i. Put $d=\det(d_{ij})$; then the degree of the mapping $T: H^{(r)}\to H^{(r)}$ given by $(x_1,\cdots,x_r)\mapsto(W_1,\cdots,W_r)$ is d^m, and if $d\neq 0$, then there are values of $x_1,\cdots,x_r$ in H such that for these values of x_i, $W_j=1$ for $j=1,\cdots,r$. This generalization of Hopf's mapping theorem [H. Hopf, Comment. Math. Helv. **13** (1940), 119–143; MR **2**, 35], whose proof relies on Hopf's original theorem, leads to two further results: If G is a finite group, W_i $(i=1,\cdots,r)$ words in $x_1,\cdots,x_r$ with coefficients in G, then the simultaneous equations $W_i=1$ have a solution in a finite supergroup of G provided $d\neq 1$. This generalizes significantly existing results by F. Levin [Bull. Amer. Math. Soc. **68** (1962), 603–604; MR **26** #212] and H. Schiek [Math. Ann. **146** (1962), 314–320; MR **25** #111]. The other result is a generalization of the "Freiheitssatz" [W. Magnus, J. Reine Angew. Math. **170** (1934), 235–240].

Hanna Neumann (Canberra)

Matsumoto, Hideya 3574

Quelques remarques sur les groupes algébriques réels.

Proc. Japan Acad. **40** (1964), 4–7.

The author announces without proof two propositions and four theorems related to work of previous authors, including Kostant, Murakami, Satake, Sugiura and Tits. His results are concerned with (i) the connectivity of algebraic groups, and (ii) the automorphisms and conjugate classes of Cartan and maximal solvable sub-algebras of semi-simple Lie algebras.

A. J. Coleman (Kingston, Ont.)

Kopfermann, Klaus 3575

Maximale Untergruppen Abelscher komplexer Liescher Gruppen.

Schr. Math. Inst. Univ. Münster No. 29 (1964), iii + 72 *pp*.

Let G be a complex Lie group. A closed complex normal subgroup N of G is called holomorphically [meromorphically] maximal in G if (1) every holomorphic [meromorphic] function on N is a constant and (2) G/N is holomorphically [meromorphically] separable. A complex manifold M is called holomorphically [meromorphically] separable if for $x,y\in M$, $x\neq y$, there is a holomorphic [meromorphic] function f on M such that (f is holomorphic at x, y and) $f(x)\neq f(y)$. If G is holomorphically maximal in G itself (e.g., a complex torus), G is called a toroidal group. The author considers the generalization of the theory of abelian functions (i.e., meromorphic functions on a complex torus) to meromorphic functions on a toroidal group.

In Chapter 1, the author proves that every connected complex abelian Lie group is the direct product of a toroidal group with $\mathbf{C}^m\times\mathbf{C}^{*n}$, and he characterizes a complex abelian Lie group to be toroidal. In Chapter 2,

the author proves that for any meromorphic function f on a toroidal group $G=\mathbf{C}^m/D$ (D is a discrete subgroup of $\mathbf{C}^m$) there exist theta functions g, h with respect to D such that $f\circ p=g/h$, where $p:\mathbf{C}^m\to G$ is the projection and where a holomorphic function g on C^m is called a theta function with respect to D if for each $c\in D$ there exists a linear polynomial L_c such that $g(z+c)=g(z)\exp(L_c(z))$ for $z\in\mathbf{C}^m$. Then, the notion of a reduced [respectively, trivial] theta function is introduced and the author proves that every theta function can be written as the product of a reduced theta function with a trivial one. The notion of a (non-degenerate) Riemann form with respect to D is also defined.

In Chapter 3, the author considers the complex torus and proves the classical theorem which asserts the equivalence of the following four conditions for a complex torus $G=\mathbf{C}^m/D$: (1) there exists a non-degenerate meromorphic function f on $\mathbf{C}^m/D$, i.e., the period group of $f\circ p$ is discrete in $\mathbf{C}^m$; (2) there exists a non-degenerate Riemann form with respect to D; (3) G is a projective-algebraic manifold; (4) the transcendence degree of the meromorphic function field of G over the complex number field (which is called the algebraic dimension of G) is m. The following theorems are also proved. If G is of algebraic dimension t, then G contains the meromorphically maximal subgroup N in G with $\dim N=\dim G-t$. For any $m>1$ and $0\leq t\leq m$ (m, t being integers), there exists an m-dimensional complex torus G of algebraic dimension t.

In Chapter 4, using the results in the preceding chapters, the author proves the existence of the meromorphically maximal subgroup in every connected complex abelian Lie group, and shows an example of a non-compact complex Lie group without non-constant meromorphic functions. *Akihiko Morimoto* (La Jolla, Calif.)

Bernat, Pierre 3576

Sur le dual d'un groupe résoluble exponentiel.

C. R. Acad. Sci. Paris **258** (1964), 5311–5314.

Let $\mathfrak{g}$ be a real solvable Lie algebra which is exponential, that is, such that the exponential mapping exp of $\mathfrak{g}$ into the corresponding simply connected Lie group G is surjective. Let f be a linear form on $\mathfrak{g}$ and $S(f)$ be the set of subalgebras $\mathfrak{h}$ of $\mathfrak{g}$ such that $f|[\mathfrak{h},\mathfrak{h}]=0$. Then any $\mathfrak{h}$ in $S(f)$ defines a character of the subgroup $\exp\mathfrak{h}$ which induces a unitary representation $\rho(f,\mathfrak{h})$ of G. The author gives the conditions for $\rho(f,\mathfrak{h})$ to be irreducible. Let $E(f)$ denote the set of $\mathfrak{h}$ in $S(f)$ with the following property: For any couple $(\mathfrak{k},\mathfrak{p})$ of a subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ and of an ideal $\mathfrak{p}$ of $\mathfrak{k}$, if $\mathfrak{k}\supset\mathfrak{h}$ and if $f|\mathfrak{p}=0$, then $\mathfrak{h}$ contains the inverse image of the ideal sum of (non-zero) minimal ideals of $\mathfrak{k}/\mathfrak{p}$ under the canonical mapping $\mathfrak{k}\to\mathfrak{k}/\mathfrak{p}$. If any element of $S(f)$ of maximal dimension belongs to $E(f)$, or if G is quasi-nilpotent, that is, $\mathfrak{g}$ is solvable exponential and for any x in $\mathfrak{g}$ the real proper values of ad x are all zero, then the set of elements $\mathfrak{h}$ of $S(f)$ such that $\rho(f,\mathfrak{h})$ is irreducible coincides with the set of elements $\mathfrak{h}$ of $S(f)$ of maximal dimension. *S. Tôgô* (Hiroshima)

Saito, Masahiko 3577

Représentations unitaires du groupe des déplacements du plan $\mathfrak{p}$-adique.

Proc. Japan Acad. **39** (1963), 407–409.

From the introduction: "Le but de cette note est une extension au cas $\mathfrak{p}$-adique des résultats de Vilenkin [Uspehi Mat. Nauk. **11** (1956), no. 3 (69), 69–112; MR **19**, 153] sur les représentations du groupe des déplacements euclidiens. En calculant explicitement les coefficients matriciels des représentations unitaires irréductibles à l'aide d'une base naturelle, on est conduit à une certaine classe de fonctions que l'on pourrait appeler fonctions de Bessel $\mathfrak{p}$-adiques. En particulier les fonctions sphériques zonales s'expriment par les fonctions de Bessel $\mathfrak{p}$-adiques d'indice 0, qui essentiellement n'est autre qu'une somme de Gauss du corps des restes." *R. Steinberg* (Los Angeles, Calif.)

Boseck, Helmut 3578

Darstellungen von Matrixgruppen über topologischen Körpern. I.

Math. Nachr. **24** (1962), 229–243.

Let G be the group PGL(2, K), where K is a discrete-valued locally compact field. G acts on K as a group of fractional linear transformations, that is, if $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ is a matrix representing $g\in G$, then $xg=(\alpha x+\beta)/(\gamma x+\delta)$. For $x\in K$ and $g\in G$ set $\beta(x,g)=|\det g|\,|\beta x+\delta|^{-2}$, where $x\to|x|$ is the normalized valuation on K. For each character χ of the multiplicative group K^* of K and $g\in G$ define a unitary operator on $L^2(K)$ (formed with respect to Haar measure on the additive group K^+ of K) by: $T_g^{(\chi)}f(x)=\chi((\det g)(\beta x+\delta)^{-2})\beta^{1/2}(x,g)f(xg)$. The maps $g\to T_g^{(\chi)}$ are unitary representations of G which the author shows, by means of the Fourier transform on K^+, are irreducible, and comprise what the author calls the first principal series of representation of G. He obtains these representations as the components in a direct integral decomposition of a quasi-regular representation of G defined as follows. Let p be a prime in K and put $Z=K(\pi)$, where $\pi^2=p$. For $z=x+\pi y\in Z$ put $Nz=x^2-py^2$ and let $Z_0=\{z=x+\pi y: y\neq 0\}$. G acts transitively as a group of fractional linear transformations on Z_0. Choose $x\in K$ and $s\in K^*$. The author calls the set

$$\omega(x,s)=\{z\in Z_0: N(z-(x+\pi s))=-ps^2\}$$

a horosphere. Let Ω_0 be the set of horospheres in Z_0. G acts transitively on Ω_0 and there is a quasi-invariant measure in Ω_0 which means that G acting on Ω_0 defines a unitary representation T on $L^2(\Omega_0)$. By use of the Fourier transform on K^* the author shows that T is the direct integral of the $T^{(\chi)}$. Similar results are given for the group PSL(2, K). *A. Kleppner* (College Park, Md.)

Nakamura, Masahiro; Umegaki, Hisaharu 3579

Heisenberg's commutation relation and the Plancherel theorem.

Proc. Japan Acad. **37** (1961), 239–242.

Let G and X be a Pontrjagin pair of locally compact abelian groups. For a Borel set $S\subset G$ let $E(S):L^2(G)\to L^2(G)$ be the operator or multiplication with the characteristic function of S. Let $U(g)$ be the left regular representation of G, and let $V(\chi)=\int\overline{(g,\chi)}\,dE(g)$. Similarly, let $E'(S')$ be the corresponding multiplication operator on X, let $V'(\chi)$ be the left regular representation of X on $L^2(X)$, and let $U'(g)=\int(g,\chi)\,dE'(\chi)$. Then both the pairs (U,V) and (U',V') satisfy the Heisenberg commutation relation $U(g)V(\chi)=(g,\chi)V(\chi)U(g)$. By the Mackey-Loomis theorem

there is a unitary transformation $T: L^2(G)\to L^2(X)$ such that $U'T=TU$ and $V'T=TV$. The Plancherel theorem is deduced from this by showing that T is essentially the Fourier transformation F when restricted to $L^1(G)\cap L^2(G)$. This is done as follows: The Heisenberg relation implies $T(\varphi*\psi)=T(\varphi)F(\psi)$. Let φ run through an approximate identity. One shows that $T\varphi$ tends to a nonzero constant. *J. Wang* (New Haven, Conn.)

Helgason, S. 3580

Some results on invariant theory.

Bull. Amer. Math. Soc. **68** (1962), 367–371.

This paper announces results that are published with proofs in the following two papers. The contents of Sections 1, 2, and 4 are published in the paper below [#3581]. The contents of Section 3 are published as Theorem 7.5 in Amer. J. Math. **86** (1964), 565–601 [MR **29** #2323]. *M. Sugiura* (Osaka)

Helgason, Sigurdur 3581

Invariants and fundamental functions.

Acta Math. **109** (1963), 241–258.

Let V be a finite-dimensional vector space over $\mathbf{R}$. Each $X\in V$ gives rise to a differential operator $\partial(X)$ on V. Let G be a subgroup of $\mathrm{GL}(V)$. Let $I(V)$ denote the set of G-invariants in the symmetric algebra $S(V)$, and let $I_+(V)$ denote the set of G-invariants without constant term. The group G acts on the dual space V^* as $(gv^*)(v)=v^*(g^{-1}v)$, and we have $I_+(V^*)\subset I(V^*)\subset S(V^*)$. An element $p\in S^c(V^*)=\mathbf{C}\otimes S(V^*)$ is called G-harmonic if $\partial(J)p=0$ for all $J\in I_+(V)$. Let $H^c(V^*)$ denote the set of G-harmonic polynomial functions. Let N_G denote the variety in V^c defined by

$$N_G=\{X\in E^c\,|\,j(X)=0 \text{ for all } j\in I_+(V^*)\}.$$

Now suppose B_0 is a nondegenerate symmetric bilinear form on $V\times V$; let B denote the unique extension of B_0 to a bilinear form on $V^c\times V^c$. If $X\in V^c$, let X^* denote the linear form $Y\to B(X,Y)$ on V^c. Let $H_1(V^*)$ denote the vector space over $\mathbf{C}$ spanned by the functions $(X^*)^n$ ($n=0,1,2,\cdots;X\in N_G$), and let $H_2(V^*)$ denote the set of G-harmonic polynomial functions which vanish identically on N_G. Then the main result of § 1 can be stated as the following theorem. Suppose that G leaves B_0 invariant and suppose that either (1) G is compact and B_0 is positive definite, or (2) G is connected and semisimple. Then (*) $S(V^*)=I(V^*)H(V^*)$ and $H^c(V^*)=H_1(V^*)+H_2(V^*)$ (direct sum). (*) was also obtained independently by B. Kostant [Amer. J. Math. **85** (1963), 327–404; MR **28** #1252].

In § 2, a result similar to (*) is obtained for the exterior algebra $\Lambda(V^*)$. In § 3, the quadric $C_{p,q}\subset\mathbf{R}^{p+q+1}$ given by the equation

$$Q(X)=x_1^2+\cdots+x_p^2-x_{p+1}^2-\cdots-x_{p+q+1}^2=-1 \qquad (p\geq 0,\ q\geq 0)$$

is considered. Let $O(p,q+1)$ be the group of linear transformations of $\mathbf{R}^{p+q+1}$ leaving Q invariant. The complex-valued continuous function f on $C_{p,q}$ is called fundamental if the vector space V_f over $\mathbf{C}$ spanned by the translations of f by the group $O(p,q+1)$ is finite-dimensional. Then the second main result of this paper can be stated as the following theorem. Let f be a fundamental function on $C_{p,q}$. Assume $(p,q)\neq(1,0)$. Then there exists a polynomial $P=P(x_1,\cdots,x_{p+q+1})$ such that $f=P$ on $C_{p,q}$. *M. Sugiura* (Osaka)

Mautner, F. I. 3582

Spherical functions over $\mathfrak{P}$-adic fields. II.

Amer. J. Math. **86** (1964), 171–200.

Pour I voir le même J. **80** (1958), 441–457 [MR **20** #82]. Soient $G=\mathrm{PGL}(2,\Omega)$ le groupe projectif sur un corps $\mathfrak{P}$-adique Ω; K son sous-groupe compact maximal formé des éléments provenant de matrices à coefficients dans l'anneau des entiers $\mathfrak{O}$ de Ω. Etant donnée une représentation unitaire irréductible $\mathfrak{u}$ de K, l'auteur considère l'ensemble $S_{\mathfrak{u}}{}^0$ des fonctions $\mathfrak{f}(g)$ sur G, dont les valeurs sont des transformations linéaires de l'espace de la représentation $\mathfrak{u}$, $\mathfrak{f}$ étant supposée continue à support compact et vérifiant la relation

$$\mathfrak{f}(kgk')=\mathfrak{u}(k)\mathfrak{f}(g)\mathfrak{u}(k')$$

pour $g\in G$, k, k' dans K; ces fonctions forment une algèbre pour la convolution. Si $\mathfrak{f}=(f_{ij})$, matrice par rapport à une base orthonormale de l'espace de $\mathfrak{u}$ (supposé de dimension r), l'application $\mathfrak{f}\to r\cdot f_{11}$ est un isomorphisme de $S_{\mathfrak{u}}{}^0$ sur l'algèbre $S_e{}^0$ des fonctions continues à support compact f sur G telles que $f(g)=\int_K e(k)f(k^{-1}g)\,dk$, où $e=r\cdot u_{11}$. L'auteur se borne à un type de représentation $\mathfrak{u}=\mathfrak{u}_\chi$, obtenu à partir d'un caractère χ de $\bar{A}$, groupe multiplicatif de $\mathfrak{O}/\mathfrak{P}^m$ pour un entier $m\geq 1$ fixé; $\bar{A}$ est un tore maximal du groupe $\bar{K}=\mathrm{PGL}(2,\mathfrak{O}/\mathfrak{P}^m)$, image canonique de K, et on a donc une représentation $\bar{\mathfrak{u}}$ de $\bar{K}$ induite par χ; $\mathfrak{u}$ est la composée de $\bar{\mathfrak{u}}$ et de l'homomorphisme canonique $K\to\bar{K}$. On considère d'autre part la "série principale" de représentations $g\to M(g,\alpha)$ de G obtenues comme représentations induites par des caractères α du tore maximal A de G (isomorphe à Ω^*); en fait, l'auteur se borne aux caractères α tels que $\alpha(\tau^n\eta)=q^{n(s-1)}\alpha_1(\eta)$ (τ uniformisante de $\mathfrak{O}$, q nombre d'éléments du corps résiduel $\mathfrak{O}/\mathfrak{P}$, η variant dans le groupe des unités $\mathfrak{U}$ de $\mathfrak{O}$, s nombre complexe), où α_1 est un caractère fixé de $\mathfrak{U}$, non réel et égal à 1 dans $1+\mathfrak{P}^m$ (même entier m que ci-dessus). Il calcule explicitement la représentation $M(g,\alpha)$ correspondante, et montre qu'elle est irréductible, et qu'elle est unitaire si et seulement si $s=\frac{1}{2}+it$ (t réel). On considère alors, pour $f\in S_e{}^0$, sa "transformée de Fourier" $\mathfrak{F}(\alpha)=\int f(g)M(g,\alpha)^*\,dg$; on montre que c'est une matrice scalaire dont on calcule explicitement la valeur $F(\frac{1}{2}+it)$ des éléments diagonaux. Si $S_e{}^2$ est l'adhérence de $S_e{}^0$ dans $L^2(G)$, on note $\mathfrak{S}_e{}^2$ le sous-espace orthogonal au noyau de l'application $f\to F(\frac{1}{2}+it)$, et on prouve (grâce à l'expression explicite de $F(\frac{1}{2}+it)$) que la restriction de $f\to F(\frac{1}{2}+it)$ à $\mathfrak{S}_e{}^2$ est un isomorphisme sur l'espace L^2 pour la mesure $c\cdot dt$, avec $c=(q^m+q^{m-1})\log q/2\pi$ sur l'intervalle $0\leq t\leq 2\pi/\log q$; on a en outre la formule d'inversion

$$f(g)=c\cdot\int_0^{2\pi/\log q}F(\tfrac{1}{2}+it)M_{11}(g,\tfrac{1}{2}+it)\,dt.$$

On peut d'ailleurs préciser encore l'intersection $\mathfrak{S}_e{}^0=\mathfrak{S}_e{}^2\cap S_e{}^0$, qui est une sous-algèbre de $S_e{}^0$, isomorphe à l'algèbre des polynômes trigonométriques $\sum_n a_n q^{nit}$; en outre, si $q>2$ et $m>1$, on montre que $\mathfrak{S}_e{}^0\neq S_e{}^0$.

J. Dieudonné (Paris)

Antoine, J. P. 3583

Représentations irréductibles du groupe SU_3. (English summary)

Ann. Soc. Sci. Bruxelles Sér. I **77** (1963), 150–162.

This is one more mathematical treatment of the representations of SU(3). *Y. Ne'eman* (Pasadena, Calif.)

Egardt, L. 3584

On the construction of invariants for SU(n).

Ark. Fys. **27** (1964), 193–194.

Let $X_1, \cdots, X_{n^2-1}$ denote a basis for the Lie algebra of SU(n), with $[X_i, X_j] = c_{ijk}X_k$ and $\frac{1}{2}\{X_i, X_j\} = \delta_{ij}I/n + d_{ijk}X_k$. Then

$$c_{ijk} = \mathrm{Tr}(X_i[X_j, X_k])$$

and

$$d_{ijk} = \mathrm{Tr}(\tfrac{1}{2}\{X_i, X_j\}X_k).$$

The author first notes that the c's and d's are related by the equation $d_{jkl}c_{ilm} + d_{kil}c_{jlm} + d_{ijl}c_{klm} = 0$. He then employs this relation to show that if Y_i and Z_i are elements of the Lie algebra chosen so that $[X_i, Y_j] = c_{ijk}Y_k$ and $[X_i, Z_j] = c_{ijk}Z_k$ (i.e., Y_i and Z_i are "vector operators"), and if V_i are defined by the relation $V_i = d_{ijk}Y_jZ_k$, then $[X_i, V_j] = c_{ijk}V_k$, so that V_i is again a vector operator.

These vector operators are of interest in constructing a complete set of independent invariants for the Lie algebra of SU(n) [cf. L. C. Biedenharn, J. Mathematical Phys. **4** (1963), 436–445; MR **26** #5097].

R. T. Prosser (Lexington, Mass.)

Hochschild, G. 3585

Ample algebras of representative functions on real analytic groups.

Topology **3** (1964), *suppl.* 1, 119–123.

Soit G un groupe analytique réel et soit $F(G)$ l'algèbre des fonctions représentatives continues à valeurs réelles sur G. Si $F(G)$ sépare les points de G, l'algèbre $F(G) \otimes \mathbf{C}$ complexifiée de $F(G)$ s'identifie à l'algèbre des fonctions représentatives holomorphes sur le groupe G^+ complexification linéaire universelle de G. Une sous-algèbre F de $F(G)$ est dite "ample" si (1) F sépare les points de G, (2) F contient les scalaires, (3) F est stable par les translations (droites et gauches) de G ainsi que par l'involution $s \to s^{-1}$. L'auteur démontre que, si $F(G)$ est engendrable par un nombre fini d'éléments et sépare les points de G, il existe une correspondance de type galoisien entre les sous-algèbres amples de $F(G)$ et les sous-groupes centraux finis Γ de G^+ qui sont stables par la conjugaison canonique dans G^+ et vérifient la condition $\Gamma \cap G = (1)$. Cette correspondance associe à un sous-groupe Γ de G^+ la sous-algèbre constituée par les $f \in F(G)$ telles que $f(xy) = f(y)$ quels que soient $x \in \Gamma$ et $y \in G^+$ lorsque f est considérée comme une fonction sur G^+. En utilisant cette correspondance, l'auteur montre qu'avec les hypothèses faites sur $F(G)$, le nombre des sous-algèbres amples est fini. *J. L. Koszul* (Grenoble)

Vainberg, Ju. R. 3586

On the reduction of formal groups with respect to a prime modulus. (Russian)

Sibirsk. Mat. Ž. **4** (1963), 1263–1270.

Lazard [C. R. Acad. Sci. Paris **239** (1954), 942–945; MR **16**, 219] has proved that every one-parameter formal Lie group $f(x, y) \in A[[x, y]]$ over an integral domain A is abelian. For short, call $f(x, y)$ a group over A. If $A = \mathbf{Q}$, and if all the coefficients of $f(x, y)$ are p-integral, reducing these coefficients modulo p yields a group $\bar{f}$ over GF(p). Considering $\bar{f}$ as a group over the algebraic closure of GF(p), $\bar{f}$ is isomorphic to exactly one of a family of groups $G_{1,m}(p)$, $m \in M = \{0, 1, 2, \cdots, \infty\}$, by a classification theorem of Lazard [Bull. Soc. Math. France **83** (1955), 251–274; MR **17**, 508] and Dieudonné [Amer. J. Math. **77** (1955), 218–244; MR **16**, 789; Math. Z. **63** (1955), 53–75; MR **17**, 174]. Let π be any set of primes; to each $p \in \pi$ assign an arbitrary $m_p \in M$. Then the author proves that there exists a group $f(x, y)$ over $\mathbf{Q}$ such that for each $p \in \pi$, f can be reduced modulo p to yield a group isomorphic to $G_{1,m_p}(p)$, while for each $p \notin \pi$ not all coefficients of $f(x, y)$ are p-integral. The proof uses the methods of the second Lazard paper to construct $f(x, y)$. The author also reformulates the definition of "group over A" in terms of a coproduct on $A[[x]]$.

W. F. Reynolds (Medford, Mass.)

FUNCTIONS OF REAL VARIABLES

See also 3598, 3852, 3854, 3858, 3913.

Rudin, Walter 3587

★Principles of mathematical analysis.

Second edition.

McGraw-Hill Book Co., New York, 1964. ix+270 pp. $7.95.

The first edition (1953) of this text was reviewed earlier [MR **14**, 1070]. The main changes appearing in this second edition are an increase by some fifty percent in the number of exercises, and an enlarged chapter on functions of several variables.

That chapter now begins with some linear algebra and uses this to introduce differentiability and to obtain the inverse and implicit function theorems. The new material of the chapter features integration in R^n, culminating in a version of Stokes's theorem.

I. Glicksberg (Seattle, Wash.)

Cinquini, Silvio 3588

Un'osservazione sopra un'estensione del lemma di Gronwall.

Ist. Lombardo Accad. Sci. Lett. Rend. A **97** (1963), 64–67.

Validity of a generalization of Gronwall's lemma given by E. Storchi [Matematiche (Catania) **16** (1961), 8–26; MR **26** #272] does not require differentiability, but only boundedness of Dini's right upper derivative at the origin. The proof reduces this apparently more general case to the classical Gronwall lemma.

E. Baiada (Modena)

Cupello, Laura 3589

Sulle costanti delle condizioni di Hölder in forma integrale.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **35** (1963), 460–462.

If $\phi(t)$ is a function of the real variable t, defined and

summable in $a \leq t \leq b$ and identically equal to zero outside the same interval, then it is known that

$$\int_a^b |\phi(t+h)-\phi(t)|\,dt = o(|h|) \qquad (h \to 0)$$

if and only if there exists a number c such that $\phi(t)=c$ almost everywhere.

Also

$$\int_a^b |\phi(t+h)-\phi(t)|\,dt = O(|h|) \qquad (h \to 0)$$

if and only if $\phi(t)$ is a function of a finite variation almost everywhere in $a \leq t \leq b$.

Here the author studies the integral

$$I(\phi) = \int_0^1 |\phi(t \pm h)-\phi(t)|\,dt, \qquad \phi(t) = |t|^{\alpha-1}\psi(t)$$

for $h \to 0^+$. Of the four theorems stated in this paper, the following is characteristic: If $\psi(t) \to 0$ as $t \to 0$, and $\psi(t)$ satisfies the condition of W. H. Young in $0<t<1$, viz., $\int_0^1 |d(u\psi(u))|$ is finite, $\int_0^\eta |d(u\psi(u))| = O(\eta)$ for $\eta \to 0^+$, then, for $\varepsilon>0$ and $0 \leq h \leq h_0(\varepsilon)$,

$$I(\phi) < \{K(\psi,\alpha)+\varepsilon\}\bar{\psi}^{1-\alpha}(\lambda)\cdot h^\alpha,$$

where $K(\psi,\alpha)=2/\alpha+H_\alpha$, $\lambda=\sup u(\bar{\psi}(u)\cdot u \leq h)$, and $\bar{\psi}(t)=\sup_{0<u<t}|\psi(u)|$, $H_\alpha=\sigma_1 \lim_{\eta\to 0} \bar{H}(\eta,\sigma)$,

$$\sigma_1 = (1+\sigma)^{1-\alpha}/\{(1+\sigma)^{1-\alpha}-1\}$$

$$\bar{H}(\eta,\sigma) = \sup_{0<u\leq\eta} V_\psi(u/(1+\sigma),u),$$

$V_\psi(a,b)$ being the total variation of $\psi(t)$ in the interval $a \leq t \leq b$, $\sigma>0$, and $0<\eta<1$. *S. K. Chatterjea* (Calcutta)

Rennie, B. C. 3590

On a class of inequalities.

J. Austral. Math. Soc. **3** (1963), 442–448.

In this paper it is shown that the problem of finding the best possible inequality in a certain class, involving k integrals, is often equivalent to the problem of determining the convex hull of a related set of points in k-dimensional space.

Let $\varphi_1,\cdots,\varphi_k$ be given continuous functions on the product space $S_1\times\cdots\times S_r$, and let $f_1,\cdots,f_r$ be measurable functions on the set E, which is either $(0,1)$ or $(-\infty,\infty)$, with respective ranges restricted to $S_1,\cdots,S_r$. For each such $f_1,\cdots,f_r$, let $u_i=\int_E \varphi_i(f_1,\cdots,f_r)\,dx$, and let C be the set of all points $(u_1,\cdots,u_k)$. It is shown first that C is convex. Next, for $E=(0,1)$, it is shown that the closure of C is the intersection of all the half-spaces $\{u; a_1u_1+\cdots+a_ku_k \geq a_0\}$ having coefficients such that $a_1\varphi_1(x)+\cdots+a_k\varphi_k(x)\geq a_0$ for all $x \in S_1\times\cdots\times S_r$. For $E=(-\infty,\infty)$, the result is the same except that a_0 is replaced by 0.

The applicability of the results is illustrated by deriving several known inequalities, in particular, the Cargo-Shisha generalization [Cargo and Shisha, J. Res. Nat. Bur. Standards Sect. B **66B** (1962), 169–170; MR **26** #5110] of the Kantorovich inequality.

E. F. Beckenbach (Los Angeles, Calif.)

Daróczy, Z. 3591

Einige Ungleichungen über die mit Gewichtsfunktionen gebildeten Mittelwerte.

Monatsh. Math. **68** (1964), 102–112.

In the interval $0<x<\infty$, let $f(x)$, $g(x)$ be positive, and let $\varphi(x)$, $\psi(x)$ be continuous and strictly monotone. For positive vectors $(a)=(a_1,a_2,\cdots,a_n)$, the author investigates the mean-value function

$$M_\varphi[a]_f = \varphi^{-1}\left[\sum_{i=1}^n f(a_i)\varphi(a_i)\Big/\sum_{i=1}^n f(a_i)\right].$$

It is noted that $M_\varphi[a]_f \leq M_\psi[a]_f$ for all positive (a) if and only if $\psi\varphi^{-1}$ is convex for ψ increasing, and is concave for ψ decreasing. Further, $M_\varphi[a]_f \leq M_\varphi[a]_g$ for all positive (a) if and only if $f(x)/g(x)$ is nonincreasing. Applications are made to power means, to the means introduced by the reviewer [Amer. Math. Monthly **57** (1950), 1–6; MR **11**, 422], to homogeneous means, and to the entropy function of information theory.

E. F. Beckenbach (Los Angeles, Calif.)

Levinson, N. 3592

On an inequality of Opial and Beesack.

Proc. Amer. Math. Soc. **15** (1964), 565–566.

Under the assumptions that $y(x)$ a complex-valued, absolutely continuous function on $(0,a)$ and $y(0)=0$, the author gives a simple proof of the inequality

$$\int_0^a |y(x)y'(x)|\,dx \leq \frac{1}{2}a\int_0^a |y'(x)|^2\,dx$$

established in another way by P. Beesack [Trans. Amer. Math. Soc. **104** (1962), 470–475; MR **25** #3137] as a generalization of an analogous inequality by the reviewer [Ann. Polon. Math. **8** (1960), 29–32; MR **22** #3772] and C. Olech [ibid. **8** (1960), 61–63; MR **22** #3773].

Z. Opial (Kraków)

Gosselin, R. P. 3593

A maximal theorem for subadditive functions.

Acta Math. **112** (1964), 163–180.

In this paper, the author first proves a maximal theorem for subadditive functions and then applies it to a variety of problems.

Let S_R be the solid sphere of radius R in E_n. The function ϕ is here said to be subadditive on S_R if there exists a constant $C>0$ such that $\phi(u+v)\leq C[\phi(u)+\phi(v)]$ for all $u, v \in S_R$, and to be subadditive-even there if, in addition, $\phi(u)\leq C[\phi(u+v)+\phi(v)]$ for all $u, v \in S_R$. The maximal function $\omega(t)$ corresponding to a given subadditive-even function ϕ on S_R is defined by $\omega(t)=\sup_{|v|\leq t}\phi(v)$, $t\in(0,R)$; then $\omega(t)$ is also subadditive-even, and is finite and nondecreasing. The maximal theorem for subadditive-even functions ϕ on S_R in E_n is expressed by the inequality

$$\int_0^R \frac{\omega^p(t)}{t^{1+p\alpha}}\,dt \leq C\int_{S_R}\frac{\phi^p(u)}{|u|^{n+p\alpha}}\,du$$

for all real α and all p, $0<p\leq\infty$.

Applications are made to known subadditive functions, to integral transforms, and to sums and integrals involving subadditive functions.

E. F. Beckenbach (Los Angeles, Calif.)

Redheffer, Raymond M. 3594

Differential and integral inequalities.

Proc. Amer. Math. Soc. **15** (1964), 715–716.

Elementary proofs are given of the following.

Let $Tu = du/dt - a(t)b(u)$, $b > 0$, a and b continuous, and let $A(t) = \int_0^t a(s)\,ds$, $B(y) = \int_\delta^y [b(s)]^{-1}\,ds$. (1) If $Tu \leq 0 \leq Tv$ and $u(0) \leq v(0)$, then $u \leq v$. (2) If $Tu \leq 0$ and $u(0) \leq v(0) = \delta$, then $u \leq B^{-1}[A(t)]$. (3) If $a \geq 0$ and b is monotone non-decreasing, then $w \leq \delta + \int_0^t a(s)b[w(s)]\,ds$. (An example shows that the monotony of b cannot be dropped.)

Let u and v be vectors with any convenient norm and let $Tu = du/dt - f(t, u)$, f a vector. (4) If $\|f(t, u) - f(t, v)\| \leq a(t)b(w)$, $w = \|u - v\|$ and for $t = 0$, $Tu = Tv$ and $\|u - v\| = \delta$, then

$$B^{-1}[-A(t)] \leq \|u - v\| \leq B^{-1}[A(t)].$$

(3) sharpens an inequality of Bihari and (4) sharpens an inequality of Bihari and Langenhop [see E. F. Beckenbach and R. Bellman, *Inequalities*, pp. 133–136, Springer, Berlin, 1961; MR 28 #1266].

A. E. Danese (Buffalo, N.Y.)

Kotljar, B. D. 3595

On quasi-smooth functions of two variables. (Russian)
Ukrain. Mat. Ž. **16** (1964), 383–385.

A function $f(x, y)$ defined in a domain D is quasi-smooth in D if for some $K > 0$

$$(*) \quad \left| f(x, y) - 2f\left(\frac{x + x'}{2}, \frac{y + y'}{2}\right) + f(x', y') \right| \leq K\sqrt{((x - x')^2 + (y - y')^2)}$$

for all (x, y), $(x', y') \in D$; it is quasi-smooth in y in D if $(*)$ holds under the restriction that $x = x'$. Similarly, the quasi-smoothness in x is defined. The result: If D is a rectangle with sides parallel to the coordinate axes and if $f(x, y)$ is bounded, then from the smoothness in both variables x and y follows the smoothness of $f(x, y)$ in D.

The same remains true if D is taken to be the entire plane. However, a similar result is not known for other types of domains. *H. Fast* (Notre Dame, Ind.)

Henkin, G. M. 3596

Linear superpositions of continuously differentiable functions. (Russian)
Dokl. Akad. Nauk SSSR **157** (1964), 288–290.

Let $p_m(x_1, x_2)$ be continuous and $q_m(x_1, x_2)$ continuously differentiable fixed functions defined on the whole plane (x_1, x_2), $m = 1, 2, \cdots, N$. Let D be a domain in that plane and $C(D)$ the space of continuous functions on D. Then (1) functions of the form $\sum_{m=1}^N p_m(x_1, x_2) f_m(q_m(x_1, x_2))$ constitute a non-dense subset of the $C(D)$ as the $f_m(t)$ run over all continuous functions of one variable; (2) for certain integers μ and ν, $\sum_{m=1}^N p_m(x_1, x_2) f_m(q_m(x_1, x_2)) \not\equiv (x_1 + \nu x_2)^\mu$, $(x_1, x_2) \in D$, as the $f_m(t)$ run over all the bounded measurable functions of one variable. The results extend those of Vituškin. Proofs are indicated.

H. Fast (Notre Dame, Ind.)

MEASURE AND INTEGRATION

See also 3589, 3855.

Fiala, Jiří 3597

Representations of generalized measures by integrals.
Comment. Math. Univ. Carolinae **4** (1963), 153–156.

Let $\phi(u)$ be an N-function, i.e., $\phi(u) = \int_0^{|u|} p(t)\,dt$, where $p(t)$ is a non-decreasing function which is positive for $t > 0$, right continuous for $t \geq 0$ and satisfies the conditions $p(0) = 0$ and $\lim_{t\to\infty} p(t) = \infty$. Further, let (X, S, μ) be a finite measure space in which there exists a decreasing sequence of sets $\{E_n\}_{n=1}^\infty$ for which $\mu(E_n) > 0$ for all n, $\bigcap_{n=1}^\infty E_n = \varnothing$, and $\lim_{n\to\infty} \mu(E_n) = 0$. Finally, let $L_\phi(X, S, \mu)$ denote the Orlicz class of real functions on X, i.e., $f \in L_\phi(X, S, \mu)$ if $\int_X \phi(f)\,d\mu < \infty$.

The author proves the following generalization of a classical result of F. Riesz [F. Riesz and B. Sz.-Nagy, *Functional analysis*, p. 75, Ungar, New York, 1955; MR 17, 175]. Theorem: A σ-finite measure ν on S can be represented $\nu(E) = \int_E f\,d\mu$ for $f \in L_\phi(X, S, \mu)$ if and only if there exists a constant C so that for each finite partition $\{E_n\}_{n=1}^N$ of X, $\mu(E_n) > 0$, the following inequality holds:

$$\sum_{n=1}^N \mu(E_n)\phi\left(\frac{\nu(E_n)}{\mu(E_n)}\right) \leq C.$$

Moreover,

$$\int_X \phi(f)\,d\mu = \sup \sum_{n=1}^N \mu(E_n)\phi\left(\frac{\nu(E_n)}{\mu(E_n)}\right),$$

where the supremum is taken over all finite partitions of X. *R. G. Douglas* (Ann Arbor, Mich.)

Ricci, Giovanni 3598

Sul teorema di Carathéodory-Bohr-Banach riguardante la copertura secondo Vitali.
Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **35** (1963), 401–408.

This paper contains what may be viewed as a generalization of the classical covering theorem of G. Vitali [Atti Accad. Sci. Torino **43** (1907/08), 229–246]. It also answers a question concerning the "parameter of regularity" of the family of closed sets from which the Vitali covering is extracted, a question related to that posed by C. Carathéodory [*Vorlesungen über reelle Funktionen*, p. 304, Teubner, Leipzig, 1918] and answered negatively by H. Bohr and by S. Banach [Fund. Math. **5** (1924), 130–136]. *T. A. Botts* (Charlottesville, Va.)

Santagati, Giuseppe 3599

Alcune osservazioni sopra il problema del prolungamento di misure relative da reticoli d'insiemi.
Matematiche (Catania) **18** (1963), 83–97.

Let $\mathfrak{R}$ be a set of subsets of a given set S which contains the empty set and is closed under the operations of taking finite unions and finite intersections. Let ϕ be a real-valued function defined on $\mathfrak{R}$. This paper contains several theorems of the following type. Assume that $\mathfrak{R}$ possesses certain properties and let $\mathfrak{R}_*$ be the smallest set of subsets of S which contains $\mathfrak{R}$ and possesses designated completeness properties. Then if ϕ possesses the additivity and continuity properties in a given class Q, it is shown that ϕ has a unique extension ϕ_* to $\mathfrak{R}_*$ which possesses properties in a class containing Q. Formulas for ϕ_* are given in terms of the components in a Jordan type decomposition of ϕ.

P. V. Reichelderfer (Columbus, Ohio)

Appling, William D. L. 3600

Interval functions and continuity.

Rend. Circ. Mat. Palermo (2) **11** (1962), 285–290.

Theorem: If n is a positive integer, $\{[t_k, u_k]\}_{k=1}^{n}$ is a sequence of number intervals, and F is a real-valued function on the interval $[t_1, u_1; \cdots; t_n, u_n]$, then the following two statements are equivalent: (1) If g is a real-valued function on $[a, b]$ having bounded variation and $\{H_k\}_{k=1}^{n}$ is a sequence of functions of subintervals of $[a, b]$ such that for each positive integer $k \leq n$, the set of values of H_k is a subset of $[t_k, u_k]$ and the Hellinger integrals $\int_{[a,b]} H_k(I)\, dg$ exist, then $\int_{[a,b]} F[H_1(I), \cdots, H_n(I)]\, dg$ exists; and (2) F is continuous. Hints to the proof. (a) (2) implies (1) for $n = 1$. Using previous results [Duke Math. J. **29** (1962), 515–520; MR **25** #4075; Proc. Amer. Math. Soc. **13** (1962), 784–788; MR **25** #5150] the author shows that if the interval function $H = H_1$ is bounded and Hellinger g-integrable, then $|H|$ is also. For any $c > 0$ there exists a function G on $[t, u]$ such that $|G(x) - F(x)| < c$ and $G(H)$ is g-integrable. The integrability of $F(H)$ follows. (b) The falsity of (2) implies that of (1). The existence of a point of discontinuity for F enables the construction of some g and $H_1, \cdots, H_n$ for which $F(H_1, \cdots, H_n)$ is not g-integrable. (c) (2) implies (1). This is established by induction on n. A similar approximation as in (b) with respect to x_{n+1} is used.

Chr. Y. Pauc (Nantes)

Ossicini, A.; Rosati, F. 3601

Sugli integrali tripli di espressioni lineari alle derivate parziali del 4° ordine.

Matematiche (Catania) **18** (1963), 40–53.

Continuant un travail antérieur [Matematiche (Catania) 17 (1962), 10–37; MR **27** #4902], les auteurs étudient à quelles conditions l'intégrale triple d'un opérateur linéaire contenant des dérivées quatrièmes depend uniquement des valeurs de la fonction et de certaines de ses dérivées aux points anguleux du domaine d'intégration.

J. Kuntzmann (Grenoble)

Barbuti, Ugo 3602

Sulla nozione di misurabilità.

Matematiche (Catania) **18** (1963), 59–72.

Pour prolonger un contenu λ fini, σ-additif, défini sur un anneau booléen $\mathfrak{A}$ d'ensembles, en une mesure m, on peut, s'inspirant de Lebesgue, prolonger d'abord λ sur $\mathfrak{A}_\sigma$ et $\mathfrak{A}_\delta$, puis définir les ensembles mesurables et leurs mesures au moyen d'insertion à ε près (ε quelconque > 0) entre un ensemble de $\mathfrak{A}_\delta$ et un de $\mathfrak{A}_\sigma$. Une autre méthode, dite de Carathéodory, définit une mesure extérieure μ^* à partir de λ sur l'ensemble des parties d'ensembles de $\mathfrak{A}_\sigma$; m est alors la restriction de μ^* aux ensembles mesurables selon Carathéodory. L'auteur envisage deux "prémesures" $\nu|\mathfrak{H}$ et $\mu|\mathfrak{K}_\sigma$ réglementées par des axiomes et en déduit ensembles mesurables I et mesure $m(I)$ tels que: pour tout $\varepsilon > 0$ il existe un $H \in \mathfrak{H}$ et un $K \in \mathfrak{K}_\sigma$ tels que $\mu(K) - \nu(H) < \varepsilon$, $m(I) = \sup\{\nu(H) : H \in \mathfrak{H}, H \subseteq I\} = \{\inf\{\mu(K) : K \in \mathfrak{K}_\sigma, K \supseteq I\}$. Les schémas de Lebesgue et de Carathéodory sont alors interprétés comme des cas particuliers de ce schéma général, par choix convenable des prémesures. D'autres exemples de prolongement d'une fonction λ définie sur un treillis y sont aussi insérés.

Chr. Y. Pauc (Nantes)

Merklen, Héctor 3603

The Lebesgue integral. (Spanish)

Univ. Nac. Ingen. Inst. Mat. Puras Apl. Notas Mat. **1** (1962/63), 49–88.

Expository.

Collins, H. S. 3604

Characterizations of convolution semigroups of measures.

Pacific J. Math. **14** (1964), 479–492.

For a compact Hausdorff S, $B(S)$ denotes the non-negative, regular Borel measures, of norm ≤ 1, $P(S)$ denotes those measures of norm 1, each provided with the weak-* topology. When S is a topological semigroup, then $P(S)$ and $B(S)$ become compact affine topological semigroups using convolution.

Let K denote a compact convex subset of some real Hausdorff linear space, let $L(K)$ denote the continuous real affine functions on K, and let $E(K)$ denote the extreme points of K. Theorem 2.1: Suppose $L(K)$ separates points of K; then K is the external image of some $P(S)$ if and only if $E(K)$ is compact. It is shown in particular (Theorem 2.2) that K is the continuous affine bicontinuous image of some $P(S)$ if and only if $E(K)$ is compact and K is a simplex (in the sense of Choquet [C. R. Acad. Sci. Paris **243** (1956), 555–557; MR **18**, 288] or Loomis [Amer. J. Math. **84** (1962), 509–526; MR **26** #2575]).

These characterizations of extremal images of $P(S)$, one-one bicontinuous images of $P(S)$, and one-one bicontinuous affine images of $B(S)$, are extended to the case of homomorphic images when S is a compact semigroup, for example (Theorem 3.1), if $L(K)$ separates points of K, then K is an extremal homomorphic image of some $P(S)$ if and only if $E(K)$ is a compact semigroup. Theorem 3.2: K is the one-one bicontinuous and isomorphic image of some $P(S)$ if and only if $E(K)$ is a compact semigroup and K is a simplex.

R. J. Koch (Baton Rouge, La.)

Kagan, A. M.; Sudakov, V. N. 3605

Separating partitions for certain families of measures. (Russian. English summary)

Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. **19** (1964), no. 3, 147–150.

Let $P = \{P_\theta; \theta \in \Theta\}$ be a family of probability measures on $\{X, A\}$, such that $P_{\theta'} = P_{\theta''}$ implies $\theta' = \theta''$. Let $A_1, A_2, \cdots, A_s$ be disjoint subsets, $A_i \in A$ for $i = 1, 2, \cdots, s$, such that their union equals X. If s equalities $P_{\theta'}(A_i) = P_{\theta''}(A_i)$ for $i = 1, 2, \cdots, s$ imply $\theta' = \theta''$, then the subsets A_i are said to form a partition of X separating the family P. The following theorems are proved. Theorem 1: If the family P is composed of a finite number s of arbitrary distributions, then partitions separating P exist composed of $r \leq s$ elements. Theorem 2: If P is a denumerable family of continuous distributions, then in order to obtain a separation of P, the sample space X need be partitioned only in two subsets. Theorem 3: If P is the family of distributions uniform on $[0, \theta]$ with $0 < \theta < 1$, then the separation of this family can be achieved with $r = 3$ subsets of the sample space but cannot be achieved with only two. Each separating partition provides means of consistently estimating the parameter point corresponding to the observable random variable.

J. Neyman (Berkeley, Calif.)

Cater, S. 3606

Some representation theorems for invariant probability measures.

Illinois J. Math. 8 (1964), 408–418.

Let X be a set, T a mapping of X into X, $\mathfrak{F}$ a σ-algebra of subsets of X such that $A \in \mathfrak{F}$ implies $T^{-1}A \in \mathfrak{F}$. A probability measure m on $\mathfrak{F}$ is invariant if $m(A) = m(T^{-1}A)$ for $A \in \mathfrak{F}$, and ergodic if $m[\bigcup_{-\infty}^{\infty} T^i A] = 0$ or 1 for $A \in \mathfrak{F}$. The author considers the problem of integral representations of invariant probability measures in terms of ergodic invariant probability measures. He gives conditions which generalize the work of the reviewer and Hanson [Pacific J. Math. **10** (1960), 1125–1129; MR **24** #A3260] and of Farrell [Illinois J. Math. **6** (1962), 447–467; MR **27** #265].

Three theorems are proved which are somewhat too involved to state here. Several examples are constructed to illustrate the applications of these theorems.

J. R. Blum (Albuquerque, N.M.)

Tsurumi, Shigeru 3607

Ergodic theorems. (Japanese)

Sûgaku **13** (1961/62), 80–88.

This is an expository article on the recent development on ergodic theorems. The author first explains briefly how the ergodic theorems of G. D. Birkhoff and J. von Neumann have been generalized by various people in the past thirty years, and then presents the detailed proof, following the paper of E. Hopf [J. Reine Angew. Math. **205** (1960/61), 101–106; MR **23** #A3234], of the generalized ergodic theorem of Chacon and Ornstein. Using the maximal ergodic lemma of Chacon and Ornstein, which appears in the proof of their generalized ergodic theorem quoted above, the author then shows the proofs of the pointwise and dominated ergodic theorems of Dunford and Schwartz. In the last section of the paper, the author presents his version of random ergodic theorem, which extends and combines all the known random ergodic theorems. A detailed proof of this generalized random ergodic theorem has been published by the author [Ergodic Theory (Proc. Internat. Sympos., Tulane Univ., New Orleans, La., 1961), pp. 259–271, Academic Press, New York, 1963; MR **28** #4086].

Yuji Itô (Providence, R.I.)

Raimi, Ralph A. 3608

Minimal sets and ergodic measures in $\beta N - N$.

Bull. Amer. Math. Soc. **70** (1964), 711–712.

Let βN be the Stone-Čech compactification of the discrete space N of positive integers. The mapping $n \to n+1$ of N extends uniquely to a continuous mapping t of βN onto βN. The restriction of t to $N^* = \beta N - N$ is a homeomorphism of N^* onto N^*. Using a theorem of Rudin [Duke Math. J. **25** (1958), 197–204; MR **20** #4774] it is shown that every set $S \subset N^*$ that is t-invariant, compact, nonempty, and minimal with respect to these properties, is the support of at least two ergodic t-invariant Borel probability measures.

J. C. Oxtoby (Bryn Mawr, Pa.)

Parry, W. 3609

Representations for real numbers.

Acta Math. Acad. Sci. Hungar. **15** (1964), 95–105.

While in previous papers on f-expansions [see C. I. Everett, Jr., Bull. Amer. Math. Soc. **52** (1946), 861–869; MR **8**, 259; B. H. Bissinger, ibid. **50** (1944), 868–876; MR **6**, 150; the reviewer, Acta Math. Acad. Sci. Hungar. **8** (1957), 477–493; MR **20** #3843] only sufficient conditions of a metric character on the function f have been given for the validity of the f-expansion, in § 1 of this paper a necessary and sufficient condition (of a topological nature) is given. § 2 deals with the f-expansions corresponding to mod 1 linear functions. Let us consider the linear mod 1 transformation $T(x) = (\beta x + \alpha)$, where $\beta > 1$, $0 \leq \alpha < 1$, and (y) denotes the fractional part of y. It is shown that, by putting

$$h(x) = \sum_{x < T^n(1)} \frac{1}{\beta^n} - \sum_{x < T^n(0)} \frac{1}{\beta^n},$$

the (signed) measure $\nu(E) = \int_E h(x)\,dx$ is invariant under T. It is shown further that if T is strongly ergodic (i.e., $T^{-1}E \subset E$ implies $l(E) = 0$ or $l(E) = 1$, where $l(E)$ denotes Lebesgue measure), then $h(x) \geq 0$. It remains an open question whether $h(x)$ ever assumes negative values. The author mentions also the following further unsolved problems. Are linear mod 1 transformations always ergodic? If there exists a finite measure invariant under T and equivalent to Lebesgue measure, is T strongly ergodic?

A. Rényi (Budapest)

Gähler, Werner 3610

Flächen in topologischen Räumen.

Math. Nachr. **26** (1963), 181–228.

In the present paper the author develops a theory leading to an abstract extension of the concept of Lebesgue area for Fréchet surfaces S in any topological space R. The general concept of Fréchet surface S is introduced, as usual, as an equivalence class of continuous single-valued mappings $f: M_\sigma \to R$ from plane, closed, simply connected Jordan regions M_σ into R (representations of S). The equivalence of two such mappings f, f' is characterized by the possibility of approaching, say f, by means of homeomorphic transformations $f'h$ of f', or equivalently, by matching f' and f by means of multiple-valued transformations. The theory is based on a subtle topological preparation. Two basic spaces are first taken into consideration. One is the family M_σ, $\sigma \in K$, of all simply connected Jordan regions M_σ of a Euclidean plane, each M_σ being endowed with the induced Euclidean topology. Another is the family $M_{(\sigma,t)}$, $(\sigma, t) \in K$, of all vertices of the triangulations T of the regions M_σ, each $M_{(\sigma,t)}$ being endowed with the discrete topology. The author then proceeds with the definition of Fréchet surfaces in R, those being of the oriented, or non-oriented type, proper surfaces S, or polyhedral surfaces P. Then a proper surface S possesses at least a representation $p \mid M_\sigma$ from a Jordan region M_σ; a polyhedral surface possesses at least a representation $p \mid M_{(\sigma,t)}$ from a discrete set $M_{(\sigma,t)}$. If $\mathfrak{S}$ is the set of all proper surfaces S, and $\mathfrak{P}$ the set of all polyhedral surfaces P, then the union $\mathfrak{R} = \mathfrak{S} \cup \mathfrak{P}$, equipped with a convenient topology, is proved to be a topological space, and even a metric space if R is metric. In any case the subset $\mathfrak{P}$ of $\mathfrak{R}$ is everywhere dense in $\mathfrak{R}$. If $\eta(P)$, $P \in \mathfrak{P}$, is any single-valued real-valued function on $\mathfrak{P}$, then an area-like functional on $\mathfrak{R}$ can be defined by taking

$$\alpha(S) = \sup_{U \in B} \inf_{P \in U \cap \mathfrak{P}} \eta(P), \qquad S \in \mathfrak{R},$$

where B is any system of neighborhoods of S in $\mathfrak{R}$. This functional $\alpha(S)$ is then lower semicontinuous in $\mathfrak{R}$. In

particular, the author considers a single-valued real-valued function $\tau(abc)$ from the triples abc of points of R into the reals, satisfying the following axioms: (1) $\tau(abc)=0$ if $a=b$; (2) $\tau(abc)=\tau(bca)$, and, for oriented surfaces only, (2') $\tau(abc)=\tau(cba)$. Then, for any polyhedral surface P an elementary area $\eta(P)$ can be defined by taking

$$\eta(P) = \sum \tau(p(e_j^1)p(e_j^2)p(e_j^3)),$$

where $\sum$ runs over all simplexes $e_j^1e_j^2e_j^3$ of the complex corresponding to a given representation $p|M_{(\sigma,t)}$ of P. This elementary area $\eta(P)$ is proved to be independent of the representation. The corresponding functional $\alpha(S)$ is said to be a generalized Lebesgue area. The functional $\alpha(S)$ is proved to be lower semi-continuous in $\mathfrak{R}$, and $\alpha(S)$ can be equivalently defined by a Lebesgue-like process. Also, $\alpha(S)$ is superadditive, that is, $\alpha(S)\geq\alpha(S_1)+\cdots+\alpha(S_n)$ for any finite subdivision of S into parts all in $\mathfrak{R}$. Finally, an oriented surface S and the corresponding non-oriented one, say S^0, have the same area $\alpha(S)=\alpha(S^0)$. Of course, for any polyhedral surface $P\in\mathfrak{P}\subset\mathfrak{R}$, we have $\alpha(P)\leq\eta(P)$, and in this generality, $\alpha(P)$ may well be smaller than P. Nevertheless, no attempt is made to discuss under which further axioms it is possible to prove the equality $\alpha(P)=\eta(P)$. A previous relevant theory for Lebesgue area for surfaces in Banach or metric spaces is due to E. Silverman [Riv. Mat. Univ. Parma **2** (1951), 47–76; MR **13**, 122].

L. Cesari (Ann Arbor, Mich.)

Silverman, Edward 3611

Geodesics and Lebesgue area.

Proc. Amer. Math. Soc. **15** (1964), 775–780.

Using the concepts of two of his previous papers [Riv. Mat. Univ. Parma **2** (1951), 47–76; MR **13**, 122; ibid. **2** (1951), 195–201; MR **13**, 731], the author defines an area for a discontinuous surface in a metric space which is an extension of Lebesgue area. This enables him to stretch a given continuous surface to its maximum extension. In the event the maximum is continuous, then its Lebesgue area is equal to that of the original surface.

If f is a surface, $f=lm$ its monotone-light factorization, and if $p, q\in$ domain f, then inf {length $l\alpha$: α is continuous on $[0,1]$ into range m, $\alpha(0)=m(p)$, $\alpha(1)=m(q)$} is the pseudo-geodesic distance between p and q under f. Now, if two continuous surfaces are related so that the pseudo-geodesic distances between corresponding points are equal, then the maximum stretched surfaces derived from these two are isometric. This implies that the Lebesgue areas of the two surfaces being stretched are equal.

W. P. Ziemer (Bloomington, Ind.)

Shapiro, Victor L. 3612

Harmonic analysis and the theory of cochains.

Bull. Amer. Math. Soc. **70** (1964), 447–467.

The following theorem is established with methods based on measure theory and Fourier analysis: There is a one-to-one correspondence between local L^1 1-cochains in E^2 and equivalence classes of local L^1 differential forms in E^2. Here a local L^1 1-cochain in E^2 is a linear function on 1-dimensional oriented polyhedral chains in E^2 whose value on 1-simplexes parallel with either the x- or y-axis does not exceed the integral over these simplexes of fixed L^1 functions of one variable, and whose value on the boundary of a 2-simplex, two of whose sides are parallel with the axes, is no more than the integral over that 2-simplex of a fixed L^1 function. A local L^1 differential form is essentially a differential 1-form whose coefficients are measurable functions dominated by fixed L^1 functions of one variable. The correspondence above associates to a local L^1 1-cochain X in E^2, the local L^1 differential form ω by requiring that for most 1-simplexes σ, $X(\sigma)=\int_\sigma\omega$. This theorem is a generalization in these dimensions of Wolfe's theorem [H. Whitney, *Geometric integration theory*, Princeton Univ. Press, Princeton, N.J., 1957; MR **19**, 309], which relates flat cochains and flat differential forms in a similar manner. The author asserts that results similar to those above hold for r-cochains in E^n.

F. J. Almgren, Jr. (Princeton, N.J.)

Marstrand, John M. 3613

The (φ, s) regular subsets of n-space.

Trans. Amer. Math. Soc. **113** (1964), 369–392.

The following theorem in geometric measure theory is established. Let φ be a measure on E_n for which closed sets are measurable. Let $s\geq 0$ and $B\subset A\subset E_n$ with $\varphi(B)>0$. If B is (φ, s) regular with respect to A, then (i) s is an integer, and (ii) φ almost all points of B are weakly tangential with respect to A. Here the (φ, s) regularity of B with respect to A means that for φ almost all points $x\in B$, $\bigcirc_n^s(\varphi, A, x)=\lim_{r\to 0} r^{-s}\varphi(A\cap\{y:|y-x|\leq r\})$ exists, and $0<\bigcirc_n^s(\varphi, A, x)<\infty$. A point $x\in B$ is weakly (φ, s, s) tangential with respect to A if

$$0 = \liminf_{r\to 0} r^{-s}$$
$$\times\varphi(A\cap\{y:|y-x|\leq r; |(y-x)\cdot v_i|>\eta r \text{ for } i=k+1,\cdots,n\})$$

for some orthonormal basis $\{v_1,\cdots,v_n\}$ of E_n and each $\eta>0$. It is not known whether the theorem remains true if "lim inf" is replaced by "lim" above. If it does remain true, then a (φ, s) regular set would be (φ, s) restricted at φ almost all points and hence (φ, s) rectifiable [H. Federer, same Trans. **62** (1947), 114–192; MR **9**, 231]. The theorem is proved by an intricate type of induction combining measure-theoretic and geometric arguments.

F. J. Almgren, Jr. (Princeton, N.J.)

FUNCTIONS OF A COMPLEX VARIABLE

See also 3658, 3659, 3665, 3802, 3967.

★**Boundary-value problems in the theory of functions of a complex variable [Краевые задачи теории функций комплексного переменного].** 3614

Izdat. Kazan. Univ., Kazan, 1962. 80 *pp.* 0.34 *r.*

This small non-periodical pamphlet contains seven papers in the theory of functions of a complex variable and in partial differential equations. The papers will be reviewed individually.

McKiernan, M. A. 3615

On the convergence of series of iterates.

Publ. Math. Debrecen **10** (1963), 30–39.

Let $f^{[0]}(x)=x$, $f^{[n+1]}(x)=f(f^{[n]}(x))$, $n=0,1,2,\cdots$. The author discusses the convergence of the series

$$(*)\qquad \sum_{n=0}^{\infty} a_n \sum_{r=0}^{n}\binom{n}{r}(-\beta)^{n-r}f^{[r]}(x)$$

when $f(z)$ is analytic in the neighbourhood of $z=0$. The results are formulated in terms of the behaviour of the series $h(z)=\sum_0^\infty a_n(z-\beta)^n$ with radius of convergence d. In stating the theorems the author makes use of the following known results on iteration [P. Montel, *Leçons sur les récurrences et leurs applications*, Gauthier-Villars, Paris, 1957; MR **19**, 427]. Let $f(z)$ be regular at $z=0$, $f(0)=0$, $f'(0)=\alpha$, $0<|\alpha|<1$. Then there is a $\rho>0$ and a unique function $F(z)$, the Schröder function for $f(z)$, which is the uniform limit of the sequence $f^{[n]}(z)/\alpha^n$ in some neighbourhood of $z=0$, with $F(0)=0$, $F'(0)=1$, and having an inverse $F^{[-1]}(z)$ in $|z|\leq\rho$ with expansion $\sum_1^\infty c_n z^n$ $(c_1=1)$, and satisfying the equation $F\{f^{[r]}(z)\}=\alpha^r F(z)$, so that $f^{[r]}(z)=\sum_1^\infty c_n\alpha^{rn}[F(z)]^n$ for all integers $r\geq 1$. Also the function $\phi(x;z)=\sum_0^\infty f^{[n]}(x)/z^{n+1}$ is regular in the whole plane except for simple poles at $z=\alpha^n$, $n=1,2,\cdots$, for which $c_n\neq 0$ and the essential singular point $z=0$, in case there are an infinite number of poles. The author has already shown that (*) converges if all the singular points of $\phi(x;z)$ are in the interior of the circle of convergence of $h(z)$ [C. R. Acad. Sci. Paris **246** (1958), 2564–2567; MR **20** #1785]. In this paper the author considers the case when these singularities lie on or outside the circle. He proves that (*) converges uniformly in $|F(x)|\leq\rho$ if $h(z)$ converges uniformly on the singularities of $\phi(x;z)$. If $z=0$ is on the circle of convergence of $h(z)$, then (*) will converge uniformly in $|F(x)|\leq\rho\alpha^{\nu+1}$ if (i) $N^{-\nu}|\sum_1^N a_n\beta^n|$ is bounded in N, and (ii) the other singularities of ϕ lie in the interior of the intersection of $|z-\beta|<|\beta|$ and the sector $\arg\beta-\frac{1}{2}\pi+\varepsilon\leq\arg z\leq\arg\beta+\frac{1}{2}\pi-\varepsilon$ for some $\varepsilon>0$. If $0<\alpha<1$ and $a_n=\binom{s}{n}$, the series (*) converges uniformly in $|F(x)|\leq\rho\alpha$ if $s\geq 0$, and in $|F(x)|\leq\rho\alpha^{1-s}$ if $s<0$, to a function $f^{[s]}(x)$ satisfying the relations $F\{f^{[s]}(x)\}=\alpha^s F(x)$ and $f^{[s]}\{f^{[t]}(x)\}=f^{[s+t]}(x)$ and reducing to the nth iterate when $s=n$ is a positive integer. Also, when $a_n=(-1)^n/n$, the series (*) converges uniformly in $|F(x)|\leq\rho\alpha^{1+\varepsilon}$, for every $\varepsilon>0$, to a function $L(x)=(\log\alpha)F(x)/F'(x)$ and $L\{f(x)\}=L(x)f'(x)$. The author gives a theorem for the series (*) when $-1<\alpha<0$ and another theorem for the series (*) to diverge in $|F(x)|\leq\rho$ except when $F(x)=0$.

V. Ganapathy Iyer (Annamalainagar)

Sunyer Balaguer, F. 3616

Generalization of the method of Wiman and Valiron to a class of Dirichlet series. (Spanish)

Rev. Acad. Ci. Zaragoza (2) **16** (1961), no. 1, 9–13.

This paper has appeared elsewhere [Actas 2.ª Reunión Mat. Españoles (1961), pp. 43–47, Sem. Mat. Zaragoza, Zaragoza, 1962; MR **28** #3145].

A. A. Armendáriz (New Orleans, La.)

Pfluger, Albert 3617

Zu einem Verzerrungssatz der konformen Abbildung.

Math. Z. **84** (1964), 263–267.

The author gives a new proof of the following theorem. Let $f(z)=z+a_2z^2+\cdots$ map the unit disc, K, univalently and conformally onto the complex w-plane. In the w-plane denote by D_n the n rays of length $(\frac{1}{4})^n$ drawn from the origin, beginning on the positive real axis, and making equal angles of $2\pi/n$ with each other. Then either at least one of these rays is contained in $f(K)$, or else $f(z)=z/\sqrt[n]{(1+z^n)^2}$, up to a rotation.

This theorem is deduced as a consequence of a more general (known) distortion theorem. The proof of this latter theorem is more analytical than existing ones. It uses a simplified form of the method of extremal length.

J. L. Kazdan (Cambridge, Mass.)

Kalandija, A. I. 3618

On approximate conformal mapping of simply connected regions. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 717–720.

Let C be a closed Jordan curve represented in polar form by the equation $\rho=\rho(\varphi)$, where $\rho(\varphi)$ is continuously differentiable in $[0,2\pi]$ and has the period 2π. Suppose $z=\omega(\zeta)$ maps the disk $|\zeta|<1$ conformally onto the interior of C such that $\omega(0)=0$, $\omega'(0)>0$. The author develops an iteration procedure for the determination of ω, similar to that of Theodorsen. Let C_0 be a closed Jordan curve, star-shaped with respect to the origin and smooth, and suppose that the mapping function $z=\omega_0(\zeta)$, with $\omega_0(0)=0$ and $\omega_0'(0)>0$, of $|\zeta|<1$ onto the interior of C_0 is known. Then, starting with $\omega_0(\zeta)$, successive approximations $\omega_n(\sigma)$ to $\omega(\sigma)$, $\sigma=e^{i\theta}$, are calculated by means of the formula

$$\omega_{n+1}(\sigma)=\omega_n(\sigma)\left\{\frac{\rho[\varphi_n(\theta)]}{|\omega_n(\sigma)|}+\frac{1}{2\pi i}\int_{-\pi}^{\pi}\frac{\rho[\varphi_n(t)]}{|\omega_n(e^{it})|}\cot\frac{t-\theta}{2}\,dt\right\},$$
$$n=0,1,2,\cdots,$$

where $\varphi_n(\theta)=\arg\omega_n(\sigma)$, and the integral is a principal value. The author does not investigate the question of the convergence of the method but remarks that results analogous to those established by the reviewer for Theodorsen's method [the reviewer, Quart. Appl. Math. **3** (1945), 12–28; MR **6**, 207] may be expected. He illustrates the application of the procedure by calculating an approximation to the mapping function of the unit disk onto the interior of an ellipse with semi-axes a and b by taking $\omega_0(\zeta)=\zeta\exp(\alpha\zeta^2+\beta)$, where

$$\alpha=\tfrac{1}{2}\ln(a/b),\qquad\beta=\tfrac{1}{2}\ln(ab)\ (a/b<e).$$

S. E. Warschawski (La Jolla, Calif.)

Kühnau, Reiner 3619

Geometrie der konformen Abbildung auf der projektiven Ebene.

Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe **12** (1963), 5–19.

Die projektive Ebene wird hier als Riemannsche Zahlenkugel mit antipodischer Punktepaarung, versehen mit der Metrik der elliptischen Geometrie, realisiert. Den konformen Abbildungen einseitiger Gebiete der projektiven Ebene in die projektive Ebene sind die konformen Abbildungen der entsprechenden Gebiete auf der Riemannschen Zahlenkugel zugeordnet, bei denen antipodische Punkte wieder in antipodische Punkte überführt werden; den konformen Abbildungen zweiseitiger Gebiete entsprechen die konformen Abbildungen von Gebieten der Zahlenkugel ohne antipodische Punktepaare in ebensolche.

Analog zum Fall der Riemannschen Zahlenkugel wird mit Hilfe der Koebe'schen Kontinuitätsmethode die Existenz (und Eindeutigkeit) der konformen Abbildung von endlichfach zusammenhängenden Gebieten der projektiven Ebene auf gewisse Normalgebiete wie Radial- und Kreisbogenschlitzgebiete nachgewiesen. Im Falle einseitiger Gebiete z.B. wird gezeigt: Sind auf der projektiven

Ebene n diametral- und drehungssymmetrische Kurvenscharen S_k, $k=1,\cdots,n$, gegeben, so existiert für jedes n-fach zusammenhängende, die Punkte 0 und z_0 enthaltende Gebiet genau eine konforme Abbildung f mit $f(0)=0$ und $f(z_0)/z_0>0$, sodass die k-te Randkomponente in einen Schlitz auf S_k, $k=1,\cdots,n$, übergeht. Mit Hilfe der Flächenstreifen- oder Extremallängenmethode ergeben sich entsprechende Extremaleigenschaften dieser Abbildungen. Auch das Koebe'sche Kreisnormierungsprinzip wird übertragen.

Es werden anschliessend eine Anzahl Extremalprobleme behandelt, die rationalen quadratischen Differentialen zugeordnet sind.
A. Pfluger (Zürich)

Shapiro, Harold S. 3620

Weakly invertible elements in certain function spaces, and generators in ℓ_1.

Michigan Math. J. **11** (1964), 161–165.

Let ℓ_1 denote the Banach algebra of complex sequences $a=\{a_n\}$ $(n=0,1,2,\cdots)$ for which $\sum|a_n|<\infty$, multiplication being defined by convolution. A sequence a is said to be a generator of ℓ_1 if the polynomials in a are dense in ℓ_1. A necessary condition for $a=\{a_n\}$ to be a generator is that $f(z)=\sum a_nz^n$ be univalent in $|z|<1$ and map $|z|<1$ onto a Jordan domain.

D. J. Newman, J. T. Schwartz, and H. S. Shapiro [Trans. Amer. Math. Soc. **107** (1963), 466–484; MR **27** #575] proved Theorem A: Let $f(z)=\sum a_nz^n$ be univalent in $|z|<1$ and map $|z|<1$ onto a Jordan domain whose boundary is rectifiable. Let $I(z)$ denote the normalized inner factor of $f'(z)$. Then $a=\{a_n\}$ is a generator of ℓ_1 if and only if $(IH_2)^\perp$ contains no non-null function whose Taylor coefficients are $O(1/n)$. D. J. Newman later showed [Trans. Amer. Math. Soc. (to appear)] that the inner factor I has, in reality, no bearing on the question by proving Theorem B: Let $f(z)=\sum a_nz^n$ be univalent in $|z|<1$ and map $|z|<1$ onto a Jordan domain whose boundary is rectifiable. Then $a=\{a_n\}$ is a generator of ℓ_1.

In the words of the author, "Newman's proof of Theorem B is somewhat complicated, and the main purpose of this note is to provide a rather simple deduction of Theorem B from Theorem A. The proof offered here is essentially a modification of Newman's proof, but has perhaps some interest in view of its simplicity." In the deduction of Theorem B from Theorem A, use is made of the concept of weakly invertible elements in the Hilbert space B_σ of analytic functions $f(z)=\sum a_nz^n$ in $|z|<1$ for which $\sum|a_n|^2/(n+1)^\sigma<\infty$.

The author also (1) proves a theorem which exhibits the quantitative interplay between the rate at which an inner function can tend to zero radially and the "evenness" with which its representing measure is distributed on the unit circle, (2) uses the idea of weak invertibility to show (in terms of a completeness theorem for ℓ_2) that for any inner function I, the Taylor coefficients of an $f\in(IH_2)^\perp$ cannot be too small, unless they vanish from some point on, and (3) points out that the results of the paper can be extended to functions on the half-line $0\le t<\infty$.
G. T. Cargo (Syracuse, N.Y.)

Hayden, T. L.; Merkes, E. P. 3621

Chain sequences and univalence.

Illinois J. Math. **8** (1964), 523–528.

Let

$$z\frac{f'(z)}{f(z)}\sim 1-\frac{a_1z^{\alpha_1}}{1}-\frac{a_2z^{\alpha_2}}{1}-\cdots-\frac{a_nz^{\alpha_n}}{1}-\cdots$$

be the C-fraction expansion determined by a fixed formal power series $f(z)=\sum_1^\infty c_nz^n$, $c_1\neq 0$, and let Π_f denote the class of formal power series $g(z)=\sum_1^\infty c_n^*z^n$ for which the C-fraction expansion of $zg'(z)/g(z)$ has exponents $\alpha_n^*=\alpha_n$ and coefficients $|a_n^*|\le|a_n|$. The radius of univalence and the starlike radius for Π_f are, respectively, $U(\Pi_f)$ and $S(\Pi_f)$, where, by definition, $U(\Pi_f)=0$ in case there is a member of Π_f which is not analytic at $z=0$. Continued-fraction convergence and value-region properties are used to show that $r_0\le S(\Pi_f)\le U(\Pi_f)$, where r_0 is the supremum of r for which $\{|a_n|r^{\alpha_n}\}_1^\infty$ is a chain sequence. It is also shown that $r_0=S(\Pi_f)=U(\Pi_f)$ in case the chain sequence has uniquely determined parameters. When $\alpha_n=\alpha$, the relation of the continued fractions to Stieltjes transforms enables the authors to obtain this last result without the unique parameter hypothesis. Results are illustrated by application to Bessel functions.
W. T. Scott (Tempe, Ariz.)

Bombieri, Enrico 3622

Sul problema di Bieberbach per le funzioni univalenti.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **35** (1963), 469–471.

In this note the author considers the classical Bieberbach conjecture concerning the coefficients of $f(z)=z+a_2z^2+\cdots$, which is assumed to be analytic and univalent in the unit disc K. He studies the limit

$$\liminf_{\mathrm{Re}\,(a_2)\to 2-}\left\{\frac{n-\mathrm{Re}\,(a_n)}{2-\mathrm{Re}\,(a_2)}\right\}\equiv\rho_n$$

and proposes that one should study the analogous limit

$$\liminf_{\mathrm{Re}\,(a_3)\to 3-}\left\{\frac{n-\mathrm{Re}\,(a_n)}{3-\mathrm{Re}\,(a_3)}\right\}\equiv\theta_n.$$

The author obtains the following results. (I) $\rho_2=1$, $\rho_3=0$, $\rho_4>0.80$, $\rho_6>0$ and $\rho_{2m+1}\le 0$. (II) There exists an absolute constant ε such that for each $f(z)=z+a_2z^2+\cdots$, analytic and univalent in K, with $2-\varepsilon<\mathrm{Re}\,(a_2)\le 2$, it follows that $\mathrm{Re}\,(a_4)\le 6$, with equality only for the Koebe function.
M. Reade (Ann Arbor, Mich.)

Lai, Wan-tsai [Lai, Wan-tsai] 3623

On starlike typically-real functions.

Acta Math. Sinica **13** (1963), 389–404 (*Chinese*); *translated as Chinese Math.* **4** (1964), 423–439.

Let $T(\mu)$ be the class of functions $w=f(z)=z+z^2+\cdots$ regular and typically real in the unit disc, and let $T^*(\mu)\subset T(\mu)$ be the subclass of functions starshaped with respect to $w=0$. In this paper the author evaluates in a manner analogous to that used by J. A. Jenkins [cf. Canad. J. Math. **13** (1961), 299–304; MR **22** #12227] the exact bounds

$$m^*(\mu,z)=\inf|f(z)|,\qquad M(\mu,z)=\sup|f(z)|$$

for fixed z ($|z|<1$) and f ranging over $T^*(\mu)$.
Z. Lewandowski (Lublin)

Waśkiewicz, Jadwiga 3624

Sur certains problèmes extrémaux dans la famille des fonctions univalentes bornées inférieurement dans le cercle $K(\infty, 1)$.

Bull. Soc. Sci. Lettres Łódź 14 (1963), *no.* 3, 30 *pp.*

Consider the class $\sum_p (m)$, $p=1, 2, 3, \cdots$, of holomorphic functions

$$\Phi_p(z) = z+\frac{B_{p-1}^{(p)}}{z^{p-1}}+\frac{B_{2p-1}^{(p)}}{z^{2p-1}}+\cdots, \qquad |z| > 1,$$

which are univalent and are bounded from below, $|\Phi(z)|>m$, $0<m<1$, and which also satisfy the symmetry condition

$$\Phi_p(ze^{2\pi i/p}) = e^{2\pi i/p}\Phi_p(z).$$

The author determines the lower bound of the functional $H(\Phi_2)=|(\Phi_2(z_1)-\Phi_2(z_2))/(z_1-z_2)|$, where $|z_1|=|z_2|>1$. Upper bounds for two other functionals and inexact lower bounds for $H(\Phi_p)$, $p=3, 4, \cdots$, are also found. Previously, these results were only known for $\Sigma_1(m)$. The proofs utilize Loewner's well-known differential equation.

J. L. Kazdan (Cambridge, Mass.)

Bazilevič, I. E. 3625

Generalization of an integral formula for a subclass of univalent functions. (Russian)

Mat. Sb. (*N.S.*) 64 (106) (1964), 628–630.

In this note the author gives a formula for a class of functions $f(z)=z+A_2z^2+\cdots$ which are analytic and univalent in the unit disc K. The formula is the following one.

$$f(z) = \left[\frac{p_0(0)}{p_1(0)}\int_0^z p_1(s)S^{p_0(0)-1}\exp\left(\int_0^s \frac{p_0(t)-p_0(0)}{t}\,dt\right)ds\right]^{1/p_0(0)},$$

where $p_0(z)$ and $p_1(z)$ are analytic and have positive real part in K, and where the proper branch of the function $f(z)$ is taken so as to have the form $f(z)=z+A_2z^2+\cdots$ near $z=0$. The author's proof of univalence depends upon the Loewner equation. The present result contains an earlier one due to the author [same Sb. (N.S.) 37 (79) (1955), 471–476; MR 17, 356].

M. Reade (Ann Arbor, Mich.)

Lewandowski, Z.; Złotkiewicz, E. 3626

Variational formulae for functions meromorphic and univalent in the unit disc.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 12 (1964), 253–254.

Let $U(p)$, $0<|p|<1$, denote the class of functions $f(z)=z+a_2z^2+\cdots$ which are meromorphic and univalent in the unit disc K and which have a simple pole at $z=p$. Let $\Sigma(p)$ denote the class of functions $F(\zeta)=\zeta+b_0+b_1/\zeta+\cdots$ which are analytic and univalent in the complement of the closure of K and which vanish at $\zeta=1/p$. It is known that $f\in U(p)$ if and only if $[1/f(1/\zeta)]\in\Sigma(p)$. Hence the authors can apply certain variational formulae for $\Sigma(p)$ due to Šlionskiĭ [Vestnik Leningrad. Univ. 13 (1958), no. 13, 64–83; MR 20 #4652] in order to obtain variational formulae for the class $U(p)$. These last are of "Schiffer type". The authors give one application, which is that of finding a differential equation for the function in $U(p)$ that yields the maximum of the absolute value of the nth Taylor coefficient at $z=0$; they recapture an earlier result due to Komatu [Proc. Japan Acad. 21 (1945), 269–277; MR 11, 170; ibid. 21 (1945), 278–284; MR 11, 170].

M. Reade (Ann Arbor, Mich.)

Aleksandrov, I. A. 3627

Extremal properties of the class $S(w_0)$. (Russian)

Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. 169 (1963), 24–58.

The author is concerned with the class $S(z_0)$ of functions $f(z)=a_1z+a_2z^2+\cdots$ regular and univalent in the unit disc which satisfy $f(z_0)=z_0$. He derives variational formulae for the class $S(z_0)$, and, using these, he determines the domains of variability of the functionals

$$I_1(f) = \mathcal{J}_1\left[\frac{f''(z)}{f'(z)}, \overline{\frac{f''(z)}{f'(z)}}\right]$$

and

$$I_2(f) = \mathcal{J}_2[f(z), \overline{f(z)}, f'(z), \overline{f'(z)}]$$

for fixed z and fixed functions $\mathcal{J}_k$ satisfying some regularity conditions. Making $z_0\to 0$, he obtains some new results for the class S.

Z. Lewandowski (Lublin)

Weyl, Hermann 3628

★The concept of a Riemann surface.

Translated from the third German edition by Gerald R. MacLane. ADIWES International Series in Mathematics.

Addison-Wesley Publishing Co., Inc., Reading, Mass.-London, 1964. xi+191 *pp.* $12.50.

Since the third edition of Weyl's celebrated monograph has already been reviewed at length [*Die Idee der Riemannschen Fläche*, dritte Auflage, Teubner, Stuttgart, 1955; MR 16, 1097], it suffices to note that the present translation into English by Professor Gerald MacLane is a lucid rendition of the original. The typography is excellent. The publication of this translation is a great service to the mathematical community in the English-speaking countries.

M. H. Heins (Urbana, Ill.)

Mori, Mineko 3629

Canonical conformal mappings of open Riemann surfaces.

J. Math. Kyoto Univ. 3 (1963/64), 169–192.

The author first proves that an arbitrary open Riemann surface of genus $g>1$ is conformally equivalent to an at most g-sheeted covering surface of the extended plane which is bounded by a set consisting of analytic curves and a totally disconnected set. The main theorems show that an arbitrary open Riemann surface R of finite genus $g>0$ can be mapped onto a $(g+1)$-sheeted covering surface of the extended plane. The total area of the projection of the boundary of the image of R under this mapping is zero. The mapping can either be selected to have all $g+1$ of its poles at a single point of R or at $g+1$ distinct points. A similar theorem is proved for the circular or radial slit mappings, in which the logarithmic area of the projection of the boundary of the image is zero.

G. Springer (Bloomington, Ind.)

Ozawa, Mitsuru 3630

Rigidity of projection map and the growth of analytic functions.

Kôdai Math. Sem. Rep. **16** (1964), 40–43.

This paper contains a proof of a composition theorem due to the reviewer [Ann. of Math. (2) **55** (1952), 296–317; MR **13**, 643] and gives examples of concrete Riemann surfaces on which the algebroid functions of low order reduce to the composition of a meromorphic function on the plane and the projection map. An example of a transcendental hyperelliptic Riemann surface on which the Denjoy-Carleman-Ahlfors theorem persists is also given.

M. H. Heins (Urbana, Ill.)

Siciak, J. 3631

Some applications of the method of extremal points.

Colloq. Math. **11** (1963/64), 209–250.

Let E be a bounded closed set in the complex plane C, and $b(z)$ a bounded real-valued function defined on E. The author uses the method of extremal points (introduced first by Fekete in 1923) to define an extremal function $\Phi(z, E, b)$, which he proves is equal to a similar function defined by Leja. "The advantage of our definition of Φ relies on the fact that it admits a straightforward generalization to the case of the space C^n of n complex variables."

Proofs of old and new properties of Φ are given, and then are applied to the effective construction of a generalized—in the sense of Kellogg and Wiener, or Perron—solution of the Dirichlet problem. Lagrange's interpolation formula and the maximum principle for subharmonic functions are the major tools.

J. L. Kazdan (Cambridge, Mass.)

Bach, W. 3632

On some extremal functions of Leja in the space.

Colloq. Math. **11** (1963/64), 251–255.

Let $\omega(p, q)$ be a continuous function defined on the Cartesian product $E^{(m)} \times E^{(m)}$, where $E^{(m)}$ is the Euclidean m-space. Suppose, moreover, that $\omega(p, q) \geq 0$, $\omega(p, p) = 0$, $\omega(p, q) = \omega(q, p)$. Let E be a compact subset of $E^{(m)}$ and suppose that the system $q^{(n)} = \{q_0, q_1, \cdots, q_n\}$ of points $q_k \in E$ maximizes the expression $V(p^{(n)}) = \prod \omega(p_j, p_k)$ for $0 \leq j < k \leq n$. In connection with some results of F. Leja ($\omega = |p-q|$, $m = 2$) the author investigates the limits $\log A_n(r)/n$, $\log B_n(r)/n$ for $n \to \infty$, where

$$A_n(r) = \max_j \prod_{\substack{k=0\\k\neq j}}^{n} \frac{\omega(r, q_k)}{\omega(q_j, q_k)},$$

$$B_n(r) = \inf_{p^{(n)} \subset E} \left\{ \max_j \prod_{\substack{k=0\\k\neq j}}^{n} \frac{\omega(r, p_j)}{\omega(p_k, p_j)} \right\}$$

with $\omega = \exp(-|p-q|^{2-m})$, $m \geq 3$.

Z. Lewandowski (Lublin)

Edrei, Albert; Fuchs, Wolfgang H. J. 3633

On the zeros of $f(g(z))$ where f and g are entire functions.

J. Analyse Math. **12** (1964), 243–255.

The authors ask the following general question, answer it under various auxiliary hypotheses, and give some striking applications. Question: If g is an entire function, let $M(r) = M(r, g)$ denote its maximum modulus for $|z| = r$. For large $|w|$ define t by $|w| = M(t, g)$. Is it possible to find a function $\varphi(t)$ such that $g(z) = w$ has a solution in $|z| < \varphi(t)$ provided $|w|$ is sufficiently large (depending on g and φ)? Answers: If g (not a polynomial) is of finite order and $\xi > 0$, we can take $\varphi(t) = t^{1+\xi}$. If g is of positive lower order and there are r_1 and B such that

$$\log M\{r(1 + (\log M(r, g))^{-1/2})\} < (\log M(r))^B \qquad (r > r_1),$$

we can take $\varphi(t) = t\{1 + 2(\log M(t, g))^{-1/2}\}$. If g is odd and not a polynomial, we can take $\varphi(t) = (1+\xi)t$. The first answer has the following applications. If f and g are entire, the zeros of f have positive exponent of convergence, and g is not a polynomial, then the zeros of $f(g(z))$ do not have finite exponent of convergence. If F is meromorphic and not of order 0, and g is entire, not a polynomial, then $F(g(z))$ is of infinite order. This last result implies, in particular, Pólya's theorem that except in the obvious cases an entire function of an entire function is of infinite order [J. London Math. Soc. **1** (1926), 12–15], providing a proof that is quite elementary (in particular, independent of Schottky's theorem). The basic tool of the paper is the following lemma. Let $g(z) = \sum c_k z^k$ be regular in $|z| < R$ and omit the value w. Then if $\max\{2|g(0)|, 4\} < 2M(r, g) \leq |w|$, we have $|c_n| r^n < 6|w|(r/R)^n \log M(r, g) + |w|^{-1} M^2(r, g)$.

R. P. Boas, Jr. (Evanston, Ill.)

Srivastava, Satya Narain 3634

Corrections to my paper entitled "On the means of an entire function and its derivatives".

Bull. Calcutta Math. Soc. **54** (1962), 105–106.

An extensive list of corrections to the author's earlier paper [same Bull. **53** (1961), 73–82; MR **26** #1460].

Roux, Delfina 3635

Su una classe di funzioni intere con il minimo modulo quasi-asintotico al massimo modulo.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **35** (1963), 12–16.

With the usual notations, the author shows that if $f(z)$ is an entire function (of order 0) such that $\sup_{x \geq r} x^{-k} n(x) = o(r^{-k} \log M(r))$, where $0 < k < 1$, then $\log M(r) \sim N(r)$ and $\log m(r)$ is quasi-asymptotic to $\log M(r)$; this last means that for each $\varepsilon > 0$ the set where $(1-\varepsilon)\log m(r) < \log M(r) < (1+\varepsilon)\log m(r)$ has linear density 1. In particular, the author's hypothesis is shown to hold if $\log M(r) = O((\log r)^2)$, or if $(\log r)^{\alpha - c} < \log M(r) < (\log r)^{\alpha}$, $\alpha > 1$, $0 < c < 1$ [condition used by S. K. Singh and K. Manjanathaiah, İstanbul Univ. Fen Fak. Mec. Ser. A **26** (1961), 9–13; MR **26** #5163], or if $n(r)$ is slowly oscillating.

R. P. Boas, Jr. (Evanston, Ill.)

Hellerstein, S.; Rubel, L. A. 3636

Subfields that are algebraically closed in the field of all meromorphic functions.

J. Analyse Math. **12** (1964), 105–111.

Suppose that the functions $f(z), f_0(z), \cdots, f_l(z)$ are meromorphic in $|z| < R \leq \infty$ and satisfy the identity

$$\sum_{j=0}^{n} f_j(z) f(z)^j = 0. \qquad (1)$$

The authors prove by induction that, with the usual notation of Nevanlinna theory,

$$T\left(r,\ \sum_{j=0}^{K} f_j(z)f(z)^j\right) = T\left[r, f_0(z)+f(z)\sum_{j=1}^{K-1} f_j(z)f(z)^{j-1}\right]$$
$$\leq \sum_{j=0}^{K} T[r, f_j]+KT[r, f]+K\log 2,$$
$$K = 0, 1, \cdots, n-1.$$

Hence (1) implies

$$T(r, f) \leq \sum_{j=0}^{n} T(r, f_j)+O(1) \quad \text{as } r \to R.$$

Thus the growth of $f(z)$—as given for instance by the order and type for functions in the plane—cannot exceed the maximal growth of the $f_j(z)$. If $f(z)$ and the $f_j(z)$ are regular, the authors use the inequality

$$T(r, f) \leq \log^+ M(r, f) \leq \frac{R+r}{R-r}\, T(R, f), \qquad 0 < r < R,$$

to obtain somewhat weaker results for the growth defined in terms of $\log M(r, f)$ instead of $T(r, f)$.

W. K. Hayman (London)

Collingwood, E. F.; Piranian, George 3637

Tsuji functions with segments of Julia.

Math. Z. **84** (1964), 246–253.

A meromorphic function w in the open unit disk D is said to be a Tsuji function if it maps the circles $\{z : |z| = r\}$, $0<r<1$, onto curves whose spherical lengths form a bounded set. A rectilinear segment S lying in D except for one endpoint $e^{i\theta}$ on the unit circle C is called a segment of Julia for w provided, in each open triangle in D having one vertex at $e^{i\theta}$ and meeting S, the function w assumes all values on the Riemann sphere except possibly two. A point $e^{i\theta}$ is a Julia point for w if each rectilinear segment lying in D except for one endpoint at $e^{i\theta}$ is a segment of Julia for w.

In a review [MR **22** #11131] of a paper by M. Tsuji [Comment. Math. Univ. St. Paul. 8 (1960), 53–55], W. Seidel asked whether a Tsuji function can have segments of Julia. The present paper displays several relevant examples, all of which are motivated by the results in Tsuji's paper. Theorem 1: There exists a Tsuji function for which each point $e^{i\theta}$ is a Julia point. Theorem 3: If E is a set of measure 0 on C, then there exists a Tsuji function of bounded characteristic for which every point of E is a Julia point. Theorem 5: There exist holomorphic Tsuji functions with segments of Julia. The paper closes with three conjectures.

{Reviewer's comments: In the Lemma on p. 247, "$z_n \to 1$" should read "$|z_n| \to 1$". Also, the estimate on line 9 of p. 248, as well as subsequent dependent estimates, needs to be altered slightly.}

G. T. Cargo (Syracuse, N.Y.)

Storvick, D. A. 3638

Radial limits of quasiconformal functions.

Nagoya Math. J. **23** (1963), 199–206.

It is known from the work of Beurling and Ahlfors on the boundary correspondence under a quasiconformal mapping that for quasiconformal functions the analogue of Fatou's theorem is false. By imposing an additional restriction on the dilatation quotient $Q[f(z)]$ of the quasiconformal function $f(z)$, the author is able to prove such an analogue: If $w=f(z)$ is a bounded K-quasiconformal function in the upper half-plane, $\operatorname{Im}(z)>0$, $|f(z)| \leq M$, and if there exists an essentially bounded measurable function $\kappa(x)$ such that for almost all $z=x+iy$,

$$Q[f(z)]-1 \leq \kappa(x)\cdot y,$$

then the limit, $\lim_{y\to 0} f(x+iy)=f^*(x)$, exists for all real numbers x except possibly for an exceptional set of linear measure zero.

Also proved is an analogue of a theorem of Privalov on bounded analytic functions: Let $w=f(z)$ be a quasiconformal function defined in $|z| \leq 1$. If the set of points $e^{i\theta}$ on $|z|=1$ at which the complement of the radial cluster set is non-empty is of second category on some arc A of $|z|=1$, then, unless $f(z)$ is identically constant, the set of radial limit values of $f(z)$ on the arc A is of positive linear measure.

A. A. Armendáriz (Lexington, Va.)

Storvick, D. A. 3639

A localization principle for a class of analytic functions.

Nagoya Math. J. **23** (1963), 207–212.

An analytic function of modulus ≤ 1 on the open unit disk is said to belong to the class U^* provided that its Fatou radial limits exist and are of modulus one save for a set of zero logarithmic capacity on $\{|z|=1\}$. The following theorem is proved. Let f be a non-constant member of U^*. Let $\{|z-a|<\rho\} \subset \{|z|<1\}$. Let G denote a component of $f^{-1}(\{|z-a|<\rho\})$. Let φ denote a univalent conformal map of the unit disk onto G. If for every w, $|w|=1$, $\operatorname{cap}\{e^{i\theta} \mid f^*(e^{i\theta})=w\}=0$, then $F=(1/\rho)(f\circ\varphi-a) \in U^*$ and, for every W, $|W|=1$, $\operatorname{cap}\{e^{i\theta} \mid F^*(e^{i\theta})=W\}=0$. Here the asterisk refers to the Fatou radial limit. The proof makes use of a theorem of Pfluger [Comment. Math. Helv. **29** (1955), 120–131, cf. p. 122; MR **16**, 810].

M. H. Heins (Urbana, Ill.)

Meier, Kurt 3640

Über die Randwerte der meromorphen Funktionen.

Math. Ann. **142** (1960/61), 328–344.

Let $f(z)$ be meromorphic in $G : \{\operatorname{Im} z>0\}$, $z=x+iy$, and let I be some finite interval on the x-axis. We denote by $s(\varphi, x)$ the set $\{\arg(z-x)=\varphi\}$, by $w(\alpha, \beta, x)$ the set $\{\alpha<\arg(z-x)<\beta\}$, $0<\varphi<\pi$, $0<\alpha<\beta<\pi$. Let G' be some subset of G for which the point $z=x$ is a limit point. The cluster set of $f(z)$ as $z\to x$, $z \in G'$, will be denoted by $H(G', x)$. Let $H_x=H(G, x)$, $S_x(\varphi)=H(s(\varphi, x), x)$, $W_x(\alpha, \beta)=H(w(\alpha, \beta, x), x)$, $\Pi_x=\bigcap S_x(\varphi)$ for $0<\varphi<\pi$, $\Lambda_x=\bigcap W_x(\alpha, \beta)$ for $0<\alpha<\beta<\pi$. The extended w-plane will be denoted by Ω. Let A be the set of those $x \in I$ for which $\Omega \setminus \Lambda_x$ consists of no more than two points, B the set of those $x \in I$ for which $\Pi_x \cup \Lambda_x=\Omega$, C the set of those $x \in I$ for which all $w(\alpha, \beta, x)$, $0<\alpha<\beta<\pi$, consists of one point, D the set of those $x \in I$ for which $\Omega \setminus H_x$ is non-empty with $\Pi_x=H_x$, F the set of those $x \in I$ for which there exists not more than one ray $s(\varphi, x)$ on which $f(z)$ is bounded.

The basic results of the paper are the following: (1) $\operatorname{mes}\{I \setminus (A \cup B \cup C)\}=0$; (2) $I \setminus (A \cup B \cup C)$ is a set of first category. If $f(z)$ is regular in G, then (3) $\operatorname{mes}\{I \setminus (C \cup F)\}=0$, and (4) $I \setminus \{D \cup F\}$ is a set of first category.

The result (1) sharpens a theorem of A. I. Plessner [see,

for example, I. I. Privalov, *Boundary properties of analytic functions*, second edition (Russian), GITTL, Moscow, 1950; MR 13, 926; German transl., VEB Deutscher Verlag, Berlin, 1956; MR 18, 727].

The proof depends in an essential way on this theorem.

A. A. Gol'dberg (RŽMat 1961 #9 B70)

Bagemihl, Frederick 3641

Analytic continuation and the Schwarz reflection principle.

Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 378–380.

Let I be an open interval on the real axis, G_1 be a region in the upper half-plane, and G_2 be a region in the lower half-plane, such that I is a free boundary arc of both G_1 and G_2. Let $f(z)=f_j(z)$, where $f_j(z)$ is holomorphic in G_j for $j=1, 2$.

In extending the work of Carleman [*L'intégrale de Fourier et questions qui s'y rattachent*, Almqvist & Wiksells, Uppsala, 1944; MR 7, 248], Wolf [Duke Math. J. **14** (1947), 877–887; MR **9**, 420], and Meier [Comment. Math. Helv. **24** (1950), 238–259; MR **12**, 490], the author proves the following theorem. Let S be a subset of I of Lebesgue measure zero and first Baire category. For every $t\in I-S$, let λ_t^1 be an arc in G_1 at t, λ_t^2 be an arc in G_2 at t, ω_t be a finite complex number, and put $\varphi(t)=\omega_t$ $(t\in I-S)$. Suppose that

(a) $\lim_{\substack{z\to t\\ z\in\lambda_t^1}} f_1(z) = \lim_{\substack{z\to t\\ z\in\lambda_t^2}} = \omega_t$ for every $t\in I-S$;

(b) φ is bounded in some neighborhood of every point of $I-S$, (c) $\bar{A}_\infty\cap(I-S)=\varnothing$, where $\bar{A}_\infty$ is the closure of the set of points of I at which at least one of the functions f_1, f_2 has ∞ as an asymptotic value. Then f is holomorphic at each point of an everywhere dense subset of I.

D. A. Storvick (Minneapolis, Minn.)

Doob, J. L. 3642

One-sided cluster-value theorems.

Proc. London Math. Soc. (3) **13** (1963), 461–470.

This paper is concerned with the relationship between the cluster values and range of a function f, meromorphic in $|z|<1$, at a point on $|z|=1$ ($z=1$, say) for approach from the interior of the disk and approach on the boundary from one side. Let C, B, R, denote the cluster-set, boundary cluster-set and range of $f(z)$ at $z=1$. The classical Gross-Iversen theorem [Noshiro, *Cluster sets*, p. 4 et seq., Springer, Berlin, 1960; MR **24** #A3295] says that (i) $B\supset\partial C$; (ii) Each point of $C\cap\mathscr{C}B\cap\mathscr{C}R$ is an asymptotic value of f at 1; (iii) $C\cap\mathscr{C}B\cap\mathscr{C}R$ contains at most two points, and will contain two points only if R is the extended plane less these two points.

The author's main theorem, Theorem G, is a generalization of this theorem for one-sided boundary approach, involving only the boundary cluster set from one side. For example, (i) above becomes $B_s\supset\partial C(0)$, where B_s is the boundary cluster set at 1 from the south and $C(0)$ is the set of tangential-from-below cluster values. Similarly (ii) and (iii) are generalized. From this elegant theorem the author deduces the classical Gross cluster-value theorem [Gross, Math. Z. **3** (1919), 43–64] and a refinement of it. Several interesting generalizations and consequences of Theorem G are also given.

J. M. Anderson (Cambridge, Mass.)

Rjabyh, V. G. 3643

Some properties of analytic functions of class H_p'. (Russian)

Dokl. Akad. Nauk SSSR **158** (1964), 528–531.

The class H_p' $(p>0)$ consists of the functions f analytic in $|z|<1$ for which $H_p'(f)$, the integral of $|f(z)|^p$ over the unit disk, is finite. The author gives a number of theorems involving H_p'. (1) If $f'\in H_1'$, then $f\in H_1$ (the converse is false). (2) If f is in every H_p' for $p<1$, then so is f'. (3) If $0<\delta<1$, there exists $f_0\in H_\delta$ for which $f_0\notin H_{\delta+\varepsilon}$ for any $\varepsilon>0$, but f_0' is in every H_p', $p<1$. (4) If $f\in H_p'$ but $f\notin H'_{p+\varepsilon}$ for any $\varepsilon>0$, and if $\max|f(z)|=O(\varphi(r))$, where φ is continuous and nondecreasing, then $\int_0^1\varphi^q(r)\,dr=\infty$ for $q>p$. (5) If $f\in H_p$, $p<1$, then $f'\in H_q'$ for $q<2p/(p+1)$. (6) If $f\in H_p'$ and a_k are its zeros, then $\sum(1-|a_k|)^{1+\varepsilon}<\infty$ for every $\varepsilon>0$ (not necessarily for $\varepsilon=0$). (7) If $H_1'(f)=2\pi$ and a_n are the Maclaurin coefficients of f, then $|a_n|\le n+2$; there is equality for $f(z)=e^{i\alpha}(n+2)z^n$.

R. P. Boas, Jr. (Evanston, Ill.)

Mikhail, M. N. 3644

The effectiveness of the inverse and the product of basic sets of polynomials.

Proc. Math. Phys. Soc. U.A.R. (Egypt) No. **25** (1961), 43–48 (1964).

A basic set of polynomials $\{p_n(z)\}$ is a sequence for which the equations $z^n=\sum\pi_{n,k}p_k(z)$, $n=0, 1, \cdots$ admit a unique solution. If $f(z)$ is analytic in $|z|<R$ with Taylor series $\sum a_nz^n$, and if the formal rearrangement $\sum b_np_n(z)$ converges uniformly on closed subsets of $|z|<R$, then $\{p_n(z)\}$ is said to be effective in $|z|<R$. The sequence $\{q_n(z)\}$, $q_n(z)=\sum\pi_{n,k}z^k$, is said to be the inverse set of $\{p_n(z)\}$. Let $\{r_n(z)\}$ be a basic set, with $r_n(z)=\sum r_{n,k}z^k$. The sequence $\{u_n(z)\}$, $u_n(z)=\sum r_{n,k}p_k(z)$, is a basic set; it is also denoted by $\{r_n(z)\}\cdot\{p_n(z)\}$ and is called the product of the two sets taken in that order.

The author claims to generalize the known conditions concerning circles in which basic sets, their inverse sets, and product sets of inverse sets are effective. A monograph by J. M. Whittaker [*Sur les séries de base de polynômes quelconques*, Gauthier-Villars, Paris, 1949; MR **11**, 344] lays the foundation for this subject and should definitely be consulted since the extremely narrow focus of this article does not permit it to communicate the interest of the subject matter.

J. L. Ullman (Ann Arbor, Mich.)

Mikhail, M. N. 3645

The effectiveness of basic sets of polynomials in general domains. I.

Proc. Math. Phys. Soc. U.A.R. (Egypt) No. **25** (1961), 49–54 (1964).

A function analytic inside a simple analytic curve C with interior D has an expansion $\sum a_nf_n(z)$, where $f_n(z)$ is the nth Faber polynomial associated with C. If a sequence of polynomials $\{u_n(z)\}$ is such that $f_n(z)=\sum\pi_{n,k}u_k(z)$ admits a unique solution, then it is called a basic set associated with D. If the series $\sum b_nu_n(z)$ obtained by formal rearrangement of the Faber expansion converges uniformly in closed subsets of D, the set is said to be effective in D. The author establishes criteria for the effectiveness of basic sets associated with D which generalize known criteria. He then investigates inverse sets and product sets which can also be defined in this context [see #3644].

J. L. Ullman (Ann Arbor, Mich.)

Mikhail, M. N. 3646

The effectiveness of the inverse and the product sets of basic sets of polynomials in a general region. II.

Proc. Math. Phys. Soc. U.A.R. (Egypt) No. 25 (1961), 55–57 (1964).

This paper continues the study of the effectiveness of basic sets associated with non-circular domains [see #3645], and the effectiveness of inverse sets of basic sets, and of product sets of basic sets. *J. L. Ullman* (Ann Arbor, Mich.)

Dineen, B. L. 3647

The deviation of analytic functions from the mean arithmetic partial sums of the Faber series. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 250–253.

Let ω be a non-decreasing function defined for $t \geq 0$, satisfying $\omega(0)=0$, $\omega(t_1+t_2) \leq M[\omega(t_1)+\omega(t_2)]$, M being a constant, let A_ω be the class of functions f defined in $[-1, 1]$ such that for each n there exists an algebraic polynomial P_n of degree n satisfying

$$|f(x)-P_n(x)| \leq \omega(n^{-1}(\sqrt{(1+x^2)}+n^{-1})).$$

Denote by $T_0(x)=\pi^{-1/2}$, $T_k(x)=(2/\pi)^{-1/2}\cos(k \operatorname{arc}\cos x)$, the sequence of Čebyšev polynomials, let

$$C_k = \int_{-1}^{1} f(t)T_k(t)(1-t^2)^{-1/2}\,dt$$

be the Fourier coefficients of f, and let

$$\sigma_n(f,x) = \sum_{k=0}^{n-1}(1-k/n)C_kT_k(x).$$

Theorem: If $f \in A_\omega$ and $x \in [-1, 1]$, then

$$|f(x)-\sigma_n(f,x)| \leq (C/n)\sum_{k=1}^{n}\omega(k^{-1}(\sqrt{(1-x^2)}+k^{-1})),$$

C being a universal constant. If f is merely continuous on $[-1, 1]$, then

$$|f(x)-\sigma_n(f,x)| \leq n^{-1}\sum_{k=1}^{n}[\omega_1(f,k^{-2})+\omega_2(f,k^{-1}\sqrt{(1-x^2)})],$$

ω_1 and ω_2 denoting the moduli of continuity of the first and the second order.

Let G be a simply connected region in the complex plane, with boundary C, let the function φ map G conformally onto the region $|w|>R$, along with the conditions $\varphi(\infty)=\infty$, $\lim_{z\to\infty} z^{-1}\varphi(z)=1$, ψ being the inverse function of φ; let $\Phi_n(z)$ be the corresponding nth Faber polynomial. If f is analytic in G and continuous on $\bar{G}$, let $a_k=(2\pi i)^{-1}\int_{|w|=R} f(\psi(w))w^{-k-1}\,dw$ be the kth Fourier coefficient with respect to the system $\{\Phi_n\}$, and let $\sigma_n(f,\Phi_n,z)=\sum_{k=0}^{n-1}(1-k/n)a_k\Phi_k(z)$ be the nth Fejér sum. If the boundary of G is sufficiently smooth and if f satisfies certain additional conditions at the angular points of the boundary (not to be stated here explicitly), then

$$|f(z)-\sigma_n(f,\Phi_n,z)| \leq (C_1/n)\sum_{k=1}^{n}\omega_1(f,\rho_{1+1/k}(z)),$$

$\rho_{1+1/k}(z)$ denoting the distance of z from the set

$$\{\zeta: |\varphi(\zeta)| = (1+1/k)R\}.$$

No proofs. *A. Alexiewicz* (Poznań)

Krikunov, Ju. M. 3648

The differentiation of singular integrals with Cauchy kernel and a boundary-value property of holomorphic functions. (Russian)

Boundary-value problems in the theory of functions of a complex variable, pp. 17–24. *Izdat. Kazan. Univ., Kazan*, 1962.

Let L be a simple open contour of continuous curvature joining points a and b. A function $f(t)$ defined on L, with discontinuities at the points a_k $(k=1,\cdots,q)$ on L, and such that, on each open arc a_ka_{k+1}, $f(t)$ is Hölder-continuous and $|f(t)| \leq B|t-a_k|^{-\alpha}$ in the neighborhood of a_k $(k=1,\cdots,q)$, where B is a positive constant and $0 \leq \alpha < 1$, is said to belong to class H^{**}.

The singular integral $\int_L \varphi(\tau)(\tau-t)^{-1}\,d\tau$, where φ is Hölder-continuous on L and $\varphi' \subset H^{**}$ on L, is then shown to have a derivative given by the formula

$$(*)\quad \frac{d}{dt}\int_L \varphi(\tau)(\tau-t)^{-1}\,d\tau = \varphi(a)(a-t)^{-1}-\varphi(b)(b-t)^{-1} + \int_L \varphi'(\tau)(\tau-t)^{-1}\,d\tau,$$

a result previously established by Struble [J. Soc. Indust. Appl. Math. 8 (1960), 305–308; MR **22** #4929] with L taken along the real axis. *J. F. Heyda* (King of Prussia, Pa.)

Zverovič, È. I. 3649

Boundary-value problems with shift on abstract Riemann surfaces. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 26–29.

The author examines several boundary-value problems of displacement on a Riemann surface R containing a domain D^+ bounded by a smooth closed contour L with exterior D^-. A typical problem is that of finding two sectionally analytic functions φ^+, φ^-, on D^+, D^-, respectively, extensible to L, and obeying on L the relation

$$\varphi^+[\alpha_+(Q)] = G(Q)\varphi^-(Q)+g(Q)$$

or

$$\varphi^+[Q] = G(Q)\overline{\varphi^-[\alpha_-(Q)]}+g(Q)$$

with α_+, α_- homeomorphisms of L onto itself preserving and reversing orientation, respectively, and $G \neq 0$ and g are Hölder-continuous functions on L.

A method is proposed for reducing these problems to Riemann problems of the form

$$\varphi^+(Q') = G(Q')\varphi^-(Q')+g(Q')$$

on a curve L' in a Riemann surface R' obtained from R by appropriate transformations based on ideas due to Bojarski [I. N. Vekua, *Generalized analytic functions* (Russian), Chapter IV, Fizmatgiz, Moscow, 1959; MR **21** #7288]; several interesting questions are discussed, and estimates on the number of solutions are given in some cases.

The paper augments the work of several authors including W. Koppelman [Comm. Pure Appl. Math. **12** (1959), 13–35; MR **26** #3916] and E. Hasabov [Izv. Vysš. Učebn. Zaved. Matematika **1963**, no. 2 (33), 124–133; MR **27** #3811]. *A. D. Solomon* (New York)

Obolašvili, E. I. 3650

A generalization of the Riemann-Schwarz symmetry principle and its applications. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 1051–1053.

The author states without proof the following generalization of the Riemann-Schwarz symmetry principle for analytic functions: Let $F(z)$ be holomorphic in the domain D and continuous in $\bar{D}$. If the boundary of D contains a segment l of a straight line or a circle on which $\operatorname{Re}[\lambda(t)F(t)]$ vanishes, where $\lambda(t)$ $(t \in l)$ is a given non-vanishing Hölder-continuous function, then there exists a holomorphic function $\chi(z)$, defined in D, for which $\chi(z)F(z)$ is analytically continuable beyond l. In particular, if one denotes the reflection of D with respect to l by D^* and the reflection of a point $z \in D$ by z_1, then the function

$$\Phi(z) = \chi(z)F(z), \qquad z \in D,$$
$$= -\overline{\chi(z_1)}\overline{F(z_1)}, \qquad z \in D^*,$$

is holomorphic in $D+D^*+l$. The function $\chi(z)$ is obtained as a solution of the boundary problem

$$\chi^+(t) = \frac{\lambda(t)}{\overline{\lambda(t)}}\chi^-(t), \qquad t \in l,$$

where $\chi^-(t) = \overline{\chi^+(t)}$.

This generalized symmetry principle is then employed to solve Riemann-Hilbert boundary problems for special domains. Thus, for the case of a semi-circle he solves the problem of determining a function $F(z)$ holomorphic in $D(|z|<1, y>0)$, continuous in $\bar{D}$ (with the possible exception of $z = \pm 1$, where it is assumed that

$$F(z) \leq \text{const}/|z \mp 1|^\alpha, \qquad \alpha < 1,$$

and satisfying the boundary conditions

$$\operatorname{Re}[\lambda(t)F(t)] = \varphi(t) \quad \text{for} \quad t \in [|t| = 1, y > 0],$$
$$\operatorname{Re}[\mu(x)F(x)] = \psi(x) \quad \text{for} \quad -1 \leq x \leq 1,$$

where $\lambda(t)$, $\mu(x)$ are given functions, non-vanishing and Hölder-continuous. *J. F. Heyda* (King of Prussia, Pa.)

Whyburn, G. T. 3651

Developments in topological analysis.

Fund. Math. **50** (1961/62), 305–318.

Recently P. Porcelli and E. H. Connell [cf. Bull. Amer. Math. Soc. **67** (1961), 177–181; MR **23** #A1010] have given classical results concerning power series developments of functions of a complex variable by topological arguments. In the present paper the author fits these results, and other recent results concerning the classical theory of functions of a complex variable secured by topological methods, into a natural sequence.

W. R. Utz (Zbl **100**, 67)

Titus, Charles J. 3652

The combinatorial topology of analytic functions on the boundary of a disk.

Acta Math. **106** (1961), 45–64.

Soit D un domaine du plan complexe, qui soit l'intérieur d'une courbe de Jordan C; une application de C dans le plan complexe est appelée frontière intérieure si elle est la restriction à C d'une application, définie et continue dans $\bar{D}$, dont la restriction à D est intérieure et conserve l'orientation; une application est dite intérieure si l'image d'un ouvert est un ouvert, et si l'image réciproque de tout point est totalement discontinue. L'auteur résout, du point de vue de la topologie combinatoire, le problème suivant: Trouver une condition nécessaire et suffisante pour qu'une représentation normale d'une courbe fermée orientée soit une frontière intérieure. La notion de représentation normale a été introduite par H. Whitney [Compositio Math. **4** (1937), 276–284]. La solution d'un problème posé par E. Picard [*Traité d'analyse*, Tome II, troisième édition revue et augmentée, p. 313, Gauthier-Villars, Paris, 1926] est un corollaire de celle du problème ci-dessus.

F. Norguet (Zbl **101**, 155)

Ullman, J. L.; Titus, C. J. 3653

An integral inequality with applications to harmonic mappings.

Michigan Math. J. **10** (1963), 181–192.

The integral inequality referred to is the following. Let $h(\theta)$ be continuous and non-decreasing on $[0, \pi]$ with $h(0) = 0$ and $h(\pi) = \pi$. Then $\int_0^\pi \exp[i(h(\theta)-\theta)]\,d\theta$ has modulus greater than 2, and this result is the best possible. Let u and v be harmonic in the open unit disk and continuous on its closure, and let $H = u+iv$ map the closed unit disk onto itself in a one-to-one and sense-preserving manner with the origin and the point 1 mapping into themselves. Let $M = u_x^2 + u_y^2 + v_x^2 + v_y^2$ and $J = u_xv_y - u_yv_x$, all derivatives being computed at the origin. The quantity $M+2J$ is related to the distortion of the mapping at the origin. Let $\lambda = \inf(M+2J)$, the infimum being taken over all mappings of the class described. It is conjectured that $\lambda = 16/\pi^2$, and the corresponding result is proved in the case in which the class of mappings is restricted to those for which $H(-z) = -H(z)$, i.e., odd mappings.

John W. Green (Los Angeles, Calif.)

Faber, Karl 3654

Konforme Abbildungen der hyperbolisch-euklidischen Ebene durch hyperkomplexe Funktionen.

Math. Z. **84** (1964), 254–262.

Hyperkomplexe Zahlen $x+jy$ und Funktionen $f(z) = u(x, y) + jv(x, y)$ $(j^2 = 1)$ werden betrachtet. Nach einer Zusammenfassung früherer Ergebnisse des Verfassers [Jber. Deutsch. Math.-Verein. **61** (1958), Abt. 1, 32–56; MR **21** #3536] wird das lokale Verhalten der Abbildung $w = f(z)$ in der Umgebung einer Stelle z_0 studiert, an der die $n-1$ ersten Ableitungen von $f(z)$ verschwinden, nicht jedoch die n-te, und wo ferner $(\partial^n u/\partial x^n)^2 - (\partial^n v/\partial x^n)^2 \neq 0$ ist. Der Fall eines nicht schlichten Bildgebiets führt zur Einführung Riemannscher Flächen, die mehrfach gefaltete Ebenen sein können. Ist $F(x, y)$ eine Lösung der Wellengleichung, so können Krümmungseigenschaften der Fläche $\zeta = F(x, y)$ mit der Abbildung $w = f(z)$ $(u = F_y, v = F_x)$ in Beziehung gebracht werden. *D. Gaier* (Pasadena, Calif.)

Balk, M. B. 3655

On bi-analytic functions with non-isolated a-points. (Russian. Armenian summary)

Izv. Akad. Nauk Armjan. SSR Ser. Fiz.-Mat. Nauk **17** (1964), no. 3, 7–19.

A bi-analytic function is a function of the form $f(z) = \phi(z) + \bar{z}\psi(z)$, where ϕ and ψ are analytic. Simple examples (e.g., $z+\bar{z}$) show that the solutions of $f(z) = a$, $a = \text{const}$, need not be isolated, further examples being provided by

the whole class of degenerate bi-analytic functions determined completely by the author [same Izv. **17** (1964), no. 2, 9–15; MR **29** #1345]. The author finds a necessary and sufficient condition for $f(z)$ to have a non-isolated a-point at z_0, and this enables him to construct non-degenerate examples. He obtains further results in the cases when ϕ, ψ are meromorphic or entire functions.

I. N. Baker (London)

POTENTIAL THEORY

See also 3591, 3653, 3712, 3756, 3757, 4107, 4206, 4599.

Januškauskas, A. 3656

On the zeros of the gradient of a harmonic function. (Russian)

Dokl. Akad. Nauk SSSR **158** (1964), 547–549.

Let h be a non-constant harmonic function in D, a domain in 3-space, and denote by N_h the set of all $x \in D$ with grad $h(x) = 0$. Using some results of F. Bruhat and H. Cartan [C. R. Acad. Sci. Paris **244** (1957), 988–990; MR **19**, 125], the author shows that, for every compact $K \subset D$, there is a neighbourhood $\mathscr{U} \supset K$ such that $N_h \cap \mathscr{U}$ is a union of a finite set and a finite number of analytic curves.

J. Král (Prague)

Selberg, Henrik L. 3657

Über eine Integralungleichung der Potentialtheorie.

Norske Vid. Selsk. Forh. (*Trondheim*) **36** (1963), 4–9.

Soient (a) S un ensemble ouvert borné de l'axe réel, composé d'un nombre fini d'intervalles qui ne s'intersectent pas, et qui contient l'origine; (b) $l(x)$ la mesure du sous-ensemble de S, formé de points à abscisse plus petite que x; $L = l(\infty)$; (c) B un domaine du plan z qui ne contient pas $z = \infty$ comme point intérieur et dont l'intersection avec Ox est S; (d) $g(z; z_0; B)$ la fonction de Green associée à B, ayant le pôle en z_0. Alors on a

$$\int_S g^2(x; z_0; B)\,dx < \frac{c}{8}\left|\int_S \sqrt{(l(x)(L - l(x)))}\, d \arg (x - z_0)\right| < \frac{cL}{2},$$

où

$$c = 8 \int_0^{\pi/4} \log \operatorname{cotan} x \, dx.$$

A. Haimovici (Iaşi)

Selberg, Henrik L. 3658

Über die Greensche Funktion eines ebenen Gebietes.

Norske Vid. Selsk. Forh. (*Trondheim*) **36** (1963), 69–71.

On démontre le théorème suivant: Si G est un domaine lisse du plan z, qui contient le point $z = 0$, et si $g(z)$ est la fonction de Green de G qui a son pôle en $z = 0$, alors

$$\lim_{z \to 0} \left\{g(z) - \log \frac{1}{|z|}\right\} \leqq 4 \log \sqrt{(l_1 l_2)},$$

où l_1 et l_2 sont les longueurs des intervalles, en nombre fini, que G a en commun avec les parties négative et respectivement positive, de l'axe réel.

A. Haimovici (Iaşi)

Selberg, Henrik L. 3659

Über die Greensche Funktion eines mehrfach zusammenhängenden Gebietes.

Norske Vid. Selsk. Forh. (*Trondheim*) **36** (1963), 177–184.

Si G est un domaine du plan z, $\zeta \in G$, et $g(z, \zeta; G)$ la fonction de Green appartenant à G, ayant le pôle en ζ, posons

$$\gamma(\zeta, G) = \lim_{z \to \zeta} \left\{g(z, \zeta; G) - \log \frac{1}{|z - \zeta|}\right\}.$$

Soit encore $A(\zeta_0, \varepsilon)$ la partie de la frontière de G située dans le cercle $|z - \zeta_0| < \varepsilon$ $(\varepsilon > 0)$. Si la capacité de $A(\zeta_0, \varepsilon)$ est strictement positive, pour chaque $\varepsilon > 0$, alors ζ_0 appartient au noyau transfini de la frontière [Myrberg, Acta Math. **61** (1933), 39–79]. Dans ce cas

$$\limsup_{\zeta = \zeta_0} e^{2\gamma(\zeta, G)} \Delta_\zeta(\zeta, G) \geq -4,$$

$$\liminf_{\zeta = \zeta_0} e^{2\gamma(\zeta, G)} \Delta_\zeta(\zeta, G) \leq -4.$$

A. Haimovici (Iaşi)

Stoddart, A. W. J. 3660

The shape of level surfaces of harmonic functions in three dimensions.

Michigan Math. J. **11** (1964), 225–229.

Let $g(P)$ be the Green's function of a region D in E_3, with pole at the origin 0. It is a well-known result of J. J. Gergen [Amer. J. Math. **53** (1931), 746–752] that if D is star-shaped relative to 0, then the regions $D_k = \{P : g(P) > k\}$ are star-shaped relative to 0. Similarly, R. M. Gabriel [J. London Math. Soc. **30** (1955), 388–401; MR **17**, 358] has shown that if D is convex, then the regions D_k are also convex.

The author now obtains corresponding results for harmonic functions, replacing the pole at the origin by a continuum, which is star-shaped relative to the origin or convex in the respective cases, and on which the function is constant.

E. F. Beckenbach (Los Angeles, Calif.)

Nifosì, Lucia 3661

Limitazioni per le derivate di funzioni armoniche positive.

Matematiche (*Catania*) **18** (1963), 98–101.

Let u be harmonic and non-negative inside a sphere of radius R in m-dimensional Euclidean space. By elementary considerations of Poisson's integral, the author finds bounds for the various partial derivatives of u at the origin in terms of $u(0)$. Typical result: $u_{x_1 x_2}(0) \leq m(m+6)u(0)/(2R^2)$. In each case, a function is given for which the equality holds

John W. Green (Los Angeles, Calif.)

Bicadze, A. V. 3662

The problem of the inclined derivative with polynomial coefficients. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 1273–1275.

On étudie le problème de la dérivée inclinée avec des coefficients polynômiaux sur la frontière S du domaine D

$$P(y) \operatorname{grad} U(y) = f(y), \qquad y = (y_1, y_2, \cdots, y_n) \in S,$$

pour la fonction harmonique régulière $U(x)$,

$$x = (x_1, x_2, \cdots, x_n),$$

dans l'espace E_n. La fonction continue $f(y)$ est donnée, $P=(p_1, p_2, \cdots, p_n)$ est un vecteur polynômial donné de variables $y_1, y_2, \cdots, y_n$ et $\operatorname{grad} U(y)=\lim_{x\to y} \operatorname{grad} U(x)$. Des résultats intéressants sont énoncés surtout pour les domaines sphériques. *P. P. Teodorescu* (Bucharest)

Dynkin, E. B. 3663

Non-negative solutions of a boundary-value problem with inclined derivative. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 1028–1030.

Let D be the interior of a smooth Jordan plane curve C and let $v(\cdot)$ be a smooth vector field on C, tangent to C at finitely many points. Let Γ be the set of points at which the projection of v on the internal normal to C changes sign. The author investigates the harmonic functions in D satisfying $\partial h/\partial v=0$ on $C-\Gamma$. These functions will be called "solutions" below. The set Γ is partitioned into Γ_- and Γ_+ according to the conduct of v near the points of Γ, and $\Gamma_+{}^0$ is a certain subset of Γ_+. If Γ_+ is empty, there are no non-constant non-negative solutions. Otherwise, every such solution has the form

$$(*)\qquad h=\sum_{\Gamma_-\cup\Gamma_+{}^0} a_e u_e+\sum_{\Gamma_+}(c_e{}^+p_e{}^+ + c_e{}^-p_e{}^-),$$

where $a_e, c_e{}^+, c_e{}^-$ are non-negative constants and $u_e, p_e{}^+, p_e{}^-$ are special solutions. For D a disk the special solutions are characterized precisely. A Martin boundary is defined (and described in detail) having the usual properties: each point of the Martin boundary is identified with a non-negative solution, certain (in this case finitely many) of these solutions are minimal, and every positive solution is a linear combination of minimal solutions. The minimal solutions are the special solutions of the representation (*). The Green's function of the problem is constructed. The probability interpretation of the problem, going back to Maljutov [same Dokl. **156** (1964), 1285–1287; MR **29** #2867] is given. Briefly, the corresponding stochastic process is a Brownian motion reflected at C in the direction v, and stopped when it hits Γ. *J. L. Doob* (Urbana, Ill.)

Zolin, A. F. 3664

Solution of boundary problems for the Laplace equation by an interpolation method. (Russian)

Issled. po Mat. Analizu i Mehanike v Uzbekistane, pp. 133–152. *Izdat. Akad. Nauk Uzbek. SSR, Tashkent*, 1960.

In this paper the author discusses the approximation by harmonic polynomials of the solutions of the Dirichlet and Neumann problems for the Laplace equation in circular and elliptical domains. The approximating polynomials in polar form are constructed by interpolating on the boundary. In the case of circular domains it is interpolation by trigonometric polynomials. For a domain bounded by an ellipse the interpolating functions are called "generalized" trigonometric polynomials. They are linear combinations of functions which constitute a so-called system of Markov (Definition 2).

The Dirichlet problem is considered first. For a circular domain (radius R), take as nodes of interpolation the equidistributed points of the circumference that correspond to the polar angles $\theta_i=2\pi i/(2n+1)$, $i=0, 1, \cdots, 2n$. The associated harmonic polynomial is denoted by $S_n(r, \theta)$. It is known from the theory of interpolation by trigonometric polynomials that for a boundary function satisfying a Dini-Lipschitz condition the sequence of interpolating polynomials $\{S_n(R, \theta)\}$ will converge uniformly to it. Thus the sequence of harmonic polynomials $\{S_n(r, \theta)\}$ will converge uniformly to the solution in the whole closed domain. Moreover, it is shown directly that the sequence $\{S_n(r, \theta)\}$ converges at interior points to the Poisson integral where the boundary function is, say, only continuous. For an elliptical domain consider the simple orientation-preserving affine stretching map that carries the unit circle into the ellipse. As nodes of interpolation on the bounding ellipse the images of the equi-distributed points on the circle as described above are chosen. Then it is shown that the corresponding sequence of harmonic polynomials will converge uniformly in the whole closed domain for a Lipschitz continuous boundary function. Furthermore, for a merely continuous boundary function the sequence of interpolating "generalized" polynomials converges in the mean square, and the associated sequence of harmonic polynomials converges uniformly in closed interior subsets of the domain. In addition, relative to a continuous boundary function, there is a rule for selecting the nodes of interpolation whereby the resulting interpolating sequence will be uniformly convergent. It is an implicit and hence non-constructive selection principle.

The last part of the paper is devoted to the Neumann problem. For this problem on an elliptical domain a conformal map is used, as opposed to the affine map referred to above. *R. K. Juberg* (Minneapolis, Minn.)

Kusunoki, Yukio 3665

On a compactification of Green spaces. Dirichlet problem and theorems of Riesz type.

J. Math. Kyoto Univ. **1** (1961/62), 385–402.

On considère, dans un espace de Green R, une famille $\mathscr{C}$ de fonctions bornées continues à chacune desquelles correspond une fonction harmonique bornée dans R telle que la différence ait une pseudo-limite nulle en presque tout point de la frontière de Martin de R. Avec la norme du suprémum, $\mathscr{C}$ forme une algèbre normée dont l'espace des idéaux maximaux est un compact (non métrisable) R^* où R est ouvert dense, et de frontière $\Delta^*=R^*-R$, applicable continûment sur la frontière de Martin de R. La "frontière harmonique" de R est un sous-compact $\Delta_1{}^*$ de Δ^*. Toute $f\in\mathscr{C}$ se prolonge continûment sur R^*, de même toute fonction surharmonique positive continue dans R. On résoud le problème de Dirichlet pour R et frontière $\Delta_1{}^*$, et l'on donne diverses applications, dont un théorème du type de Riesz pour plusieurs variables complexes. *L. Lumer-Naim* (Grenoble)

Ohtsuka, Makoto 3666

Extremal length of families of parallel segments.

J. Sci. Hiroshima Univ. Ser. A-I Math. **28** (1964), 39–51.

The definition of the extremal length $\lambda(\Gamma)$ of a family Γ of curves c in the plane [Ahlfors and Beurling, Acta Math. **83** (1950), 101–129; MR **12**, 171] is here applied to a family Γ of collections c of open segments: each c consists of mutually disjoint open segments on one vertical line, and two different collections c and c' are supposed to be disjoint. Let $l(c)$ be the total length of c and χ_c the characteristic function of c in R^2. It is shown that if $\sum_{c\in\Gamma} l(c)\chi_c$ is

Lebesgue measurable in R^2 and $l(c) < \infty$ for each $c \in \Gamma$, then $1/\lambda(\Gamma) = M(\Gamma) = \int (\sum_{c \in \Gamma_x} l^{-1}(c))\, dx$. The case where $l(c)$ might be infinite and the "non-measurable" case are then discussed. Further, a family Γ' of collections c' of curves is considered such that each $c' \in \Gamma'$ intersects each $c \in \Gamma$; under certain general conditions it is proved that $\lambda(\Gamma) \cdot \lambda(\Gamma') \geq 1$. *J. Hersch* (Zürich)

Ohtsuka, Makoto 3667

On weak and unstable components.

J. Sci. Hiroshima Univ. Ser. A-I Math. **28** (1964), 53–58.

Extending a characterization due to M. Jurchescu [Pacific J. Math. 8 (1958), 791–809; MR **21** #2047], the author defines, by means of extremal length, the weakness and the instability of a point C as a component of a bounded set E in the plane. Let D be a disk containing E; C is called a weak component of E if the family Γ of closed curves around C in $D-E$ has extremal length $\lambda(\Gamma) = 0$; if $\lambda(\Gamma) > 0$, C is called an unstable component of E. Using this definition and some results of #3666 above, sufficient conditions are given for the weakness, respectively, for the instability of the point $(0, 0)$ with respect to a set of vertical segments in the strip $0 \leq x < 1$. The conditions thus obtained extend some theorems of T. Akaza and K. Oikawa [Nagoya Math. J. 18 (1961), 165–170; MR **24** #A1387]. *J. Hersch* (Zürich)

SEVERAL COMPLEX VARIABLES

See also 3575, 3631, 4071, 4460.

Lelong, Pierre 3668

Fonctions entières (n variables) et fonctions plurisousharmoniques d'ordre fini dans C^n.

J. Analyse Math. **12** (1964), 365–407.

An elegant, new and constructive solution of the Cousin-II problem for entire divisors of finite order is given. Cousin's theorem establishes only the existence of a solution. H. Kneser [S.-B. Preuss. Akad. Wiss. Phys.-Math. Kl. **1936**, 446–462] gave a constructive solution for entire divisors of finite order in $\mathbf{C}^2$. His methods fail in $\mathbf{C}^n$. Nevertheless, the Kneser integral represents the logarithm of a solution also in the case of infinite order [the reviewer, Math. Z. 57 (1953), 211–237; MR **14**, 970]. However, the proof required the existence of a solution, and the Kneser integral only converges in a neighborhood of the origin. This paper avoids these disadvantages.

For $n > 1$, define $h_n(a, z) = |a-z|^{2-2n}$ for a, z in $\mathbf{C}^n$. For $a \neq 0$, develop h_n at zero into a series of polynomials P_ν, where P_ν is homogeneous in z, $\bar{z}$ and has degree ν. Then $e_n(a, z, q) = -h_n(a, z) + \sum_{\nu=1}^{q} P_\nu(a, z)$ is harmonic for $z \in \mathbf{C}^n - \{a\}$.

Let W be an entire divisor in $\mathbf{C}^n$, whose support $|W|$ does not contain the origin. Let $W(a)$ be the multiplicity of W at a. Let β_{n-1} be the volume element on $|W|$. Then $\sigma(t) = \int_{|W| \cap B(t)} W(a)\beta_{n-1}(a)$ is the volume of W in the ball $B(t) = \{z \mid |z| \leq t\}$. The function $\nu(t) = (n-1)!\pi^{-n+1}t^{2-2n}\sigma(t)$ increases monotonically. Suppose that

$$\operatorname{Ord} \nu = \limsup_{t \to \infty} \frac{\log \nu(t)}{\log t}$$

is finite. Let $q+1 > 0$ be the smallest integer such that $\int_0^\infty \nu(t)t^{-q-2}\, dt$ converges. Then

$$I_q(z) = \int_{|W|} W(a)e_n(a, z, q)\beta_{n-1}$$

converges and is harmonic on $\mathbf{C}^n - |W|$. Surprisingly, I_q is plurisubharmonic on $\mathbf{C}^n - |W|$. For, let W be represented by the Cousin distribution $\{U_j, f_j\}$. Then $H_j = I_q - \log|f_j|$ is harmonic on U_j. Define $A^{\mu,\nu} = I_{q z_\mu \bar{z}_\nu}$. Since $A^{\mu,\nu} = H_{j z_\mu \bar{z}_\nu}$ on U_j, the functions $A^{\mu,\nu}$ are harmonic on $\mathbf{C}^n$. A direct estimate shows that they are polynomials in z, $\bar{z}$ of at most degree $q-2$. By their definition, their partial derivatives of order $\leq q-2$ vanish. Hence $A^{\mu,\nu} \equiv 0$, and I_q is plurisubharmonic on $\mathbf{C}^n - |W|$. Now, $2\pi I_q = \log|F_0|$, where F_0 is entire with divisor W.

Obviously, the method is restricted to divisors of finite order. According to Hadamard's theorem, which was extended to several variables by H. Kneser [Jber. Deutsch. Math.-Verein. 48 (1938), 1–28], only one entire function F with divisor W exists such that $F(0) = 1$ and such that all partial derivatives of $\log F$ of order $\leq q$ vanish at 0. Hence this is the canonical function F_0 and the canonical function h as constructed by the reviewer [Math. Z. 57 (1953), 211–237; MR **14**, 970].

Since I_q exists on $\mathbf{C}^n - |W|$, the properties of F_0 can be read off directly and the classical results and methods of one variable carry over to several variables.

The results are stated and proved with the help of the theory of currents. However, both can be done without this theory. The use of currents does not only yield elegant proofs but extends to the construction of a plurisubharmonic function V of order λ such that $2i\partial_z\bar{\partial}_z V = \vartheta$, where ϑ is a given closed positive current of type $(1, 1)$ and of finite order λ.

The essential results of the paper were already published without proofs [C. R. Acad. Sci. Paris **237** (1953), 691–693; MR **15**, 415; ibid. **237** (1953), 865–867; MR **15**, 416; ibid. **237** (1953), 1379–1381; MR **16**, 123; Ann. Acad. Sci. Fenn. Ser. A I No. 250/21 (1958); MR **20** #2467].

W. Stoll (Notre Dame, Ind.)

Noverraz, Ph. 3669

Comparaison d'indicatrices de croissance pour des fonctions plurisousharmoniques ou entières d'ordre fini.

J. Analyse Math. **12** (1964), 409–418.

Recent results of P. Lelong [see the previous review #3668] permit one to extend methods and results of entire functions of one variable and finite order to plurisubharmonic functions of finite order. Some of these extensions are done in this paper. The only new element in several variables is the use of the well-known estimate (1) (Poisson inequality). The results are specialised to $V = \log|f|$, where f is an entire function of finite order. These specializations are either contained in Théorèmes 1′, 4′, 5′ or obtained trivially in Théorèmes 1′, 3′, or obtained as in one variable (Théorème 6′) from the reviewer's paper [Math. Z. 57 (1953), 211–237; MR **14**, 970] if one observes that the Nevanlinna characteristic $\lambda(\log^+ V, 0, r)$ differs from the Ahlfors-Shimizu-Weyl characteristic $T(r, r_0)$ only by a bounded function. In Proposition 1, Théorème 1 and Théorème 1′, it is not necessary to restrict p to a compact set. *W. Stoll* (Notre Dame, Ind.)

Stoll, Wilhelm 3670

The growth of the area of a transcendental analytic set. I, II.

Math. Ann. **156** (1964), 47–78; *ibid.* **156** (1964), 144–170.

The main result of this paper is the theorem that if M is a pure p-dimensional complex analytic subvariety of the space $\mathbf{C}^k$ of k complex variables, $0<p<k$, then M is algebraic if and only if the function $V(r)r^{-2p}$ is bounded, where $V(r)$ is the area of the intersection $M_r = M \cap \{z \in \mathbf{C}^k \,|\, |z| \leq r\}$ and is defined as the integral over M_r of a canonical differential form. More generally, let ζ be any continuous, positive definite, exterior differential form of bidegree (p, p) on the k-dimensional complex projective space $\mathbf{P}^k$, and let $j: \mathbf{C}^k \to \mathbf{P}^k$ be the analytic imbedding defined by $j(z_1, \cdots, z_k) = (1 : z_1 : \cdots : z_k)$; then M is algebraic if and only if $\int_M j^*(\zeta) < \infty$. The special case $p = k-1$ was treated earlier by Rutishauser [Acta Math. **83** (1950), 249–325; MR **12**, 90] and the author [Math. Z. **60** (1954), 287–304; MR **16**, 463], and the special case $p=1$ by the author [ibid. **81** (1963), 76–98; MR **29** #2430]. The proof, which is rather involved, is based upon an extension of the theory of value distribution of entire functions [cf. H. Weyl, *Meromorphic functions and analytic curves*, Princeton Univ. Press, Princeton, N.J., 1943; MR **5**, 94] to analytic varieties, following the lines of the author's work [Acta Math. **90** (1953), 1–115; MR **17**, 893; ibid. **92** (1954), 55–169; MR **17**, 894].

R. C. Gunning (Princeton, N.J.)

Königsberger, K. 3671

Mehrdeutige Funktionen zu gegebenen Divisoren auf einem Produktraum.

Math. Ann. **155** (1964), 35–52.

Soient X un espace analytique complexe (au sens de J. P. Serre), $\hat{X}$ le revêtement universel de X et D un diviseur de X. On dit que D est résoluble par une fonction multiforme s'il existe une fonction méromorphe sur $\hat{X}$ admettant pour diviseur l'image réciproque $\hat{D}$ de D par la projection canonique: $\hat{X} \to X$. On désigne par $\mathbf{O}$ le faisceau des germes de fonctions holomorphes. Soient X et Y deux espaces analytiques complexes tels que, sur X, il existe une intégrale de première espèce de périodes données. Soit D un diviseur de $X \times Y$ satisfaisant à: (1) D est résoluble sur $X \times Y$; (2) pour des points particuliers $\xi \in X$, $\eta \in Y$, les traces D_ξ et D_η de D sur les sous-espaces $\xi \times Y$ et $X \times \eta$ sont résolubles par des fonctions multiformes données g_ξ, f_η. Alors, il existe une solution de D dont les facteurs d'automorphie sont des fonctions simples des facteurs d'automorphie de g_ξ et f_η et d'une fonction holomorphe sur X qui dépend homomorphiquement des classes d'homologie rationnelles des éléments de $\pi_1(Y)$. La condition (1) est réalisée si $H^1(\widehat{X \times Y}, \mathbf{O}) = 0$. On obtient des théorèmes plus précis si on se borne aux fonctions automorphes multiplicatives [cf. K. Stein, Acta Math. **83** (1950), 165–196; MR **12**, 252] et aux fonctions théta [cf. G. de Rham et K. Kodaira, *Harmonic integrals*, Chapitre V, Inst. Advanced Study, Princeton, N.J., 1950; MR **12**, 279]. Un grand nombre de cas particuliers importants (par exemple, l'un des espaces considéré est de Stein ou est une variété kählérienne compacte) sont étudiés et les résultats de Stein (mémoire cité) généralisés. La technique utilise la cohomologie des groupes et des résultats antérieurs de l'auteur [Math. Ann. **148** (1962), 147–172; MR **26** #6442].

P. Dolbeault (Poitiers)

Norguet, François 3672

Sur la cohomologie des variétés analytiques complexes et sur le calcul des résidus.

C. R. Acad. Sci. Paris **258** (1964), 403–405.

Let X be a complex manifold, S a complex submanifold of codimension k. The author shows that every closed C^∞ form in $X-S$ is cohomologous to one having a certain type of amenable singularity on S. Specifically, a "local kernel" is defined to be a pair (U, K), U an open subset of X, K a differential form in $U-S$ given by

$$\left(\sum_{1\leq j\leq k} (s_j \bar{s}_j)\right)^{-k}\left(\bigwedge_{1\leq j\leq k} ds_j\right) \wedge \left[\sum_{1\leq h\leq k} (-1)^{h-1}\bar{s}_h\left(\bigwedge_{j\neq h} d\bar{s}_j\right)\right],$$

where s_j are holomorphic functions in U, $\bigwedge ds_j \neq 0$ in U, and $U \cap S$ is the common zero set in U of the functions s_j. A form φ, C^∞ in $X-S$, is called simple in X if there is a C^∞ form ψ on X and a family $\{(U_i, K_i)\}_{i\in I}$ of local kernels, with $\{U_i\}$ a covering of X, such that for all $i \in I$,

$$\varphi|U_i = K_i \wedge (\psi|U_i) + \theta_i, \qquad \text{where } \theta_i \text{ is } C^\infty \text{ in } U_i.$$

It is established that every C^∞ closed form in $X-S$ is cohomologous to a form simple in X. This result is used to study the "residue homomorphism" $r: H^q(X-S) \to H^{q-2k+1}(S)$ (see J. Leray [Bull. Soc. Math. France **87** (1959), 81–180; MR **23** #A3281] and the author [C. R. Acad. Sci. Paris **248** (1959), 2057–2059; MR **21** #5195]). The author shows that for cohomology classes h_1, h_2, h_3 of respective degrees p_1, p_2, p_3 of $X-S$, one has

$$(-1)^{p_1(p_3-1)} r(h_1) \cup r(h_2 \cup h_3) + (-1)^{p_2(p_1-1)} r(h_2) \cup r(h_3 \cup h_1) + (-1)^{p_3(p_2-1)} r(h_3) \cup r(h_1 \cup h_2) = 0.$$

A. Browder (Berkeley, Calif.)

Ônishi, Hidekazu 3673

Sur une extension du théorème de MM. H. Cartan et P. Thullen.

J. Math. Kyoto Univ. **3** (1963/64), 193–206.

The theorem of H. Cartan and P. Thullen [Math. Ann. **106** (1932), 617–647] is generalized to ramified coverings. Let $H(S)$ be the set of all holomorphic functions on the complex space S. If $\varnothing \neq M \subseteq S$ and $\varnothing \neq \mathfrak{F} \subseteq H(S)$, define $\|f\|_M = \sup\{f(x) \,|\, x \in M\}$ if $f \in H(S)$ and

$$\hat{M}(\mathfrak{F}) = \{x \,|\, |f(x)| \leq \|f\|_M \quad \text{for all } f \in \mathfrak{F}\}.$$

Now, let $\mathfrak{D}$ be a normal complex space. Let $\pi: \mathfrak{D} \to \mathbf{C}^n$ be an open holomorphic map such that $(\mathfrak{D}, \pi)$ is a Riemann domain in the sense of H. Grauert and R. Remmert [Math. Z. **67** (1957), 103–128; MR **19**, 317]. Let $\nu(x)-1$ be the ramification order at x. Let $\sigma = \{x \,|\, \nu(x) > 1\}$ be the set of ramification points. For $a = (a_1, \cdots, a_n) \in \mathbf{C}^n$ and $\rho > 0$ denote $C'(a, \rho) = \prod_{\nu=1}^n \{z \,|\, |z - a_\nu| < \rho\}$. For $a \in \mathfrak{D}$, at most one open connected neighborhood $C(a, \rho)$ of a exists such that $\pi: C(a, \rho) \to C'(a, \rho)$ is a surjective, proper map. For $B \subseteq \mathfrak{D}$, define

$$\delta(a, B) = \sup\{\rho \,|\, C(a, \rho) \text{ exists}, C(a, \rho) \cap B = \varnothing\}.$$

If $A \cap B = \varnothing$, define $\delta(A, B) = \inf\{\delta(a, B) \,|\, a \in A\}$. Define $\delta(a) = \delta(a, \varnothing)$ and $\delta(A) = \delta(A, \varnothing)$. If $0 < r < \delta(A)$, define $A(r) = \bigcup_{a\in A} C(a, r)$.

If f is holomorphic at a, let $(f)_a$ be the germ of f at a. If $a \in \mathfrak{D} - \sigma$, then π induces a germ $\pi^*(f)_a$ at $\pi(a)$. Now,

suppose that $\lambda \in H(\mathfrak{D})$ is given such that $L = \{x \mid \lambda(x) = 0\}$ does not contain an interior point, and such that λ vanishes at each $x \in \sigma$ with order $\nu(x)$. If $f \in H(\mathfrak{D})$, then the partial derivatives of f are defined and holomorphic on $\mathfrak{D} - \sigma$ and meromorphic on $\mathfrak{D}$ such that

$$g^{(f)}_{k_1 \cdots k_n} = \lambda^k \frac{\partial^k f}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}}, \qquad k = k_1 + \cdots + k_n,$$

is holomorphic on $\mathfrak{D}$.

Now, let $\mathfrak{F}$ be a non-empty family of holomorphic functions on $\mathfrak{D}$ such that $f \in \mathfrak{F}$ implies $g^{(f)}_{k_1 \cdots k_n} \in \mathfrak{F}$ for all $k_1 \geq 0, \cdots, k_n \geq 0$ and $c \cdot f^k \in \mathfrak{F}$ for all $c \in \mathbf{C}$ and all integers $k \geq 0$. Suppose that $\mathfrak{D}_0$ is a relatively compact open subset E of $\mathfrak{D} - \sigma$ such that for every $a \in \hat{\mathfrak{D}}_0(F) - L$, for every $f \in \mathfrak{F}$ and for $\rho = |\lambda(a)| \delta(E, L)(\|\lambda\|_{\mathfrak{D}_0})^{-1}$, one and only one holomorphic function f^* on $C'(\pi(a), \rho)$ exists such that $(f^*)_{\pi(a)} = \pi^*(f)_a$. Moreover,

$$\|f\|_{E(r')} \quad \text{if} \quad 0 < \rho' < \rho \quad \text{and if} \quad r' = \rho^{-1} \rho' \delta(E, L).$$

If $\sigma = \varnothing$, this result gives the original theorem of Cartan and Thullen. *W. Stoll* (Notre Dame, Ind.)

SPECIAL FUNCTIONS

See also 3425, 3426, 3621, 3705, 3799, 3815–3819, 3842, 4179.

Shao, T. S.; Chen, T. C.; Frank, R. M. 3674

Tables of zeros and Gaussian weights of certain associated Laguerre polynomials and the related generalized Hermite polynomials.

Math. Comp. 18 (1964), 598–616.

Mansell, W. E. 3675

★**Tables of natural and common logarithms to 110 decimals.**

Edited by A. J. Thompson. Royal Society Mathematical Tables, Vol. 8.

Published for the Royal Society at the Cambridge University Press, New York, 1964. xviii + 95 *pp.* \$7.50.

This is a remarkable volume to appear at this time, not because of its contents: log n and ln n for $n = 1(1)1000$ and log y and ln y for $y = 1 + x \cdot 10^{-q}$ for $q = 4(1)11$, $x = 1(1)9$, all to 110D, but because these were apparently computed without any mechanical aid by a retired accountant.

There is an introduction by A. J. Thompson, the author of *Logarithmetica Britannica* [Vols. I, II, Cambridge Univ. Press, New York, 1952 [published 1954]; MR **16**, 286], describing the use, construction and checking of the tables. The tables were reproduced photographically from sheets prepared on a card-controlled typewriter; a few blurred figures are clarified in an inset.

There is also a page of 110D values of relevant constants. *John Todd* (Pasadena, Calif.)

Uhde, Kurt 3676a

★**Spezielle Funktionen der mathematischen Physik. Tafeln I: Zylinderfunktionen.**

B·I-Hochschultaschenbücher, 55/55a.

Bibliographisches Institut, Mannheim, 1964. viii + 267 *pp.*

Uhde, Kurt 3676b

★**Spezielle Funktionen der mathematischen Physik. Tafeln II: Elliptische Integrale. Thetafunktionen. Legendresche Polynome. Laguerresche Funktionen. Gammafunktion. Fresnelsche Integrale. Fehlerfunktion. Integralexponentielle u. a.**

B·I-Hochschultaschenbücher, 76/76a.

Bibliographisches Institut, Mannheim, 1964. viii + 211 *pp.*

The entries in these two volumes were computed to 10S on a Siemens 2002 and then rounded, usually to 4D; a few tables are given to different precisions, e.g., to 5S. The intervals are chosen to permit linear interpolation. We shall indicate the content of the main tables, using standard notations:

Vol. I: $J_0(x)$, $J_1(x)$, $N_0(x)$, $N_1(x)$, $0 \leq x < 16$; $e^{-x}I_0(x)$, $e^{-x}I_1(x)$, $e^xK_0(x)$, $e^xK_1(x)$, ber x, bei x, $0 \leq x < 10$; her x, $0 \leq x < 6$; hei x, $0.2 \leq x < 6$; the first hundred positive zeros of J_0 and J_1 and the corresponding values of J_1 and J_0.

Vol. II: $\Gamma(1+x)$, $0 \leq x < 1$; $\mathrm{Ei}(-x)$, $0 \leq x < 5$; $\mathrm{Si}(x)$, $\mathrm{Ci}(x)$, $0 \leq x < 20$; erf x, $0 \leq x < 4$; $1 - \operatorname{erf} x$, $4 \leq x \leq 10$; $C(x)$, $S(x)$, $0 \leq x < 10$; $K(k)$, $E(k)$, $0 \leq k < 1$; $\vartheta_r(\nu, \kappa)$, $r = 1, 2, 3, 4$, $0 \leq \nu \leq 1/2$, $\kappa = K(\cos\alpha)/K(\sin\alpha)$, $\alpha = 0(1°)87°$; $P_r(x)$, $r = 2(1)7$, $0 \leq x \leq 1$; $Q_r(x)$, $r = 0(1)7$, $0 \leq x < 1$; $l_n(x) = e^{x/2}D^n(e^{-x}x^n)/n!$, $n = 0(1)5$, $0 \leq x \leq 50$; $\varphi_n(x) = (-1)^n e^{x^2/2}D^n(e^{-x^2})$, $n = 0(1)5$, $0 \leq x < 10$.

Each table is prefaced by a few lines about asymptotic formulas, recurrence relations or functional equations which will enable one to obtain the functions outside the range for which they are actually tabulated.

{The printing is tolerable, but the paper is thin and the material on the reverse shows through. There are some editorial flaws, e.g., the preface to the first volume appears again in the second, and there seems to be no key to the references.} *John Todd* (Pasadena, Calif.)

Wadsworth, D. van Z. 3677

Improved asymptotic expansion for the error function with imaginary argument.

Math. Comp. 18 (1964), 662–664.

Nagaraja, K. S. 3678

A remark on Tricomi's ψ function.

J. Math. and Phys. **43** (1964), 261–262.

In this paper, the integral

$$E_q(u) = \int_u^\infty e^{-x} x^{-q}\, dx$$

is shown to be equal to the Tricomi function $e^{-u}\psi(q, q; u)$. Also a recurrence relation for $E_{M+n}(u)$ is proved, which should be useful in the numerical evaluation of integrals of this type. *L. J. Slater* (Cambridge, England)

Vilenkin, N. Ja. 3679

The hypergeometric function and representations of the group of real second-order matrices. (Russian)

Mat. Sb. (*N.S.*) **64** (**106**) (1964), 497–520.

A representation of the group G of 2×2 real unimodular matrices is obtained by putting

$$T_\chi(g)f(x) = |\beta x+\delta|^{\omega-1}\operatorname{sgn}^\epsilon(\beta x+\delta)f((\alpha x+\gamma)/(\beta x+\delta)),$$

where $g=\begin{Vmatrix}\alpha & \beta\\ \gamma & \delta\end{Vmatrix}\in G$, χ is a pair (ω, ϵ), ω complex, $\epsilon=\pm 1$, and f is a space D_χ of functions which, with $|x|^{\omega-1}\operatorname{sgn}^\epsilon xf(-x^{-1})$, are infinitely often differentiable [Gel'fand, Graev, and Vilenkin, *Generalized functions* (Russian), No. 5, Fizmatgiz, Moscow, 1962; MR 28 #3324]. Writing

$$F_+(\lambda) = \int_0^\infty x^\lambda f(x)\,dx, \qquad F_-(\lambda) = \int_0^\infty x^\lambda f(-x)\,dx,$$

$-1<\operatorname{Re}\lambda<-\operatorname{Re}\omega$, this generates a representation $T_\chi(g)F(\lambda)=\int K(\lambda, \mu; \chi; g)F(\mu)\,d\mu$ in the space of pairs $F(\lambda)=(F_+(\lambda), F_-(\lambda))$, where K is a 2×2 matrix of functions and the integral is along the line $\operatorname{Re}\mu=a$, $-1<a<-\operatorname{Re}\omega$. In particular, for the matrices $h(\theta)=\begin{Vmatrix}\cosh\theta & \sinh\theta\\ \sinh\theta & \cosh\theta\end{Vmatrix}$ or $u(\theta)=\begin{Vmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{Vmatrix}$ the elements of K are expressible in terms of hypergeometric functions, and for the matrix $Z=\begin{Vmatrix}1 & 0\\ z & 1\end{Vmatrix}$ in terms of powers. The integral formulae corresponding to the equations $T_\chi(g)^{-1}=T_\chi(g^{-1})$, $T_\chi(gh)=T_\chi(g)T_\chi(h)$, written out explicitly, lead to inversion formulae for integral transforms on the one hand and to formulae for hypergeometric functions $F(\lambda, \mu; \nu, x)$ as Mellin transforms in x, which include several of Barnes' formulae, and to formulae for these functions as Mellin transforms with respect to the variable ν. *J. L. B. Cooper* (Pasadena, Calif.)

Vilenkin, N. Ja. 3680

Functional composition theorems for the hypergeometric function. (Russian)

Mat. Sb. (*N.S.*) **65** (107) (1964), 28–46.

The representations of the group of 2×2 real unimodular matrices described above [#3679] are used to produce numerous formulae for integrals of products of hyperbolic functions $F(\lambda, \mu; \nu; z)$ with respect to their parameters, some of which are related to formulae discovered by Ramanujan. *J. L. B. Cooper* (Pasadena, Calif.)

Wimp, Jet; Luke, Yudell L. 3681

Expansion formulas for generalized hypergeometric functions.

Rend. Circ. Mat. Palermo (2) **11** (1962), 351–366.

In this paper, the authors prove two very general theorems which express a generalized hypergeometric function $F[wz]$ in terms of the sum of a series of products of generalized hypergeometric functions $F[z]$ and $F[w]$.

They state the conditions under which their theorems are true, and they generalize their results still further to prove two theorems which express the Meijer G-function, $G[wz]$ in terms of the sum of a series of products of a generalized G-function $G[z]$ and a generalized hypergeometric function $F[w]$.

The paper concludes with several interesting examples of the application of these theorems to Laguerre polynomials, Hermite polynomials and Bessel functions.

L. J. Slater (Cambridge, England)

Sharma, K. C. 3682

Integrals involving products of G-function and Gauss's hypergeometric function.

Proc. Cambridge Philos. Soc. **60** (1964), 539–542.

Author's introduction: "The first result to be established here is

$$\int_0^1 x^{\rho-1}(1-x)^{\beta-\gamma-n}{}_2F_1(-n, \beta; \gamma; x)G^{k,l}_{p,q}\left(zx^m\middle|\begin{matrix}a_1, \cdots, a_p\\ b_1, \cdots, b_q\end{matrix}\right)dx$$

$$(1)\qquad = \frac{\Gamma(\gamma)\Gamma(\beta-\gamma+1)}{\Gamma(\gamma+n)}m^{\gamma-\beta+n-1}G^{k+m,l+m}_{p+2m,q+2m}(\mathrm{I}),$$

where (I) stands for

$$\left(z\middle|\begin{matrix}\frac{1-\rho}{m}, \ldots, \frac{m-\rho}{m}, a_1, \ldots, a_p, & \frac{\gamma-\rho}{m}, & \ldots, \frac{\gamma-\rho+m-1}{m}\\ \frac{\gamma+n-\rho}{m}, \ldots, \frac{\gamma+n-\rho+m-1}{m}, b_1, \ldots, b_q, & \frac{\gamma-\beta-\rho}{m}, & \ldots, \frac{\gamma-\beta-\rho+m-1}{m}\end{matrix}\right),$$

and where m and n both are positive integers. The result (1) holds if $\Re(\rho+mb_h)>0$, $h=1, \cdots, k$; $\Re(\beta-\gamma)>n-1$, and one of the following conditions is satisfied:

(i) $2(k+l) > p+q$, $\quad |\arg z| < (k+l-\frac{1}{2}p-\frac{1}{2}q)\pi$;

(ii) $2(k+l) \geq p+q$, $\quad |\arg z| \leq (k+l-\frac{1}{2}p-\frac{1}{2}q)\pi$ and
$\Re(\sum a_j-\sum b_j+\frac{1}{2}(q-p)+\beta-\gamma-n) > 1$.

The second result to be established is

$$(2)\quad \int_0^1 x^{\rho-1}(1-x)^{\beta-\alpha-1}{}_2F_1(\alpha, \beta; \gamma; x)G^{k,l}_{r,s}\left(zx^n(1-x)^{-n}\middle|\begin{matrix}a_1, \cdots, a_r\\ b_1, \cdots, b_s\end{matrix}\right)dx$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}(2\pi)^{1-n}n^{\beta-\alpha-1}G^{k+2n,l}_{r+2n,s+2n}(\mathrm{J}),$$

where (J) stands for

$$\left(z\middle|\begin{matrix}\frac{1-\rho}{n}, \ldots, \frac{n-\rho}{n}, a_1, \ldots, a_r, \frac{\gamma-\rho}{n}, & \ldots, \frac{\gamma-\rho+n-1}{n}\\ \frac{\beta-\rho}{n}, \ldots, \frac{\beta-\rho+n-1}{n}, & \frac{\gamma-\alpha-\rho}{n}, \ldots, \frac{\gamma-\alpha-\rho+n-1}{n}, b_1, \ldots, b_s\end{matrix}\right),$$

and where n is a positive integer, $\Re(\rho+nb_h)>0$, $h=1, \cdots, k$; $\Re(\gamma-\alpha-\rho-na_t+n)>0$, $\Re(\beta-\rho-na_t+n)>0$, $t=1, \cdots, l$; $2(k+l)>r+s$, $|\arg z|<(k+l-\frac{1}{2}r-\frac{1}{2}s)\pi$."

The proofs of the above results start from an integral representation of the G-function, from a formula for $\Gamma(mz)$ and from an integral of the hypergeometric function multiplied by polynomials. These four formulas are taken from two handbooks [Erdélyi et al., *Higher transcendental functions*, Vol. I, McGraw-Hill, New York, 1953; MR 15, 419; *Tables of integral transforms*, Vol. II, McGraw-Hill, New York, 1954; MR 16, 468]. The proofs proper are then effected by substitutions. *M. J. O. Strutt* (Zürich)

Carlitz, L. 3683

Another Saalschützian theorem for double series.

Rend. Sem. Mat. Univ. Padova **34** (1964), 200–203.

In this paper, the author proves a second summation theorem for a double Appell F_1 series of the Saalschützian type. He uses this result to express a series of Appell F_1 functions as a simple ${}_3F_2(x)$ ordinary hypergeometric series. *L. J. Slater* (Cambridge, England)

Saxena, R. K. 3684

Integrals involving products of Bessel functions.

Proc. Glasgow Math. Assoc. **6**, 130–132 (1964).

From the author's introduction: "In this paper certain infinite integrals involving products of four Bessel functions of different arguments are evaluated in terms of Appell's function F_4 by the methods of the operational calculus. The results obtained are believed to be new."

Suzuki, Yasutaka 3685

Some formulae on Bessel and Legendre functions.

J. College Arts Sci. Chiba Univ. **3** (1961/62), 441–446.

The author lists several properties of the Bessel and Legendre functions re-expressed in forms involving the Dirac δ-function. The properties include ordinary and partial differential equations, Fourier transforms and definite integrals. *F. W. J. Olver* (Washington, D.C.)

Singh, B. 3686

On certain expansions and integrals involving $\tilde{\omega}_{\mu,\nu}(x)$.

Proc. Rajasthan Acad. Sci. **9** (1962), 9–22.

The author evaluates a great many series and integrals involving

$$\tilde{\omega}_{\mu,\nu}(x) = \sqrt{x}\int_0^\infty J_\nu\left(\frac{x}{t}\right)J_\mu(t)\,\frac{dt}{t}.$$

Two of the simpler are:

$$\tilde{\omega}_{0,\nu}(x)+2\sum_{\mu=1}^{\infty}\tilde{\omega}_{2\mu,\nu}(x) = \frac{\sqrt{x}}{\nu}, \qquad \Re(\nu) < 0,$$

$$\int_0^\infty \frac{x^{3/2}}{(a^2+x^2)^{3/2}}\,\tilde{\omega}_{1/2,1}(x)\,dx = \sqrt{\frac{2}{a}}\sin\left(\sqrt{(2a)}\right)e^{-\sqrt{(2a)}}.$$

Stanley Katz (New York)

Kovalenko, A. D. 3687

A generalization of Lommel's solutions. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 442–446.

Particular solutions are found for the equation

$$\left[z\frac{d}{dz}\prod_{k=1}^{q}\left(z\frac{d}{dz}+\beta_k-1\right)-z\prod_{k=1}^{q}\left(z\frac{d}{dz}+\alpha_k\right)\right]W+Az^\lambda = 0$$

for $\lambda=0,-1,-2,\cdots$, and β_k, α_k, and A are constants. *S. Drobot* (Columbus, Ohio)

Khandekar, P. R. 3688

A note on the associated Legendre polynomials.

Proc. Glasgow Math. Assoc. **6**, 156–160 (1964).

The aim of this paper is to give a new Rodrigues' formula for the associated Legendre polynomials $P_n^m(x)$ if m is an even positive integer. In the first part, starting from Gegenbauer polynomials, the function $P_n^{2k}(x)$ is expressed by a generalized hypergeometric function. In the second part, this expression is modified so as to obtain the above Legendre function as the nth derivative of a generalized hypergeometric function. The third and fourth parts are concerned with integrals between the limits -1 and $+1$ of a polynomial in x and the product of two Legendre functions of equal k, as well as different k. These integrals are also expressed by generalized hypergeometric functions. *M. J. O. Strutt* (Zürich)

Piefke, Gerhard 3689

Asymptotische Näherungen der modifizierten Mathieuschen Funktionen. (English and Russian summaries)

Z. Angew. Math. Mech. **44** (1964), 315–327.

The solutions of the modified Mathieu equation

$$f''(\xi)+[2h^2\cosh 2\xi-\vartheta]f(\xi) = 0,$$

where ϑ is one of the periodic or half-periodic eigenvalues $a_m(h^2)$, $b_{m+1}(h^2)$, $m=0,1,2,\cdots$, are studied for real ξ and large real values of the parameter h. The procedure consists essentially in writing $f(\xi)=\exp[ht(\xi,h)]$ and in expanding $t(\xi,h)$ in descending powers of h. First approximations of the solutions in $-\infty<\xi<\infty$ are given. The quality of the approximation is demonstrated in eighteen figures. Better approximations have already been obtained by Goldstein [Trans. Cambridge Philos. Soc. **23** (1927), 303–336]. *J. Meixner* (Aachen)

Müller, H. J. W. [Müller, Harald J. W.] 3690

On asymptotic expansions of Mathieu functions.

J. Reine Angew. Math. **211** (1962), 179–190.

Mathieu's differential equation

$$y''(x)+[\lambda-2h^2\cos 2x]y(x) = 0$$

is studied for large positive values of h in the interval $0\le x\le 2\pi$. The substitutions $\omega(x)=4h^{1/2}\cos(\pi/4\pm x/2)$ and $X(\omega)=\exp(\pm 2h\sin x)y(x)$ are made, with independent $\pm$ signs. The solutions $X(\omega)$ are expanded in terms of Hermite polynomials of the variables ω and $i\omega$, respectively. The first coefficients in these expansions are explicitly given. *J. Meixner* (Aachen)

Müller, Harald J. W. 3691

Asymptotic expansions of prolate spheroidal wave functions and their characteristic numbers.

J. Reine Angew. Math. **212** (1963), 26–48.

The differential equation of prolate spheroidal wave functions

$$(1-u^2)Y''(u)-2uY'(u)$$
$$+[\Lambda+4h^2(1-u^2)-m^2(1-u^2)^{-1}]Y(u) = 0$$

is studied for large positive values of h in the interval $-1\le u\le 1$. The substitutions $u=\cos x$ and $X(x)=\exp(\pm 2h\sin x)(\sin x)^{1/2}Y(u)$ yield a differential equation whose solutions are expanded in terms (1) of the functions

$$A_q(x) = \cos^{(q-1)/2}\left(\frac{\pi}{4}+\frac{x}{2}\right)\Big/\cos^{(q+1)/2}\left(\frac{\pi}{4}-\frac{x}{2}\right),$$

(2) of the Hermite polynomials $H_{(q-1)/2}(z)$, where $z=4h^{1/2}\cos(\pi/4+x/2)$, and (3) of the Hermite polynomials $H_{(q-1)/2}(i\zeta)$, where $\zeta=z(-x)$. The various expansions which are suitable in different but overlapping parts of the interval $0\le x\le\pi$ are linked together. In all expansions only a few terms are explicitly evaluated. But a general procedure is given for the evaluation of all coefficients in the expansion of the eigenvalues. *J. Meixner* (Aachen)

Dingle, R. B.; Müller, H. J. W. 3692

The form of the coefficients of the late terms in asymptotic expansions of the characteristic numbers of Mathieu and spheroidal-wave functions.

J. Reine Angew. Math. **216** (1964), 123–133.

In a series of papers [Proc. Roy. Soc. London Ser. A **244** (1958), 456–475; MR **21** #2145; ibid. **244** (1958), 476–483; MR **21** #2146; ibid. **244** (1958), 484–490; MR **21** #2147; ibid. **249** (1959), 270–283; MR **21** #2148a; ibid. **249** (1959), 284–292; MR **21** #2148b; ibid. **249** (1959), 293–295; MR **21** #2148c], R. B. Dingle developed a method for replacing

the divergent remainders of asymptotic expansions by series involving convergent factors, and applied his method to a variety of special functions. In another series [H. J. W. Müller and R. B. Dingle, J. Reine Angew. Math. **211** (1962) 11–32; MR **26** #1513; Müller, ibid. **211** (1962), 33–47; MR **26** #1514; Müller, #3690 and #3691 above], H. J. W. Müller, and the authors jointly, applied Dingle's method to Mathieu and spheroidal wave functions. The present work establishes the form of the coefficients of the late terms in the asymptotic expansions of the characteristic numbers of these functions, by means of the asymptotic solution of a partial difference equation. No error estimates are given, but some numerical evidence is furnished that the authors' approximations yield a correct solution. *I. Marx* (Lafayette, Ind.)

Hirschman, I. I., Jr. 3693

Extreme eigenvalues of Toeplitz forms associated with orthogonal polynomials.

J. Analyse Math. **12** (1964), 187–242.

Let $\Omega(dx)$ be a non-negative Borel measure on the real line with infinite spectrum $\sigma(\Omega)$ and for which moments of all orders exist. Let $P(n, x)$, $n = 0, 1, \cdots$, be the orthogonal polynomials associated with Ω, with

$$\int P(n, x)P(m, x)\Omega(dx) = \delta_{nm}h(n).$$

Given a real bounded continuous function $f(x)$ on $\sigma(\Omega)$, define

$$c(p, q) = \{h(p)h(q)\}^{-1/2}\int f(x)P(p, x)P(q, x)\Omega(dx).$$

Under study are the extreme eigenvalues of the Toeplitz matrix $[c(p, q)]_{p,q=0,\cdots,n}$ for the systems of Jacobi, Laguerre, and Hermite polynomials. Let the eigenvalues be $\lambda_{n,1} \geq \lambda_{n,2} \geq \cdots \geq \lambda_{n,n+1}$, and assume that f attains its maximum M at exactly one point x_0 of $\sigma(\Omega)$; if x_0 is interior to $\sigma(\Omega)$, assume $f''(x_0) \neq 0$, and if x_0 is an endpoint of $\sigma(\Omega)$, assume $f'(x_0) \neq 0$. It is shown that for fixed $\nu = 1, 2, \cdots$,

$$\lambda_{n,\nu} = M - \alpha_\nu n^{-e} + o(n^{-e}),$$

where the α_ν are explicit constants depending only on $f''(x_0)$ or $f'(x_0)$ and the polynomial system under consideration, and e is 1 or 2. The author has subsequently treated the more general case where f has arbitrary order of contact at its maximum for the systems of Jacobi polynomials [Pacific J. Math. **14** (1964), 107–161; MR **28** #4391] and ultraspherical polynomials [J. Math. Mech. **13** (1964), 249–282; MR **28** #3351]. *H. Widom* (Ithaca, N.Y.)

Makai, E.; Turán, P. 3694

Hermite expansion and distribution of zeros of polynomials. (Russian summary)

Magyar Tud. Akad. Mat. Kutató Int. Közl. **8** (1963), 157–163.

The purpose of this paper is to prove a theorem in the theory of Hermite expansions corresponding to the Landau-Fejér-Montel theory of polynomial expansions, this theorem being that the Hermite trinomial equation

$$f(z) = 1 + H_1(z) + \xi H_n(z) = 0,$$

where ξ is arbitrary and complex, always has at least one of its zeros in the strip $|y| \leq A$, A being a positive constant. This theorem is a consequence of the theorem: The above equation has for $n \geq 36$ at least one zero in the strip $|y| \leq e^3$. The proof of the latter theorem is involved. It starts from recurrence formulas for Hermite functions, from inequalities of G. Szegö and van Veen, and proceeds to a lemma giving an inequality for the absolute value of a Hermite function on the circle $|z - \xi_n| = \xi_n - \xi_{n-1}$, where ξ_n is the maximal zero of $H_n = 0$. This inequality is transformed somewhat and then leads to a proof of the second theorem above. Considering the case $n \leq 35$, a proof of the first theorem is obtained. *M. J. O. Strutt* (Zürich)

Chatterjea, S. K. 3695

On a paper of Banerjee.

Boll. Un. Mat. Ital. (3) **19** (1964), 140–145.

After the author points out that a paper of Banerjee [Proc. Nat. Acad. Sci. India Sect. A **29** (1960), 83–86; MR **26** #1505] contains some incorrect results and others which are already known, he goes on to evaluate the Christoffel-Darboux expression

$$\frac{\sigma_{n(x,z)}}{x - z} = y_{n+1}(x)y_n(z) - y_{n+1}(z)y_n(x),$$

where $y_n(x)$ is the nth Bessel polynomial of Krall and Frink [Trans. Amer. Math. Soc. **65** (1949), 100–115; MR **10**, 453]. The author fails to note that his formulas are special cases of some results for the generalized Bessel polynomials which the reviewer has already stated [Duke Math. J. **26** (1959), 519–539; MR **22** #120]. *W. A. Al-Salam* (Lubbock, Tex.)

Chatterjea, S. K. 3696

Integral representation for the product of two Jacobi polynomials.

J. London Math. Soc. **39** (1964), 753–756.

The author proves the following formula.

$$P_m^{(\alpha,\beta)}(x)P_n^{(\alpha',\beta')}(y) = \frac{2^{m+n+\alpha+\alpha'}}{\pi^2}$$

$$\times \frac{\Gamma(\alpha+\alpha'+\beta+\beta'+m+n+2)}{\Gamma(\alpha+\beta+m+1)\Gamma(\alpha'+\beta'+n+1)}\,\frac{\Gamma(\alpha+m+1)\Gamma(\alpha'+n+1)}{\Gamma(\alpha+\alpha'+m+n+1)}$$

$$\times \int_0^1 \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} t^{m+\alpha+\beta}(1-t)^{n+\alpha'+\beta'} e^{(m-n)\theta i+(\alpha-\alpha')\phi i}$$

$$\times \cos^{m+n}\theta \cos^{\alpha+\alpha'}\phi$$

$$\times P_{m+n}^{(\alpha+\alpha',\beta+\beta'+1)}\left\{1 - \frac{[xte^{(\phi-\theta)i} + y(1-t)e^{(\theta-\phi)i}]2\cos\phi}{\cos\theta}\right\} d\phi\, d\theta\, dt$$

$$(\beta, \beta' > -1;\ \alpha+\alpha' > -1).$$

The method is similar to the method used by the reviewer [Boll. Un. Mat. Ital. (3) **17** (1962), 25–28; MR **25** #1318] in obtaining a double integral representation for the product of two Laguerre polynomials. *L. Carlitz* (Durham, N.C.)

Chatterjea, S. K. 3697

On a generalization of Laguerre polynomials.

Rend. Sem. Mat. Univ. Padova **34** (1964), 180–190.

If k is a postive integer,

$$\frac{1}{n!}x^{-\alpha}e^{px^k}\frac{d^n}{dx^n}(x^{\alpha+n}e^{-px^k})$$

is a polynomial of degree n; it is denoted by $T_{kn}^{(\alpha)}(x,p)$. $T_n^{(\alpha)}(x,1)=L_n^{(\alpha)}(x)$, the Laguerre polynomial. An explicit formula, an operational formula, a generating function, and various recurrence formulas are obtained, which are generalizations of the corresponding formulas for Laguerre polynomials. Of particular interest are the following relations:

$$(1)\qquad T_n^{(\alpha)}(x,m)=L_n^{(\alpha)}(mx),$$

$$(2)\qquad T_{kn}^{(\alpha+\beta+1)}(x,p+q)=\sum_{m=0}^{n}T_{km}^{(\alpha)}(x,p)T_{k(n-m)}^{(\beta)}(x,q),$$

$$(3)\qquad H_{kn}^{(\alpha)}(x,p)=(-1)^n n!\,T_{kn}^{(\alpha-n)}(x,p),$$

where $H_{kn}^{(\alpha)}(x,p)$ are the generalized Hermite polynomials defined by Gould and Hopper [Duke Math. J. **29** (1962), 51–63; MR **24** #A2689].

A. E. Danese (Buffalo, N.Y.)

Abdul-Halim, N.; Al-Salam, W. A. 3698

A characterization of the Laguerre polynomials.

Rend. Sem. Mat. Univ. Padova **34** (1964), 176–179.

Two proofs are given of the result that the only orthogonal polynomials of the form

$${}_{p+1}F_q\left[\begin{matrix}-n,\alpha_1,\alpha_2,\cdots,\alpha_p;\\ \beta_1,\beta_2,\cdots,\beta_q;\end{matrix}\; x\right],$$

where n is a non-negative integer, and the α's and β's are independent of x and n, are ${}_1F_1[-n;\beta;x]$. (These are the Laguerre polynomials $L_n^{(\beta-1)}(x)$ with normalization $L_n^{(\beta-1)}(0)=1$.)

One of the proofs follows directly from the general recurrence relation for orthogonal polynomials and the definition of the confluent hypergeometric function; the other proof is more elegant and is established from current results.

A. E. Danese (Buffalo, N.Y.)

Gatteschi, Luigi 3699

Proprietà asintotiche di una funzione associata ai polinomi di Laguerre e loro utilizzazione al calcolo numerico degli zeri dei polinomi stessi.

Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **98** (1963/64), 113–124.

Taking his clue from the asymptotic representation of Laguerre polynomials $L_n^{(\alpha)}(x)$ as $\nu=4n+2\alpha+2\to\infty$ while α is fixed and $t=x/\nu$ is restricted to a closed subinterval of $(0,1)$, the author writes, for $0<t<1$,

$$L_n^{(\alpha)}(\nu t)=A_n^{(\alpha)}(t,\nu)\sin[\theta(t)-\nu^{-1}A(t)-\eta_n^{(\alpha)}(t)],$$

where θ and A are certain elementary functions known from the asymptotic representation. Let $\nu\tau_{n,r}^{(\alpha)}$ be the rth zero of $L_n^{(\alpha)}(x)$. The author shows that for fixed r and s, $\eta_n^{(\alpha)}(\tau_{n,r}^{(\alpha)})$ and $\eta_n^{(\alpha)}(\tau_{n,n-s+1}^{(\alpha)})$ approach finite limits as $n\to\infty$, and expresses these limits in terms of the rth zero of $J_\alpha(x)$ and the sth zero of $\mathrm{Ai}(-x)$, respectively. He then describes the application of his results to the computation of the zeros of Laguerre polynomials.

A. Erdélyi (Edinburgh)

Parodi, Maurice 3700

Sur quelques propriétés des zéros des polynomes de Laguerre.

C. R. Acad. Sci. Paris **258** (1964), 5303–5304.

Starting from the differential equation of Laguerre polynomials, the author transforms it in such a way as to eliminate the first-order term. This equation is then compared with an equation, a particular solution of which may be expressed by a Bessel function of the first kind of order unity. Considering the first two consecutive (real) zeros of the latter solution, the application of Sturm-Liouville properties yields the situation that the first zero of the Laguerre polynomial of order $n\geq 6$ is between the two consecutive zeros. Thus bounds for this first zero appear. Also, bounds for generalized Laguerre polynomials are derived in this way.

M. J. O. Strutt (Zürich)

Rangarajan, S. K. 3701

Generalised Angelescu polynomials: Some properties.

Proc. Indian Acad. Sci. Sect. A **60** (1964), 65–73.

The author defines a set of polynomials $\{\pi_n^\alpha(x)\}$ by means of

$$\frac{\pi_n^\alpha(x)}{(1+\alpha)_n}=\sum_{m=0}^{n}(-1)^m\binom{n}{m}\frac{A_m(x)}{(1+\alpha)_m},$$

where $A_m(x)\equiv(a_0,a_1,\cdots,a_m,x,1)^m$. Thus for $\alpha=0$, $\pi_n^\alpha(x)$ reduces to Angelescu's polynomial $\pi_n^\alpha(x)$ [C. R. Acad. Sci. Roumaine **2** (1938), 199–201]. The author obtains various results concerning $\pi_n^\alpha(x)$, thus generalizing known properties of $\pi_n(x)$. For example, $\pi_n^\alpha(x)=n!L_n^\alpha(x)$, where $L_n^\alpha(x)$ is the Laguerre polynomial, and

$$\pi_n^{\alpha+\beta+1}(x+y)=\sum_{m=0}^{n}\binom{n}{m}m!L_m^\alpha(x)\pi_{n-m}^\beta(y).$$

The general identity

$$\sum_{r=0}^{n}\frac{\pi_r^\alpha(x)t^rf^{(r)}(z)}{\Gamma!(1+\alpha)_r}=\sum_{r=0}^{n}\frac{(-1)^rA_r(x)f^{(r)}(z+t)}{\Gamma!(1+\alpha)_r},$$

where $f(z)$ is an arbitrary polynomial of degree n, is proved, and some special cases noted.

L. Carlitz (Durham, N.C.)

ORDINARY DIFFERENTIAL EQUATIONS

See also 3821, 3903, 3904, 3906, 3910, 3911, 4067, 4197, 4199, 4228, 4242, 4252, 4254, 4628, 4630, 4632, 4633.

Wilcox, Calvin H. (Editor) 3702

★Asymptotic solutions of differential equations and their applications.

Proceedings of a Symposium Conducted by the Mathematics Research Center, United States Army, at the University of Wisconsin, Madison, May 4–6, 1964. Publication No. 13 of the Mathematics Research Center, United States Army, The University of Wisconsin.

John Wiley & Sons, Inc., New York-London-Sydney, 1964. x + 249 pp. $4.95.

The papers presented at the colloquium described in the heading will be reviewed individually.

Bandić, Ivan 3703

Sur une classe d'équations différentielles non-linéaires à plusieurs dimensions. (Serbo-Croatian summary)

Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hrvatske Ser. II **18** (1963), 61–68.

The author considers differential equations of the form

$$f_n(x, y) + \sum_{k=0}^{n-1} f_k(x, y) y^{(n-k)} = 0,$$

where the $f_k(x, y)$ are entire rational functions of dimension m_k, in the case where x has dimension λ and y has dimension ν. Thus, for example, $x(y+x^2)$ has dimension 3 whenever $\lambda = 1$ and $\nu = 2$. It is shown that if λ, ν and m_k satisfy certain conditions, then an equation of order n can be reduced to an equation of order $n-1$.

N. P. Bhatia (Cleveland, Ohio)

Hukuhara, Masuo 3704

On the zeros of solutions of linear ordinary differential equations. (Japanese)

Sûgaku **15** (1963), 108–109.

The author proved the following result: Consider an nth-order linear ordinary differential equation

$$\text{(A)} \qquad y^{(n)} + p_1(x) y^{(n-1)} + \cdots + p_n(x) y = 0,$$

where the coefficients $p_j(x)$ are bounded and measurable functions on an x-interval I. Then the length of the interval I should not be less than the positive root of the algebraic equation

$$\text{(B)} \qquad M_1 \|p_1\| \left(\frac{x}{2}\right) + M_2 \|p_2\| \left(\frac{x}{2}\right)^2 + \cdots + M_n \|p_n\| \left(\frac{x}{2}\right)^n = 1,$$

where $M_{2k} = 1/(2(k!)^2)$, $M_{2k+1} = 1/((2k+1)(k!)^2)$, and $\|p_k\| = \sup_{x \in I} |p_k(x)|$, in order that the differential equation (A) admits a solution which has at least n zeros in the interval I.

This result is a refinement of the result due to T. Sato [Kansu-Hoteisiki **22** (1940), 39–43]. T. Sato assumed that the $p_j(x)$ are continuous, and he used the algebraic equation

$$\|p_1\| \left(\frac{x}{2}\right) + \|p_2\| \left(\frac{x}{2}\right)^2 + \cdots + \|p_n\| \left(\frac{x}{2}\right)^n = 1$$

in place of (B). On the other hand, M. Tumura [ibid. **30** (1941), 20–35] also previously proved a result in which he assumed the continuity of the p_j, and he used the algebraic equation

$$\frac{(n-1)^{n-1}}{n^n \cdot n!} \|p_n\| x^n + \sum_{k=1}^{n-1} \frac{(n-k)}{n \cdot k!} \|p_k\| x^k = 1$$

in place of (B). The author gives no definite comparison between Tumura's result and his. (Cf. W. B. Fite [Ann. of Math. (2) **18** (1917), 214–220], M. Nagumo [Japan. J. Math. **5** (1928), 225–238], and C. de La Vallée Poussin [J. Math. Pures Appl. (9) **8** (1929), 125–144].) The author also considers the case when the p_j are Lebesgue integrable on I. In this case he uses the algebraic equation

$$M_0 \|p_1\| \left(\frac{x}{2}\right) + \cdots + M_{n-1} \|p_n\| \left(\frac{x}{2}\right)^n = 1$$

in place of (B), where $\|p_j\| = \int_I |p_j(x)|\, dx$.

Y. Sibuya (Minneapolis, Minn.)

Gray, H. L. 3705

Application of the Holmgren-Riesz transform.

Ann. Scuola Norm. Sup. Pisa (3) **18** (1964), 57–65.

The Holmgren-Riesz transform is applied to the equation

$$(a_2 x^2 + b_2 x + c_2) y'' + (a_1 x + b_1) y' + c_0 y = 0,$$

where a_1, a_2, b_1, b_2 and c_0 are real constants. Starting from appropriate definitions of the Holmgren-Riesz transform, the equivalent HR form of this equation is given, introducing an exponential function. The proof of this HR form follows by its expansion. Defining the first $n-1$ arbitrary constants of the HR form to be zero, the general solution of the above differential equation is obtained. Specializing, a "generalized" Rodrigues formula is defined. This is then applied to the differential equation of Laguerre functions, thus obtaining generalized Laguerre polynomials, as is proved by direct series expansion. Next, the differential equation of Legendre functions is considered. Application of the HR transform gives its solution, and suggests some formulas for these functions. The proofs again follow by direct series expansion. Finally, the solution of the hypergeometric differential equation is obtained in the same way.

M. J. O. Strutt (Zürich)

Aniščenko, R. I. 3706

On a boundary-value problem. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 481–492.

The differential equation considered in this paper is

$$\varphi'' = f(x, \varphi) \psi(x)$$

with conditions

$$\varphi(0) = y_0, \qquad R\varphi'(R) - \varphi(R) = q$$

for $0 \le x \le R$, where y_0, R, q are given real numbers $y_0 > 0$, $R > 0$. The solution sought, $\varphi(x)$, is to be nonnegative. A number of conditions have been imposed on $f(x, \varphi)$ and ψ. Problems of this type arise in the statistical theory of the atom, and these conditions follow naturally from the physical setting of the problems. Continuous dependence of the interval of existence of the solution and the continuous dependence of the solution itself on the boundary values have been established.

P. K. Ghosh (Calcutta)

Kibenko, A. V. 3707

The Green's function for an ordinary first-order differential equation with a parameter. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 310–314.

Given the boundary-value problem

$$\frac{dx}{dt} = A(t) x + B(t) \lambda + f(t),$$

$$Cx(t_0) + Dx'(t_0) = \xi,$$

$$Fx(t_1) + Rx'(t_1) = \eta,$$

where $A(t)$, $B(t)$ are matrices which are continuous in $J = \langle t_0, t_1 \rangle$, C, D, F, R are constant matrices, $f(t)$ is a vector function which is continuous in J, λ is a vector parameter, and ξ, η are constant vectors. The author proves theorems on the existence of a Green's function for this system.

M. Ráb (Brno)

Coffman, Charles V. 3708
On the uniqueness of solutions of a non-linear boundary value problem.
J. Math. Mech. **13** (1964), 751–763.

The article deals with the boundary-value problem $v''+f(x,v)=0$, $v(0)=v'(b)=0$ $(b>0)$, where (i) $f(x,v)=vg(x,v)$, with $g>0$ and in C^1 on $x>0$, $|v|<\infty$; (ii) $f_v>0$ and $vg_v<0$ for $v\neq 0$; (iii) $xf_v(x,0)\leq M$ on $0<x<\delta$; and (iv) $\lim_{v\to\infty} g(x,v)=\lim_{v\to-\infty} g(x,v)$. With more special assumptions on f, the problem has been studied by I. I. Kolodner [Comm. Pure Appl. Math. 8 (1955), 395–408; MR 17, 154] and by G. H. Pimbley, Jr. [J. Math. Mech. 11 (1962), 121–138; MR 25 #2262a]. The question asked is whether a solution $v_n(x)$ of the boundary-value problem is uniquely determined by the requirement that it have $n-1$ zeros on $(0,b)$ and satisfy $v_n'(0)>0$. A sufficient condition is given for uniqueness, one involving a certain ordering of the zeros of several functions associated with the problem. For $n=1$ the condition reduces to a classical one of E. Picard (originally established under more restrictive assumptions). The general condition is satisfied also when f has the special form

$$f(x)=\phi\left(v\exp 2\int_1^x h^{-1}(t)\,dt\right),$$

where $h(x)+x$ is non-positive and monotone non-increasing. The last generalizes a result of Kolodner [loc. cit.].

Robert McKelvey (Boulder, Colo.)

Everitt, W. N. 3709
Singular differential equations. I. The even order case.
Math. Ann. **156** (1964), 9–24.

The author extends his earlier results [J. London Math. Soc. 37 (1962), 372–384; MR 25 #2261; Math. Ann. 149 (1962/63), 320–340; MR 27 #2668] on the use of boundary functions in boundary-value problems for differential equations with complex coefficients on finite intervals, to the case where the differential equation has a singularity at one end of the finite interval or where the interval is a half-line. The treatment is similar in both cases, and only the second case is considered. The differential equation is of the form $L\phi=\lambda\phi$, where λ is complex and L is an even-order, linear, formally self-adjoint differential operator with complex coefficients. There are obtained certain extensions of the limit-point, limit-circle theory (due to Weyl) for the second-order real case.

C. R. Putnam (Lafayette, Ind.)

Javrjan, V. A. 3710
The spectral shift function for Sturm-Liouville operators. (Russian. Armenian summary)
Akad. Nauk Armjan. SSR Dokl. **38** (1964), 193–198.

Consider

$$(1)\qquad H_2y=-y''+q(r)y,\quad y(0)=0,\quad 0\leq r<\infty,$$

with the domain of the operator H_2 in $L_2(0,\infty)$. Let $\psi(r,k)$ be a solution of the problem

$$-\psi''+q\psi-\lambda\psi=0,\quad \lambda=k^2,\quad \psi(0,k)=0,\quad \psi'(0,k)=1.$$

Put $M(k)=1+\int_0^\infty e^{ikr}q(r)\psi(r,k)\,dr$, $\operatorname{Im} k\geq 0$, $A(k)=|M(k)|$, $\delta(k)=\arg M(k)$ for $\operatorname{Im} k=0$. If q is real-valued and satisfies

$$(2)\qquad \int_0^1 r|q(r)|\,dr+\int_1^\infty |q(r)|\,dr<\infty,$$

then the following asymptotic formulas are valid:

$$\psi(r,k)\sim -\frac{M(k)}{2ik}e^{-ikr},\qquad r\to\infty,\quad \operatorname{Im} k>0,$$
$$\sim A(k)\sin(kr-\delta(k)),\qquad r\to\infty,\quad \operatorname{Im} k=0.$$

V. S. Buslaev and L. D. Faddeev [Dokl. Akad. Nauk SSSR 132 (1960), 13–16; MR 22 #11171] proved that under the supposition $\int_0^\infty r|q(r)|\,dr<\infty$ the spectral shift function for (1) is of the form

$$(3)\qquad \xi(\lambda)=\frac{1}{\pi}\delta(\sqrt{\lambda}),\qquad \lambda>0,$$
$$=-\int_{-\infty}^{\lambda}\sum_l \delta(t-\lambda_l)\,dt,\qquad \lambda<0,$$

where λ_l denotes points of the negative spectrum of H_2 and $M(\sqrt{\lambda})=\det(E+qR_\lambda)-R_\lambda$ is the resolvent of H_0. The author proves that the formulas (3) are valid under supposition (2) and that $M(\sqrt{\lambda})=\det(E+R_\lambda^{1/2}qR_\lambda^{1/2})$, $\arg\lambda\neq 0$.

M. Ráb (Brno)

Valakmadze, T. S. 3711
Multi-point linear boundary-value problems. (Russian. Georgian summary)
Soobšč. Akad. Nauk Gruzin. SSR **35** (1964), 29–36.

The author uses the method of Š. E. Mikeladze [Izv. Akad. Nauk SSSR Otd. Mat. Estest. Nauk 1935, 255–300] for the approximate solution of the boundary-value problem

$$y^{(n)}+X_1(x)y^{(n-1)}+\cdots+X_n(x)y=X_0(x),$$

where X_i are continuous in $\langle 0,l\rangle$ and

$$(1)\qquad \sum_{s=1}^{m}\sum_{j=0}^{n-1}a_{ij}^{(s)}y^{(j-1)}(x_s)=\gamma_i\qquad (i=1,2,\cdots,n),$$
$$0\leq x_1<x_2<\cdots<x_m\leq l,$$

under the supposition that this problem has a unique solution. The existence and uniqueness of the solution is proved when the boundary conditions (1) are of the form $y^{(j-1)}(x_i)=\gamma_{ij}$ $(i=1,\cdots p;\ j=1,\cdots,n_i;\ \sum_{i=1}^p n_i=n)$.

M. Ráb (Brno)

Ahmedova, A. M. [Guseinbekova, A. M.] 3712
Asymptotic expansion of eigenfunctions and eigenvalues in problems with a narrow deep potential well. (Russian. Azerbaijani summary)
Akad. Nauk Azerbaĭdžan. SSR Dokl. **20** (1964), no. 6, 3–7.

The purpose of the paper is to establish expansions in powers of $\varepsilon\to 0+$ for the eigenvalues and eigenfunctions of the operator

$$L_\varepsilon y=-y''+c_\varepsilon(x)y$$

for the boundary conditions $y(0)=y(+\infty)=0$ and the

transition conditions $y(\varepsilon^\beta-)=y(\varepsilon^\beta+)$, $y'(\varepsilon^\beta-)=y'(\varepsilon^\beta+)$, the coefficient $c_\varepsilon(x)$ having the form

$$c_\varepsilon(x) = -\varepsilon^{-2\beta}c_-(x\varepsilon^{-\beta}), \qquad 0 < x < \varepsilon^\beta,$$
$$= \varepsilon^{-2\alpha}c_+(x\varepsilon^{-\beta}), \qquad x > \varepsilon^\beta,$$

where $c_-\geq 0$ and $c_+>0$ are suitable functions defined in $(0,1)$ and $(1,\infty)$, respectively. Using simple transformations, the author reduces the problem to a similar problem discussed in her earlier paper [A. M. Guseinbekova, Izv. Akad. Nauk Azerbaĭdžan. SSR Ser. Fiz.-Mat. Tehn. Nauk **1960**, no. 6, 49–67; MR **24** #A2701].

J. Král (Prague)

Titchmarsh, E. C. 3713

On the relation between the eigenvalues in relativistic and non-relativistic quantum mechanics. II.

Quart. J. Math. Oxford Ser. (2) **15** (1964), 193–207.

Part I appeared in Proc. Roy. Soc. Ser. A **266** (1962), 33–46 [MR **24** #A1447]. In Dirac's equation for the quantum mechanics of a particle, assume the vector potential to be zero and the scalar potential V to be a function of the radial coordinate r alone. If $V(r)\to\infty$ as $r\to\infty$ (details are worked out only for $V(r)=r^2$), the spectrum is purely continuous. However, it is "almost discrete" at those points ("resonances") which are actually discrete in the corresponding non-relativistic case. In this posthumous paper the author shows that the resolvent of the relativistic operator, analytically continued across the spectrum as a cut, has non-real poles near these points. The difference between such a pole and the corresponding non-relativistic eigenvalue is $O(c^{-2})$, where c is the velocity of light.

J. M. Cook (Argonne, Ill.)

Sahnovič, L. A. 3714

Analytic properties of the discrete spectrum of the Schrödinger equation. (Russian)

Mat. Sb. (*N.S.*) **64** (**106**) (1964), 185–204.

In the present paper the author studies the Schrödinger equation (radial)

$$(1) \qquad \frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\left[2E+\frac{2A}{r}-\frac{l(l+1)}{r^2}-V(r)\right]R = 0,$$

where the potential is a superposition of the power potentials r^{-t} $(1<t<2)$.

The function studied in detail is

$$C(\varepsilon,\mu) = \lim_{r\to\infty} R(\varepsilon,\mu,r)e^{-\varepsilon r}r^{A/\varepsilon+1},$$

where $\mu=l+1$, $\varepsilon=\sqrt{(-2E)}$ $(\mathrm{Re}\,\varepsilon>0)$, $R(\varepsilon,\mu,r)$ being the solution of (1) such that

$$\lim_{r\to\varepsilon}\frac{R(\varepsilon,\mu,r)}{r^{\mu-1}} = 1.$$

The function $C(\varepsilon,\mu)$ can be continued analytically from the half-plane $\mathrm{Re}\,\varepsilon>0$ to the entire ε-plane with a cut along the line $\varepsilon=iy$ $(y\geq 0)$. The present paper is restricted to the study of the discrete spectrum of (1), taking $A>0$ for this. The possibility of continuing analytically the eigenvalues as functions of μ is considered, whence it is shown to follow that these functions of μ are uniquely determined by their values at integral points $\mu=1,2,\cdots$. Some of the results of this paper were given without proof in an earlier note by the author [Dokl. Akad. Nauk SSSR **152** (1963), 286–289; MR **29** #301].

P. K. Ghosh (Calcutta)

Kostin, O. V. 3715

Asymptotic series in the theory of non-linear systems of ordinary differential equations. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 461–464.

On traite le système non-linéaire

$$(4) \qquad \frac{dy_i}{dt} = q_i(t)+\sum_{k=1}^{n}p_{ik}(t)y_k + \sum_{k_1+\cdots+k_n=2}^{\infty}p_{ik_1,\cdots,k_n}y_1^{k_1}\cdots y_n^{k_n}$$
$$(t\geq T;\ i=1,\cdots,n),$$

dont les coefficients peuvent être représentés dans un certain sens par les séries asymptotiques

$$(1) \qquad \sum_{s=0}^{\infty}F_\alpha^{s}[f_1(t)]^{k_0}[f_1'(t)]^{k_1}\cdots[f_1^{(s-1)}(t)]^{k_{s-1}}\cdots$$
$$\times[f_\nu(t)]^{l_0}[f_\nu'(t)]^{l_1}\cdots[f_\nu^{(s-1)}(t)]^{l_{s-1}},$$
$$\alpha=(k_0,\cdots,l_{s-1}).$$

On a montré qu'il est possible, sous certaines conditions, de trouver la solution formelle de ce système, les composantes de laquelle ayant aussi la forme (1).

Le rapport entre la solution formelle ainsi obtenue et la solution de (4) est discutée. Les résultats sont donnés dans le texte de trois théorèmes, complétant les recherches de G. Birkhoff, P. Costomaroff, V. P. Bassoff, A. G. Hiuhine. Le théorème 2 est fondé sur une méthode spéciale des approximations successives (O. Perron, O. V. Costine).

M. Bertolino (Belgrade)

Grobman, D. M. 3716

Asymptotic behaviour of almost linear systems of differential equations. (Russian)

Dokl. Akad. Nauk SSSR **158** (1964), 774–776.

The author extends and makes more precise earlier work [same Dokl. **86** (1952), 19–22; MR **14**, 274] concerning the asymptotic equivalence of solutions of the systems (1) $x'=Ax+F(t,x)$ and (2) $y'=Ay$, where $F(t,0)=0$ and F satisfies a Lipschitz condition in x with Lipschitz "constant" $g(t)$. Solutions $x(t)$ and $y(t)$ of (1) and (2), respectively, are said to be analogous if $|x(t)-y(t)|\,|y(t)|^{-1}$ converges to 0 as $t\to\infty$. Theorem: Let $\int_{t_0}^{\infty}e^{\alpha\tau}\tau^\beta g(\tau)\,d\tau<\infty$ for some constant β and positive constant α. Then there is a homeomorphism Φ of the x-space onto the y-space such that Φ and Φ^{-1} satisfy a Lipschitz condition and such that for large enough t^*, the solution of (1) passing through a given point x^0 of the x-space at $t=t^*$ is analogous to the solution of (2) passing through Φx^0 at $t=t^*$. Moreover, the deviation of these two solutions is $O(e^{-\alpha t}t^{m-\beta})$, where m is a certain integer. A second similar theorem is stated and two examples are given illustrating the force of the hypothesis on the integral. No proofs are given.

C. S. Coleman (Claremont, Calif.)

Leontovič, E. A. [Andronova-Leontovič, E. A.] 3717

Letter to the editor concerning the paper of N. F. Otrokov, "Multiple limit cycles". (Russian)

Mat. Sb. (N.S.) **64 (106)** (1964), 140–144.

In a paper by Otrokov [Mat. Sb. (N.S.) **41 (83)** (1957), 417–430; MR **19**, 278] there was discussed the question of the generation of limit-cycles from a multiple limit-cycle of a structurally stable planar system $\dot{x}=P(x,y)$, $\dot{y}=Q(x,y)$. Lemma 2 of that paper played an important role in the argument. The present author shows by an example that this lemma is incorrect. {References: Otrokov [Vestnik Leningrad. Univ. **16** (1961), no. 19, 23–44; MR **25** #4181], Andronov and Leontovič [Mat. Sb. (N.S.) **40 (82)** (1956), 179–224; MR **19**, 36], Neĭmark [Izv. Vysš. Učebn. Zaved. Radiofiz. **1** (1958), no. 5–6, 146–165; Dokl. Akad. Nauk SSSR **129** (1959), 736–739; MR **24** #A2101; ibid. **148** (1963), 281–283; MR **26** #1584].}

S. Lefschetz (Princeton, N.J.)

Leighton, Walter 3718

Behavior of solutions of a linear differential equation of second order.

Proc. Nat. Acad. Sci. U.S.A. **52** (1964), 830–832.

In this paper an affirmative answer is given to the question of whether or not all solutions of the equation (1) $y''+p(x)y=0$ tend to zero as $x\to\infty$ if $p(x)\to\infty$ as $x\to\infty$. The following theorem is proved. If $p(x)$ is of class C' and nondecreasing on the interval $I: x_0\leq x<\infty$, and if $p(x)\to\infty$ as $x\to\infty$, every solution of (1) tends to zero as $x\to\infty$. By a change of the independent variable in the equation (2) $(r(x)y')'+p(x)y=0$, the following corollary is obtained. If $r(x)$ and $p(x)$ are of class C' on the interval $I: x_0\leq x<\infty$, with $r(x)>0$ there, if $r(x)p(x)$ is nondecreasing on I, $r(x)p(x)\to\infty$ as $x\to\infty$, and if $\int_{x_0}^{\infty} dx/r(x)=\infty$, then every solution of (2) tends to zero as $x\to\infty$.

An outline of the theorem's proof is as follows. Introducing $u(x)=(y_1^2(x)+y_2^2(x))^{1/2}$, where y_1 and y_2 are linearly independent solutions of (1), and noting that one can write $y_1=u\sin v$, $y_2=u\cos v$, where $v'=k^2/u^2$ ($k=$ const), it is observed that y_1 and y_2 lie between the curves u and $-u$ and are tangent to them. The assumption that $u\geq m$, m an arbitrary positive constant, is shown to contradict Osgood's theorem concerning the monotonic decrease of the relative maxima of any solution of (1) as $x\to\infty$ [cf. Osgood, Bull. Amer. Math. Soc. **25** (1919), 216–221]. For the remaining case, in which u continually rises above and falls below the value m, an analysis (which again uses the Osgood theorem) shows that u may be made arbitrarily small as $x\to\infty$.

H. C. Howard (College Park, Md.)

Chen, Xiang-yan [Chen, Hsiang-yen] 3719

Periodic solutions and limit cycles of differential equations containing a parameter.

Acta Math. Sinica **13** (1963), 607–619 (*Chinese*); *translated as Chinese Math.* **4** (1964), 661–674.

In the second-order system

$$(*)\qquad \dot{x}=P(x,y,\alpha),\qquad \dot{y}=Q(x,y,\alpha)$$

let the functions P, Q be continuous and have partial derivatives of any order required for all $(x,y,\alpha)\in R\times I$, where R is a region in the (x,y)-plane and I an interval of variation of α having $\alpha=0$ as an interior point. Assume that for $\alpha=0$ the system (*) has the periodic solution $L_0: x=x_0(t)$, $y=y_0(t)$. The paper is specifically directed to the search for the conditions for the non-vanishing of L_0 when the parameter α in (*) is slightly varied. In the special case where L_0 is a limit cycle and (*) a rotating vector field this problem has been investigated by Duff [Ann. of Math. (2) **57** (1953), 15–31; MR **14**, 751] and subsequently generalized by Seifert [*Contributions to the theory of non-linear oscillations*, Vol. IV, pp. 125–139, Princeton Univ. Press, Princeton, N.J., 1958; MR **21** #4278], Urabe [J. Sci. Hiroshima Univ. Ser. A **18** (1954), 183–219; MR **17**, 264] and others. In the paper under review the author considers the two cases: (i) L_0 a limit cycle, and (ii) L_0 lying in a periodic annulus. For the case (i) he obtains results which generalize those of Duff and Seifert mentioned above. For the case (ii) his results generalize those of Malkin [*Some problems of the theory of nonlinear oscillations* (Russian), GITTL, Moscow, 1956; MR **18**, 396], Pontrjagin [Ž. Èksper. Teoret. Fiz. **4** (1934), 883–885] and Zhang Zhi-fen [Degree Thesis, Moskow Univ., Moscow, 1957 (unpublished)]. The precise statements are too involved to be reproduced here.

J. O. C. Ezeilo (Ibadan)

Faure, Robert 3720

Solutions périodiques d'équations différentielles et méthode de Leray-Schauder. (Cas des vibrations forcées).

Ann. Inst. Fourier (Grenoble) **14** (1964), *fasc.* 1, 195–204.

A theorem of Leray and Schauder on functional equations and its extension is used to prove the existence of periodic solutions of some special nonlinear differential equations and systems of equations of the second order.

M. Zlámal (College Park, Md.)

Yamaguchi, Mikiko 3721

On the existence of a periodic solution for a certain nonlinear equation. (Japanese)

Sûgaku **15** (1963/64), 165–167.

The author proves that the conditions $C>AB$, $A-(AB/C)D<0$ and $A-\{1-(AB/C)\}D<0$ are sufficient for the existence of stable periodic solutions of the system of two differential equations

$$dx/dt = -A(x+B)+C(1-y)+Dy^*x,\qquad dy/dt = y^*x,$$

where A, B, C, D are positive constants and y^* is the function which is defined for all x and $0<y<1$ by $y^*=y$ $(x\leq 0)$ and $=1-y$ $(x\geq 0)$. If $A=2$, $B=375$, $C=2000$, and $D=20$, the author's sufficient conditions are satisfied. This special case is studied first and then the general case is studied. These sufficient conditions guarantee the existence of an unstable critical point around which we can construct a ring-shaped region R so that no solution starting from a point of R may not leave R. The uniqueness of periodic solutions is not discussed.

Previously, in connection with the study of electrochemical oscillations, U. F. Franck and R. FitzHugh [Z. Electrochemie **65** (1961), 156–168] treated the same problem by the use of analogue computers. The motivation of the present paper is to give a rigorous proof to this previous result.

Y. Sibuya (Minneapolis, Minn.)

Kiguradze, I. T. 3722

On non-oscillatory solutions of the equation

$$u'' + a(t)|u|^n \operatorname{sign} u = 0.$$

(Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **35** (1964), 15–22.

Der Verfasser untersucht die angegebene Differentialgleichung, wobei $n>1$ und die Funktion $a(t)$ nichtnegativ und in jedem endlichen Teilintervall der Halbachse $t\geq 0$ summierbar sei. Er setzt sich das Ziel, Funktionsklassen $a(t)$ zu suchen, für die alle Lösungen $u(t)$ der Gleichung nichtoszillatorisch sind. Z.B. werden folgende Sätze bewiesen: (1) Die Gleichung besitzt dann und nur dann Lösungen $u_i(t)$, $i=1, 2$, für die $u_i(t)\sim c_i t^{i-1}$, $c_i\neq 0$ gilt, wenn $\int_0^\infty t^n a(t)\,dt<\infty$ ist. (2) Jede nicht-triviale Lösung genügt der Beziehung $u(t)\sim c_1+c_2 t$, $c_1^2+c_2^2>0$, wenn $a(t)$ eine absolutstetige, positive Funktion ist und

$$\int_0^\infty t^n a(t) \exp\left\{\frac{n-1}{4}\int_0^t [|a'(\tau)|+a'(\tau)]\,d\tau\right\} dt < \infty$$

gilt. Zwei weitere Sätze beruhen auf komplizierteren Koeffizientenbedingungen. *R. Reissig* (Berlin)

Corduneanu, C. 3723

The stability theorem of Liapunov for the first approximation. (Romanian. Russian and French summaries)

Gaz. Mat. Fiz. Ser. A **15** (**68**) (1963), 681–686.

Après une brève introduction historique, l'auteur donne une démonstration du théorème de stabilité de Liapounoff, d'après la première approximation, en utilisant le théorème de point fixe de Banach. *A. Haimovici* (Iaşi)

Wasow, Wolfgang 3724

Asymptotic expansions for ordinary differential equations: Trends and problems.

Asymptotic Solutions of Differential Equations and Their Applications (Proc. Sympos., Math. Res. Center, U.S. Army, Univ. Wisconsin, Madison, Wis., 1964), *pp.* 3–26. *Wiley, New York*, 1964.

A review covering mainly asymptotic expansions for solutions in the cases (1) linear systems near an essential singularity, (2) turning-point problems for systems which involve a parameter ε and degenerate for $\varepsilon=0$, (3) singular perturbation boundary-value problems for such systems. The systems considered are sometimes of arbitrary order, sometimes of second order only. *T. M. Cherry* (Melbourne)

Leighton, Walter 3725

Morse theory and stability by Liapunov's direct method.

Proc. Nat. Acad. Sci. U.S.A. **52** (1964), 42–43.

This paper states some theorems on the relationship between stability in autonomous ordinary differential equations and Morse critical point theory of an auxiliary real-valued function. *S. Smale* (New York)

Ezeilo, J. O. C. 3726

An elementary proof of a boundedness theorem for a certain third order differential equation.

J. London Math. Soc. **38** (1963), 11–16.

A previous result of the author [Proc. London Math. Soc. (3) **9** (1959), 74–114; MR **21** #3620] is proved in a simpler way. *M. M. Peixoto* (Providence, R.I.)

Hayart, René 3727

Extension des théorèmes de Liapounoff et de Chetayev relatifs à la stabilité.

C. R. Acad. Sci. Paris **259** (1964), 38–41.

Take an n-vector system

$$(1)\qquad \dot{x} = X(x, t), \qquad X(0, t) = 0, \qquad 0 \leq t \leq T,$$

and let Ω be the set $\|x\|\leq a$, $t\in[0, T]$. It is assumed that standard existence and unicity properties hold in Ω. Let also $0\leq\alpha<\beta$. Definition of (α, β, T) practical stability: It holds if there exists $\lambda\in[\alpha, \beta]$ such that if $x_0=x(0)$ and $\|x_0\|<\lambda$, then $\|x(t)\|<\beta$ for $t\in[0, T]$.

One defines as usual the Liapunov function $V(x, t)$ as positive definite if there exists $W(x)$, continuous and positive in $\|x\|<a$, such that $W\leq V$ in Ω. Theorem (Generalized Liapunov stability): The solutions of (1) are (α, β, T) practically stable if there exists $V(x, t)$, $W(x)$ such that $\sup(V|\ \|x\|\leq\lambda)<\inf(W|\ \|x\|=\beta)$, this for some $\lambda\in[\alpha, \beta]$. A somewhat complicated extension of Četaev's instability theorem is also given. *S. Lefschetz* (Princeton, N.J.)

Brauer, Fred 3728

Nonlinear differential equations with forcing terms.

Proc. Amer. Math. Soc. **15** (1964), 758–765.

The basic problem is the comparison of the solutions of the linear equations $x'=Ax+p(t)$ and $y'=Ay+f(t, y)+p(t)$, where x, p, y, f are vectors and A is a square matrix. The author seeks conditions under which $y\to x$ as $t\to\infty$. This is a strong result, and, as might be expected, requires a strong condition on f: $\int_0^\infty (|f|/|y|)\,dt<\infty$.

Conditions that x and y tend to 0 as $t\to\infty$ are given, and are of the usual form. Conditions are also given that $|y|$ be below a certain bound for $0\leq t<\infty$. *E. Pinney* (Berkeley, Calif.)

Rozov, N. H. 3729

On the asymptotic theory of relaxation oscillations in systems with one degree of freedom. II. Calculation of the period of the limit cycle. (Russian. English summary)

Vestnik Moskov. Univ. Ser. I Mat. Meh. **1964**, *no.* **3**, 56–65.

Author's summary: "This is a continuation of the paper published in the previous issue of this journal [same Vestnik **1964**, no. 2, 70–82; MR **29** #333]. As is known, under certain conditions in the system $\varepsilon\dot{x}=f(x, y)$, $\dot{y}=g(x, y)$, where ε is a small positive parameter, there may occur periodic oscillations that are close to discontinuous. The author assumes the presence of an irregular point of order $m>2$ on the curve $f(x, y)=0$. Under this assumption, an asymptotic formula to quantities of the order $O(\varepsilon^{(3m)/(2m-1)})$ is obtained for the period T_ε of the limit cycle Z_ε (that is, close to a discontinuous periodic solution Z_0). This formula permits an estimate to be made of the period without integrating the system of differential equations. It is found, in particular, that the difference between the period T_ε and the period T_0 of the discontinuous solution Z_0 has the order $O(\varepsilon^{n/(2n-1)})+O(\varepsilon^{(k+1)/(2k-1)})$, where n is maximal even, and k is maximal odd order of irregular points on the solution Z_0." *R. Conti* (Florence)

Aeppli, A.; Markus, L. 3730

Sur l'équivalence des systèmes différentiels.

Ann. Inst. Fourier (Grenoble) **14** (1964), *fasc.* 1, 83–86.

This short paper is principally a summary of known results about the equivalence (up to homotopy or homeomorphy) of line element fields on the two-dimensional torus. One new result states that, in most circumstances, if two systems are homotopic, they may be joined by a path in the space of systems containing only a finite number of bifurcation points. No indication of a proof is given. *R. L. Reinhart* (College Park, Md.)

Glužko, K. S. 3731

A possible generalization of the differential equations of motion for non-holonomic mechanical systems. (Russian. Uzbek summary)

Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk **1964**, no. 2, 14–22.

Bibikov, Ju. N. 3732

Convergence in second-order dissipative systems. (Russian. English summary)

Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. **19** (1964), no. 1, 15–25.

In a previous paper [same Vestnik **18** (1963), no. 4, 14–26; MR **28** #1345] the author investigated the system

$$(1)\qquad \dot{x} = ax+f_2(y)+P(t), \qquad \dot{y} = f_1(x)+by+Q(t),$$

where the functions f_i are of class C^1. It was assumed that (2) $a+b<0$; (3) $ab-h_1(x)h_2(y)>\varepsilon(x)+\varepsilon(y)$, $h_i(z)=f_i(z)/z$, for $|x|>\bar{a}>0$, $|y|>\bar{b}>0$ and $\varepsilon(z)>0$ with $z\varepsilon(z)\to+\infty$ with z. It was shown that there exists a region Ω to which all solutions tend, remaining in it as $t\to+\infty$. Systems with such a property are said to be dissipative. For such systems, if P and Q have period ω, then (1) has a solution of the same period. The author is particularly interested in "convergent" systems, i.e., stable in the large. If $ab>0$, sufficient convergence conditions were found to be

$$(4)\qquad a+b < 0, \qquad ab-\sup|f_1'(x)|\cdot\sup|f_2'(y)| \geq 0.$$

Note that (4) will hold for the system (1) in which a, b are replaced by $\mu_1 a$, $\mu_2 b$ with $a, b<0$ and f_i satisfy (3) with the $\mu_i>0$ and such that $\mu_1\mu_2>-ab/(\sup|f_1(x)f_2(y)|)$, $(x, y)\in\Omega$. Similar results hold for $ab<0$, with Theorem 1: If (3) holds and $a+b=-K$, where $K>0$ and is bounded away from zero, then convergence takes place.

Analogous results are obtained for a system such as (1) with P, Q of the form $P(x, y, t)$, $Q(x, y, t)$ both bounded. Application is made to the Cartwright-Littlewood equation (5): $\ddot{x}+\mu f(x)\dot{x}+g(x)=\mu p(t)$ by the following theorem. Let $F(x)=\int_0^x f(x)\,dx$, $G(x)=\int_0^x g(x)\,dx$, $P(t)=\int_0^t p(t)\,dt$. Suppose (a) $f(x)>0$ and $g'(x)>0$ are continuous, and $|F(x)|$ and $G(x)\to\infty$ with $|x|$; (b) $p(t)$ is continuous with period ω and $P(t)$ is bounded; (c) $\mu>0$ and is bounded away from zero. Then the system (5) is convergent.

[Additional references: The reviewer, *Differential equations: Geometric theory*, second edition, Interscience, New York, 1963; MR **27** #3864; Krasovskiĭ, Prikl. Mat. Meh. **17** (1953), 651–672; MR **15**, 624; Cartwright, *Contributions to the theory of nonlinear oscillations*, pp. 149–241, Princeton Univ. Press, Princeton, N.J., 1950; MR **11**, 722; Reuter, Proc. Cambridge Philos. Soc. **47** (1951), 49–54; MR **12**, 827.] *S. Lefschetz* (Princeton, N.J.)

Brjuno, A. D. 3733

The normal form of differential equations. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 1276–1279.

Theorem: There exists an invertible transformation $x=\xi(y)$ of the differential system $\dot{x}=\varphi(x)$ into the system $\dot{y}=\psi(y)$, where ξ, φ, and ψ are formal power series without free terms, $(\partial x_i/\partial y_j|_0)=(\lambda_i\delta_{ij})$, $\sum_k(\partial\xi_i/\partial y_k)\psi_k(y)=\varphi_i(\xi(y))$, $\psi_i(y)=y_i\sum_Q g_{iQ}y^Q$, $Q=(q_1,\cdots,q_n)$, $y^Q=y_1^{q_1}\cdots y_n^{q_n}$, $g_{iQ}\neq 0$ if and only if $\sum_i q_i\lambda_i=0$ and Q ranges over a certain subset of the integer lattice points of E^n. This theorem, stated, proved, and discussed in this paper, is the latest in a long series of results concerning the equivalence of differential systems beginning with Poincaré's thesis [Fac. Sci. Paris, Gauthier-Villars, Paris, 1879]. More recently Sternberg [Amer. J. Math. **79** (1957), 809–824; MR **20** #3335] and Chen [ibid. **85** (1963), 693–722; MR **28** #3224] have worked on similar problems.

C. S. Coleman (Claremont, Calif.)

Yoshizawa, Taro 3734

Extreme stability and almost periodic solutions of functional-differential equations.

Arch. Rational Mech. Anal. **17** (1964), 148–170.

The author combines the idea of using Ljapunov functions defined for pairs of solutions instead of single solutions, previously introduced by Hale [same Arch. **15** (1964), 289–304; MR **29** #1395] for ordinary differential equations, with his own work on functional-differential equations. He gives an appropriate definition of uniform-asymptotic stability and proves that if such stability obtains and certain boundedness conditions are satisfied, there exists a Ljapunov function with the desired properties, including a negative-definite "total derivative" along the pair of solutions envisaged. This existence theorem is then applied to prove the existence and local uniqueness of almost periodic solutions of almost periodic functional-differential equations. The precise statements of the theorems are too complicated to reproduce here.

J. J. Schäffer (Pittsburgh, Pa.)

Blaż, J. 3735

Sur l'existence et l'unicité de la solution d'une équation différentielle à argument retardé.

Ann. Polon. Math. **15** (1964), 9–14.

The standard method of successive approximations is adopted to establish an existence and a uniqueness theorem for the equation

$$(1)\qquad \phi(t) = \omega(t) \quad \text{for } t\leq 0;$$

$$\phi'(t) = \int_0^\infty f[t, \phi(t-s)]\,dr(t, s)+g(t) \quad \text{for } t\in[0, a], \quad s\geq 0.$$

The author claims an improvement over the result of A. Bielecki and M. Maksym [Folia Soc. Sci. Lublinensis **2** (1962), 74–78]. *N. P. Bhatia* (Cleveland, Ohio)

Dłotko, T.; Kuczma, M. 3736

Sur une équation différentielle fonctionnelle à argument accéléré.

Colloq. Math. **12** (1964), 107–114.

Let $H(t, x, y, z)$ be continuous and let $h(t)$ be continuous, where all the variables are real. Consider the differential

equation (1) $H(t, \varphi(t), \varphi(h(t)), \varphi'(t)) = 0$. Two special cases of (1) are considered, namely, (2) $\varphi(h(t)) = g(t, \varphi(t), \varphi'(t))$ and (3) $\varphi'(t) = f(t, \varphi(t), \varphi(h(t)))$. Assume that $h(t) \geq t$. The authors then give sufficient conditions so that the following two problems have a solution. (I) To find a function $\varphi(t)$ satisfying (2) in $a \leq t < b$ such that the initial condition, $\varphi(t) = \alpha(t)$ for $a \leq t \leq h(a)$, is satisfied. (II) To find a function $\varphi(t)$ satisfying (3) in $a \leq t \leq b$ (or $a \leq t < \infty$) such that the initial condition $\varphi(a) = \varphi_0$ is satisfied. The method of proof (for II) is to use the Schauder fixed-point theorem. {The authors do not observe that their theorems are true for systems where x, y, and z are in R^n.}

G. R. Sell (Minneapolis, Minn.)

Borisovič, Ju. G.; Kibenko, A. V. 3737

A one-sided bound for ordinary differential equations with time lag. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 853–856.

On traite le système différentiel "à retard" $dx/dt = f(t, x_t)$ sous la forme vectorielle. Trois théorèmes concernent le prolongement des solutions sur le demi-axe, et l'unicité aux estimations unilatérales. Les fonctionnelles spéciales sont définies et utilisées dans les inégalités différentielles. L'idée des estimations en question tire son origine de N. N. Krasovskiĭ. Un groupe suivant de travaux concerne les équations différentielles ordinaires (aussi dans l'espace de Banach).

M. Bertolino (Belgrade)

PARTIAL DIFFERENTIAL EQUATIONS

See also 3487, 3614, 3649, 3900, 3901, 3902, 3905, 3907, 3916, 4093, 4094, 4207, 4208, 4210, 4211, 4213, 4214, 4228, 4320, 4427.

Friedman, Avner 3738

A new proof and generalizations of the Cauchy-Kowalewski theorem.

Trans. Amer. Math. Soc. **98** (1961), 1–20.

The author gives a proof of the Cauchy-Kowalewski theorem by a method different from those already known. The proof is based on a priori bounds of the successive derivatives of a system under the hypothesis that these derivatives exist. Thus he constructs a solution in the form of power series in the corresponding variables. In the proof it is not necessary to reduce the system to one of first order. He also considers the Goursat problem (the case of several "time" variables) and also a mixed problem.

V. N. Maslennikova (RŽMat **1962** #10 B191)

Friedman, Avner 3739

Simplifying the structure of second order partial differential equations.

Trans. Amer. Math. Soc. **99** (1961), 303–307.

Let D be a bounded or an unbounded region in Euclidean n-space with compact boundary. In the equation

$$(1) \qquad \Delta_2 A + cA = 0$$

let Δ_2 denote the Laplace-Beltrami operator describing a strong positive definite tensor. The author shows that, by a conformal mapping and a change of the unknown variable, equation (1) can be reduced to the form $\bar{\Delta}_2\bar{A} + c_0\bar{A} = 0$, where c_0 is a constant which can be taken to be 0 if $c > 0$. A similar result obtains for the corresponding parabolic equation whenever it is possible to set $c_0 = 0$ independently of the sign of c.

Gröbner, W. 3740

Lösung der allgemeinen partiellen Differentialgleichung 1. Ordnung mittels Lie-Reihen.

Monatsh. Math. **68** (1964), 113–124.

This paper is concerned with the solution of general first-order partial differential equations by using Lie-Reihen for integrating the associated characteristic differential equations. Three theorems are proven for a single partial differential equation; a fourth theorem deals with a pair of equations, and it is indicated that the methods could be extended to a system of equations. Although the functions occurring are commonly required to be holomorphic in the region under consideration, it is possible by this Lie-Reihen method to consider functions which are finitely differentiable.

W. Sangren (San Diego, Calif.)

Tutschke, W. 3741

Parameterabhängige Pfaffsche Formen in mehrfach-zusammenhängenden ebenen Gebieten im Zusammenhang mit globalen Normalformen partieller Differentialgleichungen.

Monatsb. Deutsch. Akad. Wiss. Berlin **6** (1964), 129–134.

Let G be a multiply connected domain of the (x_1, x_2)-plane and ω_ρ a Pfaffian whose coefficients depend on a function ρ and its derivatives $\partial\rho/\partial x_1$, $\partial\rho/\partial x_2$. The paper deals with the periods $\pi_k[\omega_\rho] = \int_{(\gamma)_k} \omega_\rho$ and their dependence with respect to the boundary values of ρ. The result is that, apart from the trivial case $\rho \equiv 0$, the function ρ does not vanish in the (closed) domain $\bar{G}$. Similarly, the existence of functions $\rho > 0$ with constant boundary values can be proved having a closed Pfaffian ω_ρ with periods arbitrarily prescribed. Finally, a multiplicator μ exists with constant boundary values which transforms the differential operator

$$\sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right) + \sum_k e_k \frac{\partial}{\partial x_k}$$

into Bianchi's canonical form. It follows that $\mu \neq 0$ in $\bar{G}$.

M. Pini (Cologne)

Kuks, L. M. 3742

On the oscillation of solutions of partial differential equations in complex regions. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 1279–1284.

The article is concerned with the existence of zeros of solutions w (or of some expressions depending on w) of $\partial^2 w/\partial z_1^2 + \partial^2 w/\partial z_2^2 + j(z_1, z_2) = 0$, where z_1, z_2 are complex variables and j is analytic. Results concerning elliptic equations with two real independent variables and ordinary differential equations in a complex independent variable are then obtained by specialization. The technique used is based upon integration by parts (Stokes's theorem).

A. Friedman (Evanston, Ill.)

Riekstiņš, E. 3743

★**Matemātiskās fizikas vienādojumi** [The equations of mathematical physics].

Latvijas Valsts Izdevniecība, Riga, 1964. 530 *pp.* 1.00 *r.*

A standard solid treatment of the classical partial differential equations of mathematical physics. Some modern topics are touched upon (for instance, distributions).

L. Bers (New York)

Yang, Guang-jun [Yang, Kuang-chün] 3744

The Dirichlet problem for a class of equations of degenerating elliptic type.

Acta Math. Sinica **12** (1962), 40–46 (*Chinese*); *translated as Chinese Math.* **3** (1963), 42–48.

Consider the equation

$$(*)\qquad M[u] = u_{xx}+y^m u_{yy}+au_x+bu_y+cu = 0,\qquad c\le 0,\ m\ge 1,$$

in a domain D in $y>0$ bounded by a segment γ of $y=0$ and a smooth curve τ in $y>0$. It is known that the Dirichlet problem in D can be solved if $b(x,0)<1$ when $m=1$, $b(x,0)\le 0$ when $1<m<2$, and $b(x,0)<0$ when $m\ge 2$. If these conditions are not satisfied, then one can specify data only on τ. If one can find a function Ω such that

$$\lim_{y\to 0}\,(b+2y^m\Omega_y/\Omega) \begin{cases} <1 & \text{if } m=1,\\ \le 0 & \text{if } 1<m<2,\\ <0 & \text{if } m\ge 2\end{cases}$$

with $M(\Omega)/\Omega<0$, then one can set $u=\Omega V$ and obtain a problem of the first type for V. u itself will then satisfy the modified Dirichlet problem $\lim_{(x,y)\to P} u/\Omega=\varphi$ for $P\in\gamma+\tau$. The author establishes the existence of Ω for all m with certain restrictions on b.

R. C. MacCamy (Pittsburgh, Pa.)

Sikora, B. S. 3745

The index and normal solvability of a boundary-value problem for an elliptic system of equations of higher order. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 26–30.

The author studies the elliptic system

$$Lu = \sum_{k+l\le n} A_{kl}(z)\partial^{k+l}u/\partial x^k\partial y^l = F(z),\quad z=(x,y),$$

in a region D with boundary conditions

$$\Lambda u = \sum_{k+l\le m} a_{kl}(t)\partial^{k+l}u/\partial x^k\partial y^l = f(t),$$

where the matrices $A_{kl}(z)$ and $a_{kl}(t)$ are suitable (A_{kl} is square of order p and a_{kl} is of dimension $r\times p$ with $r=\frac12 np$). Conditions for normal solvability and a formula for the index are obtained.

R. Carroll (Urbana, Ill.)

Aruffo, Giulio 3746

Sistemi ellittici di equazioni lineari del primo ordine in domini a connessione multipla.

Ricerche Mat. **13** (1964), 80–91.

Let $T\subset \mathbf{R}^n$ be bounded by an external $(n-1)$-cycle C and by p $(n-1)$-cycles γ_i interior to C. The problem is to find v such that

$$\sum_{i,k}\frac{\partial}{\partial x_i}\left(a_{ik}\frac{\partial v}{\partial x_k}\right) = -\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}$$

with $\int_{\gamma_i}\omega_{n-1}=0$, where $\sum a_{ik}(x)\xi_i\xi_k$ is positive definite and

$$\omega_{n-1} = \sum_i(-1)^i\left(f_i+\sum_k a_{ik}\frac{\partial v}{\partial x_k}\right)dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge dx_n.$$

It is shown that if $g, g_1, \cdots, g_p$ are suitable functions given on $C, \gamma_1, \cdots, \gamma_p$, then in a suitable function space there exists a unique solution of the problem satisfying the conditions of taking on C the value g and taking values on γ_i such that $v-g_i=\lambda_i$ with the constants λ_i arbitrary (not determined). The solution v of the homogeneous problem is found in the subspace V of $H'(T)$ consisting of functions v such that $v=0$ on C, $v=\lambda_i$ on γ_i (λ_i arbitrary), and the region T is supposed to be such that $\|v\|=\sum_1^n(\int_T(\partial v/\partial x_i)^2\,dx)^{1/2}$ is a norm equivalent to the $H'(T)$ norm.

R. Carroll (Urbana, Ill.)

Ossicini, Alessandro; Rosati, Francesco 3747

Sulla regolarità alla frontiera di soluzioni di equazioni ellittiche.

Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **98** (1963/64), 893–907.

A bounded domain Ω in R^m is in $H_{\delta,\beta}$ ($\delta>0$, $0<\beta<\pi/2$) if for each $x^0\in\partial\Omega$ there exists a cone with vertex x^0 of angular opening $\pi/2-\beta$ and height δ which lies outside Ω. Let $n(x^0)$ be the unit vector penetrating Ω in the direction of the axis of the cone. Let $d(x)=\operatorname{dist}(x,\partial\Omega)$ and set $T(x^0,t)=\{x\in\Omega: t|x-x^0|<d(x)\}$, $x^0\in\partial\Omega$, $0<t<1$. Let $\gamma(t)$ be the supremum of the angles between the vectors $x-x^0$ and $n(x^0)$ for all $x\in T(x^0,t)$. It follows that $\gamma(t)\le\beta+\arccos t$ for t close to 1. Let $\Omega_1=\{x\in\Omega: d(x)<\delta_1\}$, $0<\delta_1<\delta$.

Let $u\in C^2(\Omega)\cap C(\Omega^-)$ be a solution of the uniformly elliptic equation $\sum A_{ij}(x)u_{x_ix_j}+\sum B_i(x)u_{x_i}+C(x)u=f(x)$, where $0<\alpha\le\sum A_{ij}(x)\xi_i\xi_j\le 1$, $|\xi|=1$. Let $B(x)$ be the vector $(B_1(x),\cdots,B_m(x))$ and assume that the lower-order coefficients satisfy for all $x^0\in\partial\Omega$ the relations: (i) $d(x)|\langle n(x^0),B(x)\rangle|\le 8^{-1}\alpha(1-\lambda)$, $x\in T(x^0,\frac12)\cap\Omega_1$; (ii) $d(x)|B(x)|\le K$ and $|C(x)|d(x)^{2-\lambda}\delta_1^{\lambda}\le K_1$, $x\in\Omega_1$; (iii) $d(x)|B(x)|\le 3\alpha(1-\lambda)/16$ and $|C(x)|d(x)^2\le K_1$, $x\in\Omega-\Omega_1$, where $0<\lambda<1$ and $K_1=2^{-4-\lambda}\alpha\lambda(1-\lambda)$. Let $|u(x^1)-u(x^0)|\le h|x^1-x^0|^{\lambda}$ for $x^0, x^1\in\partial\Omega$. Let

$$F=\sup\{d(x)^{2-\lambda}|f(x)|: x\in\Omega\}$$

and $U=\sup\{|u(x^0)|: x^0\in\partial\Omega\}$. Then there exists $\beta^*>0$ such that if Ω is in $H_{\delta,\beta}$ for $\beta\le\beta^*$, $|u(x)-u(x^0)|\le\tau(h+F+K_1\delta_1^{-\lambda}U)|x-x^0|^{\lambda}$ for $x\in\Omega$, $x^0\in\partial\Omega$, where β^* and τ depend only on α, K, λ, m, and δ.

J. Douglas, Jr. (Houston, Tex.)

Akô, Kiyoshi 3748

On Perron's process associated with second order elliptic differential equations. II.

J. Fac. Sci. Univ. Tokyo Sect. I **10**, 81–87 (1964).

This note is a completion and correction of a previous paper of the author [same J. **9** (1962), 165–202; MR **25** #5267].

L. M. Graves (Chicago, Ill.)

olemjak, T. B. 3749

Boundary value problems for a class of quasi-linear equations and systems of elliptic type. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1959**, no. 5 (12), 184–196.

n a bounded domain Ω with sufficiently smooth boundary ' the author considers the equation

$$Lu = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i(x, u, p) + b(x, u, p) = f(x),$$

$$x = (x_1, \cdots, x_n) \in \Omega,$$

$= (p_1, \cdots, p_n)$, $p_i = \partial u/\partial x_i$, where $\sum_{ij} \partial a_i \xi_i \xi_j/\partial p_j \geq m \sum_i \xi_i^2$. 'or $u, \Phi \in W_2^1(\Omega)$ put

$$[Lu, \Phi] = \sum_i \int_\Omega a_i \frac{\partial \Phi}{\partial x_i} d\Omega + \int_\Omega b\Phi \, d\Omega$$

nd suppose that

A) $$[Lu - Lv, u - v] \geq \gamma \sum_i \|\partial(u-v)/\partial x_i\|_{L_2}^2,$$

B) $$[Lu - Lv, \Phi] \leq C\|u - v\|_{W_2^1} \|\Phi\|_{W_2^1}$$

or every $u, v, \Phi \in W_2^1$. Assume $f = f_1 - \sum_i \partial f_{2i}/\partial x_i$, with $_1, f_{2i} \in L_2$, then for the boundary-value problem $Lu = f$, $u|_\Gamma = 0$ there exists a unique generalized solution in $\overset{\circ}{W}{}_2^1$, nd an estimate is given for it. The proof is an application f the continuation method to the problem $-\gamma \Delta u + \lambda(Lu + \gamma \Delta u) = f$, $u|_\Gamma = 0$, $0 \leq \lambda \leq 1$. For equations of the orm

$$Lu = -\sum_{ij} \frac{\partial}{\partial x_i} (a_{ij}(x, u, p)p_j) + b(x, u, p) = f,$$

vith $\gamma \sum_i \xi_i^2 \leq \sum_{ij} a_{ij} \xi_i \xi_j \leq C \sum_i \xi_i^2$ the author considers lso the second boundary-value problem

$$Lu = f, \quad f \in L_2, \quad \sum_{ij} a_{ij} p_i \cos(\nu, x) + \sigma u|_\Gamma = 0.$$

Jnder the conditions (A), (B) and $\sigma \geq 0$, $\sigma \not\equiv 0$, he proves he existence and uniqueness of a generalized solution of his problem and gives an estimate for it. Same results for $\sigma \equiv 0$ if the condition (A) is replaced by

A') $$[Lu - Lv, u - v] \geq \gamma \|u - v\|_{W_2^1}^2.$$

'he approach is the same followed for the first problem. 'he author applies his results to the non-linear problem f plasticity. *L. Cattabriga* (Zbl 107, 306)

lrowder, Felix E. 3750

Variational boundary value problems for quasi-linear elliptic equations of arbitrary order.

Proc. Nat. Acad. Sci. U.S.A. **50** (1963), 31–37.

.et $H^m(\Omega)$ be the Hilbert space of (classes of) L^2 functions n the open subset Ω of R^n whose derivatives of order $\leq m$, in the distributions sense, belong to $L^2(\Omega)$; let $H_0^m(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $H(\Omega)$; $\| \|_m$ is the norm n $H^m(\Omega)$. Consider the form on $H^m(\Omega) \times H^m(\Omega)$, antilinear n v but non-linear in u,

$$a(u, v) = \sum_{|\alpha|, |\beta| \leq m} (a_{\alpha\beta}(x, u, \cdots, D^m u) D^\alpha u, D^\beta v)_{L^2}.$$

'he "coefficients" $a_{\alpha\beta}$ are continuous functions of all heir arguments and, for u in a bounded subset of $H^m(\Omega)$, they remain in a bounded subset of $L^\infty(\Omega)$. The crucial assumption is then the following:

$$\mathrm{Re}[a(u, u - u_1) - a(u_1, u - u_1)] \geq$$
$$g(\sup(\|u\|_m, \|u_1\|_m))\|u - u_1\|_m^2,$$

where $g(t)$ is a non-decreasing function in $1 < t < \infty$ such that $\int_1^\infty g(t)\,dt = +\infty$, and u, u_1 are arbitrary elements of a closed subspace V of $H^m(\Omega)$. As usual, V contains $H_0^m(\Omega)$.

The main result of the paper is then the following: Under the above assumptions, for every $f \in L^2(\Omega)$, there is a unique element $u \in V$ such that $a(u, v) = (f, v)_{L^2}$ for all $v \in V$.

The main tool in the proof is the following result of non-linear functional analysis. Let H be a Hilbert space, D an open subset of H, G a (non-linear) mapping of D into H such that, for any strongly convergent sequence $u_k \to u$ in D, $G(u_k) \to G(u)$ weakly in H, and such that, for some constant $c > 0$,

$$\mathrm{Re}(G(u) - G(u_1), u - u_1) \geq c(u - u_1, u - u_1),$$

where (,) is the inner product in H. Then the image of D under G is open. *F. Treves* (New York)

Browder, Felix E. 3751

Variational boundary value problems for quasi-linear elliptic equations. II.

Proc. Nat. Acad. Sci. U.S.A. **50** (1963), 592–598.

This paper improves and extends the results presented in an article previously reviewed [#3750 above]. The operators studied this time are of the form

$$Au = \sum_{|\alpha| \leq m} D^\alpha [A_\alpha(x, u, Du, \cdots, D^m u)].$$

A suitable assumption is made so as to insure that $A_\alpha(x, u, \cdots, D^m u) \in L^2(\Omega)$ for each $u \in H^m(\Omega)$ (Ω is the region in R^n where the boundary-value problem is posed). The generalized Dirichlet form $a(u, v)$ associated with A is supposed to satisfy a condition

$$\mathrm{Re}[a(u, u - u_1) - a(u_1, u - u_1)] \geq$$
$$c(\sup(\|u\|_m, \|u_1\|_m))\{\|u - u_1\|_m^2$$
$$-k\|u - u_1\|_{m-1}\|u - u_1\|_m\} - k\|u - u_1\|_{m-1}^2,$$

with $k \geq 0$, $c(r) > 0$ non-increasing such that $\int_1^\infty c(r)\,dr = +\infty$. This replaces the positivity assumption of Part I. The reasonings are based on arguments of functional analysis in Hilbert spaces and make use of the notion of "hemi-continuity", introduced here by the author. In Part I, the mappings considered, from an open subset of a Hilbert space H into H, were continuous from strong topology into weak topology on H. Now, it is only required that their restriction to any line segment be continuous (the space of values H still being equipped with the weak topology). Various existence and uniqueness theorems are obtained (for the weak solution to the boundary-value problem). *F. Treves* (New York)

Browder, Felix E. 3752

Variational boundary value problems for quasi-linear elliptic equations. III.

Proc. Nat. Acad. Sci. U.S.A. **50** (1963), 794–798.

The same operators, as in Part II [#3751 above], are

studied, but the basic assumption on the Dirichlet form associated with the operator is replaced by the following:

$$\operatorname{Re}[a(u, u-u_1)-a(u_1, u-u_1)] \geq -k\|u-u_1\|_{m-1}^2.$$

Then an existence theorem can be proved, but the author does not prove uniqueness (for the weak solution to the boundary-value problem associated with the form $a(u, v)$). The functional analysis arguments make use of the notion of hemicontinuity, introduced in Part II.

F. Treves (New York)

Vaghi, Carla 3753

Soluzioni C-quasi-periodiche dell'equazione non omogenea delle onde.

Ricerche Mat. **12** (1963), 195–215.

Let Ω be a bounded domain of the n-dimensional Euclidean space, normal with respect to the Dirichlet problem. (A domain is called normal if the Dirichlet problem for Laplace's equation is solvable for any continuous boundary function.) The author considers for the hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} = Au+f(x,t) \qquad (x \in \Omega,\ 0 \leqq t \leqq l)$$

the following mixed boundary-value problem:

$$u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x) \qquad (x \in \Omega),$$

$$u(x,t)|_{x\in\partial\Omega} = 0 \qquad (\partial\Omega = \text{boundary of } \Omega).$$

A is the operator $Au = \sum_{i,j}^{1,\cdots,n} \partial(a_{ij}(x)\partial u/\partial x_j)/\partial x_i - a(x)u$, whose coefficients are real, measurable and bounded in Ω. Further,

$$a_{ij}(x) = a_{ji}(x), \qquad a(x) \geqq 0,$$

$$\sum_{i,j}^{1,\cdots,n} a_{ij}(x)\xi_i\xi_j \geqq \nu \sum_{i=1}^n \xi_i^2 \qquad (\nu = \text{const} > 0).$$

Il'in [Uspehi Mat. Nauk **15** (1960), no. 2 (92), 97–154; MR **22** #9721] has given conditions on the operator A and the functions f, φ, ψ in order that the above mixed problem be solvable in the classical sense. On the other hand, Amerio [Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **28** (1960), 147–152; ibid. (8) **28** (1960), 322–327; ibid. (8) **28** (1960), 461–466; MR **22** #6927] has obtained for the wave equation results about the almost periodicity of generalized solutions of the mixed boundary-value problem, assuming that $f(x,t)$ is almost periodic. Using these results of Il'in and Amerio, the author obtains conditions (generalizing those given by Il'in) on the operator A and the functions f, φ, ψ so that, for $f(x,t)$ almost periodic, the above boundary-value problem has a classical solution $u(x,t)$, which, together with $u_t(x,t)$ and $u_{tt}(x,t)$, is $C_{\bar\Omega}$ almost periodic (a function $v(x,t)$ is called $C_{\bar\Omega}$ almost periodic if it is almost periodic with respect to t and continuous with respect to $x \in \bar\Omega$). Likewise, it is proved that if Ω' is a domain such that $\bar\Omega' \subset \Omega$, then the functions $u_{x_i}(x,t)$, $u_{x_i t}(x,t)$ and $u_{x_i x_j}(x,t)$ (and hence the derivatives of first and second order with respect to all the variables) are $C_{\bar\Omega'}$ almost periodic.

J. Nieto (College Park, Md.)

Resende, Eliseu 3754

Solution of an initial-boundary value problem for the wave equation.

An. Acad. Brasil. Ci. **36** (1964), 1–6.

Author's introduction: "By using Busemann's conical flow method [A. Busemann, Schr. Deutsch. Akad. Luftfahrtforschung **7B** (1943), 105–121; MR **8**, 415], J. B. Keller and A. Blank [Comm. Pure Appl. Math. **4** (1951), 75–94; MR **13**, 304] obtained an explicit closed expression in terms of elementary functions for the solution of the problem of diffraction and reflection of pulses by perfectly conducting wedges and corners. In this work the author formulates another initial-boundary value problem for the wave equation which is transformed into a characteristic-boundary-value problem and then solved by using the same method."

Phariseau, P. 3755

On the Green's function for the Helmholtz equation.

Simon Stevin **37** (1963/64), 71–74.

The Green's function for $\Delta\psi + k^2\psi = 0$ in a half-space, with $\partial\psi/\partial n = \omega\psi$ (ω constant) and a radiation condition at ∞, is found.

P. Ungar (New York)

Arscott, F. M. 3756

Paraboloidal co-ordinates and Laplace's equation.

Proc. Roy. Soc. Edinburgh Sect. A **66** (1962/63), 129–139 (1964).

Author's summary: "In this paper we examine the general paraboloidal co-ordinate system, in which the normal surfaces are elliptic or hyperbolic paraboloids, including as special cases the 'parabolic plate' and the 'plate with a parabolic hole'. We then show that normal solutions of Laplace's equation in these co-ordinates are given as products of three Mathieu functions, and apply this to the solution of boundary-value problems for Laplace's equation in these co-ordinates."

I. Marx (Lafayette, Ind.)

Gouyon, René 3757

Sur le problème de Dirichlet pour l'équation $\Delta U = \varphi(U)$.

C. R. Acad. Sci. Paris **251** (1960), 26–28.

L'auteur cherche dans l'intérieur D d'un contour plan C la solution $u = u(x,y) = u(z)$ du problème (1) $\Delta u = \varphi(u)$, $u^* = \psi(s)$ où Δ est le laplacien, $\varphi(u)$ est une série entière $\varphi(u) = a_0 + a_1 u + \cdots$ absolument convergente sur $|u| < R$, u^* est la valeur de u sur C et $\psi(s)$ est une fonction continue de l'arc s de C. Le problème modifié suivant (2) $\Delta u = \lambda\varphi(u)$, $u^* = \lambda\varphi(s)$ équivaut à la résolution de l'équation intégrale (3) $u(z) = -(\lambda/2\pi)\iint_D g(z,\zeta)\varphi[u(\zeta)]\,d\sigma + \lambda v(z)$ où $g(z,\zeta)$ est la fonction de Green du domaine D et $v(z)$ est la solution du problème de Dirichlet $\Delta v = 0$, $v^* = \psi(s)$. Cherchons les solutions analytiques en λ, $u = u_0 + \lambda u_1 + \lambda^2 u_2 + \cdots$ de l'équation (3). Puisque celle qui correspond à $\lambda = 0$ est nulle, on a (4) $u = \sum_{n=1}^\infty \lambda^n u_n$ et par suite pour $p = 1, 2, \cdots$ et $\alpha_k = 1, 2, \cdots$,

$$u^p = \sum_{n=p}^\infty \lambda^n s_n^{(p)}, \qquad \text{où } s_n^{(p)} = \sum_{\alpha_1+\cdots+\alpha_p = n} u_{\alpha_1}u_{\alpha_2}\cdots u_{\alpha_p}.$$

En portant la série $\varphi(u) = a_0 + \sum_{n=1}^\infty \lambda^n \sum_{p=1}^n a_p s_n^{(p)}$ dans l'équation (3) on trouve les formules

$$u_1 = -\frac{a_0}{2\pi}\iint_D g(z,\zeta)\,d\sigma + v(z),$$

$$u_{n+1} = -\frac{1}{2\pi}\iint_D g(z,\zeta)\sum_{p=1}^n a_p s_n^{(p)}(\zeta)\,d\sigma$$

pour les termes de la solution (4). La suite de la note est consacrée à l'étude de la validité de la solution formelle (4) se réduisant dans le cas $\lambda = 1$ à la solution du problème (1). *F. Leja* (Zbl 95, 83)

Jakubov, S. Ja. 3758
Investigation of the Cauchy problem for evolution equations of hyperbolic type. (Russian. Azerbaijani summary)
Akad. Nauk Azerbaĭdžan. SSR Dokl. **20** (1964), no. 4, 3–6.

The equation $u''+Au=f(t,u)$ with $u(0)=u_0$, $u'(0)=u_1$ is considered, where $t\to u(t)$ takes values in a Banach space and A is a closed operator. One requires solutions to be twice continuously differentiable on $[0,T]$ with Au and $A^{1/2}u'$ continuous. Some standard results for first-order equations are invoked to obtain existence and uniqueness theorems under suitable, frequently used types of hypotheses. *R. Carroll* (Urbana, Ill.)

Mamedov, Ja. D. 3759
Some properties of the solutions of non-linear equations of hyperbolic type in Hilbert space. (Russian)
Dokl. Akad. Nauk SSSR **158** (1964), 45–48.

The author considers equations of the form $y''+A(t)y+B[t,y']+P(t,y)=0$ and $x''+A(t)x+P(t,x)=0$, where $A(t)$ is a self-adjoint positive operator in a real Hilbert space H, $D(A(t))$ is independent of t, $t\to A(t)$ is strongly differentiable, $(A'(t)x,x)\le\alpha(t)(A(t)x,x)$, $B[t,\cdot]$ is an unbounded operator with $(B[t,x],x)\ge-\frac{1}{2}\alpha(t)\|x\|^2$ for $x\in D(B)$, and $(P(t,x),h)=\lim(1/\lambda)(F(t,x+\lambda h)-F(t,x))$ for $x\in D(A^{1/2})$ defines P, where F is suitable. If F_t exists with $F_t(t,x)\le\alpha(t)F(t,x)$ $(x\in D(A^{1/2}))$, then for $x(0)$, $y(0)$ in $D(A^{1/2}(0))$ estimates are obtained, for example, of the form

$$\|y'\|^2+\|A^{1/2}(t)y\|^2+2F(t,y)\le$$
$$\left(\|y(0)\|^2+\|A^{1/2}(0)y(0)\|^2+2F(0,y(0))\right)\exp\left(\int_0^t\alpha(s)\,ds\right)$$

and various consequences are derived. Then $x''+A(t)x=f(t,x)$ is considered, where $\|f(t,x)-f(t,y)\|\le k(t,r)\|A^{1/2}(t)(x-y)\|$ when $x,y\in S_r$, $S_r=(x:\|A^{1/2}x\|\le r)$ with $\int_0^\infty k(t,r)\,dt<\infty$ and $A(t)$ is as above. For $x_0-y_0\in D(A^{1/2}(0))$ it is asserted that

$$\|x(t)-y(t)\|_E\le c\|x(0)-y(0)\|\exp\left(\int_0^t|\alpha(s)|\,ds\right),$$

where $\|x\|_E=\|x'(t)\|+\|A^{1/2}(t)x(t)\|$, and applications are indicated. Existence questions are then treated under various hypotheses for the above equations. *R. Carroll* (Urbana, Ill.)

Kuznecov, N. N.; Ci, Čžun-Tao [Chi, Chung-tao] 3760a
A uniqueness theorem in the theory of quasi-linear hyperbolic equations. (Russian. English summary)
Vestnik Moskov. Univ. Ser. I Mat. Meh. **1964**, no. 3, 25–30.

Kuznecov, N. N.; Ci, Čžun-tao [Chi, Chung-tao] 3760b
On the uniqueness of the generalized solution of the Cauchy problem for a hyperbolic system of two quasi-linear equations. (Russian. English summary)
Vestnik Moskov. Univ. Ser. I Mat. Meh. **1964**, no. 4, 3–6.

Ces deux notes considèrent le système:

$$\frac{\partial u_i}{\partial t}+\frac{\partial\varphi_i(u,t,x)}{\partial x}=f_i(u,t,x),$$

où φ est deux fois, et f, une fois, continûment dérivable sur $W=K\times[0,T]\times\mathbf{R}$ (K convexe et borné). Le système vérifie sur W une condition d'hyperbolicité un peu plus forte que le fait que les valeurs propres de la matrice d'éléments $\partial\varphi_i/\partial u_k$ sont réelles et distinctes.

Dans la première note, les auteurs démontrent l'unicité pour le problème de Cauchy en ce qui concerne les solutions continûment différentiables par morceaux, restant dans K, à dérivées localement bornées et vérifiant une condition qui concerne la position de chaque ligne de discontinuité par rapport aux caractéristiques.

La seconde note étend ce résultat aux solutions possédant en outre des points singuliers isolés centres d'ondes de détente et de compression (une définition mathématique de ce phénomène aérodynamique est donnée) pourvu que le système soit d'ordre deux et vérifie une condition supplémentaire. *M. Zerner* (Marseille)

Kolesov, Ju. S. 3761
Some existence criteria for stable periodic solutions of quasi-linear parabolic equations. (Russian)
Dokl. Akad. Nauk SSSR **157** (1964), 1288–1290.

Consider the equation $\partial u/\partial t=Lu+f(t,x,u)$, $x\in\Omega$ (bounded), where L is a second-order time-independent elliptic operator and f is periodic in t. Various existence, uniqueness, and stability theorems for periodic solutions $u(x,t)$ vanishing on the boundary Γ are given. The hypotheses on f involve inequalities of the type $f\le au+a_1$ and/or $f\ge bu+b_1$, with a and b compared to the least eigenvalue of $-L$ (with $x=0$ on Γ), and convexity or concavity conditions on f as a function of u. Proofs are omitted. *P. C. Fife* (Minneapolis, Minn.)

Sobolevskiĭ, P. E. 3762
Coerciveness inequalities for abstract parabolic equations. (Russian)
Dokl. Akad. Nauk SSSR **157** (1964), 52–55.

Inégalités de coercivité pour les équations paraboliques abstraites. L'auteur étudie le problème

$$(1)\qquad \frac{dv}{dt}+Av=f,\qquad v(0)=v_0$$

dans l'intervalle $(0,T)$ en supposant que $-A$ est le générateur infinitésimal d'un semi-groupe fortement continu d'opérateurs dans l'espace de Banach E. Pour $\alpha\in]0,1[$, $C_0{}^\alpha(T)$ désigne l'espace des fonctions $t\to w(t)$ continues dans $[0,T]$ à valeurs dans E et telles que

$$\|w\|_{C_0{}^\alpha(T)}=\max_{0\le t\le T}\|w(t)\|_E+\sup_{0<t<t+\Delta t\le T}t^\alpha(\Delta t)^{-\alpha}\|w(t+\Delta t)-w(t)\|_E<+\infty.$$

Le problème (1) est dit coercif dans $C_0^{\alpha}(T)$ si pour tout $f \in C_0^{\alpha}(T)$ et pour tout $v_0 \in D(A)$ il existe une solution v du problème (1), telle que

$$\left\|\frac{dv}{dt}\right\|_{C_0^{\alpha}(T)} + \|Av\|_{C_0^{\alpha}(T)} \leq C(\alpha, T)(\|f\|_{C_0^{\alpha}(T)} + \|Av_0\|_E).$$

L'auteur prouve que le problème (1) est coercif dans $C_0^{\alpha}(T)$ si et seulement si $-A$ est fortement positif, i.e., si $(A+\lambda I)$ est inversible et $\|(A+\lambda I)^{-1}\|_{E\to E} \leq C(|\lambda|+1)^{-1}$ pour $\operatorname{Re}\lambda \geq \sigma_0$. Des notions analogues de coercivité dans des espaces différents de $C_0^{\alpha}(T)$ sont étudiées. L'auteur énonce également des résultats concernant le problème $dv/dt + A(t)v = f$, $v(0) = v_0$ lorsque $D(A(t))$ ne dépend pas de t.

P. Grisvard (Nancy)

Krasnosel'skiĭ, M. A.; Sobolevskiĭ, P. E. 3763

The structure of the set of solutions of an equation of parabolic type. (Russian. English summary)

Ukrain. Mat. Ž. **16** (1964), 319–333.

The authors present proofs of results announced earlier [Dokl. Akad. Nauk SSSR **146** (1962), 26–29; MR **27** #2730].

The main result of the paper can be formulated as follows: If the mixed boundary-value problem for a nonlinear parabolic equation has two solutions, it has a continuum of solutions.

W. Bogdanowicz (Washington, D.C.)

Browder, Felix E. 3764

Strongly non-linear parabolic boundary value problems.

Amer. J. Math. **86** (1964), 339–357.

The main results of this paper were communicated in the research note [Bull. Amer. Math. Soc. **69** (1963), 858–861; MR **27** #6049]. The equation under consideration is

$$\frac{\partial u}{\partial t} + \sum_{|\alpha|\leq m} D^{\alpha}A_{\alpha}(x, t, u, \cdots, D^m u) = f(x, t).$$

Extending his earlier results on existence of weak solutions by Hilbert-space "monotonicity" methods (in the case where the A_α have at most linear growth as the arguments tend to infinity), the author now attacks the case of "at most polynomial" growth by corresponding L^p-space methods.

Notation: Ω is an open set in R^n, and D_j is $i^{-1}\partial/\partial x_j$; also, u, v are functions over Ω with range in C^r (i.e., complex vector-valued functions). The symbol $\langle u, v\rangle$ stands for $\int_\Omega \sum_{j=1}^r u_j \bar{v}_j$. Also

$$A(t)u = \sum_{|\alpha|\leq m} D^{\alpha}A_{\alpha}(x, t, u, \cdots, D^m u).$$

In the Sobolev space $W^{m,p}(\Omega)$, V stands for a closed linear subspace with $C_0^{\infty}(\Omega) \subset V$, where C_0^{∞} are the C^{∞} functions over Ω satisfying the "boundary condition" of compact support in Ω. The (non-linear) Dirichlet form h is defined by

$$h(u, v) = \sum_{|\alpha|\leq m} \int_{t_0}^{t_1} \langle A_{\alpha}(x, t, u, \cdots, D^m u(t)), D^{\alpha}v(t)\rangle\, dt.$$

The closed line-segment $[t_0, t_1]$ in R^1 is called S, and for given p, the relation $p^{-1} + q^{-1} = 1$ defines q.

It is assumed that A satisfies, besides measurability and continuity hypotheses, the following growth condition on the coefficients: for some $p > 1$, there exist $C > 0$ and function $g \in L^q$ such that for all (x, t) in $\Omega \times R^1$ and eac complex vector $\zeta = \{\zeta_\beta : |\beta| \leq m\}$,

$$|A_{\alpha}(x, t, \zeta)| \leq C \sum_{|\beta|\leq m} |\zeta_\beta|^{p-1} + g(x, t).$$

The main theorem then states that if $A(t)$ also satisfi the "monotonicity" requirement (a) for all u, v in $L^p(S, V$ $\operatorname{Re}\{h(u, u-v) - h(v, u-v)\} \geq 0$, and the "coercivenes requirement (b) there exists a function $c(r) > 0$ wit $c(r) \to +\infty$ as $r \to +\infty$ such that $\operatorname{Re} h(u, v) \geq c(\|u\|)\|u\|$ f all u in $L^p(S, V)$, then the system $\partial u/\partial t + A(t)u = f$ h one and only one weak solution for given $f \in (L^p(S, V))$ The solution is continuous from S to $L^2(\Omega)$ and $u(t_0) =$

The proof of this theorem rests on a rather complicate "monotonicity/complete continuity" theorem in an a stract reflexive separable Banach space; this theorem wi not be quoted here, and is closely related to a theorem Leray and Lions, quoted (without reference) in C. Morrey's multiprinted lecture notes "Multiple integrals i the calculus of variations" for the Amer. Math. So Colloq. Lectures given August 1964 at Amherst, Mas The reader interested in the regularity of the wea solution should refer to Morrey's notes.

G. J. Minty (New Yor

Perel'man, T. L.; Ryvkin, V. B. 376

Uniqueness of the solution of a conjugate problem i heat transfer. (Russian)

Dokl. Akad. Nauk BSSR **8** (1964), 365–368.

The following problem is considered. The functions $\theta(x,$ and $t(x, y)$ are unknown with $u\theta_x = \chi\theta_{yy}$, $0 \leq x <$ $0 \leq y < \infty$, u and χ constant, $t_{xx} + t_{yy} = -(1/k_s)Q(x, y$ $0 \leq x < \infty$, $-\infty < y \leq 0$, where also k_s is constant. Fo $y = 0$, $x \geq 0$, one sets $\theta|_{y=0^+} - t|_{y=0^-} = f(x)$, an $-k_f\theta_y|_{y=0^+} = -k_s t_y|_{y=0^-} + q(x)$ with $\theta|_{x=0} = 0$, $\theta|_{y=\infty} =$ and $t|_{x=0} = 0$. Some uniqueness theorems are indicated.

R. Carroll (Urbana, Ill

Van Tun 376

Theory of the heat potential. I. Level curves of hea potentials and the inverse problem in the theory of th heat potential. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 660–670.

Let $G(x, t; \xi, \tau)$ denote the fundamental solution o $u_{xx} - u_t = 0$. $W(x, t) = \int_0^t \mu(t)[\partial G(x, t; \varphi(\tau), \tau)/\partial\xi]\, d\tau$ is th double-layer potential corresponding to a density $\mu(t)$ o a curve $x = \varphi(t)$. The author proves that if $\varphi(t)$, $\mu(t)$ ar sufficiently smooth, then the same is true of $W(\varphi(t), t)$ Also, if $\varphi(t)$ and $W(x, t)$ are sufficiently smooth (uniforml for $x > \varphi(t)$), then the same is true of $\mu(t)$. Similar result are established for single-layer potentials.

A. Friedman (Evanston, Ill.

Arima, Reiko; Hasegawa, Yôjiro 376

On the existence of a global true solution in the mixe problem concerning a certain type of semi-linear partia differential equation. (Japanese)

Sûgaku **15** (1963/64), 161–165.

Let $x = (x_1, x_2, x_3) \in R^3$ and $t \in R^1$. In this paper, th following mixed problem for a scalar function $u(x, t)$ i

considered in the domain: $0 \leq x_1 < +\infty$, $-\infty < x_2, x_3 < +\infty$, $0 \leq t < +\infty$:

$$u_{tt} - (\Delta u)_t = f(u)u_t + g(u)$$

with initial conditions $u(x, 0) = u_0(x)$, $u_t(x, 0) = u_1(x)$ and boundary conditions: $u(0, x_2, x_3, t) = \psi(x_2, x_3, t)$. The authors prove the following result. Under the conditions given below, the mixed problem given above has a true solution $u(x, t)$ on the domain: $0 \leq x_1 < +\infty$, $-\infty < x_2, x_3 < +\infty$, $0 \leq t < +\infty$, and $u(x, t)$ satisfies for $0 \leq t < +\infty$ the conditions:

$$t \to u(x, t), \quad u_t(x, t) \in \mathscr{B}_+{}^1(x) \cap \mathscr{D}^2_{L^2+}(x),$$
$$t \to u_{x_ix_i}(x, t), \quad u_{tx_ix_i}(x, t) \in \mathscr{B}_+{}^0(x),$$

where $\mathscr{B}_+{}^k(x)$ is the set of all functions whose derivatives with respect to x of orders $\leq k$ are uniformly continuous and bounded for $x_1 \geq 0$, $-\infty < x_2, x_3 < +\infty$, $\mathscr{D}^k_{L^2+}(x)$ is the set of all functions whose derivatives with respect to x of orders $\leq k$ are in $L^2(x)$ in the same domain, and $t \mapsto u(t, x) \in \mathscr{B}_+{}^k(x) \cap \mathscr{D}^k_{L^2+}(x)$ means that the mapping $t \mapsto u(t, x)$ is continuous with respect to the topologies of $\mathscr{B}_+{}^k(x)$ and $\mathscr{D}^k_{L^2+}(x)$. The conditions on f and g are that (i) $f, g \in C^1$ and $g(0) = 0$; (ii) $f(u) \leq L$, $|f(u)| \leq k_1(u^2+1)$,

$$\text{(iii) } |g(u)| \leq k_2(u^2 + |u|), \qquad G(u) = \int_0^u g(z)\,dz \leq k_3 u^2,$$

where L, k_i are positive constants; the conditions on the initial values are that (i) $u_0, u_1 \in \mathscr{B}_+{}^1(x) \cap \mathscr{D}^2_{L^2+}(x)$, (ii) $u_{0x_ix_i}, u_{1x_ix_i} \in \mathscr{B}_+{}^0(x)$; and the conditions on the boundary values are that

(i) $\quad t \to \psi_t(x_2, x_3, t) \in \mathscr{B}^2(x_2, x_3) \cap \mathscr{D}_{L^2}{}^2(x_2, x_3)$,

(ii) $\quad t \to \psi_{tt}(x_2, x_3, t) \in \mathscr{B}^0(x_2, x_3) \cap \mathscr{D}_{L^2}{}^0(x_2, x_3)$.

The compatibility conditions are that (i) $u_0(0, x_2, x_3) = \psi(x_2, x_3, 0)$, (ii) $u_1(0, x_2, x_3) = \psi_t(x_2, x_3, 0)$, (iii) $\psi_{tt}(x_2, x_3, 0) - \Delta u_1(0, x_2, x_3) = f(\psi(x_2, x_3, 0))\psi_t(x_2, x_3, 0) + g(\psi(x_2, x_3, 0))$. Another similar result is also proved for the case $x \in R^1$ (the one-dimensional case).

The authors first prove the existence of a local solution by the use of successive approximations which can be constructed by using the elementary solution of the heat equation. Then estimates of the local solution and its derivatives with respect to t are derived. By virtue of these estimates, the existence of the global solution is proved.

A particular example of the mixed problems of this kind is given by

$$u_{tt} - u_{txx} = -\mu(1 + u + \varepsilon u^2)u_t - u,$$
$$u(x, 0) = 0, \qquad u_t(x, 0) = 0, \qquad u(0, t) = \psi(t) \in C^\infty,$$

where $x \in R^1$, $\mu > 0$, and $3/16 > \varepsilon > 0$. This particular problem is used to simulate the propagation of the stimulus in the nervous system. The authors' results guarantee the existence of a global solution of this problem.

Y. Sibuya (Minneapolis, Minn.)

Smirnov, M. M. 3768

Some boundary-value problems for an equation of mixed composite type. (Russian)

Sibirsk. Mat. Ž. 5 (1964), 923–928.

Two problems are posed for the equation

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y^2} + \alpha\frac{\partial u}{\partial y}\right) = 0$$

($\alpha > 1$) in a domain bounded by the closed segment from $A(-1, 0)$ to $B(1, 0)$ and a curve Γ joining A to B which lies in the upper half-plane and has its maximum ordinate at $N(0, h)$. In the first, values of u are assigned as functions of x on AB, Γ, and of y on ON; in the second, $\lim_{y=0} y^\alpha \partial u/\partial y$ is given as a function of x in $(-1, 1)$, with u as before prescribed on Γ and ON. For each case uniqueness is proved with the use of a maximum principle and existence (for the normal contour Γ_0: $x^2 + 4y = 1$, $y \geq 0$) by reduction to Fredholm integral equations.

R. N. Goss (San Diego, Calif.)

Mel'nik, Z. O. 3769

On a general mixed problem. (Russian)

Dokl. Akad. Nauk SSSR 157 (1964), 1039–1042.

The author claims an existence and uniqueness theorem for a mixed initial-boundary-value problem for a linear hyperbolic integro-differential equation of the form ($a_{m0} = 1$)

$$\sum_{i+j\leq m} a_{ij}(x, t)\frac{\partial^{i+j}u(x, t)}{\partial t^i \partial x^j}$$
$$+ \int_0^t \sum_{i+j\leq m-1} b_{ij}(x, t, \tau)\frac{\partial^{i+j}u(x, \tau)}{\partial t^i \partial x^j}\,d\tau = f(x, t).$$

Initial-values are given on $\{(x, 0); 0 \leq x \leq 1\}$ and boundary conditions on $\{(0, t); 0 \leq t \leq T\}$ and $\{(1, t); 0 \leq t \leq T\}$. The number of boundary conditions on $x = 0$ and $x = 1$ are determined by the number of $\lambda_j(x, t)$ in

$$\sum_{i+j=m} a_{ij}(x, t)\lambda^i \xi^j = \prod_{j=1}^m (\lambda - \lambda_j(x, t)\xi)$$

which are negative and positive, respectively. The boundary conditions themselves are linear integro-differential equations of order $m - 1$ which are assumed to satisfy linear independence conditions of conventional type. The proof (not given in detail in the note) is by reduction to a first-order system and thereafter by the method of characteristics.

V. Thomée (Göteborg)

Finn, Robert 3770

Estimates at infinity for stationary solutions of the Navier-Stokes equations.

Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) 3 (51) (1959), 387–418.

The author studies the behaviour in the neighbourhood of infinity of the solutions of the three-dimensional, stationary non-linear system of Navier-Stokes equations

$$(1) \qquad \Delta \mathbf{w} - w_k\frac{\partial \mathbf{w}}{\partial x_k} - \operatorname{grad} p = 0, \qquad \operatorname{div} \mathbf{w} = 0.$$

The basic results may be formulated in terms of the following two theorems.

Theorem A. Let $\mathbf{w}(\mathbf{x})$, $p(\mathbf{x})$ be a solution of (1) defined in a neighbourhood of infinity, and let there exist a constant vector $\mathbf{w}_0 \neq 0$ such that $|\mathbf{w}(\mathbf{x}) - \mathbf{w}_0| \to 0$ as $|\mathbf{x}| \to \infty$. Furthermore, let $m(r) = \max_{|\mathbf{x}|\geq r} |\mathbf{w}(\mathbf{x}) - \mathbf{w}_0|$,

$$\gamma(r) = \max_{|\mathbf{x}|\geq r} |\nabla \cdot \mathbf{w}|, \qquad \zeta(r) = \max_{|\mathbf{x}|\geq r} |\nabla p|, \qquad r \gg 1.$$

Then $\gamma(r)$ and $\zeta(r)$ are finite and satisfy the inequalities

$$\gamma(r) < C_1m(r-1), \qquad \zeta(r) < C_2m(r-1),$$

$$p(\mathbf{x}) = O(\log r), \qquad r \to \infty.$$

In addition, the author obtains integral representations for the velocity $\mathbf{w}(\mathbf{x})$ and the pressure $p(\mathbf{x})$ in terms of the fundamental tensor of Oseen.

Theorem B. Suppose that there exist a vector $\mathbf{w}_0 \neq 0$ and constants C and $\varepsilon > 0$ such that $|\mathbf{w}(\mathbf{x}) - \mathbf{w}_0| < Cr^{-1/2-\varepsilon}$, $r = |\mathbf{x}| \to \infty$. Then the velocities of the flow tend to the limit $\mathbf{w}_0$ in the following way. There exists a paraboloidal region with axis $\mathbf{w}_0$ inside which $|\mathbf{w}(\mathbf{x}) - \mathbf{w}_0| < C_1r^{-1}$ (this bound cannot be improved), and outside a certain semi-infinite cone with axis $\mathbf{w}_0$ containing the above paraboloid the stronger inequality $|\mathbf{w}(\mathbf{x}) - \mathbf{w}_0| < C_2r^{-2}$ holds. In addition, the author obtains asymptotic representations for the velocity $\mathbf{w}(\mathbf{x})$ and the pressure $p(\mathbf{x})$, whence it is found that the principal terms in the asymptotic development are determined by the fundamental tensor of Oseen.

The results obtained in Theorem B are used to determine the forces exerted on the surface Σ by the flow, whose velocity tends to the limit $\mathbf{w}_0$ as $|\mathbf{x}| \to \infty$, where these forces are also determined completely by the solutions of the linearized Oseen equations

$$\Delta\mathbf{u} - (\mathbf{w}_0)_k \frac{\partial \mathbf{u}}{\partial x_k} - \operatorname{grad} p = 0, \qquad \operatorname{div} \mathbf{u} = 0, \; \mathbf{u} = \mathbf{w} - \mathbf{w}_0.$$

A. Oskolkov (RŽMat **1963** #11 B308)

Finn, Robert 3771

Estimates at infinity for steady state solutions of the Navier-Stokes equations.

Proc. Sympos. Pure Math., Vol. IV, *pp.* 143–148. *American Mathematical Society, Providence, R.I.*, 1961.

This is a summary of the results in a paper of the author [#3770 above], which, though dated earlier, apparently appeared later than the present summary. As descriptions of the proofs are also given, it is recommended for the reader who wishes to obtain a knowledge of the wealth of material in that paper without perusing the details.

P. C. Fife (Minneapolis, Minn.)

Finn, Robert 3772

An energy theorem for viscous fluid motions.

Arch. Rational Mech. Anal. **6** (1960), 371–381.

This paper treats bounded solutions $\mathbf{w}$ of the steady-state Navier-Stokes system $\Delta\mathbf{w} - \mathbf{w}\cdot\nabla\mathbf{w} - \nabla p = 0$, $\nabla\cdot\mathbf{w} = 0$ in a neighborhood $\mathscr{E}$ of infinity in 3-dimensional space. The primary concern is whether the kinetic energy of disturbance $K(\mathbf{w}_0) = \int_{\mathscr{E}} |\mathbf{w} - \mathbf{w}_0|^2\,dv$ is finite for some constant $\mathbf{w}_0$. If it is, then the Dirichlet integral is finite; and if, furthermore, $\mathbf{w}_0 \neq 0$, then $|\mathbf{w} - \mathbf{w}_0| < C|\mathbf{x}|^{-3/2+\varepsilon}$ for every $\varepsilon > 0$. The main result of the paper is simply stated: if $\mathbf{w}$ vanishes on the boundary of $\mathscr{E}$ but is not identically zero, then $K(\mathbf{w}_0) = \infty$ for every choice of $\mathbf{w}_0$. A further result is a Liouville theorem stating that an entire bounded solution with finite $K(\mathbf{w}_0)$ is a constant.

P. C. Fife (Minneapolis, Minn.)

Finn, Robert 3773

On the steady-state solutions of the Navier-Stokes equations. III.

Acta Math. **105** (1961), 197–244.

L'autore presenta un esteso studio del sistema stazionario di Navier-Stokes:

$$(*) \qquad \mu\Delta\mathbf{w} - \rho(\mathbf{w}\cdot\nabla)\mathbf{w} - \nabla p = 0, \qquad \operatorname{div} \mathbf{w} = 0.$$

Nella prima parte l'autore dà valutazione per l'integrale di Dirichlet (problema interno ed esterno) e dimostra teoremi di esistenza. I risultati trovati sono vicini a quelli dati da O. A. Ladyženskaja [Uspehi Mat. Nauk **14** (1959), no. 3 (87), 75–97; MR **22** #10437], per quanto riguarda il problema interno; nel caso del problema esterno (cfr. il lavoro di Fujita [J. Fac. Sci. Univ. Tokyo Sect. I **9** (1961), 59–102; MR **24** #A2152] e #3770) viene portato questo miglioramento: l'esistenza sussiste se il dato sulla frontiera ha flusso non nullo, purchè sufficientemente piccolo. L'autore, non fa uso di soluzioni generalizzate e dimostra direttamente l'esistenza di soluzioni di tipo classico, il cui gradiente soddisfa ad una condizione di Hölder uniforme. Fra l'altro, viene esposta diffusamente la costruzione di una funzione $w = \operatorname{rot}\psi$ che assume prescritti valori sulla frontiera del dominio.

L'autore esamina poi il comportamento all'infinito delle soluzioni, portando qualche perfezionamento ai suoi precedenti risultati [Arch. Rational Mech. Anal. **3** (1959), 381–396; MR **21** #6167]. Segue un esame critico dei risultati e l'enunciazione di problemi che rimangono aperti. (Tra questi, come è noto, è il problema della unicità della soluzione: per quanto l'unicità di soluzioni "grandi" sia scarsamente attendibile, nessun esempio contrario è attualmente conosciuto.) L'autore nota anche che tutte le dimostrazioni finora date del teorema di esistenza per il problema esterno sono, in certo senso, innaturali perchè ottenute mediante passaggio al limite, a partire da soluzioni di problemi interni.

L'ultima parte del lavoro riguarda l'esame del divario che sussiste tra le soluzioni del sistema di Navier-Stokes e quelle del sistema linearizzato

$$(**) \qquad \mu\Delta\mathbf{u} - \nabla q = 0, \qquad \operatorname{div} \mathbf{u} = 0$$

assumenti gli stessi valori al contorno. Indicando con $\mathbf{w}(\mathbf{x}, \lambda)$ la soluzione (unica se λ è piccolo) del sistema (*) assumente il valore $\lambda\mathbf{w}^*$ sul contorno, con $\mathbf{u}_0$ la soluzione del sistema (**) assumente il valore $\mathbf{w}^*$ sul contorno, si trova, sotto certe condizioni di regolarità per i dati:

$$\left|\frac{1}{\lambda}\mathbf{w}(\mathbf{x}, \lambda) - \mathbf{u}_0(x)\right| < c\lambda \qquad \text{(essendo } c \text{ una costante).}$$

Una analoga limitazione, ma meno forte, vale per il problema esterno.

G. Prodi (Trieste)

Fujita, Hiroshi; Kato, Tosio 3774

On the Navier-Stokes initial value problem. I.

Arch. Rational Mech. Anal. **16** (1964), 269–315.

Many theories of the initial-value problem for the Navier-Stokes system have been developed. The present paper is an especially careful and readable account of the authors' version, which is similar in approach and in results to that of Sobolevskiĭ [Dokl. Akad. Nauk SSSR **128** (1959), 45–48; MR **22** #1763; ibid. **131** (1960), 758–760; MR **25** #3282; ibid. **155** (1964), 50–53; MR **28** #4258; ibid. **156** (1964), 745–748; MR **29** #1462]. The essence of the

authors' theory, applied only to a slightly more abstract form of the Navier-Stokes system, was given in Rend. Sem. Mat. Univ. Padova 32 (1962), 243–260 [MR 26 #495]; for the basic existence and uniqueness results, see that review. The present paper simplifies the proofs, presents them in greater detail, extends the results to the system in its classical form, and provides detailed results concerning regularity in the interior, up to the boundary, and as $t \to 0$. For example, if the forcing term is a C^∞ function of t with range in $L_q(D)$ (D is the space domain and $q > 3$), then the solution u and ∇u are in C^∞ with respect to the variable t. A future paper is promised which will include an extension to the case of unbounded D (with bounded complement), and also an extension from this L_2 to an L_p theory. *P. C. Fife* (Minneapolis, Minn.)

Guberman, I. Ja. 3775

On the existence of several solutions of the Dirichlet problem for an equation with a Monge-Ampère operator. (Russian)

Leningrad. Gos. Ped. Inst. Učen. Zap. **238** (1962), 132–140.

The Monge-Ampère equation $rt - s^2 = (1 + p^2 + q^2)^\alpha \times f(x, y, z, p, q)$ with boundary condition $z = 0$ on Γ, where $0 \leq \alpha \leq 1$, Γ a convex curve, and where f is a non-negative and bounded function (with respect to z), was considered by Bakel'man and Krasnosel'skiĭ [Dokl. Akad. Nauk SSSR **137** (1961), 1007–1010; MR **25** #4265]. The author studies the case $\alpha > 1$ and shows by methods similar to those in the cited paper the existence of at least countably many solutions by imposing certain conditions on the growth of the function $f(z)$. Related references are given in Bakel'man [ibid. **114** (1957), 1143–1145; MR **20** #1983; Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. **13** (1958), no. 1, 25–38; MR **20** #3384].

V. Linis (Ottawa, Ont.)

Bakel'man, I. Ja. 3776

A variational problem associated with the Monge-Ampère equation. (Russian)

Leningrad. Gos. Ped. Inst. Učen. Zap. **238** (1962), 119–131.

The author studies the variational problem for the functional

$$K(u) = \iint_\Omega [u_x^2 u_{yy} - 2u_x u_y u_{xy} + u_y^2 u_{xx} + 6\phi(x, y)]\, dx dy$$

for which the corresponding formal Euler's equation reduces to the Monge-Ampère equation $u_{xx}u_{yy} - u_{xy}^2 = \phi(x, y)$. In the case $\phi(x, y) > 0$ in Ω (a region with sufficiently smooth boundary of positive curvature bounded from below), the Monge-Ampère equation is of elliptic type; the solutions are convex and unique (in a half-space). The author remarks that $K(u)$ is unbounded in the class of functions $C^2(\Omega)$ and that the Euler's equation is degenerate, its order being two instead of four. The method for obtaining the solution consists of considering $K(u)$ in the region $\Omega - \Omega_\varepsilon$ obtained from Ω by deleting an ε-wide strip along the boundary, and then letting $\varepsilon \to 0$. For the class of totally additive non-negative functions defined on Borel sets in Ω and vanishing on Ω, there exists two solutions of the variational problem; both are convex and one each in opposite half-spaces [cf. the author, Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. **13** (1958), no. 1, 25–38; MR **20** #3384; Uspehi Mat. Nauk **15** (1960), no. 1 (91), 163–170; MR **22** #A1918].

V. Linis (Ottawa, Ont.)

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

See also 3615, 3835, 4631.

Kuczma, M. 3777

Sur une équation aux différences finies et une caractérisation fonctionnelle des polynômes.

Fund. Math. **55** (1964), 77–86.

Let $\Delta_h^n f(x)$ denote the usual nth difference of a function f, with difference interval h, and let $[x_1, x_2, \cdots, x_{n+1}; f]$ denote the divided difference of order n. Let $J_+^n(a, b)$ denote the class of functions such that $\Delta_h^{n+1} f(x) \geq 0$ for $h > 0$ and $x \in (a, b)$, $x + (n+1)h \in (a, b)$. Similarly, define $J_-^n(a, b)$ by $\Delta_h^{n+1} f(x) \leq 0$, and define $J^n(a, b)$, called the class of convex functions of order n in the sense of Jensen, by the relation $J^n = J_+^n \cup J_-^n$. Let $M_+^n(a, b)$ denote the class of functions such that $[x_1, \cdots, x_{n+2}; f] \geq 0$ for all distinct $x_1, \cdots, x_{n+2}$ in (a, b), and let $M_-^n(a, b)$ denote the class such that $[x_1, \cdots, x_{n+2}; f] \leq 0$. The class $M^n(a, b) = M_+^n(a, b) \cup M_-^n(a, b)$ was introduced by Popoviciu [Mathematica (Cluj) 8 (1934), 1–85], and is here called the class of convex functions of order n.

The author now proves that if $\varphi \in M^n(a, \infty)$, $n \geq 0$, and if φ satisfies $\lim_{x\to\infty} \Delta_1^n \varphi(x) = 0$, then for each pair of real numbers x_0, y_0 there is exactly one solution g of (1) $g(x+1) - g(x) = \varphi(x)$ such that g satisfies the condition $g(x_0) = y_0$ and g is in class $M^n(a, \infty)$. A formula for g is given. This theorem generalizes previous results of the author and of W. Krull [Math. Nachr. **1** (1948), 365–376; MR **11**, 112]. From this theorem is derived a functional characterization of the class P^n of polynomials of degree at most n, namely, that P^n is identical with the class of functions in M^n for which $\Delta_1^{n+1} f(x) = 0$ for $x \in (a, \infty)$.

K. L. Cooke (Claremont, Calif.)

Ta Van Din' [Ta Wang Ting] 3778

On the correctness of difference problems. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 308–312.

A connection is established between the correctness of posing a problem and the Fillipov stability of non-linear and linear difference equations.

H. P. Thielman (Alexandria, Va.)

Howroyd, T. D. 3779

The solutions of some functional equations.

Canad. Math. Bull. 7 (1964), 279–282.

From the author's introduction: "In this paper we are concerned with functional equations of the type

$$(1) \qquad f(x+y) = F[f(x), f(y), f(x-y)]$$

in which x, y do not appear explicitly. J. Aczel [*Vorlesungen über Funktionalgleichungen und ihre Anwendungen*, Chapter 2, Birkhäuser, Basel, 1961; MR **23** #A1959] has given a method for finding real solutions of some of

these equations. We prove a theorem which can sometimes be used to solve problems concerning the uniqueness of solutions of such equations."

Mitrinović, Dragoslav S. 3780

Sur les équations fonctionnelles linéaires paracycliques de seconde espèce. (Serbo-Croatian summary)

Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hrvatske Ser. II 18 (1963), 177–182.

The author considers the functional equation

$$(*)\quad f_1(x,y,u)+f_2(y,x,v)+f_3(x,u,v)+f_4(y,v,u) = 0,$$

where $f_1, \cdots, f_4$ are 3-place functions from an arbitrary set E to an Abelian group M which has the property that the equation $mX=A$ has a unique solution X for any A in M and any integer m. He shows that the general solution of (*) has the form

$$\begin{aligned} f_1 &= F_1(I,K)+G_1(J,K)+H_1(I,J),\\ f_2 &= F_2(J,K)+G_2(I,K)-H_1(J,I),\\ f_3 &= -F_1(I,J)-F_2(I,K)+H_2(J,K),\\ f_4 &= -G_1(I,K)-G_2(I,J)-H_2(K,J), \end{aligned}$$

where $F_1, \cdots, H_2$ are arbitrary 2-place functions from E to M and I, J, K are the 3-place selectors on E, i.e., for any u, v, w in E, $I(u,v,w)=u$, $J(u,v,w)=v$, $K(u,v,w)=w$. He also shows that if $f_1=f_2=f_3=f_4=f$, then $f=g(I,J)-g(J,I)+g(J,K)-g(K,J)+g(K,I)-g(I,K)+h(I)-2h(J)+h(K)$, where g and h are arbitrary functions from E to M. *B. Schweizer* (Tucson, Ariz.)

Vincze, E. 3781

Eine allgemeinere Methode in der Theorie der Funktionalgleichungen. III, IV.

Publ. Math. Debrecen 10 (1963), 191–202; *ibid.* 10 (1963), 283–318.

Part II appeared in same Publ. 9 (1962), 314–323 [MR 27 #2749]. The following functional equations are solved.

$$(1)\quad F(z_1+z_2) = G(z_1)H(z_2)+K(z_1)L(z_2);$$

$$(2)\quad F(z_1+z_2) = F(z_1)+F(z_2)+G(z_1)H(z_2)+G(z_2)H(z_1);$$

$$(3)\quad F(z_1+z_2) = F(z_1)+F(z_2)+G(z_1)G(z_2)+H(z_1)H(z_2).$$

Here, all functions have as their domain an abelian group and have complex values. The solutions are expressed in terms of the solutions of

$$\begin{aligned} \phi(z_1+z_2) &= \phi(z_1)+\phi(z_2),\\ \psi(z_1+z_2) &= \psi(z_1)+\psi(z_2). \end{aligned}$$

Equations (1), (2), (3) have, respectively, 6, 5, and 6 essentially different types of solutions. *A. Nijenhuis* (Philadelphia, Pa.)

Aczél, J.; Vincze, E. 3782

Über eine gemeinsame Verallgemeinerung zweier Funktionalgleichungen von Jensen.

Publ. Math. Debrecen 10 (1963), 326–344.

En partant d'une méthode simple qu'ils utilisent pour la résolution de l'équation:

$$(a)\quad f(x+y)+f(x-y)-2f(x) = 2|y|g(x),$$

les auteurs étudient par la même méthode l'équation plus générale

$$(b)\quad f(x+y)+f(x-y)-2f(x) = 2g(x)h(y).$$

Cette dernière équation est étudiée aussi dans toute sa généralité quand les fonctions f et g sont définies sur un groupe abélien, et ont des valeurs complexes. L'équation (b) contient deux cas particuliers de J. L. W. V. Jensen

$$f\left(\frac{u+v}{2}\right) = \frac{f(u)+f(v)}{2}$$

et

$$f(x+y)+f(x-y)-2f(x) = 2f(y).$$

A. Haimovici (Iaşi)

Sakovič, G. N. 3783

Functional equations for sums of exponentials. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR 1964, 714–718.

Functions of the form $f(x)=\sum P_i(x)e^{a_ix}$, where $P_i(x)$ are polynomials, can be characterized as the solutions of linear differential equations with constant coefficients. The present author points out that these functions can also be characterized as the solutions of some functional equations, of which $f(x+y)=\sum_{i,j=1}^{n} a_{ij}f_i(x)f_j(y)$ is a typical example. {It is not stated under what regularity conditions this is true; some conditions, though very weak, are necessary. E.g., it would be enough to assume that $f(x)$ is measurable.} The proof is outlined.

The author's references might be supplemented by W. Eichhorn [Arch. Math. 14 (1963), 266–270; MR 27 #3956] and F. Radó [Mathematica (Cluj) 4 (27) (1962), 131–143; MR 27 #2667]. The latter paper covers one of the four cases considered by the present author. *M. Kuczma* (Katowice)

SEQUENCES, SERIES, SUMMABILITY

See also 3417, 3615, 3644–3646, 3683, 3855, 4175.

Bhatnagar, P. L.; Srinivasienagr, C. N. [Srinivasiengar, C. N.] 3784

★**The theory of infinite series.**

National Publishing House, Delhi, 1964. vii+196 pp. Rs. 10.00.

This monograph is at about the same level as, but on the whole more detailed, in those topics that both books cover, than Hirschman's *Infinite series* [Holt, Rinehart and Winston, New York, 1962; MR 26 #510]. It includes a number of rather refined theorems that do not usually appear in a short monograph. The style is rather old-fashioned; more advantage could have been taken of modern terminology and points of view. Contents: Infinite sequences; infinite series; series of positive terms; series in general; derangement of series; double series; multiplication of series; uniform convergence; real power series; infinite products; Tannery's theorems; complex series and products; transformation of a convergent series into a more rapidly convergent series; summability processes. The phrase "Tannery's theorems" will be unfamiliar to most readers nowadays. These theorems

deal with limits of uniformly convergent sums and products of a special form.

R. P. Boas, Jr. (Evanston, Ill.)

Diananda, P. H. 3785

On rearrangement of series. IV.

Colloq. Math. **12** (1964), 85–86.

Part III appeared in same Colloq. **10** (1963), 287–288 [MR **27** #5060]. This paper is an addendum to Part II of the author's paper of the same title [ibid. **9** (1962), 277–279; MR **27** #5059]. He denotes there by (1), (2), (3), (5) and (6) the following relations: (1) $na_n \to 0$ as $n \to \infty$; (2) $(a_{N_1} + \cdots + a_{N_n}) - (a_1 + \cdots + a_n) \to 0$ as $n \to \infty$; (3) $\sum_{r > n \geq N_r} 1/N_r = O(1)$ as $n \to \infty$; (5) $\sum_{r \leq n < N_r} 1/N_r \to 0$ as $n \to \infty$; (6) $\sum_{r > n \geq N_r} 1/N_r \to 0$ as $n \to \infty$. In this addendum he states the following theorem: A necessary and sufficient condition (NSC) that (5) be true for every (a) series, (b) divergent series, satisfying (6) and (7) $\sum_{r \leq n < N_r} a_{N_r} = o(1)$ as $n \to \infty$ is (3). If (5) is replaced by (1) an NSC is (8) $\sum_{r > n \geq N_r} 1/N_r = o(1)$ as $n \to \infty$. If (6) and (7) are replaced by (2) an NSC is (3). Also (1) is necessarily satisfied. Relation (8) is an NSC that (1) be true for every (a) series, (b) divergent series, satisfying (6).

J. W. Andrushkiw (S. Orange, N.J.)

Banerjee, C. R.; Lahiri, B. K. 3786

On subseries of divergent series.

Amer. Math. Monthly 71 (1964), 767–768.

The following theorem is proven. Let $u(1) + u(2) + u(3) + \cdots$ be a divergent series of positive terms for which $\lim u(n) = 0$. Let P be a positive number. Then there exists a subseries $u(n_1) + u(n_2) + u(n_3) + \cdots$ $(n_1 < n_2 < n_3 < \cdots)$ which converges to P. It is also shown how to construct the convergent subseries.

J. W. Andrushkiw (S. Orange, N.J.)

Seall, Robert; Wetzel, Marion 3787

Quadratic forms and chain sequences.

Proc. Amer. Math. Soc. 15 (1964), 729–734.

Sequences $\{a_n\}$, $\{b_n\}$ form a double chain sequence if, for $n = 1, 2, \cdots$,

$$a_n = 4(1 - g_{2n-2})(1 - g_{2n-1})g_{2n-1}g_{2n},$$
$$b_n = 2(1 - g_{2n-2})(1 - g_{2n-1}) + 2g_{2n-2}g_{2n-1} - 1,$$

where $0 \leq g_{n-1} \leq 1$, $g_{-1} = 1$, $g_0 = 0$. The authors used double chain sequences in an earlier paper [Pacific J. Math. **9** (1959), 861–873; MR **23** #A1966] to characterize the monotone moment problem on $[-1, 1]$ in terms of Jacobi-type continued fractions. In the present paper a direct inductive proof is given that the quadratic forms $\sum_1^n (1 + \varepsilon b_p)x_p^2 - 2\sum_1^{n-1} a_p x_p x_{p+1}$ $(n = 1, 2, \cdots; \varepsilon = \pm 1)$, are positive semidefinite if and only if $\{a_n\}$, $\{b_n\}$ is a double chain sequence. Formulas are given for the minimal parameters of the double chain sequence and extension to double chain sequences of some results for ordinary chain sequences is indicated. *W. T. Scott* (Tempe, Ariz.)

Ishiguro, Kazuo 3788

On the Sonnenschein methods of summability.

Math. Z. **84** (1964), 374–377.

The following two theorems are proved. (1) The generalized Euler summation method is the only method, regular or not, which is common to the $[F, d_n]$ and Sonnenschein summation schemes. (2) Let $J(x) = \sum_0^\infty p_k x^k$, $p_k \geq 0$, be an entire function and $s_0, s_1, \cdots$ a given sequence. Define $J_s(x) = \sum_0^\infty s_k p_k x^k$. If $J_s(n)/J(n) \to s$ as $n \to \infty$ $(n = 1, 2, \cdots)$, then $\{s_k\}$ is said to be summable J^* to s. The method generated by $J(x) = a^x$, $a > 1$ (equivalent to Borel's exponential means if n is replaced by x), is the only method which is of both J^* and Sonnenschein types.

{Reviewer's remarks: (i) Result (2) is rephrased from the paper where it is stated imprecisely; (ii) the references made by the author to L. Lorch alone (p. 374 and p. 377[4]) should be jointly to L. Lorch and D. J. Newman [Comm. Pure Appl. Math. **15** (1962), 109–118; MR **26** #2805].} *L. Lorch* (Edmonton, Alta.)

Ishiguro, Kazuo 3789

On the Lebesgue constants for quasi-Hausdorff methods of summability. I, II.

Proc. Japan Acad. **40** (1964), 188–191; *ibid.* **40** (1964), 192–195.

The quasi-Hausdorff methods considered here are those generated by weight function $\psi(r)$, $0 \leq r \leq 1$, with $\psi(r)$ a step function and $\psi(1) = 1$, $\psi(0) = \psi(+0) = 0$, which suffices to assure regularity. Under these assumptions it is shown that the nth Lebesgue constant for the application of such a method to Fourier series is $C^*(\psi) \log n + o(\log n)$. $C^*(\psi)$ depends only on ψ; its value is given explicitly. This result subsumes in part an earlier one on the circle or (γ, r) method of summation (a particular quasi-Hausdorff method) obtained first by the present author [same Proc. **36** (1960), 470–474; MR **23** #A1205], and subsequently rederived with other methods by the reviewer and D. J. Newman [Canad. Math. Bull. **6** (1963), 179–182; MR **27** #6083]. {Unfortunately, the author omits Newman's name in referring to this article, p. 188, line 8 from the bottom, p. 195, reference [4].} The present result is analogous to a special case of one obtained for all regular Hausdorff methods by the reviewer and D. J. Newman [Canad. J. Math. **13** (1961), 283–298; MR **27** #502] and is established in a similar way. {In this last reference $\psi(r)$ is not restricted to be a step function, although only this case is cited (as Theorem 2) in the present paper.}

L. Lorch (Edmonton, Alta.)

Sen, Mira 3790

Extension of a theorem of Hyslop on absolute Cesàro summability.

Proc. Japan Acad. **40** (1964), 183–187.

Let $\sum u_n^{(\delta)}$ denote the Cesàro (C, δ)-transform of $\sum u_n$, $\delta > -1$. By definition, absolute summability $|C, \delta|$ of $\sum u_n$ means absolute convergence of $\sum u_n^{(\delta)}$. Hyslop proved that, for $0 < k < 1$, absolute convergence of $\sum n^k u_n^{(p)}$ is equivalent to $|C, p|$ summability of $\sum n^k u_n$ (here p is a positive integer).

In this short but important paper a monotonic increasing sequence $\{\lambda_n\}$, $\lambda_n \to \infty$, replaces $\{n^k\}$ of Hyslop's theorem. The last is now split into two theorems. The first one formulates the properties of $\{\lambda_n\}$ which make the convergence of $\sum \lambda_n |u_n^{(p)}|$ a sufficient condition of the summability $|C, p|$ of the series $\sum \lambda_n u_n$. The second theorem

gives the properties of $\{\lambda_n\}$ under which the convergence of $\sum \lambda_n |u_n^{(p)}|$ is a necessary condition of the summability $|C, p|$ of $\sum \lambda_n u_n$. *E. Kogbetliantz* (New York)

Slepenčuk, K. M. 3791

Some special summation methods for infinite products. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1963**, no. 6 (37), 133–137.

Let $\alpha \geq 0$, $1 \leq k_1 \leq k_2 \leq \cdots$, and $\lim_{n\to\infty} k_n = \infty$. Given an infinite product $\prod (1+u_i)$, the sequence $\{P_n\}$ defined by

$$P_n = \prod_{i=1}^{n} \left(1 + \left(1 - \frac{k_i^{\alpha}}{k_n^{\alpha+1}}\right)^{k_i} u_i\right), \qquad n = 1, 2, \cdots,$$

is called the (K, α) transform of $\prod (1+u_i)$. If $\lim_{n\to\infty} P_n = \sigma \neq 0$, the infinite product $\prod (1+u_i)$ is said to be (K, α) summable to σ. The author proves the following regularity theorem: In order that every convergent infinite product should be (K, α) summable to the same value it is necessary and sufficient that the sequence

$$\sum_{i=1}^{n} (k_i/k_n)^{\alpha+1}, \qquad n = 1, 2, \cdots,$$

be bounded. *R. Bojanic* (Columbus, Ohio)

Dawson, David F. 3792

Some rate invariant sequence transformations.

Proc. Amer. Math. Soc. **15** (1964), 710–714.

The author uses the following basis for comparing convergence rates of series: If $\sum a_p$ and $\sum b_p$ are convergent series, then "$\sum a_p$ converges faster than $\sum b_p$" means that

$$\lim_{n\to\infty} \sum_{p=n}^{\infty} a_p \Big/ \sum_{p=n}^{\infty} b_p = 0.$$

An equivalent definition for sequences is given. Also, the relative rate of divergence of two series (or sequences) is defined similarly. The following definition is also used: "A sequence-to-sequence transformation A is rate (of convergence) invariant" means that if each of x and Ax is a convergent sequence, then neither sequence converges faster than the other.

The following theorems are proved: Theorem 1: If $\sum |a_p|$ converges and $d_p \to k$, then the convergence-preserving matrix $A = (a_{pq})$ defined by $a_{pq} = 0$ if $q > p$, $a_{pq} = d_p$ if $p = q$, $a_{pq} = a_q$ if $q < p$, speeds up the convergence of some null sequence if and only if $k = 0$. Theorem 2: If $\sum_{i=N}^{\infty} |1 - p_{i+1}/p_i|$ converges for some N, then for each convergent series $\sum a_i$, the series $\sum a_i$ and $\sum p_i a_i$ are rate invariant.

Another theorem similar to Theorem 1 is also given, as are two partial converses to Theorem 2. Finally, two theorems on rate of divergence are proved.

E. R. Hansen (Palo Alto, Calif.)

Golubov, B. I. 3793

On the summability of sequences. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, no. 4 (41), 47–55.

D'après R. C. Buck et H. Pollard [Bull. Amer. Math. Soc. **49** (1943), 924–931; MR **5**, 117] il existe une correspondance biunivoque entre les nombres de $(0, 1)$ et la suite des nombres naturels de la manière suivante. Soient $x \in (0, 1]$ et $x = 0$, $\alpha_1(x)\alpha_2(x)\cdots$ la représentation de x par une fraction duale (avec une infinité des chiffres 1), alors $x \sim \{n_k\}$ où n_k est tel que $\alpha_{n_k}(x) = 1$, $\alpha_n(x) = 0$, $n \neq n_k$ $(k = 1, 2, \cdots)$. La réciproque est aussi vrai: A toute suite de nombres naturels strictement croissants $\{n_k\}$ correspond, par les mêmes lois, un point $x \in (0, 1]$.

Soient $\{s_n\}$ une suite, $s(n, x)$ sa suite partielle définie par $x \in (0, 1]$, s_n' l'ensemble des points limites de s_n, $T = \|a_{n,k}\|$ un procédé régulier de sommation, $\sigma_n(s) = \sum_{k=1}^{\infty} a_{n,k} s(k, x)$, alors l'auteur démontre: Pour toute suite $\{s_n\}$ on peut trouver un ensemble $Q \in (0, 1]$ de mesure positive et de deuxième catégorie sur $(0, 1]$ tel que $\{s(n, x)\}' = \{s_n\}'$, $x \in Q$. Si $|s_n| \leq C$ pour tout procédé régulier T il existe $Q \in (0, 1)$ de deuxième catégorie sur $(0, 1]$ tel que $\{\sigma_n(x)\}' \supset \{s_n\}'$, $x \in Q$. L'auteur donne des résultats analogues concernant les réarrangements des séries à termes constants et à termes variables.

M. Tomić (Belgrade)

Meir, Amram 3794

Tauberian constants for a family of transformations.

Ann. of Math. (2) **78** (1963), 594–599.

Let $\sum u_n$ be a series of complex terms, let $s_n = u_0 + u_1 + \cdots + u_n$, and let the linear transformation $t_p = \sum_{k=0}^{\infty} c_{pk} s_k$ satisfy the conditions (i) there exist a positive constant a and a positive increasing function $q = q(p)$ such that, for every fixed δ $(\frac{1}{2} < \delta < \frac{2}{3})$,

$$c_{pk} = (a/\pi q)^{1/2} \exp\{-a(k-q)^2/q\} \times \left(1 + O\left(\frac{|k-q|+1}{q}\right) + O\left(\frac{|k-q|^3}{q^2}\right)\right)$$

as $p \to \infty$, uniformly in k for $|k-q| \leq q^{\delta}$; (ii) $\sum k c_{pk} = O\{\exp(-q^{\eta})\}$, where η is some positive number independent of p and where the summation is taken over all k satisfying $|k-q| > q^{\delta}$.

The author's main result is that if $n(\beta)$ and $p(\beta)$ increase monotonically and tend to $+\infty$ with β, and if

$$\limsup_{\beta\to\infty} |n-q| \cdot q^{-1/2} = M < \infty,$$

then the least constant A_M such that $\limsup_{\beta\to\infty} |s_n - t_p| \leq LA_M$ for every series $\sum u_n$ which satisfies the Tauberian condition $\limsup |n^{1/2} u_n| = L < \infty$, is given by

$$A_M = (a\pi)^{-1/2}\left(e^{-aM^2} + 2aM \int_0^M e^{-ax^2}\, dx\right).$$

The case of the Borel transform, in which $c_{pk} = e^{-p} p^k / k!$ and $a = \frac{1}{2}$, $q = p$, is due to Agnew [Math. Z. **67** (1957), 51–62; MR **18**, 732]. *P. Heywood* (Edinburgh)

APPROXIMATIONS AND EXPANSIONS

See also 3647, 3692, 3810, 3815, 4176, 4189.

Rice, John R. 3795

★**The approximation of functions. Vol. I: Linear theory.**

Addison-Wesley Publishing Co., Reading, Mass.-London, 1964. xi + 203 pp. \$8.75.

This volume is centered on the approximation of real continuous functions by approximating functions which

depend linearly on a finite number of parameters. The book will be summarized by a selective listing of its contents.

Chapter One: Fundamentals. Various norms which are common in approximation theory are introduced and appraised, and the theorem of Pólya on the convergence of best L_p approximations to a best Tchebycheff approximation as $p \to \infty$ is proved. There follows a useful heuristic discussion on "choice of form and norm". The chapter concludes with the basic theorem on the existence of best approximations.

Chapter Two: Least Squares and Orthogonal Functions. The explicit solution of the problem of least squares approximation to a function on an interval by a linear combination of a finite number of linearly independent functions is given first. There follows a brief discussion of orthogonal polynomials on an interval and on a finite point set. Approximation on an interval as the limit of approximation on a finite point set comes next. The final topic in this chapter is the Gram-Schmidt orthogonalization technique.

Chapter Three: Tchebycheff Approximation. Standard characterization and uniqueness results are given for uniform approximation by a linear combination of functions on an interval and related to results on a finite point set. Tchebycheff-type theory is then extended to approximation by unisolvent functions and rational functions. There follows a discussion of the limits of Tchebycheff-type theory. These latter sections contain much material due to the author.

Chapter Four: Approximation in the L_1 Norm. The characterization of best L_1 approximations is followed by a proof of Jackson's uniqueness theorem. A brief discussion of L_1 approximation on a finite point set concludes this chapter.

Chapter Five: The Weierstrass theorem is given, and Jackson's theorem on degree of convergence is proved.

Chapter Six: Computational Methods. Transformations of known expansions are first discussed, then telescoping procedures. Descent methods are thoroughly studied as well as ascent methods for Tchebycheff approximation. Finally, connections between approximation techniques and linear programming are treated.

Each chapter is followed by a generous section of problems. Some of the problems are substantial pieces of work, and they are starred. As the summary shows, the author has covered a lot of ground in a small volume. He has everywhere been willing to give the germinal ideas behind proofs and methods. For this heuristic quality the reader will be grateful. However, the book also gives the impression of having been composed in great haste. Definitions are sometimes careless and imprecise. For example: the word "parameter" on page 2 is used both for the individual components of a vector and the vector itself. According to Definition 3-3, 0 is a simple zero of x^3. Among other instances of error are Problem 3-2, which is quite wrong (although the student who straightens it out will never forget the property of the Tchebycheff polynomials involved), and Problem 3-20a. The value 100 for the constant in Jackson's theorem on degree of convergence is not obtained in the proof given, despite the assertion that it is on page 139. Some purely typographical errors were also noted. A list of these may be obtained from the author.

T. J. Rivlin (Yorktown Heights, N.Y.)

Lambert, Pol V. 3796

Two theorems in the approximation of functions of two variables by polynomials of the Bernstein-type.

Simon Stevin **36** (1962/63), 122–130.

Let $f(x, y)$ be a real function defined and bounded on the triangle $S=\{(x, y): x \geq 0, y \geq 0, x+y \leq 1\}$. For $n=1, 2, \cdots$, let

$$B_n(x, y) = \sum_{\substack{k\geq 0,\, m \geq 0,\\ k+m\leq n}} f\left(\frac{k}{n}, \frac{m}{n}\right) n! \times [k!m!(n-k-m)!]^{-1} x^k y^m (1-x-y)^{n-k-m}.$$

Then for every $(x, y) \in S$ one has $|B_n(x, y)-f(x, y)| \leq 1.5\,\omega(n^{-1/2}, n^{-1/2})$, where for every positive δ_1, δ_2,

$$\omega(\delta_1, \delta_2) = \sup_{\substack{|x-x'|<\delta_1,\, |y-y'|<\delta_2\\ (x,y)\in S,\, (x',y')\in S}} |f(x', y')-f(x, y)|.$$

The author also gives an analogue of an asymptotic formula due to E. V. Voronovskaja [Dokl. Akad. Nauk SSSR (A) **1932**, 79–85]. *O. Shisha* (Dayton, Ohio)

Grigor'eva, I. A. 3797

An application of a method of Chebyshev and Bernstein to a class of extremal functions satisfying certain relations which are linear with respect to the coefficients. (Russian. English summary)

Ukrain. Mat. Ž. **16** (1964), 283–291.

The author indicates a way to determine the solution of the problem $\int_{-1}^{+1} p(x) y_n(x)\,dx = \min$, where $y_n(x) = \sum_0^n p_k x^k$ is a polynomial of degree n, positive on an interval $[-a, a]$, whose coefficients satisfy the conditions $\sum_{k=0}^n a_{kj} = A_j$, $j=1, \cdots, s$. She gives applications to extremal problems for multiply monotone polynomials.

G. G. Lorentz (Syracuse, N.Y.)

Pogodičeva, N. A. 3798

Lebesgue functions for certain linear methods of approximation by ordinary polynomials on a finite interval. (Russian)

Ukrain. Mat. Ž. **15** (1963), 100–101.

Let $P_n(f, x)$ be a linear combination of the first n Chebyshev polynomials that interpolates to a continuous function $f(x)$ over $[-1, 1]$. Let $V_n(f, x, \lambda)$ be $P_n(f, x)$ modified by the inclusion of arbitrary but bounded constants as multipliers of the coefficients. The author gives an asymptotic relation for large n for $\sup |V_n(f, x, \lambda)|$ taken over $|f(x)| \leq 1$. *R. G. Langebartel* (Urbana, Ill.)

Szász, Paul 3799

On generalized quasi-step and almost-step parabolas, respectively.

Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **6** (1963), 13–15.

Given a function $f(x)$ continuous in the closed interval $[-1, 1]$, the author constructs a polynomial $S_n(x)$, of degree not exceeding $2n+1$, from the zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ and states conditions under which $S_n(x)$ converges (and converges uniformly) to $f(x)$. Here $0 \leq \alpha < 1$, $0 \leq \beta < 1$. He states an analogous result for the case in which $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{2}$. No proofs are supplied.

L. Lorch (Edmonton, Alta.)

Bajšanski, B.; Bojanić, R. 3800

A note on approximation by Bernstein polynomials.

Bull. Amer. Math. Soc. **70** (1964), 675–677.

Let $f=f(x)$ be a function defined on the interval $I=[0, 1]$, and consider the corresponding Bernstein polynomial of degree n: $B_n(f;x)=\sum_{\nu=0}^{n}\binom{n}{\nu}x^\nu(1-x)^{n-\nu}f(\nu/n)$. If f has a continuous derivative f' on the interval I, then we have the following inequality, established by G. G. Lorentz [*Bernstein polynomials*, Univ. Toronto Press, Toronto, Ont., 1953; MR 15, 217]:

$$|f(x)-B_n(f;x)| \leq 0.75n^{-1/2}\omega_1(n^{-1/2}),$$

where $\omega_1(\delta)$ is the modulus of continuity of f'. In particular for $f'\in \mathrm{Lip}\,1$ it follows that we have $|B_n(f;x)-f(x)|=O(n^{-1})$. It is known that $O(n^{-1})$ cannot be improved by increasing the smoothness of f, because of the well-known theorem of Voronovskaja. In Lorentz's book it has been conjectured that a continuous function f defined on I can satisfy $|B_n(f:x)-f(x)|=o(n^{-1})$ uniformly on some interval only if f is linear on that interval. Let $0\leq\alpha<\beta\leq 1$. In this paper the authors prove the following theorem: If f is continuous on the interval I and if $B_n(f;x)-f(x)=o(n^{-1})$ holds for each fixed $x\in(\alpha,\beta)$, then f is a linear function on $[\alpha,\beta]$. This theorem complements the important results obtained by K. de Leeuw [J. Analyse Math. 7 (1959), 89–104; MR 22 #3916] in connection with this kind of problem.

D. D. Stancu (Cluj)

Dzjadyk, V. K. 3801

Approximation of non-periodic functions by polynomials on a system of segments. (Russian)

Ukrain. Mat. Ž. **15** (1963), 88–94.

The author extends his earlier theorem on polynomial approximation [Dokl. Akad. Nauk SSSR **121** (1958), 403–406, MR **21** #249], which assumed a single interval as the domain, to the case where the domain consists of a finite number of disjoint intervals.

R. G. Langebartel (Urbana, Ill.)

Šabunin, M. I. 3802

Determination of the convergence class of interpolation series for the Abel-Gončarov and Gel'fond problems. (Russian)

Dokl. Akad. Nauk SSSR **149** (1963), 272–275.

Evgrafov [same Dokl. **115** (1957), 31–33; MR **20** #987] gave a theorem justifying the expansion of an entire function in terms of a class of linear functionals of the given function and polynomials bi-orthogonal to these. The present article shows that the hypotheses of Evgrafov's theorem are satisfied by the functionals with the Abel-Gončarov kernel $(z-\lambda_n)^{-n-1}$ and the Gel'fond kernel

$$\prod_{k=0}^{n}(z-\lambda_{n,k})^{-1}.$$

R. G. Langebartel (Urbana, Ill.)

Olegov, V. B. 3803

Generating functions for sequences of Euler-Bernstein polynomials. (Russian. Estonian and English summaries)

Eesti NSV Tead. Akad. Toimetised Füüs.-Mat. Tehn.-tead. Seer. **12** (1964), 115–120.

The author finds generating functions for the extremal cyclic polynomials $C_n(x)$, $S_n(x)$ of Euler and Bernstein [S. Bernštein, *Collected works* (Russian), Vol. II, pp. 493–516, Izdat. Akad. Nauk SSSR, Moscow, 1954; MR **16**, 433], and derives several identities.

G. G. Lorentz (Syracuse, N.Y.)

Moursund, D. G. 3804

Chebyshev approximations of a function and its derivatives.

Math. Comp. **18** (1964), 382–389.

Let X be a compact subset of the real line. Let $n\geq 0$, $r\geq 1$ be fixed integers. Let $f(x)$ (the function to be approximated) and the base functions $\phi_0(x),\cdots,\phi_n(x)$ be of class C^r in an interval $[a,b]$ containing X. Finally, let

$$M[g] = \max_{k=0,\cdots,r}\|w_k(x)D^kg(x)\|,$$

where $\|\cdot\|$ denotes the uniform norm on X. The problem is to find the coefficients $a_0,\cdots,a_n$ such that $M[\sum a_i\phi_i(x)-f(x)]$ is a minimum. Beginning with a numerical example for $\phi_i(x)=x^i$, $r=1$ and X a set of five points, while $n=5$, 6, 7, 8, and 9, the general discussion leads to a characterization theorem for solutions. We quote only the last of the four theorems. Let $X=[-1,1]$, $r=1$, $\phi_i(x)=x^i$, $i=0,\cdots,n$, and assume $w_0(x)$ and $w_1(x)\in C^1$, $f(x)\in C^2$. Then either the best approximation P is unique, or else it is unique up to an appropriate additive constant. In the latter case, $DP(x)=P'(x)$ is the best Chebyshev approximation of degree $n-1$ to $f'(x)$ for the weight function $w_1(x)$.

I. J. Schoenberg (Madison, Wis.)

Mamedov, R. G. 3805

Some general results on the asymptotic value and order of approximation of functions by a family of positive linear operators. III. (Russian. Azerbaijani summary)

Izv. Akad. Nauk Azerbaĭdžan. SSR Ser. Fiz.-Mat. Tehn. Nauk **1963**, no. 1, 3–13.

Part II appeared in same Izv. **1962**, no. 6, 3–13 [MR **27** #2770]. The problem is to determine the rate of convergence of approximations to a continuous function of period 2π, assuming that the approximating functions are obtained by a positivity-preserving linear process. In polynomial and trigonometric approximation, the rate of convergence is known to be determined by the degree of differentiability of the function approximated. The author now subsumes a number of such results in a general theorem which expresses the rate of convergence to a $2m$-times differentiable function in terms of its left and right derivatives of order $2m+1$ at a point. The principal hypotheses are differentiability conditions on the functions produced by the approximation process. The general problem of relating rate of convergence to local conditions (such as differentiability) is also considered.

L. de Branges (Lafayette, Ind.)

Berg, Lothar 3806

Asymptotische Entwicklungen für Parameterintegrale. III.

Math. Nachr. **27** (1963/64), 265–275.

The author further generalizes his previous work on

asymptotic expansions of integrals of the form $\int_{a(s)}^{b(s)} G(s,t)\,dt$ [same Nachr. 24 (1962), 181–192; MR 26 #6668; ibid. 27 (1964), 133–143; MR 28 #5283], by weakening the differentiability assumptions and allowing $G(s,t)$ to have many zeros in the interval of integration.

R. R. Goldberg (Evanston, Ill.)

Schmidt, Hermann 3807

Elementarer Beweis für eine asymptotische Entwicklung aus dem Gebiet der ζ-Funktion.

Math. Z. 84 (1964), 271–276.

The author derives asymptotic expansions of

$$\alpha S_{n-1}(\alpha) = \alpha \sum_{\nu=1}^{n-1} \nu^{\alpha-1} \qquad (\alpha \text{ complex})$$

and related expressions, for large n, by a method based on infinite series without the use of integration.

A. E. Ingham (Cambridge, England)

Fedorjuk, M. V. 3808

The stationary-phase method. Near-by saddle points in the higher-dimensional case. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. 4 (1964), 671–682.

Consider the integral

$$\Phi(k, \alpha) = \int_D e^{ikf(x,\alpha)}\varphi(x, \alpha)\,dx,$$

$x = (x_1, \cdots, x_n)$, $D \in E^n$, $k \to \infty$, under the assumption that the function $f(x, \alpha)$ has two nearby stationary points, x_1 and x_2, that coalesce for $\alpha = 0$. The principal result obtained is the following (not all the hypotheses are stated in detail): if $f(x, \alpha)$ and $\varphi(x, \alpha)$ are regular in a neighborhood of x^0 for $|x_j - x_j^0| < \delta'$, $|\alpha| < \delta'$, and infinitely differentiable for $|x_j - x_j^0| < \delta$, $|\alpha| < \delta$, where D is now the region $|x_j - x_j^0| \leq \delta$, $1 \leq j \leq n$, then

$$\Phi(k, \alpha) \sim k^{(n-1)/2} \exp[ikF(\alpha)] \sum_{m=0}^{\infty} \Phi_m(k, \alpha, \zeta)k^{-m},$$

where

$$\Phi^m(k, \alpha, \zeta) = \sum_{j=1}^{4} w_j(\zeta, k) \sum_{s=0}^{m} \frac{a_{js}(\alpha)}{k^{2s}} + O(k^{-2m-2}),$$

$$w_j(\zeta, k) = v(\zeta)/k^{(3j-1)/6} \quad \text{for } j \text{ odd},$$
$$= v'(\zeta)/k^{(3j-2)/6} \quad \text{for } j \text{ even},$$

$$F(\alpha) = \tfrac{1}{2}[f(x_1(\alpha), \alpha) + f(x_2(\alpha), \alpha)],$$

$$\zeta^{3/2} = -\tfrac{3}{4}k[f(x_1(\alpha), \alpha) - f(x_2(\alpha), \alpha)],$$

$v(\zeta)$ is an Airy integral, and a_{js} are regular functions of α for small α. In addition, a formula is given for the first two terms of the above representation in terms of the square root of

$$\det \left\| \frac{\partial^2 f(x^j(\alpha), \alpha)}{\partial x_k \partial x_l} \right\|, \qquad j = 1, 2.$$

R. N. Goss (San Diego, Calif.)

Ionescu, D. V. 3809a

La représentation de la différence divisée d'une fonction de deux variables par une intégrale double.

Mathematica (Cluj) 3 (26) (1961), 59–78.

Ionescu, D. V. 3809b

La représentation de la différence divisée d'une fonction de deux variables par une intégrale double. II.

Mathematica (Cluj) 3 (26) (1961), 231–271.

These papers obtain a representation for the divided difference of order (m, n) of a function $f(x, y)$ of two variables as a double integral

$$\begin{bmatrix} x_0, x_1, \cdots, x_m \\ y_0, y_1, \cdots, y_n \end{bmatrix}; f \bigg] = \iint_D \phi(x, y) \frac{\partial^{m+n} f}{\partial x^m \partial y^n}\,dx\,dy,$$

where D is the rectangle $x_0 \leq x \leq x_m$, $y_0 \leq y \leq y_n$. It is assumed that the nodes in the divided difference are simple. The function $\phi(x, y)$ is uniquely determined and the main purpose of these papers is to study its properties in detail. Much information is obtained about $\phi(x, y)$ which we cannot summarize here. One important result is that $\phi(x, y)$ has a constant sign of $(-1)^{m-n}$ on D. This result is important in numerical analysis for the study of the error in certain approximations and generalizes a similar result for a single variable previously obtained by the author [*Numerical quadrature* (Romanian), pp. 171–180, Editura Tehnică, Bucharest, 1957; MR 22 #B3149].

A. H. Stroud (Lawrence, Kans.)

FOURIER ANALYSIS

See also 3579, 3612, 3664, 3693, 3789, 3832, 3903, 3906, 4642.

Berman, D. L. 3810

Some remarks on the problem of speed of convergence of polynomial operators. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika 1961, no. 5 (24), 3–5.

If $f(x)$ is a continuous periodic function and $u_n(f, x)$, $v_n(f, x)$ are two trigonometric polynomials of order n approximating f and converging uniformly to f as $n \to \infty$ and which coincide if f is a trigonometric polynomial of degree $\leq n$, then

$$C_1\|u_n(f, x) - f(x)\| \leq \|v_n(f, x) - f(x)\| \leq C_2\|u_n(f, x) - f(x)\|,$$

where C_1, C_2 are constants and $\|f\| = \max |f(x)|$.

R. G. Langebartel (Urbana, Ill.)

Xie, Ting-fan [Hsieh, Ting-fan] 3811

The best approximation of periodic differentiable functions by trigonometric polynomials.

Acta Math. Sinica 13 (1963), 162–169 (*Chinese*); *translated as Chinese Math.* 4 (1963), 179–187.

We denote by $C_{2\pi}$ the set of all continuous functions with period 2π, by $L_{2\pi}$ the set of all L-integrable functions with period 2π, and by T_{n-1} the set of all trigonometrical polynomials of order $n-1$. For $f(x) \in C_{2\pi}$, we set

$$E_n(f) = \min_{T_{n-1}} \max_x |f(x) - T_{n-1}(x)|,$$

and for $f(x) \in L_{2\pi}$, we set

$$E_n(f)_L = \min_{T_{n-1}} \int_0^{2\pi} |f(x) - T_{n-1}(x)|\,dx.$$

We write $K_r(t)=\sum k^{-r}\cos(kt-\frac{1}{2}r\pi)$ $(r>0)$. When $f(x)\in C_{2\pi}$ can be expressed as

$$f(x)=\frac{1}{2}a_0+\frac{1}{\pi}\int_0^{2\pi}K_r(x-t)\varphi(t)\,dt,\qquad \varphi(t)\in C_{2\pi},$$

we say that the function $f(x)$ has an rth derivative in the sense of Weyl, $f^r(x)=\varphi(x)$, where $f(x)$ is the rth-order integral of $\varphi(x)$. In this case, the conjugate function $\bar{f}(x)$ can be expressed as

$$\bar{f}(x)=\frac{1}{\pi}\int_0^{2\pi}\bar{K}_r(x-t)\varphi(t)\,dt,$$

in which

$$\bar{K}_r(t)=\sum k^{-r}\sin(kt-\tfrac{1}{2}\pi r).$$

If k is a natural number, we set

$$\omega_k(\delta,f)=\max_{|h|\le\delta}\max_x\left|\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}f(x+ih)\right|.$$

In this note, the following three theorems are proved. Theorem 1: If $r>0$, and the function $f(x)$ has its rth derivative $f^{(r)}(x)\in C_{2\pi}$, then

$$E_n(f)\le\frac{4}{\pi}\frac{A_r}{n^r}E_n(f^{(r)}),\qquad E_n(\bar{f})\le\frac{4}{\pi}\frac{\bar{A}_r}{n^r}E_n(f^{(r)})$$
$$(n=1,2,\cdots),$$

in which A_r, $\bar{A}_r$ are constants dependent only on r.

Theorem 2: If $r>0$, $f(x)\in C_{2\pi}$, $\sum n^{r-1}E_n(f)<+\infty$, then $f(x)$, $\bar{f}(x)$ both have continuous rth derivatives $f^{(r)}(x)$, $\bar{f}^{(r)}(x)$ and

$$E_n(f^{(r)})\le C_r\left\{n^rE_n(f)+\sum_{\nu=n+1}^{\infty}\nu^{r-1}E_\nu(f)\right\},$$
$$E_n(\bar{f}^{(r)})\le\bar{C}_r\left\{n^rE_n(f)+\sum_{\nu=n+1}^{\infty}\nu^{r-1}E_\nu(f)\right\}$$
$$(n=1,2,\cdots),$$

in which C_r and $\bar{C}_r$ are constants dependent only on r.

Theorem 3: Under the hypotheses of Theorem 2, we have

$$\omega_k\left(\frac{1}{n},f^{(r)}\right)\le C_{k,r}\left\{\sum_{\nu=1}^{n}\nu^{k+r-1}E_\nu(f)+\sum_{\nu=n+1}^{\infty}\nu^{r-1}E_\nu(f)\right\},$$
$$\omega_k\left(\frac{1}{n},\bar{f}^{(r)}\right)\le\bar{C}_{k,r}\left\{\sum_{\nu=1}^{n}\nu^{k+r-1}E_\nu(f)+\sum_{\nu=n+1}^{\infty}\nu^{r-1}E_\nu(f)\right\}$$
$$(n=1,2,\cdots),$$

in which $C_{k,r}$ and $\bar{C}_{k,r}$ are constants dependent only on r and k. *F. C. Hsiang* (Taipei)

Krylov, V. I.; Janovič, L. A. 3812

On the convergence of a trigonometric interpolation for analytic periodic functions. (Russian)

Dokl. Akad. Nauk BSSR 7 (1963), 649–652.

Let z_n^k, $k=0,1,\cdots,2n$, $n=1,2,\cdots$, be arbitrary complex numbers in a closed region D contained in $|\mathrm{Im}(z)|\le h$, $0\le\mathrm{Re}(z)<2\pi$. The authors determine the smallest region containing D in which a 2π-periodic function f must be holomorphic in order that the sequence of trigonometric polynomials $\{T_n\}$ defined by $T_n(z_n^k)=f(z_n^k)$, $k=0,1,\cdots,2n$, $n=1,2,\cdots$, converges uniformly to f in the region D^* obtained by 2π-translations of D, for any choice of the nodes in D. A typical result is the following: If $D=\{z:|\mathrm{Im}(z)|\le d,\ 0\le\mathrm{Re}(z)<2\pi\}$, then a 2π-periodic function f should be holomorphic at least in the region $|\mathrm{Im}(z)|\le d+2\log(\cosh d+\sqrt{(1+\cosh^2 d)})$, in order that the sequence of trigonometric polynomials converges uniformly in $|\mathrm{Im}(z)|\le d$, for any choice of the nodes in D.

Similar results for polynomial interpolation have been established previously by V. I. Krylov [same Dokl. 78 (1951), 857–859; MR 13, 637]. *R. Bojanic* (Columbus, Ohio)

Pavlovskiĭ, N. M. 3813

A method of approximating differentiable functions by trigonometric polynomials. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, no. 4 (41), 118–125.

Let $MW^{(r)}$ be the class 2π-periodical continuous functions having absolutely continuous $(r-1)$st derivatives, and rth derivatives almost everywhere such that $|f^{(r)}(x)|\le M$. For $f(x)$ (with respect to its Fourier sum $S_n(f,x)$) the author defines

$$\sigma_n^N(f,x)=\frac{1}{N(n+1)}\sum_{k=1}^{N}f(x_k)\left[\frac{\sin\frac{1}{2}(n+1)(x-x_k)}{\sin\frac{1}{2}(x-x_k)}\right]^2,$$

where $x_k=2k\pi/N$, N is any positive integer satisfying $N\ge 2n+1$ (for $N=2n+1$, $\sigma_n^N(f,x)$ becomes the Fejér interpolation sum for $f(x)$). Following the method of Nikol'skiĭ [Trudy Mat. Inst. Steklov. 15 (1945), p. 25; MR 7, 435], the author shows that

$$\sup_{f\in MW^{(r)}}\sup_x|f(x)-\sigma_n^N(f,x)|=\frac{MC_r}{n}+O\left(\frac{1}{n^r}\right)\qquad(r\ge 2),$$
$$=\frac{2M}{\pi}\frac{\log n}{n}+O\left(\frac{1}{n}\right)\quad(r=1),$$

where

$$C_r=\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^{k(r-1)}}{(2k+1)^r}.$$

The O-term is due to an inequality of Jackson [Dissertation, Göttingen, 1911]. A more general result is obtained by utilizing an inequality of v. Sz. Nagy [Ber. Verh. Akad. Wiss. Leipzig 90 (1938), 103–134]. *C. V. Stanojevic* (Detroit, Mich.)

Igari, Satoru 3814

Sur les facteurs de convergence des séries de Walsh-Fourier.

Proc. Japan Acad. **40** (1964), 250–252.

An outline is given of proofs of a result that was stated by Paley [Proc. London Math. Soc. (2) **34** (1932), 241–264; ibid. (2) **34** (1932), 265–279] that for almost all x, $(\log k)^{-1/p}$, $1\le p\le 2$, is a factor of convergence of the Walsh series for $f\in L_p[0,1]$. For $p=2$, the proof had been given by Yano [Tôhoku Math. J. (2) **3** (1951), 223–242; MR 13, 550]. *P. Civin* (Eugene, Ore.)

Szegő, Gábor 3815

On bi-orthogonal systems of trigonometric polynomials. (Russian summary)

Magyar Tud. Akad. Mat. Kutató Int. Közl. **8** (1963), 255–273 (1964).

This article, dedicated to Karl Loewner, introduces a set of trigonometric polynomials biorthogonal with respect to a prescribed weight function $w(\theta)$. They are generalizations of orthogonal polynomials defined on a finite interval and give rise to a theory generalizing classical Fourier series (for which $w(\theta)=1$). For this system, the author establishes recurrence relations, properties of zeros, a mechanical quadrature theorem, asymptotic behavior, a theorem on equiconvergence with ordinary Fourier series, etc. He initiates also a consideration of the analogous problems on the sphere, arising from the corresponding generalization of surface harmonics, and calls attention to the simplifications afforded by the "zonal" case, where the weight function depends on only one variable.

The function $w(\theta)$ is non-negative, of period 2π, Lebesgue integrable, and not equivalent to the zero function. If, in addition, $w(\theta)$ is even, then the bi-orthogonal system is composed of cosine and sine polynomials, respectively, and many of the results assume especial neatness. Another special case leading to noteworthy results is also discussed, namely, when $w(\theta)$ is the reciprocal of a positive trigonometric polynomial. *L. Lorch* (Edmonton, Alta.)

Sneddon, I. N.; Srivastav, R. P. 3816

Dual series relations. I. Dual relations involving Fourier-Bessel series.

Proc. Roy. Soc. Edinburgh Sect. A **66** (1962/63), 150–160 (1964).

The dual series

$$\text{(A)}\quad \sum_{n=1}^{\infty} \lambda_n^{-p} a_n J_\nu(\rho\lambda_n) = f_1(\rho), \qquad 0 \le \rho < 1,$$
$$\sum_{n=1}^{\infty} a_n J_\nu(\rho\lambda_n) = f_2(\rho), \qquad 1 < \rho \le a,$$

where $\{\lambda_n\}$ are the positive zeros of the Bessel function $J_\nu(a\lambda)$ in increasing order, $-1 \le p \le 1$, $\nu \ge 0$, $f_1(\rho)$ and $f_2(\rho)$ prescribed, are solved for the unknown sequence $\{a_n\}$. This solution is the sum of the solution of (A) with $f_1(\rho)=0$ and of the solution of (A) with $f_2(\rho)=0$. The latter solutions are obtained by first giving an integral representation of a_n in terms of a function $g(t)$. It is then shown that $g(t)$ satisfies a Fredholm integral equation of the second kind. Solution of this equation yields the solution of the given problem. *A. E. Danese* (Buffalo, N.Y.)

Srivastav, R. P. 3817

Dual series relations. II. Dual relations involving Dini Series.

Proc. Roy. Soc. Edinburgh Sect. A **66** (1962/63), 161–172 (1964).

The dual series

$$\alpha c_0 \rho^\nu + \sum_{n=1}^{\infty} c_n \lambda_n^{\,p} J_\nu(\lambda_n\rho) = f_1(\rho), \qquad 0 \le \rho < c,$$
$$c_0 \rho^\nu + \sum_{n=1}^{\infty} c_n J_\nu(\lambda_n\rho) = f_2(\rho), \qquad c < \rho \le 1,$$

where $\{\lambda_n\}$ are the positive roots of $zJ_\nu'(z)+HJ_\nu(z)=0$, H is a real constant, $H+\nu \ge 0$ with $c_0=0$ if $H+\nu\ne 0$, $-1 \le p \le 1$, $\nu > -\frac12$, are solved for the unknown sequence $\{c_n\}$ using the same technique as in #3816 above.

A. E. Danese (Buffalo, N.Y.)

Srivastav, R. P. 3818

Dual series relations. III. Dual relations involving trigonometric series.

Proc. Roy. Soc. Edinburgh Sect. A **66** (1962/63), 173–184 (1964).

Dual series involving trigonometric series such as

$$\sum_{n=1}^{\infty} n^p a_n \sin nx = f_1(x), \qquad 0 \le x < c,$$
$$\sum_{n=1}^{\infty} a_n \sin nx = f_2(x), \qquad c < x \le \pi,$$

and

$$\sum_{n=1}^{\infty} [(n-\tfrac12)^p] a_n \cos(n-\tfrac12)x = f_1(x), \qquad 0 \le x < c,$$
$$\sum_{n=1}^{\infty} a_n \cos(n-\tfrac12)x = f_2(x), \qquad c < x \le \pi,$$

$0<c<\pi$, $-1<p<1$, are solved for $\{a_n\}$ using the same techniques as in #3816. Instances where the resulting integral equations have explicit analytical solutions are treated in detail. *A. E. Danese* (Buffalo, N.Y.)

Srivastav, R. P. 3819

Dual series relations. IV. Dual relations involving series of Jacobi polynomials.

Proc. Roy. Soc. Edinburgh Sect. A **66** (1962/63), 185–191 (1964).

The dual equations

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha+n+1)\Gamma(\beta+n+\frac12)} P_n^{(\alpha,\beta)}(\cos\theta) = f_1(\theta), \qquad 0 \le \theta < \varphi,$$
$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\beta+n+1)\Gamma(\alpha+n+\frac12)} P_n^{(\alpha,\beta)}(\cos\theta) = f_2(\theta), \qquad \varphi < \theta \le \pi,$$

where $\alpha > -\frac12$, $\beta > -1$, and $P_n^{(\alpha,\beta)}$ are the Jacobi polynomials, are solved for $\{A_n\}$ by essentially the same technique used in #3816. *A. E. Danese* (Buffalo, N.Y.)

Ul'janov, P. L. 3820

Series with respect to a Haar system with monotone coefficients. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 925–950.

(I) Es sei $\{c_n\} \in A$, d.h. es sei $\{c_n\}$ eine positive, monoton abnehmende Folge. Ist

$$P(t) = \lim_{N\to\infty}\left(\sup_{2\le\nu\le N} \sum_{m=2}^{\nu} c_m\chi_m(x)\right)$$

integrierbar in $[0,1]$, dann ist die Haarsche Reihe $\sum c_m\chi_m(x)$ die Entwicklung einer Funktion $f(t)$, für die $f\in L^p[0,1]$ bei jedem $p\ (\ge 1)$ besteht. (II) $\bar{A}$ bezeichnet die Klasse der Folgen $\{b_m\}$ mit $\max_{2^n<m\le 2^{n+1}}|b_m| \le C\min_{2^{n-1}<m\le 2^n}|b_m|$ $(n=1,2,\cdots)$. Ist $\{b_m\}\in\bar{A}$ und gilt für die Partialsummen $s_n(x)$ der Reihe $\sum b_m\chi_m(x)$,

$$\limsup_{n\to\infty} |s_{q_n}(t)| < \infty$$

in einer Menge vom positiven Mass, wobei $\{q_n\}$ eine im strengen Sinne wachsende Folge der natürlichen Zahlen ist, dann gilt $\sum b_m^2 < \infty$. (III) Es sei $\{a_m\}\in A$. Ist die Reihe

$\sum a_m\chi_m(x)$ in einer Menge vom positiven Mass $|C_1 1|$-summierbar, dann gilt $\sum (a_m/\sqrt{m}) < \infty$.

K. Tandori (Szeged)

Freud, G.; Sallay, M. 3821

Sur la vitesse de convergence du développement selon des fonctions propres de Sturm-Liouville. (Russian summary)

Magyar Tud. Akad. Mat. Kutató Int. Közl. **6** (1961), 271–279.

Suppose $q(x)$ is continuous on $[0, \pi]$, and consider in that interval the differential equation $y''+[\lambda-q(x)]y=0$ with the boundary conditions (1) $a_0y(0)+b_1y'(0)=0$; $a_2y(\pi)+b_2y'(\pi)=0$. Denote by $\lambda_0, \lambda_1, \lambda_2, \cdots$ the eigenvalues, $v_0, v_1, v_2, \cdots$ the normalized eigenfunctions and $s_n(x, f)=\sum_{k=0}^{n} a_kv_k(x)$, where $a_k=\int_0^\pi f(x)v_k(x)\,dx$. Denote by $w_2(f, \delta)=\max_{|h|\le\delta, x\in(0,\pi)} |f(x+h)-2f(x)+f(x-h)|$.

Theorem 1: If $f(x)$ is continuous in $[0, \pi]$, satisfies conditions (1) and (2) $\begin{vmatrix} a_1 & b_1 \\ a_2 & a_2+b_2 \end{vmatrix} \ne 0$, then

$$|f(x)-s_n(x, f)| \le K_1 \log n[w_2(f, n^{-1})+K_2n^{-1} \max_{i=0,1} \|\delta^{(i)}f\|],$$

where

$$\begin{aligned} \delta^{(0)}f &= 0 && \text{for } b_1 = 0,\\ &= \frac{f(\pi/n)-f(0)}{\pi/n}-f'(0) && \text{for } b_1 \ne 0,\\ \delta^{(1)}f &= 0 && \text{for } b_2 = 0,\\ &= \frac{f(\pi)-f(\pi-\pi/n)}{\pi/n}-f'(\pi) && \text{for } b_2 \ne 0. \end{aligned}$$

(In case $b_1=b_2=0$, there is no requirement that f be differentiable at 0 and π.) K_1 and K_2 are constants which are independent of x, n, and f.

Suppose $f(x)$ and $q(x)$ possess continuous rth derivatives and denote by $L^{(i)}: i=0, 1, \cdots, k=[\frac{1}{2}(r-1)]$ the operators $L^{(0)}f=f$; $L^{(1)}f=f''-qf$; $\cdots$; $L_f^{(i)}=L^{(1)}(L^{(i-1)}f)$. Theorem 2: If f and q have continuous rth derivatives, $L^{(i)}f$, $i=1, 2, \cdots, k=[\frac{1}{2}(r-1)]$ satisfy conditions (1) and (2) holds, then

$$|f(x)-s_n(x, f)| \le O\left(\frac{\log n}{n^r}\right)[w_2(f^{(r)}, n^{-1}) + \frac{1}{n^2}\sum_{m=0}^{r}\|f^{(m)}\| + \frac{1}{n}\max_{i=0,1}\sum_{m=1}^{r}\|\delta^{(i)}f^{(m)}\|].$$

D. H. Tucker (Salt Lake City, Utah)

Aljančić, S. 3822

Sur le module de continuité des séries de Fourier transformées par des multiplicateurs convexes.

Bull. Acad. Serbe Sci. Arts Cl. Sci. Math. Natur. Sci. Math. **31** (1963), *no.* 4, 41–51.

This is a somewhat concise expository note on the author's paper [Math. Z. 81 (1963), 215–222; MR 27 #1774].

G. Sunouchi (Sendai)

Sinha, S. R. 3823

A theorem on the absolute Cesàro summability of Fourier series.

Proc. Nat. Inst. Sci. India Part A **29** (1963), 302–308.

Let $F_k(t)$ denote the kth integral arithmetic mean in $(0, t)$ of $F(t)=\frac{1}{2}[f(x+t)+f(x-t)]$, k being a positive integer. The $(C, h>0)$ summability of the Fourier series of $F_k(t)$ at the point $t=0$ is known (Bosanquet) to be a necessary and sufficient condition for the Fourier series of $f(t)$ to be summable $(C, k+h)$ at $t=x$. The author proves an analogous result in which ordinary Cesàro summabilities (C, h) and $(C, k+h)$ are replaced by absolute Cesàro summabilities $|C, h|$ and $|C, k+h|$.

It is probable that in both cases the limitation $k=E(k)$ is inessential, and it should be possible to prove the same results for any $k>0$.

E. Kogbetliantz (New York)

Singh, T. 3824

Nörlund summability of Fourier series and its conjugate series.

Ann. Mat. Pura Appl. (4) **64** (1964), 123–132.

If

$$\varphi(t) = f(x+t)+f(x-t)-2f(x), \quad \Phi(t) = \int_0^t |\varphi(u)|\,du,$$

the author shows the following theorem. If (N, p_n) is a regular Nörlund method defined by a real, non-negative, monotonic non-increasing sequence $\{p_n\}$ such that $P_n=\sum_{k=0}^{n} p_k\to\infty$, then, if

$$\Phi(t) = o\left(\frac{p(t^{-1})}{P(t^{-1})}\right)$$

as $t\to 0$, the Fourier series of $f(t)$ at $t=x$ is summable (N, p_n) to $f(x)$. A corresponding theorem is also proved for the conjugate series. These generalize the earlier results.

G. Sunouchi (Sendai)

Lal, Shiva Narain 3825

On the absolute Nörlund summability of a Fourier series.

Arch. Math. **15** (1964), 214–221.

Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let p_n be a sequence of constants, real or complex, and let us write

$$P_n = p_0+p_1+\cdots+p_n, \qquad P_{-1} = p_{-1} = 0.$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n}\sum_{\nu=0}^{n} p_{n-\nu}s_\nu, \qquad P_n \ne 0,$$

defines the sequence $\{t_n\}$ of Nörlund means of the sequence $\{s_n\}$ generated by the sequence of coefficients p_n. The series $\sum a_n$ is said to be summable $|N, p_n|$ if the sequence $\{t_n\}$ is of bounded variation, that is, the series $\sum |t_n-t_{n-1}|$ is convergent.

Let $f(t)$ be a periodic function with period 2π and integrable in the sense of Lebesgue in $(-\pi, \pi)$. Let its Fourier series be

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n \cos nx+b_n \sin nx).$$

The author proves the following theorem. Let (N, p_n) be a Nörlund transformation with p_n non-negative and non-increasing, $\lim_{n\to\infty} p_n=0$, $|\Delta p_n|$ non-increasing, and satisfying the condition

$$\text{(i)} \qquad \sum_{n=1}^{\infty} P_n{}^2n^{-2} \le C.$$

If $f(x)$ is a continuous, periodic function with period 2π, and if

$$\text{(ii)} \qquad \sum_{n=1}^{\infty} P_n^{-1}\omega(n^{-1})n^{-1/2} < \infty,$$

$$\text{(iii)} \qquad \sum_{n=1}^{\infty} n^{-1}\omega(n^{-\delta}) < \infty \qquad (0 < \delta < 1),$$

then the Fourier series of $f(x)$ is summable $|N, p_n|$, where $\omega(\eta)$ is the modulus of continuity of $f(x)$.
F. C. Hsiang (Taipei)

Kachroo, I. C. 3826
The harmonic summability of a series associated with Fourier series.
Univ. e Politec. Torino Rend. Sem. Mat. **22** (1962/63), 159–164 (1964).
Let $s_n(x)$ be the partial sums of the Fourier series of $f(x)$. Then the author shows that if $\int_0^t |(1/u)\int_0^u \varphi(v)\,dv|\,du = o(t/\log(1/t))$ and $\int_0^\pi \varphi(t) \operatorname{cosec} \frac{1}{2}t\,dt$ exists, then the series $\sum ((s_n(x)-s)/n)$ is harmonically summable at the point x, where $\varphi(t) = f(x+t)+f(x-t)-2s$. *G. Sunouchi* (Sendai)

Kachroo, I. C. 3827
On the matrix summability of the derived Fourier series.
Univ. e Politec. Torino Rend. Sem. Mat. **22** (1962/63), 165–170.
The author assumes that the function

$$g(t) = \psi_x(t)/4(\sin \tfrac{1}{2}t) - c$$

is of bounded variation in the neighbourhood of $t=0$ and absolutely continuous in (η, π) for any $0<\eta<\pi$. The function $g(t)-g(+0)$ is evidently absolutely continuous $(0,\pi)$ and hence the derived Fourier series is convergent. So the author's theorem is trivial. The author says that his theorem contains Rath's result, but the reviewer cannot agree with this, because Rath assumes only that $g(t)$ is of bounded variation in $(0, \pi)$. *G. Sunouchi* (Sendai)

Dolcher, Mario 3828
Su un criterio di convergenza uniforme per le successioni monotone di funzioni quasi-periodiche.
Rend. Sem. Mat. Univ. Padova **34** (1964), 191–199.
Oggetto della nota è un teorema di L. Amerio [Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **28** (1960), 147–152; ibid. (8) **28** (1960), 322–327; ibid. (8) **28** (1960), 461–466; MR **22** #6927], il quale afferma, sotto opportune ipotesi, la convergenza uniforme di una successione monotona di funzioni quasi-periodiche. L'autore, tenuto presente che tale teorema è del tipo di U. Dini, dà una nuova dimostrazione, utilizzando la così detta compattizzazione di Bohr della retta reale. *S. Cinquini* (Pavia)

Ponomarenko, Ju. O. [Ponomarenko, Ju. A.] 3829a
On linear summability methods for double Fourier series and best approximations of continuous functions of two variables. (Ukrainian. Russian and English summaries)
Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 38–41.

Griffin, V. B. 3829b
On linear summability methods for Fourier series and best approximations of periodic functions of two variables. (Ukrainian. Russian and English summaries)
Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 151–155.

Both papers give estimates for the degree of approximation to a continuous function by the partial sums of its double Fourier series, transformed by a matrix $(\lambda_k^{(m)}{}_j^{(n)})$ in convergence-factor fashion. The (rather complicated) estimates are expressed in terms of the best approximations to the function by trigonometric polynomials and differences through fourth order of the λ's. The second paper also contains a necessary and sufficient condition for the summability of the Fourier series of a continuous function by the summation process generated by the λ's.
R. P. Boas, Jr. (Evanston, Ill.)

INTEGRAL TRANSFORMS, OPERATIONAL CALCULUS

See also 3686.

Hochstadt, Harry 3830
Laplace transforms and canonical matrices.
Amer. Math. Monthly **71** (1964), 728–736.
From the author's introduction: "The purpose of this article is to demonstrate the intimate connection between systems of differential equations and canonical matrices. In particular it will be shown, using only some basic concepts of differential equations and linear algebra, that for every matrix a similarity transformation can be found which will put the given matrix into the Jordan canonical form. Subsequently some special cases will be treated, in particular normal matrices. The method leads to a canonical form naturally and is also a constructive method. Although the basic results are not new, it is believed that the treatment is original."

Abdi, Wazir Hasan 3831
Application of q-Laplace transform to the solution of certain q-integral equations.
Rend. Circ. Mat. Palermo (2) **11** (1962), 245–257.
In the words of the author, "The object of this paper is to make use of the q-Laplace transform in the solution of certain q-integral equations similar to the classical Fredholm and Volterra-type integral equations. It is not proposed to study here the general theory of such equations, but it is only intended to show as to how the q-Laplace operator can sometimes be used with advantage in the solution of such equations". *V. P. Mainra* (Pilani)

Feinberg, Irwin 3832
Sur une équation de convolution.
C. R. Acad. Sci. Paris **258** (1964), 6331–6333.
Given $P(x) = \sum_{-N+1}^{N+1} P_k e^{2\pi ikx}$, $P_{-k} = P_k \geq 0$, the author considers the problem of finding $f(x) \geq 0$ such that $f(x) = \sum_{-N+1}^{N+1} f_k e^{2\pi ikx}$, $f_{-k} = \bar{f}_k$ and $|f_k|^2 = P_k$. This last condition is equivalent to $P(k) = f * \bar{f}$ and the positivity of f is equivalent to requiring a minimum for $|\int_{-1/2}^{1/2} f(x)\,dx|$. The minimization problem is taken up in further detail and

generalizations to distributions and n dimensions discussed.
J. Blackman (Syracuse, N.Y.)

Kierat, W. 3833

Une remarque sur les logarithmes unilatéraux.

Studia Math. **24** (1964), 281–283.

The method employed by J. Mikusiński [same Studia **12** (1951), 208–224; MR **13**, 740] in proving the theorem: "Let $\alpha_1, \cdots, \alpha_n$ be distinct real numbers and $\beta_1, \cdots, \beta_n$ be non-zero complex numbers; in order for the exponential function $\exp[(\beta_1 s^{\alpha_1} + \cdots + \beta_n s^{\alpha_n})\lambda]$ to exist for $\lambda \geq 0$, it is necessary and sufficient that there exist the individual exponential functions $\exp(\beta_1 s^{\alpha_1}\lambda), \cdots, \exp(\beta_n s^{\alpha_n}\lambda)$" applies only when the functions are considered for all real λ and fails when these functions exist only for $\lambda < 0$. The author gives a modified argument which treats both these cases.
E. J. Scott (Urbana, Ill.)

Vich, Robert 3834

★Z-Transformation. Theorie und Anwendung.

Theoretische Grundlagen der Technischen Kybernetik.

VEB Verlag Technik, Berlin, 1964. x+150 pp. *DM* 16.00.

Die schon von Laplace benutzte und in neuerer Zeit von den Ingenieuren wiederentdeckte Z-Transformation ordnet einer Originalfolge f_n die Bildfunktion $F^*(z) = \sum_{n=0}^{\infty} f_n z^{-n} \equiv Z\{f_n\}$, also eine außerhalb eines Kreises inklusive $z = \infty$ analytische Funktion zu. Verfasser stellt zunächst die theoretischen Grundlagen der Transformation (Abbildungsgesetze, Umkehrung) zusammen und behandelt dann folgende Anwendungen: (1) Lösung von linearen Differenzengleichungen mit konstanten Koeffizienten (als Beispiele Kettenleiter aus Vierpolen und die Summierung von Reihen). (2) Abtastsysteme für die beiden Fälle, daß der Eingang eine Impulsfolge und eine Folge von Rechteckpulsen ist, wobei als Übertragungsfunktion (Systemfunktion) im Sinne der Laplace-Transformation eine gebrochen rationale Funktion zugrunde gelegt wird. (3) Numerische Methoden: Wird ein Integral $y(t) = \int_0^t x(\tau)\,d\tau$ durch die Summe $y_1(nT) = T\sum_{k=0}^{n} x(kT)$ approximiert, so ist $Z\{y_1(nT)\} = TX^*(z)/(z-1)$. Diese einfache Bildfunktion führt zu einem übersichtlichen Algorithmus für die Summation. Hiervon werden Anwendungen gemacht auf die praktische Berechnung von Faltungsintegralen, wie sie in der Systemtheorie auftreten (der Ausgang eines Übertragungssystems ist gleich der Faltung von Eingang und Gewichtsfunktion), sowie auf die umgekehrte Laplace-Transformation von rationalen Funktionen ohne Kenntnis per Pole. Der Anhang enthält Tabellen für die Z-Bilder von Operationen an Folgen und die Z-Bilder von speziellen Folgen, sowie ein ausführliches Literatur-Verzeichnis. Alle theoretischen Entwicklungen werden durch numerische Beispiele illustriert, die vollständig durchgerechnet werden. Was die Ausstattung angeht, ist der Text in Maschinenschrift geschrieben. Das Buch ist die erste zusammenfassende Darstellung der Z-Transformation in deutscher Sprache.
G. Doetsch (Freiburg)

Kober, H. 3835

On functional equations and bounded linear transformations.

Proc. London Math. Soc. (3) **14** (1964), 495–519.

From the author's introduction: "Functional equations of certain types are of importance in the theory of linear transformations, in analogy to the remarkable part which other types play in conformal mapping. Dirichlet's singular integral and Hilbert transforms, for instance, are operations W of the 'closed cycle': if $W(f) = W\{f(t); x\} = g(x)$, then

$$W\{f(t+a); x\} = g(x+a) \qquad (-\infty < a < \infty).$$

When $W\{f(t); s\} = g(s)$ is the Mellin transform,

$$W\{f(at); s\} = a^{-s}g(s) \qquad (0 < a < \infty).$$

The Fourier transformation $g(x) = T_0 f$ satisfies a number of equations,

$$(\alpha) \qquad T_0\{f(t+a); x\} = e^{iax}g(x),$$

$$(\beta) \qquad T_0\{f(at); x\} = |a|^{-1}g(x/a) \qquad (-\infty < a < \infty).$$

Conversely, if a Wf satisfies certain continuity conditions and the equation (α), then it is representable in the form $W\{f(t); x\} = \psi(x)T_0 f$, and if it satisfies both equations (α) and (β), then $Wf = cT_0 f$, where c is a complex constant, as will be shown (see § 9 of the paper).

"For both the Riemann-Liouville and the Weyl fractional integral W of order μ

$$(1) \qquad W\{f(at); x\} = a^{-\mu}g(ax) \qquad (0 < a < \infty),$$

and a similar equation holds for the modified fractional integrals; see earlier papers of the author [Quart. J. Math. Oxford Ser. **11** (1940), 193–211; MR **2**, 191; Proc. London Math. Soc. (3) **11** (1961), 434–456; MR **24** #A1025]; while for the transformations of the Fourier class, i.e., the sine transform and related ones

$$(2) \qquad W\{f(at); x\} = a^{-\nu}g(x/a) \qquad (0 < a < \infty).$$

There are numerous further examples.

"M. Plancherel [Proc. Cambridge Philos. Soc. **33** (1937) 413–418] used equation (2), with $\nu = 1$, which includes the 'general transforms', to deduce a group property of the operations concerned, provided that $f, g \in L^2(0, \infty)$. In a recent paper I have dealt with two wide classes of transformations which comprise most of the familiar operations and satisfy (1) for $\mu = 0$ or (2) for $\nu = 1$, and I have resorted to these equations in some proofs. In the present paper the general equations (μ, ν real) are discussed and are used systematically to obtain essential properties of the transformations. It will be shown that (1) and (2) imply functional equations between the transformations and their adjoints and that the latter equations lead to two-mappings rules. Again these turn out to be of considerable value for applications. A good insight is gained into the nature of well-known results, which incidentally are completed, and a number of further theorems are deduced, among them some on Stieltjes transforms."
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The author generalizes to L^p ($p > 1$) some results previously known for L^2 concerning the integral transform

$$\{B_a f\}(x) = \pi^{-1}\int_{-\infty}^{\infty} (t-x)^{-2} f(t) \sin a(t-x)\, dt$$

[see P. Heywood, same J. 28 (1963), 162–168; MR 26 #6680; and also R. Goldberg, ibid. 25 (1960), 200–204; MR 22 #1797]. In addition, he obtains some new formulas relating B_s not only to the Hilbert transform (as was done in the references cited above) but also to Dirichlet's singular integral and the integral transform

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as well. The methods involve use of Hille's theory of H_p functions on the real axis.

R. R. Goldberg (Evanston, Ill.)

Varma, V. K. 3837

On further generalisation of the new transform.

Bull. Calcutta Math. Soc. 55 (1963), 79–87.

The author generalizes the Hankel transform given in the form

$$(1)\qquad f(x)=\int_0^\infty \psi^{\xi}_{\nu,\lambda}(xy)g(y)\,dy,$$

where

$$(2)\quad \psi^{\xi}_{\nu,\lambda}(x)=(1/\mu\mu')\int_0^\infty\int_0^\infty \sqrt{(xy/z)}J_\nu(xy/z)z^{1/\mu-3/2} \times y^{1/\mu'-1/2}\times J_\xi(y^{1/\mu'})J_\lambda(z^{1/\mu})\,dy\,dz,$$

$$0<\mu\leq 1,\quad 0<\mu'\leq 1,$$

still further and discusses special cases, as well as conditions under which the kernels enjoy the self-reciprocal property. The formulas involved are too complicated to be reproduced here.

E. J. Scott (Urbana, Ill.)

Chatterjee, Arun Kumar 3838

On SA-transform.

J. Sci. Res. Banaras Hindu Univ. 14 (1963/64), 68–75.

The transform is defined by

$$f_{k,\mu}(p)=\int_0^\infty (t+p)^{-\sigma}e^{(p+t)/2}W_{k,\mu}(p+t)\Phi(t)\,dt.$$

Some formulae connected with the various f found from differing σ, k and μ are obtained by the use of known formulae for the W-functions.

J. L. Griffith (Lawrence, Kans.)

Chatterjee, Arun Kumar 3839

Certain properties of the WA-transform.

J. Sci. Res. Banaras Hindu Univ. 14 (1963/64), 151–157.

The author obtains a number of series formulae connected with the transform

$$\Phi(p)=\int_0^\infty W_{k,m}(2pt)f(t)\,dt.$$

Due to the large number of printing errors and some incomplete symbolism, the reviewer is unable to follow most of the arguments put forward.

J. L. Griffith (Lawrence, Kans.)

Gutiérrez Suárez, Juan José 3840

Characterization of functions representable by the generalized Whittaker transform. (Spanish)

Collect. Math. 15 (1963), 179–191.

Die verallgemeinerte Whittaker-Transformation ist definiert durch

$$(1)\quad f(z)=\int_0^\infty e^{-zt/2}W_{\beta/2-\alpha+1,(\beta-1)/2}(zt)(zt)^{\beta/2-\gamma-1}\alpha(t)\,dt.$$

Für $q=1,2,\cdots$ wird der Operator eingeführt

$$V_q[f(u)]=\frac{u^{-1}q^{\alpha+\gamma-\beta+1}}{\Gamma(q+1)}[U_q[f(z)]]_{z=q/u}$$

mit

$$U_q[f(z)]=(-1)^q z^{\alpha-\gamma-2-q}D^q[z^{\gamma+2-\alpha}f(z)] \text{ und } D\equiv x^2\frac{d}{dx}.$$

Theorem: Die notwendigen und hinreichenden Bedingungen dafür, daß $f(x)$ durch (1) darstellbar ist mit $\beta-\alpha-\gamma>-1$ und beschränktem $\alpha(t)$ sind: (1) $f(x)\in C^\infty$ in $(0,\infty)$, (2) $f(x)=O(1/x)$ für $x\to\infty$, (3) $|V_q[f(x)]|\leq K$ $(0<x<\infty)$ für $q=1,2,\cdots$.

G. Doetsch (Freiburg)

Liverman, T. P. G. 3841

★Generalized functions and direct operational methods. Vol. I: Non-analytic generalized functions in one dimension.

Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964. xii+338 pp. $10.60.

The object of this book is to present an account of generalized functions and of their applications accessible to students with not much more than a knowledge of ordinary calculus. The present volume treats functions of one variable. The generalized functions considered form a space $\mathfrak{T}_+'$ of functionals on a space $\mathfrak{T}_-$ consisting of infinitely differentiable functions with support bounded above, and are defined, modifying procedures of Mikusiński and Temple, as weak limits of the functionals generated by sequences of piecewise continuous functions with support bounded below. The properties of these spaces necessary for the study of ordinary differential equations are developed, and important elementary examples of generalized functions—principal values, etc.—are treated in detail. A thorough account of the treatment of single ordinary differential equations, as well as systems, is given. There follows an account of the structure of generalized functions, leading to the proof that they are locally of finite order, and to proofs of the equivalence of the definitions given with those of Schwartz. Two final chapters deal with the Laplace transforms of generalized functions and with generalized Fourier series.

The author's exposition is careful both as regards rigour and in the selection of methods of proof adapted to his intended audience: The sections of the book directed to methods of solution of differential equations should be accessible to engineering students, and the whole requires nothing more than elementary real variable analysis; no use is made of complex function theory. The book should serve as an excellent introduction to its subject. It is accompanied by numerous exercises both on the text itself and on other lines of development. {Note a minor error in the determinantal formulae on p. 153.}

J. L. B. Cooper (Pasadena, Calif.)

Dahiya, R. S. 3842

A theorem in operational calculus. (German, French, and Italian summaries)

Mitt. Verein. Schweiz. Versich.-Math. 64 (1964), 175–181.

generalizations to distributions and n dimensions discussed.
J. Blackman (Syracuse, N.Y.)

Kierat, W. 3833

Une remarque sur les logarithmes unilatéraux.

Studia Math. **24** (1964), 281–283.

The method employed by J. Mikusiński [same Studia **12** (1951), 208–224; MR **13**, 740] in proving the theorem: "Let $\alpha_1, \cdots, \alpha_n$ be distinct real numbers and $\beta_1, \cdots, \beta_n$ be non-zero complex numbers; in order for the exponential function $\exp[(\beta_1 s^{\alpha_1} + \cdots + \beta_n s^{\alpha_n})\lambda]$ to exist for $\lambda \geq 0$, it is necessary and sufficient that there exist the individual exponential functions $\exp(\beta_1 s^{\alpha_1}\lambda), \cdots, \exp(\beta_n s^{\alpha_n}\lambda)$" applies only when the functions are considered for all real λ and fails when these functions exist only for $\lambda < 0$. The author gives a modified argument which treats both these cases.
E. J. Scott (Urbana, Ill.)

Vich, Robert 3834

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Für $q=1,2,\cdots$ wird der Operator eingeführt

$$V_q[f(u)]=\frac{u^{-1}q^{\alpha+\gamma-\beta+1}}{\Gamma(q+1)}[U_q[f(z)]]_{z=q/u}$$

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G. Doetsch (Freiburg)

Liverman, T. P. G. 3841

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J. L. B. Cooper (Pasadena, Calif.)

Dahiya, R. S. 3842

A theorem in operational calculus. (German, French, and Italian summaries)

Mitt. Verein. Schweiz. Versich.-Math. **64** (1964), 175–181.

Derivation of a rule for one-sided Laplace integrals $\phi(p)=p\int_0^\infty \exp(-pt)f(t)\,dt$ according to which the transform ϕ of the function $x^{2\mu-\lambda}f(x)$ can be connected with that of $f(x)$ itself. The transform in question proves to be proportional to an integral over an hypergeometric function ${}_3F_2$ (Pochhammer's notation) with a factor containing five gamma functions in front of it. The derivation is given in a very concise form. Two applications of this rule lead to identities connecting the hypergeometric function ${}_3F_2$ with the other hypergeometric function ${}_2F_1$, and with the Legendre function $Q^n_{2\mu+n+1}$, respectively.

H. Bremmer (Eindhoven)

INTEGRAL EQUATIONS

See also 3648, 3769, 3816, 3831, 3884, 3889, 3911, 3912, 4341.

Hutson, V. 3843

Asymptotic solutions of integral equations with convolution kernels.

Proc. Edinburgh Math. Soc. (2) **14** (1964/65), 5–19.

The main portion of this paper is devoted to finding approximate solutions to integral equations of the form

$$(*)\qquad f(x) = \lambda\int_0^a k(x-t)f(t)\,dt,$$

where a is a large parameter. The basic assumptions are, roughly, that k is real, even, and exponentially small at infinity, and that the Fourier transform of k is strictly decreasing for positive values of the argument and has non-vanishing second derivative at 0. It is then shown that if λ lies in any finite range, then there is a τ_1 such that, uniformly on $0\le x\le a$,

$$f(x) = \phi(\min(x, a-x))+O(e^{-a\tau_1/2}),$$

where ϕ is the solution of the corresponding Wiener-Hopf equation ((*) with a replaced by ∞). An equation for the eigenvalues is also found, with a similar exponentially small error. These striking results justify the idea of Carrier [*50 Jahre Grenzschichtforschung*, pp. 13–20, Vieweg, Braunschweig, 1955; MR **18**, 693] that the solutions of (*) may be approximated by the solutions of the Wiener-Hopf equations, and contain as particular cases theorems of the reviewer [Trans. Amer. Math. Soc. **88** (1958), 491–522; MR **20** #4782; ibid. **100** (1961), 252–262; MR **25** #2420], which give weaker asymptotic results concerning only the largest eigenvalues of (*) and the corresponding eigenfunctions. The procedure is based on the observation of Latter [Quart. Appl. Math. **16** (1958), 21–31; MR **21** #2890] that, after taking Fourier transforms, (*) is equivalent to a pair of integral equations, and the asymptotic results are obtained by adroit but simple movement of the contours of integration.

H. Widom (Ithaca, N.Y.)

Williams, W. E. 3844

The solution of dual series and dual integral equations.

Proc. Glasgow Math. Assoc. **6** 123–129 (1964).

The equations

$$(1)\qquad L_1(x,\lambda)A(\lambda) = \varphi_1(x),\qquad \alpha<x<\beta,$$

$$(2)\qquad L_2(x,\lambda)A(\lambda) = \varphi_2(x),\qquad \beta<x<\gamma,$$

are symbolic equations for dual integral equations with $L_i(x,\lambda)=\int K_i(x,\lambda)A(\lambda)\,d\lambda$ and for dual series with $L_i(x,\lambda)=\sum_\lambda K_i(x,\lambda)A(\lambda)$, $i=1,2$, with φ_i, K_i, $i=1,2$, known functions. If L_1 and L_2 are operators with the properties (a) L_2^{-1} exists; (b) there exist operators M_1 and M_2 such that $M_1(x,u)f(u)=\int_a^x N_1(x,u)f(u)\,du$, $M_2(x,u,\delta)f(u)=\int_\delta^x N_2(x,u)f(u)\,du$, $\alpha<\delta\le\gamma$, N_1 and N_2 known functions, such that

$$L_1(x,\lambda)L_2^{-1}(\lambda,t)g(t) = M_1(x,u)M_2(u,v)g(v),\quad \alpha<x<\gamma,$$

then the determination of the solution of (1) and (2) formally reduces to the successive solution of two integral equations. (We have an analogous situation if L_1^{-1} rather than L_2^{-1} exists.) The author claims that all known types of dual integral and series equations are such that they determine a factorization of the above form and that this is the basis of all known methods (a variety of which exist as indicated in the references) for solving (1) and (2).

A. E. Danese (Buffalo, N.Y.)

Rakovščik, L. S. 3845

On calculating the index of a system of almost difference integral equations. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 904–909.

This work is directly related to the author's earlier papers [same Ž. **3** (1962), 250–255; MR **25** #2401; Vestnik Leningrad. Univ. **16** (1961), no. 13, 52–72; MR **25** #2453]. By imposing certain natural restrictions on the integral equations considered, the author shows that certain hypotheses used for the proof of one of his earlier theorems are superfluous.

H. P. Thielman (Alexandria, Va.)

Žilelkin, Ja. M. 3846

On the approximate solution of integral equations. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 749–753.

The author considers the Fredholm integral equation $u(P)=\lambda\int_{G_s}K(P,Q)u(Q)\,dQ+f(P)$, with G_s the s-dimensional unit hypercube, f and K periodic with period 1 in each variable, f from a given class, K with a given singularity at $P=Q$. At the points of a mesh, the integrals in the first l terms of the Neumann series for u (the iterated kernels are supposed to be known) are replaced by complicated sums of order N which are of the type considered by Korobov [*Number-theoretic methods in approximate analysis* (Russian), Fizmatgiz, Moscow, 1963; MR **28** #716]. The remainder is shown to satisfy, with an error of $O(N^{-\alpha+\varepsilon})$, $\varepsilon>0$, a certain linear algebraic system of order N_1 when l and N_1 are chosen in dependence of N; α is the exponent which determines the decrease of the Fourier coefficients of f.

H. J. Stetter (Munich)

Lando, Ju. K. 3847

A boundary-value problem for linear integro-differential equations of Volterra type in the case of disjoint boundary conditions. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1961**, no. 3 (22), 56–65.

The author applies a result of M. V. Keldyš [Dokl. Akad. Nauk SSSR **77** (1951), 11–14; MR **12**, 835] to obtain completeness and expansion theorems for the eigenfunctions

and associated functions of integro-differential equations of the form

$$u^{(n)}(x)+\sum_{i=0}^{n-2} a_i(x)u^{(i)}(x)+\sum_{i=0}^{n-1}\int_0^x A_i(x,t,\rho)u^{(i)}(t)\,dt + \rho^{n+1}\int_0^x u(t)\,dt = f(x),$$

under boundary conditions of the form

$$\sum_{k=0}^{n-1} \alpha_{jk}u^{(k)}(0) = 0 \qquad (j = 1, 2, \cdots, m),$$

$$\sum_{k=0}^{n-1} \beta_{jk}u^{(k)}(1) = 0 \qquad (j = m+1, m+2, \cdots, n).$$

F. Smithies (Cambridge, England)

Dombrovskaja, I. N.; Ivanov, V. K. 3848

Ill-posed linear equations and exceptional cases of equations of convolution type. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, no. 4 (41), 69–74.

The integral equation

$$(*)\quad \lambda\varphi(x)+\int_0^\infty k_1(x-t)\varphi(t)\,dt+\int_{-\infty}^0 k_2(x-t)\varphi(t)\,dt = f(x),$$

where $k_1(x), k_2(x) \in L(-\infty, \infty)$; $f(x), \varphi(x) \in L_2(-\infty, \infty)$ and $\lambda = \lambda_1$, $x > 0$, $\lambda = \lambda_2$, $x < 0$, was considered by Gahov and Smagina [Izv. Akad. Nauk SSSR Ser. Mat. **26** (1962), 361–390]. For such an equation the coefficient in the associated Riemann boundary problem may have zeros or poles on the real axis, in which case it will no longer be sufficient, for the solvability of (*), to have $f(x)$ orthogonal to all the solutions of the corresponding homogeneous adjoint equation (Noether's theorem). The authors confirm this fact employing methods of functional analysis and go on to supply an algorithm for approximating the solution of (*) in terms of Laguerre polynomials.

J. F. Heyda (King of Prussia, Pa.)

Gorbačuk, V. I. 3849

An integral representation of Hermitian indefinite kernels (the case of several variables). (Russian)

Ukrain. Mat. Ž. **16** (1964), 232–236.

On considère le domaine $G = G^{(1)} \times \cdots \times G^{(n)}$, $G^{(i)} = (a_i, b_i)$. Un noyau hermitien $K(x, y)$ est indéfini, à k carrés négatifs, si pour chaque m, la forme $\sum_{j,k=1}^m K(x_j, x_k)\xi_k\bar{\xi}_j$ $(x_j \in G^{(j)})$ a au plus k carrés négatifs, et au moins pour un m, exactement k carrés négatifs.

On désigne (a) par $L^{(i)}$ les expressions:

$$L^{(i)}u = \sum_{k=1}^{r_i} a_{k,i}d^ku/dt^k, \qquad t \in [a_i, b_i];$$

(b) par $P_i(t)$ les polynômes minimaux de $L^{(i)}$; (c) par

$$\Phi_i(x, y) = [P_i(L^{(i)})]_{x_i}[P_i(L^{(i)})]_{y_i}K(x, y) \quad (i = 1, 2, \cdots, n);$$

(d) par $\Phi(x, y) = \sum_1^n \Phi_i(x, y)$; (e) par $\chi_s^{(j)}(x_j, \lambda_j)$ $(s = 0, 1, \cdots, r_{j-1})$ un système fondamental de solutions de $L^{(j)}u = \lambda_j u$ $(x_j \in G^{(j)})$ satisfaisant à $\partial^k\chi_m(x_j, \lambda_j)/\partial x_j^k|_{x_j=a_j} = \delta_{mk}$; (f) par

$$X_\alpha(x, \lambda) = \chi_{\alpha_1}(x_1, \lambda_1)\cdots\chi_{\alpha_n}(x_n, \lambda_n).$$

Sous certaines conditions restrictives pour $L^{(i)}$, $K(x, y)$ admet la représentation:

$$K(x, y) = T_\rho(x, y) + \int_{E_n} \frac{\sum_{\alpha,\beta\in A}[X_\alpha(x, \lambda)\bar{X}_\beta(y, \lambda) - S_\rho^{(\alpha,\beta)}(\lambda; x, y)]}{\sum_1^n |P_i(\lambda_i)|^2}\, d\sigma_{\alpha\beta}(\lambda),$$

où $S_\rho^{(\alpha,\beta)}$ est une correction qui fait l'expression sous l'intégrale régulière et T_ρ une solution de

$$\sum_1^n [\bar{P}_i(L^{(i)})]_{x_i}[P_i(\bar{L}^{(i)})]_{y_i}u(x, y) = 0,$$

$d\sigma_{\alpha\beta}(\lambda)$ une matrice positive définie et A un domaine d'indices, convenablement choisis.

A. Haimovici (Iaşi)

Mullikin, T. W. 3850

Chandrasekhar's X and Y equations.

Trans. Amer. Math. Soc. **113** (1964), 316–332.

The problem of radiative transfer in homogeneous plane-parallel atmospheres leads to certain X and Y functions, determined by two nonlinear integral equations. The solutions of these equations may be expressed by those of an auxiliary linear integral equation. A system of two simultaneous linear equations for X and Y is derived under specified assumptions, having a unique solution. It is then shown that X and Y satisfy a system of two singular linear integral equations. These are equivalent to a system of two Fredholm equations. These may be solved by interation, and this solution is elaborated. The solutions are expressed by means of an H-function, which may be computed.

M. J. O. Strutt (Zürich)

FUNCTIONAL ANALYSIS

See also 3597, 3604, 3620, 3693, 3710, 3750–3752, 3805, 3841, 3849, 4420–4426, 4428–4430, 4432, 4450.

Kelley, J. L.; Namioka, Isaac 3851

★Linear topological spaces.

With the collaboration of W. F. Donoghue, Jr., Kenneth R. Lucas, B. J. Pettis, Ebbe Thue Poulsen, G. Baley Price, Wendy Robertson, W. R. Scott, Kennan T. Smith. The University Series in Higher Mathematics. *D. Van Nostrand Co., Inc., Princeton, N.J.*, 1963. xv + 256 *pp.* \$8.00.

This book is the result of the efforts of many authors who happened to agree (in 1953) on how to arrange the theory of topological linear spaces in such a way as to make the more recent (at that time) results on duality appear as the natural consequences of the preliminary work. For example, having observed how a criterion of Shmulian's was used [the reviewer, Duke Math. J. **14** (1947), 787–794; MR **9**, 241] to improve a theorem of Mackey's, they make it one of the initial propositions of Chapter 5 (on duality, which occupies almost half the book) and use it systematically. By thus boldly, but properly, permuting the original order of entrance of basic ideas, they produced a body of ideas knit together by a few basic notions but possessing conspicuous concavities. Some of these they filled with such ideas as hypercompleteness and other concepts not

observed by the reviewer in any other text. Incidentally, this concept is due to J. L. Kelley, but characteristically, the text does not say so. To tell the truth, it is unfortunate that the authors found no recruit to take charge of documentation.

In a future edition we should have at least a reference from which Grothendieck's paper could be located [C. R. Acad. Sci. Paris 230 (1950), 605–606; MR 12, 715], and a reference (as given above) locating Mackey's and Shmulian's theorems. Then among other benefits, the reader would appreciate the authors' skill in reorganization and reformulation.

It is remarkable that such a joint effort should have produced such a valuable book, which will probably be the best text for a graduate course covering all that material, excepting operators in Hilbert space and topological linear algebras, which an aspirant to functional analysis ought to learn. This is in a large part due to the excellent and non-routine exercises, mainly added when the manuscript was revised in 1961.

R. Arens (Los Angeles, Calif.)

Moreau, Jean Jacques 3852

★Étude locale d'une fonctionnelle convexe.

Université de Montpellier, Montpellier, 1963. 25 *pp.*

These mimeographed notes are a discussion of a general convex functional and its "gradient" in a linear topological space, sometimes taken as locally convex. The "gradient" is sometimes the strong or weak differential, or more generally a multiple-valued function defined by the slopes of supporting hyperplanes to the set of points above the graph. A number of "rules of calculation" of the gradient are given, including theorems on its existence and relationships with the conjugate (dual) convex functional of Fenchel. Special attention is given to the gradient of the sum of two convex functions, the Γ-convolution of two convex functions (i.e., the dual of the sum of their duals), and the inf-convolution, defined as

$$(f\nabla g)(x) = \inf_{u+v=x} [f(u)+g(v)],$$

which appears often in the theory of dynamic programming.

The spirit of the notes is a development in fuller detail and in slightly greater generality of earlier work of the author reported in a series of notes [cf., e.g., C. R. Acad. Sci. Paris 256 (1963), 5047–5049; MR 27 #4102].

G. J. Minty (New York)

Brøndsted, Arne 3853

Conjugate convex functions in topological vector spaces.

Mat.-Fys. Medd. Danske Vid. Selsk. 34, *no.* 2, 27 *pp.* (1964).

Let X and Y be paired topological linear spaces so that each is the dual of the other. For a real-valued function f with domain D in X one defines D' in Y as the set of all Y such that $xy-f(x)$ is bounded on D. One defines $f'(y)$ as $\sup(xy-f(x): x\in D)$. Then f' is the conjugate of f. The theory of convex conjugate functions was developed for finite-dimensional spaces by W. Fenchel, and extended independently by W. L. Jones and J. J. Moreau to infinite-dimensional spaces. Their investigations are continued, and to some extent, their results are included in the present paper.

R. Arens (Los Angeles, Calif.)

Diaz, J. B.; Výborný, R. 3854

A mean value theorem for strongly continuous vector valued functions.

Czechoslovak Math. J. 14 (89) (1964), 322–323.

In this announcement the authors state the following mean value theorem, which is a generalization of the theorem proved by Aziz and Diaz [Contributions to Differential Equations 1 (1963), 251–269; MR 27 #247]. Suppose f is defined on $[a, b]$ with values in a normed vector space and suppose f is strongly continuous. Then there is a number $c\in(a, b)$ such that either, whenever both $h>0$ and $a\le c+h\le b$, one has

$$\left\|\frac{f(b)-f(a)}{b-a}\right\| \le \left\|\frac{f(c+h)-f(c)}{h}\right\|,$$

or, whenever both $h>0$ and $a\le c-h\le b$, one has

$$\left\|\frac{f(b)-f(a)}{b-a}\right\| \le \left\|\frac{f(c)-f(c-h)}{h}\right\|.$$

W. P. Ziemer (Bloomington, Ind.)

Edwards, R. E. 3855

Weak convergence of vector-valued series and integrals.

J. Austral. Math. Soc. 3 (1963), 159–166.

Let E, F, G be separated locally convex topological vector spaces such that $F\subset G$ and the injection map $i: F\to G$ is continuous. Let E', F', G' denote the respective duals of E, F and G. The paper discusses weak convergence of vector-valued series and integrals. The problem discussed in the case of series is concerned with the convergence of series of the type (1) $\sum_k\langle x, e_k'\rangle g_k$, where $x\in E$, $\{e_k'\}$ and $\{g_k\}$ are sequences of elements of E' and G, respectively. Under certain conditions, it is shown that the convergence of (1) for each $x\in E$ and for the weakened topology $\sigma(G, G')$ of G, each sum being in F, implies the convergence for the weakened topology $\sigma(F, F')$ of F.

Ky Fan (Evanston, Ill.)

Garling, D. J. H. 3856

Locally convex spaces with denumerable systems of weakly compact subsets.

Proc. Cambridge Philos. Soc. 60 (1964), 813–815.

Einige von Dieudonné [Proc. Amer. Math. Soc. 8 (1957), 367–372; MR 18, 746] stammende Sätze werden verschäft. Beispiel: Jeder quasitonnelierte lokalkonvexe Raum, in dem es ein abzählbares Fundamentalsystem von kompakten Teilmengen gibt, ist ein Montel-(DF)-Raum.

A. Pietsch (Berlin)

Hayes, A. 3857

Sequentially pointwise continuous linear functionals.

Fund. Math. 55 (1964), 67–75.

Let E be a linear subspace of R^X, the linear space of all real-valued functions on a non-empty set X (R is the real line). A consequence of the duality theory of linear spaces is that each linear functional L on E which is continuous with respect to pointwise convergence, on X, of nets in E is of the form $L(f)=\sum_{i=1}^n \lambda_i f(x_i)$, where λ_i are n real numbers for some n, $f\in L$ and $x_i\in X$. The author shows that a similar result is true for linear lattices $E\subset R^X$ and linear functionals L which are continuous with

respect to pointwise convergence, on X, of sequences in E (or sequentially continuous). More specifically, he proves the following. Let L be a sequentially continuous linear functional on a linear lattice $E \subset R^X$. Then there exists a unique linearly independent family $S(L)$ of sequentially continuous lattice linear functionals, or the negative of lattice linear functionals, such that for each $f \in E$, $L'(f) = 0$ for all but a finite number of L' in $S(L)$ and $L(f) = \sum\{L'(f): L' \in S(L)\}$. He also shows that under strengthened conditions $S(L)$ is finite. Whether or not $S(L)$ is infinite is not known. Similar results for lattice algebras are also derived. As the author points out, results closely related to these have also been obtained by J. R. Isbell and E. S. Thomas, Jr. [Proc. Amer. Math. Soc. **14** (1963), 644–647; MR **27** #1816] under stronger hypotheses.

T. Husain (Hamilton, Ont.)

Keller, H. H. 3858

Differenzierbarkeit in topologischen Vektorräumen.

Comment. Math. Helv. **38** (1964), 308–320.

This paper is concerned with the generalization of the notions of the Gâteaux and Fréchet derivatives to functions defined between Hausdorff real linear topological spaces which are usually (but not always) locally convex. Several notions of differentiability are considered pertaining to different conditions on the approach of the remainder term to zero. The interrelation between these types of remainder terms is analyzed, and the usual formulas and results concerning differentiability are extended under various conditions which are too complicated to describe here. *R. G. Bartle* (Urbana, Ill.)

Kruse, Arthur H. 3859

Badly incomplete normed linear spaces.

Math. Z. **83** (1964), 314–320.

A normed linear space is said to be subcomplete provided it is the image of a Banach space under a continuous linear transformation (or, equivalently, a biunique continuous linear transformation). And Y is said to be badly incomplete provided every subcomplete subspace is of smaller dimension than Y. The paper is motivated by an earlier observation that if X is a separable normed linear space and L is an infinite-dimensional Banach space, then there exists a biunique continuous linear transformation of X into L. Of course, this may fail when "Banach" is replaced by "normed linear", for then it may happen that $\dim X = c > \dim L$. The author's Theorem A shows that the failure may occur even when $\dim L = c$. Indeed, each separable infinite-dimensional Banach space Y contains 2^c linear subspaces L such that whenever T is a continuous linear transformation of a Banach space X into Y, then (a) $\dim T(X) < \aleph_0$ if $T(X) \subset L$, and (b) $\dim (L \cap T(X)) = c$ if $\dim T(X) \geq \aleph_0$. In particular, $\dim L = c$ and L is badly incomplete. Theorem A is generalized in various ways which are too technical to describe here.

Some of the author's constructions, lemmas and corollaries are of independent interest. In particular, he shows that a linear space Y is a Banach space (or alternatively, a Hilbert space) under some norm if and only if $\dim Y < \aleph_0$ or $(\dim Y)^{\aleph_0} = Y$. This implies a theorem of Goffman [Bull. Amer. Math. Soc. **49** (1943), 611–614; MR **5**, 149] to the effect that if the generalized continuum hypothesis holds, then a linear space Y is Banach-normable if and only if $\dim Y$ is not the limit of a strictly increasing sequence of cardinal numbers.

Victor Klee (Seattle, Wash.)

Mirman, B. A. 3860

A basis criterion in Hilbert space. (Russian)

Sibirsk. Mat. Ž. **4** (1963), 1433–1435.

The author proves various forms of a criterion stating that a minimal double total set $\{\varphi_i\}$ [see S. Kaczmarz and H. Steinhaus, *Theorie der Orthogonalreihen*, Monogr. Mat., Band 6, Warsaw, 1935] of elements in a separable Hilbert space H be a basis. A criterion due to M. M. Grinbljum [Dokl. Akad. Nauk SSSR **31** (1941), 428–432; MR **3**, 49] is used in the proofs. A formulation of the author's criterion is of interest, which consists in a translation of that criterion to the uniform boundedness of secants of (maximal) angles between the closed subspaces L_n and G_n of H determined by the n first elements of $\{\varphi_i\}$ and the n first elements of the corresponding biorthogonal set $\{\psi_i\}$. This work may be applied to simplify some proofs in the papers of N. K. Bari [Moskov. Gos. Univ. Učen. Zap. **148** (1951), Mat. 4, 69–107; MR **14**, 289] and K. I. Babenko [Dokl. Akad. Nauk SSSR **62** (1948), 157–160; MR **10**, 249]. A closely related result may be found in the paper by M. M. Grinbljum and L. A. Gurevič [ibid. **30** (1941), 287–289; MR **2**, 313].

G. K. Kalisch (Minneapolis, Minn.)

Orihara, Masae 3861

Linear functionals on a Banach space with semi-norms.

Yokohama Math. J. **11** (1963), 13–22.

A "many-norms space" (as defined in an earlier paper [same J. **10** (1962), 1–4; MR **27** #2830]) is a Banach space $\langle X, \|\ \|\rangle$ provided with a family $\|\ \|_\alpha$ $(\alpha \in \Lambda)$ of semi-norms such that $\|x\| = \sup \|x\|_\alpha$ for every x in X. The author continues the work and asserts fourteen theorems and propositions.

{Reviewer's comments: Both papers contain a peculiar mixture of known facts and false statements. Specifically, Proposition 2 of the paper under review (asserting that if $\|x_n - x\|_\alpha^* \to 0$ for each $\alpha \in \Lambda$, then the norms $\|x_n\|$ are bounded) is false, and the other theorems and propositions are known (and proved in essentially the same form in papers quoted by the author and in papers by A. Grothendieck [C. R. Acad. Sci. Paris **230** (1950), 605–606; MR **12**, 715] and A. Alexiewicz and the reviewer [Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **6** (1958), 135–139; MR **20** #6643]). The full list of errors is too long to be quoted here.} *Z. Semadeni* (Poznań)

Ruckle, William H. 3862

The infinite sum of closed subspaces of an F-space.

Duke Math. J. **31** (1964), 543–554.

The first part of this paper studies various conditions under which the sum of a collection of closed linear subspaces of an F-space will be closed. The second part is a discussion of Schauder decompositions of an F-space. A sequence $\{M_i\}$ of closed linear subspaces of an F-space X with $\sum_{i=1}^\infty M_i = X$ is called a Schauder decomposition of X if $x_i \in M_i$ $(i = 1, 2, \cdots)$ and $\sum_{i=1}^\infty x_i = 0$ imply $x_i = 0$ for each i. The final part of the paper deals with boundedly

complete Schauder decompositions, i.e., Schauder decompositions $\{M_i\}$ of X having the property that for each sequence $\{x_i\}$ with $x_i \in M_i$ ($i = 1, 2, \cdots$), the convergence of the series $\sum_{i=1}^{\infty} x_i$ in X is equivalent to the boundedness of the sequence of the partial sums $\sum_{i=1}^{n} x_i$ ($n = 1, 2, \cdots$).

Ky Fan (Evanston, Ill.)

Simons, S. 3863

Boundedness in linear topological spaces.

Trans. Amer. Math. Soc. **112** (1964), 169–180.

The author introduces and proves equivalent two classifications of locally bounded linear topological spaces, i.e., spaces in which there is a bounded neighbourhood of the origin. First, if $0 < k \leq 1$, a (non-trivial) non-negative function f defines a k-pseudometric on a vector space X (on putting $\rho(x, y) = f(x-y)$) if and only if $f(x+y) \leq f(x)+f(y)$ and $f(\lambda x) = |\lambda|^k f(x)$ for all scalars λ and all x, y in X. It is proved that X is locally bounded if and only if its topology is defined by a k-pseudometric; the upper bound p of such k is called the type of the space X. Secondly, a (non-trivial) non-negative function g on a vector space X is a quasi-seminorm if and only if $g(\lambda x) = |\lambda| g(x)$ and, for some fixed b, $g(x+y) \leq b(g(x)+g(y))$ for all scalars λ and all x, y in X. The smallest such b is the multiplier of g. Again, X is locally bounded if and only if its topology is defined by a quasi-seminorm; the lower bound β of multipliers b of such quasi-seminorms is called the multiplier of X. The author shows that $\beta = 2^{(1/p)-1}$.

A linear topological space is called an upper bound space if and only if its topology is the upper bound of a family of locally bounded topologies, or, equivalently, defined by a family (f_j) of continuous k_j-pseudometrics. It is shown that such a space X has the useful property that a subset B is bounded if and only if each $f_j(B)$ is bounded.

A. P. Robertson (Glasgow)

Stampacchia, Guido 3864

Formes bilinéaires coercitives sur les ensembles convexes.

C. R. Acad. Sci. Paris **258** (1964), 4413–4416.

A generalization of a theorem of Lax and Milgram [*Contributions to the theory of partial differential equations*, pp. 167–190, Princeton Univ. Press, Princeton, N.J., 1954; MR **16**, 709] is presented. Let U be a closed convex subset of a real Hilbert space X and for $u \in U$ let $V(u) = \{v \in X : u + \varepsilon v \in U$ for some $\varepsilon > 0\}$. If $a(u, v)$ is a real continuous bilinear form on $X \times X$ which satisfies $a(u, u) \geq c\|u\|^2$, $c > 0$, for all $u \in X$, and if f is in the strong dual of X, then there exists a unique $u \in U$ such that $a(u, v) \geq f(v)$ for all $v \in V(u)$. Applications are given in the completion of the space of continuously differentiable functions on $\bar{\Omega}$, the closure of a bounded open set in R_n, with respect to the norm determined by the sum of the $L_2(\Omega)$ norms of functions and their first partial derivatives.

T. R. Jenkins (Los Altos, Calif.)

Veic, B. E. 3865

Some characteristic properties of unconditional bases. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 509–512.

Let $\{x_k\}$ be a basis of a Banach space E; let $f_i \in E'$ be such that $f_i(x_k) = \delta_{ik}$. The author states (partly with short proofs) various conditions which are necessary and sufficient for $\{x_k\}$ to be an unconditional basis. Some of them are (1) for any $n_1 < n_2 < \cdots$, the closed subspace generated by $\{x_{n_k}\}$ is complemented; (2) for some M, $\|\sum_1^{\infty} f_k(x) x_k\| \leq M \|\sum_1^{\infty} |f_k(x)| x_k\|$ for all $x \in E$; (3) for some m, M,

$$m\left\|\sum_1^n |f_k(x)| x_k\right\| \leq \left\|\sum_1^n f_k(x) x_k\right\| \leq M\left\|\sum_1^n |f_k(x)| x_k\right\|$$

for all $x \in E$ and all $n = 1, 2, \cdots$; (4) $\{x_k\}$ is "Φ-stable", which means that $\{u_k\}$ is a basis whenever $\sum \|u_k - x_k\| \|f_k\|$ converges and u_k are independent in the sense that $\sum c_k u_k$ cannot converge to 0 unconditionally unless $c_k = 0$, $k = 1, 2, \cdots$.

M. Katětov (Prague)

Mitjagin, B. S.; Švarc, A. S. 3866

Functors in categories of Banach spaces. (Russian)

Uspehi Mat. Nauk **19** (1964), no. 2 (116), 65–130.

The categories $\mathfrak{K}$ studied here all have Banach spaces as the objects of the category and the linear continuous mappings as the mappings of the category. $\mathfrak{B}$, $\mathfrak{R}$, and $\mathfrak{H}$ are, respectively, the categories of all Banach spaces, all reflexive Banach spaces, and all Hilbert spaces. If X and $Y \in \mathfrak{K}$, then $(X \to Y)$ will represent the Banach space of all linear continuous mappings from X into Y. Attention is restricted to covariant functors of one variable. Examples are (1) Ω_A defined by: for each X in K, $\Omega_A(X) = (A \to X)$; for each α in $(X \to Y)$ and γ in $(A \to X)[\Omega_A(\alpha)](\gamma) = \alpha \circ \gamma$ (functional composition); (2) Σ_A defined by: for each X in K $\Sigma_A(X) = A \hat{\otimes} X$ (tensor product completed under the greatest cross norm); for each α in $(X \to Y)$ $\Sigma_A(\alpha) = 1_A \hat{\otimes} \alpha$; (3) l^p defined by: for each X in K $l^p(X)$ is the space of sequences $x = \{x_n\}$ of elements of X such that $\|x\| = (\sum \|x_n\|^p)^{1/p} < \infty$; for each α, $[l^p(\alpha)](x) = \{\alpha(x_n)\}$.

A category $\mathfrak{K}$ is regular if $X \in \mathfrak{K}$ implies $X^* \in \mathfrak{K}$. $\mathfrak{K}$ is reflexive if also each object X of $\mathfrak{K}$ is a reflexive Banach space, $X^{**} = X$.

A mapping T between functors F and H on a category $\mathfrak{K}$ is a family of linear continuous functions T_X, one for each object X of the category, such that $\sup_X \|T_X\| < \infty$, for each X in $\mathfrak{K}$, $T_X \in (F(X) \to H(X))$, and for each α in $(X \to Y)$ the diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{T_X} & H(X) \\ {\scriptstyle F(\alpha)}\downarrow & & \downarrow{\scriptstyle H(\alpha)} \\ F(Y) & \xrightarrow{T_Y} & H(Y) \end{array}$$

(For example, every τ in $(A \to B)$ determines and is determined by a mapping T between the functors Σ_A and Σ_B.)

Functors F and G are called isometric [isomorphic] if every T_X is an isometry [isomorphism]. A functor G is dual to F, $G = \mathcal{D}F$, if for each A in K, $G(A) = (F \to \Sigma_A)$ and if for each γ in $(F \to \Sigma_A)$, α in $(A \to B)$ and object X in K,

$$[G(\alpha)\gamma]_X \simeq [1_X \hat{\otimes} \alpha]\gamma.$$

Theorem 1: For each two functors F and G the spaces of mappings $(F \to \mathcal{D}G)$ and $(G \to \mathcal{D}F)$ are isometric.

A functor F is reflexive if the natural isomorphism of F into $\mathcal{D}\mathcal{D}F$ is onto. Direct and inverse limits of "spectra" of functors are defined. Theorem 4: If F is the inductive limit of $(F_\alpha, \Pi_\alpha^\beta)$, then $\mathcal{D}F$ is the projective limit of $(\mathcal{D}F_\alpha, \partial\Pi_\alpha^\beta)$. Theorem 5 [6]: Each [reflexive] functor is an

inductive limit of functors of type Ω_A [type Σ_A]. Theorem 9: In a regular category, for each functor F, $\mathcal{D}F$ is a subfunctor of U, where, for each X in K, $U(X)=[F(X^*)]^*$.

This takes us by sampling through § 5 of the paper. § 6 is concerned with ideals in the commutative normed algebra m of bounded numerical sequences, with the functors defined by these ideals of m, such as l^p, and with cases where the dual of such a sequence functor is again a sequence functor.

§ 7, functors in categories of Hilbert spaces, starts with a fixed Hilbert space and discusses "normed ideals" Ω in the ring $\Omega_H(H)=(H\to H)$. The first part of the section matches ideals in m with symmetric ideals in $\Omega_H(H)$, so that l^1 corresponds to the class of nuclear operators and l^2 to the Hilbert-Schmidt class. In general, left normed ideals determine and are determined by functors and (Theorem 16) the dual to a functor determined by a left ideal Φ is the functor determined by an appropriately defined dual ideal Φ'.

§ 8 has results paralleling those of § 6, but beginning with Lebesgue measurable functions instead of sequences. § 9 describes the spaces of mappings of some of these functors into themselves. § 10 discusses Hilbert functors, which are defined in terms of the tensor square of T. Examples are l^2 and L^2 from §§ 6 and 8. § 11 tells of some interesting problems which still remain open.

M. M. Day (Urbana, Ill.)

Mac Nerney, J. S. 3867

Note on successive approximations.

Rend. Circ. Mat. Palermo (2) **12** (1963), 87–90.

This note proves a fixed-point theorem in an abstract setting. M is a complete metric space with metric Δ; S is a nondegenerate linearly ordered space; $D(c)$ is a class of functions γ on S to the non-negative reals satisfying the condition $\gamma(x)\ge\gamma(y)\ge\gamma(c)=0$ if $x\ge y\ge c$ or $x\le y\le c$; $F(c,A)$ is a class of functions on S to M such that $F(A)=c$, closed under uniform convergence on every interval of S and such that for each pair $u(x)$, $v(x)$ of $F(c,A)$ there exists a $\gamma(x)$ in $D(c)$ with $\Delta(u(x),v(x))\le\gamma(x)$ for x in S; $K[u]$ is an operation on $F(c,A)$ to $F(c,A)$ satisfying a Lipschitz-type condition, i.e., there exists a $\lambda(x)$ in $D(c)$ such that $\Delta(u(x),v(x))\le\gamma(x)$ implies $\Delta(K[u,x],K[v,x])\le L\int_c^x\gamma\,d\chi$, for each x in S, with $L\int$ the left Cauchy-Stieltjes integral; then there exists a unique U in $F(c,A)$ such that $K[U]=U$. The U is obtained by iteration. A generalization of the existence theorem for systems of linear differential equations, the theorem extends results of the author [Illinois J. Math. 7 (1963), 148–173; MR **26** #1726].

T. H. Hildebrandt (Ann Arbor, Mich.)

Merklen, Héctor 3868

Tensor product of ordered vector spaces. (Spanish)

Univ. Nac. Ingen. Inst. Mat. Puras Apl. Notas Mat. **2** (1964), 41–57.

Let X and Y be vector spaces over the real field preordered by positive cones ξ and η, respectively. A preorder ζ in the tensor product $Z=X\otimes Y$ is compatible with (X,ξ) and (Y,η) provided ζ is $w(Z,Z^*)$-closed and $\pi(\xi\times\eta)\subset\zeta$, where π is the canonical mapping $(x,y)\to x\otimes y$. The cone ζ_π made up of the weakly closed conical hull of $\xi\otimes\eta$ is called the projective cone of $X\otimes Y$. If ξ^* and η^* are the sets of positive linear forms on X and Y, then the biprojective cone ζ_ϵ of Z is defined to be the weakly closed conical hull of the set

$$\{\sum x_i\otimes y_i\,|\,\sum f(x_i)g(y_i)\ge 0,\quad \forall_{f\in\xi^*},\quad \forall_{g\in\eta^*}\}.$$

Both of these cones are compatible. The author proves a number of immediate results concerning these cones. He also shows that the biprojective cone of $X^*\otimes Y^*$ is related in a very natural way to the projective cone of $X\otimes Y$. He notes that the space of linear transformations $T:X\to Y$ with finite range can be identified with $X^*\otimes Y$ and shows that in $X^*\otimes Y$ the natural positive cone, that is, the one composed of T such that $T(\xi)\subset\eta$, is the biprojective cone of $X^*\otimes Y$. Similar results are found for operators $T:X\to Y^*$ and $T:X^*\to Y$. Finally, if Q_i $(i=1,2)$ is a compact Hausdorff space and $C(Q_i)$ is the ordered vector space of continuous functions on Q_i, then the order induced on $C(Q_1)\otimes C(Q_2)$ by that in $C(Q_1\times Q_2)$ is the biprojective order.

B. Brainerd (Toronto, Ont.)

Gel'fand, I. M. [**Gel'fand, I. M.**]; 3869
Shilov, G. E. [**Šilov, G. E.**]

★**Generalized functions. Vol. I: Properties and operations.**

Translated by Eugene Saletan.

Academic Press, New York-London, 1964. xviii+423 *pp.* $12.00.

A translation of the original Russian edition of 1958 [Fizmatgiz, Moscow, 1958; MR **20** #4182]. Correction of typographical errors has been made and references to the Western literature have been added.

Granirer, Edmond 3870

A theorem on amenable semigroups.

Trans. Amer. Math. Soc. **111** (1964), 367–379.

A left- [right-] invariant mean for a semigroup G is a positive linear functional of norm 1 on the Banach space $m(G)$ of bounded real functions on G with the supremum norm that is invariant under left [right] translations. The main result of the paper is that if the two linear manifolds spanned respectively by the left- and the right-invariant means for G are non-trivial and finite-dimensional, then each is of dimension 1, G contains a finite group that is a two-sided ideal, and the left-invariant mean M is of the form $f\to\sum_{g\in G}\alpha(g)f(g)=M(f)$, where $f\in m(G)$ and $\alpha:G\to\mathbf{R}$ is such that $\sum_{g\in G}|\alpha(g)|<\infty$. This sharpens a result of Luthar [Illinois J. Math. **3** (1959), 28–44; MR **21** #2184]. For countable semigroups the author has deduced the same conclusions from weaker assumptions [ibid. 7 (1963), 32–48; MR **26** #1744]. In the spirit of a conjecture of Civin and Yood [Pacific J. Math. **11** (1961), 847–870; MR **26** #622] it is deduced that, for any commutative semigroup G without finite ideals, the Banach algebra $m(G)^*$ (the second dual of the convolution algebra $l_1(G)$) has infinite-dimensional radical.

D. A. Edwards (Oxford)

Persson, Arne 3871

Compact linear mappings between interpolation spaces.

Ark. Mat. **5**, 215–219 (1964).

Two Banach spaces E_0 and E_1 form an interpolation pair if they are continuously imbedded in some separated

linear topological space $\mathscr{E}$. If A_0, A_1 and E_0, E_1 are interpolation pairs in $\mathscr{A}$ and $\mathscr{E}$, respectively, then A_θ, E_θ are called interpolation spaces of exponent θ $(0<\theta<1)$ with respect to A_0, A_1 and E_0, E_1 if the topological inclusions $A_0\cap A_1\subset A_\theta\subset A_0+A_1$ and $E_0\cap E_1\subset E_\theta\subset E_0+E_1$ hold and if each linear mapping T from $\mathscr{A}$ into $\mathscr{E}$ which maps A_i continuously into E_i $(i=0,1)$ maps A_θ continuously into E_θ such that

$$\|T\|_{A_\theta\to E_\theta}\leq\|T\|_{A_0\to E_0}^{1-\theta}\cdot\|T\|_{A_1\to E_1}^{\theta}.$$

For given pairs A_0, A_1 and E_0, E_1 there exist respective minimal and maximal interpolation spaces $\underline{A}_\theta$, $\underline{E}_\theta$ and $\bar{A}_\theta$, $\bar{E}_\theta$ [see J. L. Lions and J. Peetre, Inst. Hautes Études Sci. Publ. Math. No. 19 (1964), 5–68; MR 29 #2627]. The following condition plays an important rôle in the main theorem: (H) There exists to each compact set $K\subset E_0$ a constant C and a set $\mathscr{P}$ of linear operators $P:\mathscr{E}\to\mathscr{E}$ which map E_i into $E_0\cap E_i$ $(i=0,1)$ and are such that $\|P\|_{E_i\to E_i}\leq C$ $(i=0,1)$. Furthermore, there exists to each $\varepsilon>0$ a $P\in\mathscr{P}$ such that $\|Px-x\|_{E_0}<\varepsilon$ for all $x\in K$.

The following theorem is proved. Let A_0, A_1 and E_0, E_1 be interpolation pairs, A_θ, E_θ interpolation spaces of exponent θ $(0<\theta<1)$ with respect to these pairs and $A_\theta\subset\bar{A}_\theta$. If, furthermore, E_0, E_1 satisfy (H), $T:A_0\to E_0$ is compact and $T:A_1\to E_1$ is bounded, then $T:A_\theta\to E_\theta$ is compact. It is also shown that the interpolation pairs L^{p_0}, L^{p_1} $(p_0<\infty)$ and L_0^{∞}, L^{p_1} satisfy an approximation hypothesis even stronger than (H). Here L^p denotes the Banach space of all measurable functions on a locally compact space for which $\int|f|^p\,d\mu(x)<\infty$ (μ a positive measure) and L_0^{∞} is the closed subspace of L^{∞} which consists of all bounded measurable functions vanishing at infinity. *G. Goes* (Chicago, Ill.)

Pták, Vlastimil 3872

An extension theorem for separately continuous functions and its application to functional analysis.

Comment. Math. Univ. Carolinae **4** (1963), 109–116.

Let S, T be two completely regular spaces. A real-valued function f on $S\times T$ is said to satisfy the double limit condition if it is impossible to find two sequences $\{s_i\}\subset S$ and $\{t_j\}\subset T$ such that both $\lim_i\lim_j f(s_i,t_j)$ and $\lim_j\lim_i f(s_i,t_j)$ exist and are different. Let $C(S)$ be the Banach space of all real-valued, bounded continuous functions on S, and denote by $C(S)'$ the dual space of $C(S)$ topologized with the weak star topology. As S may be identified with a subset of $C(S)'$, $S\times T$ is imbedded in the topological vector space $C(S)'\times C(T)'$. The main theorem states that a real-valued, bounded, separately continuous function f on $S\times T$ can be extended to a separately continuous bilinear form on $C(S)'\times C(T)'$, if and only if f satisfies the double limit condition. The proof is based on a combinatorial lemma on convex means given in another paper [Proc. Sympos. Pure Math., Vol. VII, pp. 437–450, Amer. Math. Soc., Providence, R.I., 1963; MR 28 #4337], where the author proved a weaker version of the above theorem and discussed its relationship to a number of known results (e.g., theorems of Krein and Eberlein) on weak compactness. An earlier version of the combinatorial lemma (also due to the author) and its use in a proof of Krein's theorem are well known and can be found in the book of G. Köthe [*Topologische lineare Raume*, I, pp. 329–333, Springer, Berlin, 1960; MR 24 #A411]. *Ky Fan* (Evanston, Ill.)

Riedrich, Thomas 3873

Die Räume $L^p(0,1)$ $(0<p<1)$ sind zulässig.

Wiss. Z. Techn. Univ. Dresden **12** (1963), 1149–1152.

The author defines a sequence of mappings of $L^p(0,1)$ $(0<p<1)$ to its finite-dimensional subspaces which converges uniformly to the identical mapping on every compact subset. This shows that the metric linear spaces $L^p(0,1)$ are admissible [cf. V. Klee, Math. Ann. **141** (1960), 286–296; MR 24 #A1004]. *I. G. Amemiya* (Sapporo)

Riedrich, Thomas 3874

Der Raum $S(0,1)$ ist zulässig.

Wiss. Z. Techn. Univ. Dresden **13** (1964), 1–6.

Let E be a Hausdorff linear space, T a topological space, and f a mapping of T into E. Then f is called "compact" provided fT is a relatively compact subset of E, "finite" provided f is compact and fT lies in a finite-dimensional linear subspace of E, and "uniformly finitely approximable" provided f can be approximated uniformly on T by means of finite mappings. The space E is called "admissible" (zulässig) provided the injection in E of any compact subset of E is uniformly finitely approximable. Every locally convex space is admissible [Nagumo, Amer. J. Math. **73** (1951), 497–511; MR 13, 150]. For an arbitrary E, the Leray-Schauder theory can be developed for mappings $\Phi:X\to E$ such that the mapping $I_X-\Phi$ is compact and uniformly finitely approximable (where $X\subset E$ and I_X is the injection of X into E) [the reviewer, Math. Ann. **141** (1960), 286–296; MR 24 #A1004; ibid. **145** (1961/62), 464–465; MR 25 #1437]; the second condition is redundant if E is admissible. Thus there is considerable interest in deciding whether every space E is admissible, and in determining the admissibility of various spaces E which are not locally convex. This was accomplished earlier by the author for the spaces $L^p[0,1]$ $(0<p<1)$ [see #3873 above]; here he treats the space $S[0,1]$ of all measurable functions on the unit interval, topologized by means of convergence in measure ($d(f,g)=\int_0^1[|f(t)-g(t)|/(1+|f(t)-g(t)|)]\,dt$).

For integers $1\leq k\leq n$, let $I(k,n)$ denote the interval $[(k-1)/n,k/n]$ and $\chi_{k,n}$ the characteristic function of $I(k,n)$. Let F_n denote the linear hull of the set $\{\chi_{1,n},\chi_{2,n},\cdots,\chi_{n,n}\}$. For non-negative $f\in S[0,1]$, let

$$s_{k,n}(f)=\left(\int_{I(k,n)}[f(t)/(1+f(t))]\,dt\right)\Big/\left(\int_{I(k,n)}[1/(1+f(t))]\,dt\right)$$

and let $s_n(f)=\sum_{k=1}^n s_{k,n}(f)\chi_{k,n}\in F_n$. Finally, for arbitrary $f\in S[0,1]$ let $s_n(f)=s_n(f^+)-s_n(f^-)$. The author shows that $S[0,1]$ is admissible by proving that s_n is a retraction of $S[0,1]$ onto F_n, that the sequence $s_1,s_2,\cdots$ is equicontinuous, and that for each f the sequence $s_1(f)$, $s_2(f),\cdots$ converges to f. *Victor Klee* (Seattle, Wash.)

Robert, Jacques 3875

Continuité d'un opérateur non linéaire sur certains espaces de suites.

C. R. Acad. Sci. Paris **259** (1964), 1287–1290.

Let l be an F-space (in the sense of Banach) of sequences $X=\{x_n\}$ with an F-norm $|X|$. Let us write $E_n=\delta_n{}^i$, where $\delta_n{}^i=0$ for $n\neq i$ and $\delta_i{}^i=1$. The author calls the space "F-monotone" if the following conditions are satisfied. (A_1) all finite sequences belong to l; (A_2) $|X-x_nE_n|\leq|X|$

for all $X \in l$; (A_3) for every n, $|aE_n|$ is a nondecreasing function of $|a|$. Now, a map Φ of a space l into a space m by means of a sequence $\{f_n\}$ of real functions of a real variable is defined by the formula $\Phi\{x_n\} = \{f_n(x_n)\}$. Assuming the functions $f_n(u)$ to be continuous at $u = x_n$, the author gives a necessary and sufficient condition for the continuity of Φ (Theorem 3). This theorem is applied to maps Φ of l^1 into l^1 and to maps of an Orlicz space of sequences l_{M_1} into another Orlicz space of sequences l_{M_2}. As an example, we quote Theorem 5.b. Let l_{M_1} and l_{M_2} be two Orlicz spaces of sequences defined by the functions $M_1(u)$ and $M_2(u)$, respectively. Let the functions $f_n(u)$ be continuous and equal to zero at $X = 0$. Then the map Φ of l_{M_1} into l_{M_2} is continuous at $X = 0$ if and only if for every $\alpha > 0$ there exist positive constants β, ε and b and a sequence $\{a_n\} \in l^1$ such that $M_2(\alpha|f_n(u)|) \le a_n + bM_1(\beta|u|)$ for all indices n and all $|u| \le \varepsilon$. *J. Musielak* (Poznań)

v. Waldenfels, Wilhelm 3876

Positive Halbgruppen auf einem n-dimensionalen Torus.
Arch. Math. **15** (1964), 191–203.

Let T^n be a n-dimensional torus, and $C^q = C^q(T^n)$ the Banach space of real-valued q-times continuously differentiable functions, normed in the usual way. Let $\{U(t);\ t \ge 0\}$ be a semi-group of positive linear operators defined on C^0 into C^0 satisfying the conditions: there exists a $q \ge 2$ such that, for fixed t, $U(t)$ maps C^q continuously into C^q, and, for fixed $f \in C^q$, the mapping $t \to U(t)f$ is continuous in the C^q-norm. Then, for any $f \in C^2$, $\lim_{t \downarrow 0} t^{-1}(U(t)f - f)(x)$ exists uniformly in $x \in T^n$ to

$$Af(x) = c(x)f(x) + \sum_i m_i(x)\frac{\partial f}{\partial x_i}(x) + \frac{1}{2}\sum_{i,k}\sigma_{ik}(x)\frac{\partial^2 f}{\partial x_i \partial x_k}$$

$$+ \int Q(x, dy)F_x(y)\Big(1 - \prod_i \cos(y_i - x_i)\Big),$$

where

$$F_x(y) = f(y) - f(x) - \sum_i \sin(y_i - x_i)\frac{\partial f}{\partial x_i}$$

$$- \frac{1}{2}\sum_{i,j}\sin(y_i - x_i)\sin(y_j - x_j)\frac{\partial^2 f}{\partial x_i \partial x_j}$$

with coefficients $c(x)$, $m_i(x)$ and $\sigma_{ij}(x)$ in C^0 such that the matrix $(\sigma_{ij}(x))$ is symmetric and positive definite, and the operation $g(x) \to Qg(x) = \int Q(x, dy)g(y)$ is continuous on C^0 into C^0. *K. Yosida* (Tokyo)

Peressini, A. L. 3877

Concerning the order structure of Köthe sequence spaces. II.
Michigan Math. J. **11** (1964), 357–367.

Teil I ist in demselben J. **10** (1963), 409–415 [MR **28** #454] erschienen. Für zwei reelle Folgenräume λ und μ wird der lineare Raum $L(\lambda, \mu)$ aller schwach stetigen linearen Abbildungen von λ in μ untersucht, auf dem sich in natürlicher Weise eine Ordnungsstruktur erklären läßt. Es werden Bedingungen dafür angegeben, daß $L(\lambda, \mu)$ ein Verband ist. Außerdem wird für gewisse Topologien die Normalität des Kegels aller positiven Abbildungen nachgewiesen. *A. Pietsch* (Berlin)

Marinescu, G. 3878

★Espaces vectoriels pseudotopologiques et théorie des distributions.
Hochschulbücher für Mathematik, Band 59.
VEB Deutscher Verlag der Wissenschaften, Berlin, 1963. viii + 232 *pp.* *DM* 35.00.

This is an attractive introduction to the theory of distributions and relevant parts of functional analysis. In the past there have been two divergent approaches to the subject. Schwartz's approach is by means of the general theory of locally convex spaces developed by Bourbaki, while the formulations of Gel'fand and Šilov, Mikusiński, König, Sikorski, Korevaar, Sebastião e Silva, Urbanik and others are based on simpler topological tools and rely on convergence of sequences whenever possible. Schwartz's approach has the advantage of generality and elegance, while the others are more intuitive, more readily accepted by physicists, and better adapted to certain applications. The present author draws from both approaches in order to obtain a large share of the advantages of each. This is done efficiently and with good taste, so that his contribution is considerably more than a mere amalgamation of what has been done before. His main departure from the general theory of locally convex spaces lies in the fact if E is the space of all continuous linear transformations of one locally convex space into another, or is any of several other important linear spaces whose structure cannot be described by means of a metrizable locally convex topology, then he regards E as a pseudotopological union of locally convex spaces (in the sense described below) and employs true topologies only in an auxiliary fashion. In this respect, he follows the ideas of Gel'fand and Šilov, but he is able to go beyond their treatment by providing a satisfactory theory of vector-valued distributions.

Chapter 1 (58 pages) starts at an elementary level and proceeds to the key notions of a polynormed space and a pseudotopological union of topological linear spaces. (Both of these notions have been studied previously by the author. The second notion has been used in distribution theory by Schwartz and especially by Gel'fand and Šilov. Polynormed spaces have been studied by Mikusiński as "réunions d'espaces de Banach" and by Sebastião e Silva and Waelbroeck as "espaces à bornés".) Suppose that $\{E_\lambda\}_{\lambda \in L}$ is a family of linear subspaces of a linear space E, and that each space E_λ is equipped with a topology. Then E is said to be the "pseudotopological union" of the spaces E_λ provided $E = \bigcup_{\lambda \in L} E_\lambda$, and whenever $\lambda', \lambda'' \in L$ there exists $\lambda \in L$ such that $E_{\lambda'} \cup E_{\lambda''} \subset E_\lambda$ and the given topologies of the spaces $E_{\lambda'}$ and $E_{\lambda''}$ are finer than those induced by the topology of E_λ. Notions such as the limit of a net (generalized sequence), continuity of a function, etc., are introduced in terms of the pseudotopological structure of E by referring directly to the spaces E_λ and without introducing a genuine topology in the space E itself. The pseudotopological union $E = \bigcup_{\lambda \in L} E_\lambda$ is said to be "polynormed" if each E_λ is a normed space. The structure of a polynormed space E can always be described by a family $\{\|\ \|_\alpha\}_{\alpha \in A}$ of generalized norms (functions satisfying the usual requirements for a norm except that they may assume the value $+\infty$) such that for each $x \in E$ there exists $\alpha \in A$ with $\|x\|_\alpha < \infty$, and whenever $\alpha', \alpha'' \in A$, there exists $\alpha \in A$ such that

$$\|\ \|_\alpha \le \inf(\|\ \|_{\alpha'}, \|\ \|_{\alpha''}).$$

Let $D_{[-r,r]^n}$ denote the space of all infinitely differentiable functions x on R^n such that x and all of its derivatives vanish everywhere on the complement of the cube $[-r,r]^n \subset R^n$. This is a locally convex space, denumerably normed by an increasing sequence of norms corresponding to the partial derivatives of various orders. The pseudotopological union of the family $\{D_{[-r,r]^n}\}_{r>0}$ is the space D_{R^n} of all infinitely differentiable functions with compact support on R^n. Let Z_r^n denote the set of all analytic entire functions on C^n (where C is the complex plane) such that for each n-tuple $q=(q_1,\cdots,q_n)$ of non-negative integers there exists a constant M_q such that $|s_1^{q_1}\cdots s_n^{q_n}z(s)| \leq M_q e^{r|\mathrm{Im}\, s|}$ for all $s=(s_1,\cdots,s_n)\in C^n$ (where $|\mathrm{Im}\, s| = (\sum_{i=1}^n (\mathrm{Im}\, s_i)^2)^{1/2}$). Each space Z_r^n is (in a natural way) a denumerably normed locally convex space, and the pseudotopological union Z^n of the family $\{Z_r^n\}_{r>0}$ is a space which was introduced by Gel'fand and Šilov. Chapter 1 culminates with a description of the Fourier transform and a proof that it determines an isomorphism between the spaces D_{R^n} and Z^n, both algebraically and for their structures as pseudotopological unions of locally convex spaces.

Chapter 2 (38 pages) is concerned with the space of all continuous linear transformations of a space E into a space F. Starting with the standard case of normed spaces, the author proceeds to the case in which E and F are both locally convex spaces and finally to the case in which both are pseudotopological unions of locally convex spaces. Standard results on the existence and extension of continuous linear forms are proved and are then extended to the more general situations required in the author's theory. A key result asserts that if a linear space E is the union of an increasing sequence $E_1 \subset E_2 \subset \cdots$ of bornological spaces such that E_n is always a closed subspace of E_{n+1}, then E admits a bornological topology such that each E_n is a closed subspace of E, every bounded subset of E is a bounded subset of some E_n, and a linear transformation of E into an arbitrary locally convex space is continuous if and only if its restriction to E_n is always continuous. Dual spaces are defined and the dual of a locally convex space is represented as a polynormed space in a natural way. A distribution is defined as a continuous linear form on the space D_{R^n}. Derivatives and primitives of distributions are defined, as well as the convolution of two distributions, and the basic properties of these notions are established.

Chapter 3 (42 pages) is devoted to tensor products. Nuclear operations and spaces are defined, and the nuclearity of the spaces $D_{[-r,r]^n}$ is established. The chapter culminates with Schwartz's "théorème de noyaux", asserting that the space of all weakly continuous linear transformations of D_{R^n} into $D^*_{R^n}$ is isomorphic to the space D^{*n+m}. This is proved in the manner of Ehrenpreis [Proc. Amer. Math. Soc. 7 (1956), 713–718; MR 18, 584] and Gask [Math. Scand. 8 (1960), 327–332; MR 23 #A2740].

Chapter 4 (57 pages) treats vector-valued functions, measures, integrals and distributions. The differential calculus of vector-valued functions is based on the author's notion of Fréchet differential [Marinescu, Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) 1 (49) (1957), 77–86; MR 20 #1188], which is intimately related to the pseudotopological structure of the space of linear transformations. The approach to vector-valued measures and integrals is due jointly to the author and to Dinculeanu, and appeared first in the author's earlier book [*Topological and pseudo-topological vector spaces* (Romanian), Editura Acad. R. P. Romîne, Bucharest, 1959; MR 21 #6525]. It is much closer to the approach of Bochner than to that of Bourbaki. The treatment of vector-valued distributions is less general than that of Schwartz [Ann. Inst. Fourier (Grenoble) 7 (1957), 1–141; MR 21 #6534; ibid. 8 (1958), 1–209; MR 22 #8322], but is deemed to be adequate for most applications. It draws on the paper of Sebastião e Silva [Portugal. Math. 19 (1960), 1–80; errata, ibid. 19 (1960), 243–244; MR 25 #453].

Chapter 5 (16 pages) is concerned with applications, its sections having the following titles: (1) Représentations intégrales de certaines opérations linéaires; (2) Décompositions intégrales des espaces nucléaires suivant les opérateurs propres d'un opérateur symétrique; (3) Équations différentielles et aux dérivées partielles; (4) Processus stochastiques generalisés. (The last two areas are only touched upon.)

There are useful bibliographical notes at the end of each chapter, and a bibliography of about 300 items. However, most of these items are not mentioned in the text and many of them are related to the subject of the book only in that they are somehow concerned with topological linear spaces. In the reviewer's opinion, the bibliography would be much more useful if it were more selective and if the author had added a few more pages as a guide to those items which are not mentioned elsewhere in the text. Curiously, the bibliography includes three papers by "Gelfand und Schilow" and three others by "Ghelfand et Šilov".

Inevitably, there are a few minor ambiguities, slips or misprints in the text, but none of those noted by the reviewer should cause any serious difficulty. The book is well written and attractively printed. It appears to be suitable both for a formal course on distribution theory (prerequisite: a course in real analysis) and for individual study. *Victor Klee* (Seattle, Wash.)

Semjanistyi, V. I. 3879

Some integral transformations and integral geometry in an elliptic space. (Russian)

Trudy Sem. Vektor. Tenzor. Anal. **12** (1963), 397–441.

Proofs are given for the elliptic case of theorems announced by the author [Dokl. Akad. Nauk SSSR **136** (1961), 288–291; MR **24** #A2842]. Explicit formulas for the inversion of Radon-type transforms in the space of generalized functions on projective n-space are obtained by examining their extensions to Euclidean space as even generalized functions, homogeneous of arbitrary (complex) order. For n odd, some of the formulas were first found by Helgason [see Section 4 of the paper reviewed below #4068]. *L. W. Green* (Minneapolis, Minn.)

Ding, Guang-gui [Ting, Kuang-kuei] 3880

The resonance theorem for a noncomplete space.

Acta Math. Sinica **13** (1963), 216–222 (*Chinese*); *translated as Chinese Math.* 4 (1963), 235–241.

Let $\{T_\lambda\}$ be a family of bounded linear transformations of a normed space E into (perhaps different) normed spaces F_λ, and let $M(x)=\sup_\lambda \|T_\lambda(x)\|$, $x\in E$. It is well known that if E is complete, then $M(x)<\infty$ everywhere implies M is bounded, and that the result fails in general if E is

incomplete. In this note it is shown that $M < \infty$ everywhere and M upper semicontinuous (at one point) imply M is bounded, whether E is complete or not. The proofs are elementary and are couched in the terminology of convex functionals. *A. Brown* (Ann Arbor, Mich.)

Hildebrandt, Stefan; Wienholtz, Ernst 3881
Constructive proofs of representation theorems in separable Hilbert space.
Comm. Pure Appl. Math. **17** (1964), 369–373.

Let $\{e_1, e_2, e_3, \cdots\}$ be a linearly independent sequence of vectors that span a Hilbert space H and let H_n be the span of $\{e_1, \cdots, e_n\}$, $n = 1, 2, 3, \cdots$. If B is a bounded bilinear form and L is a bounded linear form on H, and if there exists a uniformly bounded sequence Q_n of operators on H_n such that $|B(Q_n x, x)| \geq \alpha\|x\|^2$ (for some positive α, all x, and all sufficiently large n), then there exists a unique y in H such that $B(x, y) = L(x)$ for all x; the vector y is, in fact, the limit of the sequence whose nth term is $\sum_{k=1}^n \beta_{kn} e_k$, where the β's are the solutions of the equations $\sum_{k=1}^n \beta_{kn} B(e_i, e_k) = L(e_i)$, $i = 1, \cdots, n$. The authors call this a "constructive" version of that corollary of the Riesz representation theorem which is known to students of partial differential equations as the Lax-Milgram lemma. They derive some elementary consequences and special cases. *P. R. Halmos* (Ann Arbor, Mich.)

Hildebrandt, Stefan 3882
The closure of the numerical range of an operator as spectral set.
Comm. Pure Appl. Math. **17** (1964), 415–421.

If T is a linear transformation on a Hilbert space with numerical range W $(= \{(Tx, x) : \|x\| = 1\})$ and if $\bar{W}$ is a spectral set of T (that is, $\|f(T)\| \leq \sup\{|f(\lambda)| : \lambda \in \bar{W}\}$ for all rational functions f with no poles in $\bar{W}$), then (i) $\sup\{|(Tx, x)| : \|x\| = 1\} = \|T\|$; (ii) the set E of all extreme points of $\bar{W}$ is included in the approximate point spectrum of T, and $E \cap W$ is the same as the set of all proper values of T in E; (iii) $\bar{W}$ is the convex hull of the spectrum of T; and (iv) a necessary and sufficient condition that W be closed is that E be included in the point spectrum of T.

The paper consists of this main theorem together with a few examples, corollaries, and related remarks.
P. R. Halmos (Ann Arbor, Mich.)

Kardžiev, N. 3883
Calculation of the eigenvalues of a self-adjoint operator. (Russian. German summary)
C. R. Acad. Bulgare Sci. **16** (1963), 793–796.

The paper exploits a variation of the Galerkin method to obtain upper and lower bounds for the eigenvalues of a self-adjoint completely continuous operator in Hilbert space. The orthogonalization is not with respect to the coordinate elements z_i but with respect to elements which are linear transforms of the z_i. *H. J. Stetter* (Munich)

Hopf, Eberhard 3884
Remarks on my paper "An inequality for positive linear integral operators".
J. Math. Mech. **12** (1963), 889–892.

The author considers linear integral operators

$$Kf(x) = \int_Y K(x, y) f(y) \mu(dy),$$

$x \in X, y \in Y$, in which μ is a given σ-finite measure defined in a σ-field of subsets of Y. The kernel is supposed to be essentially positive, that is, $\varphi \geq 0$, $\varphi \not\equiv 0$ imply $K\varphi > 0$. Under the additional hypothesis that the kernel has bounded cross-ratio, that is, that there exists a finite constant $k \geq 1$ such that

$$\frac{K\varphi(x)}{K\varphi(x')} \bigg/ \frac{K\psi(x)}{K\psi(x')} \leq k^2$$

holds for all functions $\varphi \geq 0$, $\psi \geq 0$, both $\not\equiv 0$, and for all points $x \in X$, $x' \in X$, the author [same J. **12** (1963), 683–692; MR **29** #2614] established a certain inequality involving the oscillations of f/p and of Kf/Kp for such an operator K.

The second part of the earlier paper dealt with applications of the inequality. The first of these consisted in a simple rederivation of classical results of Frobenius-Perron (matrices) and Jentzsch (integral operators) about the proper values of $Kf = \lambda f$. The second application was the proof of a new inequality about the proper values λ different from the absolutely largest one $\lambda_0 > 0$. The author re-examines in the present paper the hypotheses about K under which those applications are valid and substantially reduces the somewhat too restrictive hypothesis made in the earlier paper. It is again supposed that $Y = X$ and that the kernel is μ-measurable in x for fixed y. This insures that a μ-measurable function $f(x)$ over X goes over again into a μ-measurable function $Kf(x)$ if $Kf(x)$ is well-defined for every $x \in X$.
R. V. Chacon (Providence, R.I.)

Lučka, A. Ju.; Kurpel', N. S. 3885
On a non-stationary iteration method for the approximate solution of linear operator equations. (Russian)
Ukrain. Mat. Ž. **16** (1964), 389–395.

Let A be a bounded linear operator in a Banach space E and let $f \in E$. It is desired to solve the linear equation (*) $u = f + Au$ by using a sequence (A_n) of operators in E such that $(I - A_n)^{-1}$ exists. Let $L_n = (I - A_n)^{-1}(A - A_n)$ and let u_0 be arbitrary. Suppose that

$$\lim_{n \to \infty} \|L_n L_{n-1} \cdots L_2 L_1\| = 0,$$

and consider the iteration $u_n = (I - A_n)^{-1} f + L_n u_{n-1}$, $n \geq 1$. It is asserted (but not proved) that this iteration procedure converges to the unique solution u^* of (*). Moreover, if $\|L_n\| < 1$, then

$$\|u^* - u_n\| < \|L_n\|(1 - \|L_n\|)^{-1}\|u_n - u_{n-1}\|.$$

Modifications of this procedure are considered where the A_n are projections of A into subspaces of E.
R. G. Bartle (Urbana, Ill.)

Putnam, C. R. 3886
On the structure of semi-normal operators.
Bull. Amer. Math. Soc. **69** (1963), 818–819.

A bounded operator T in a Hilbert space $\mathfrak{H}$ is said to be semi-normal if $D = TT^* - T^*T \geq 0$ or ≤ 0. In this paper, some previous results of the author concerning commutators [Amer. J. Math. **86** (1964), 310–316; MR **29** #1552]

are applied to the study of semi-normal operators. It is immediately shown, for instance, that Re T and Im T are absolutely continuous on the smallest closed subspace of $\mathfrak{H}$ reducing T and containing the range of D. An estimate of $\|D\|$ by certain quantities related to the spectra of Re T and Im T is also given.

S. T. Kuroda (Tokyo)

Marčuk, G. I. 3887

On the formulation of certain inverse problems. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 503–506.

The author considers an equation in linear operator form, $L\varphi=f$, and its conjugate equation $L^*\varphi_p{}^*=p$ and the linear functional $J_p(\varphi)=(p,\varphi)=(f,\varphi_p{}^*)$. The problem is to determine L from $J_p(\varphi)$, given by physical indications. He states how to approximate the solution of the problem starting from a solution of

$$L_1\varphi_1 = f_1,$$
$$L^*\varphi_{1p}{}^* = p$$

which corresponds to the prognostic values of the parameters sought. A perturbation method is used.

The note is very interesting inasmuch as the problem is related to many problems of technology, as is indicated by the author.

P. K. Ghosh (Calcutta)

Rejto, P. A. 3888

On gentle perturbations. II.

Comm. Pure Appl. Math. **17** (1964), 257–292.

Part I appeared in same Comm. **16** (1963), 279–303 [MR **27** #4089]. If S and T are selfadjoint transformations in a Hilbert space such that $(S-w)^{-1}-(T-w)^{-1}$ is of trace class for some, and hence every, non-real number w, the absolutely continuous parts of S and T are known to be unitarily equivalent. It is now shown that the spectrum of T is absolutely continuous in an interval (a,b) if the spectrum of S is absolutely continuous in (a,b) and if the perturbation is "smooth". By using a canonical model of the perturbation situation (see, for example, the reviewer [Amer. J. Math. **84** (1962), 543–560; MR **27** #4083]), the author is able to assume that the undisturbed transformation S is multiplication by x in some L^2 space of vector-valued functions. In this preliminary paper he assumes that $T-S$ is bounded and has finite-dimensional range, but the method is quite general. Smoothness is interpreted as a condition on the kernel representing $T-S$, but since the canonical model is a unitary equivalent of the perturbation situation, the hypothesis will no doubt eventually be reformulated as a condition on resolvents. The paper is of interest for introducing local questions in perturbation theory and opening up a new area for research.

L. de Branges (Lafayette, Ind.)

Ringrose, J. R. 3889

On the resolvent and the principal vectors of a compact linear operator.

Proc. Cambridge Philos. Soc. **60** (1964), 525–531.

This is a continuation of the author's previous paper [Proc. London Math. Soc. (3) **12** (1962), 367–384; MR **25** #458]. Let T be a compact linear operator in a complex Hilbert space $\mathfrak{H}$. In the preceding paper, generalizing the theorem of von Neumann and of Aronszajn and Smith [Ann. of Math. (2) **60** (1954), 345–350; MR **16**, 488], the author showed that there is a simple resolution $\{E_\lambda\}_{0\le\lambda\le1}$ of the identity in $\mathfrak{H}$ such that $E_\lambda TE_\lambda=TE_\lambda$. In the present paper, he gives constructions for the resolvent of T, and for a complete set of principal vectors, in terms of T, $\{E_\lambda\}$, and the diagonal coefficients (α_λ) of T. Moreover, those results are applied to Volterra integral operators.

S. Sakai (Berkeley, Calif.)

Thyssen, M. 3890

Une condition d'isométrie pour un opérateur linéaire.

Bull. Soc. Roy. Sci. Liège **33** (1964), 141–142.

T. Nieminen raised the following question [Ann. Acad. Sci. Fenn. Ser. A I No. 316 (1962); MR **25** #2452]: If T is a bounded operator on a Hilbert space, the spectrum of which is a subset of $|z|=1$ and whose resolvent satisfies the inequality $\|R_z\|\le\big||z|-1\big|^{-1}$ on the resolvent set, does it follow that T is unitary?

In this paper the author makes an unsuccessful attempt at settling this question in the negative. This question, however, has been already settled in the affirmative by W. F. Donoghue [Inst. Hautes Études Sci. Publ. Math. No. 16 (1963), 31–33; MR **27** #2804], whose note is listed in the bibliography of the paper.

R. G. Douglas (Ann Arbor, Mich.)

Taldykin, A. T. 3891

Eigenelements and adjoint elements of linear operators. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **2** (1962), 165–169.

Part of the material has been discussed earlier [Izv. Vysš. Učebn. Zaved. Matematika **1959**, no. 6 (13), 174–188; MR **24** #A3499]. The statement of the main theorem has changed slightly, but its proof has not, and continues to harbor a fatal flaw.

A. Brown (Ann Arbor, Mich.)

Brown, T. A. 3892

Analysis of a new formalism in perturbation theory.

Proc. Nat. Acad. Sci. U.S.A. **50** (1963), 598–601.

Given the equation $u=f+\lambda T(u)$, where f is a known function, λ is a parameter, and T is a linear transformation, find the unknown function u. The iterative technique of Bellman and Richardson [same Proc. **48** (1962), 1913–1915; MR **26** #416] (based on continued fraction expansion) is suggested and certain convergence proofs presented.

L. W. Ehrlich (Silver Spring, Md.)

Zabreĭko, P. P.; Krasnosel'skiĭ, M. A. 3893

Calculation of the index of a fixed point of a vector field. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 509–531.

Les auteurs exposent un algorithme à l'aide duquel on réduit le calcul de l'indice d'un point fixe, à l'étude d'un champ de vecteurs dans un espace à un nombre fini de dimensions. Les résultats s'obtiennent de quelques théorèmes dont voici les premiers. (1) Si (a) P est un opérateur linéaire de projection de l'espace E sur un sous-espace E_0 à un nombre fini de dimensions, $Q=I-P$, (b) A est un opérateur complètement continu, qui admet

la représentation $A = P + C_2 + C_3 + \cdots + C_m + D$ où C_i satisfont, entre autres restrictions, à la condition $C_i(\lambda x) = \lambda^i C_i(x)$ et $\|Dx\| \leq \beta(\rho)\|x\|^m$ ($\|x\| \leq \rho$), β étant une fonction non négative; (3) $PC_i = 0$ ($i = 2, 3, \cdots, k-1$) sur E et $PC_k \neq 0$, $QC_i P = 0$ ($i = 2, 3, \cdots, s$) sur E; (4) il existe un nombre naturel $r \leq m$, et un nombre $\rho_0 < 1$ tels que:

$$\|P(C_k x + C_{k+1} x + \cdots + C_r x)\| \geq \alpha\|x\|^r,$$

où $x \in E_m$, $\|x\| \leq \rho_0$, $\alpha > 0$, $r = s + k$, alors θ est un point fixe isolé du champ vectoriel complètement continu $\Phi = I - A$. Si γ_0 est l'indice du point fixe isolé θ de $P(C_k + C_{k+1} + \cdots + C_r)$, alors l'indice de θ pour Φ est γ_0. La théorie est appliquée à l'étude de l'équation des ondes symétriques sur les surfaces d'un fluide lourd.

A. Haimovici (Iași)

Blattner, Robert J. 3894

A theorem on induced representations.

Proc. Amer. Math. Soc. **13** (1962), 881–884.

In this paper the author gives a substantial improvement, both in content and method of proof, of a theorem he established in an earlier work of his [Amer. J. Math. **83** (1961), 79–98; MR **22** #A2757] concerning the intertwining number of two induced representations of a locally compact group.

P. C. Deliyannis (Chicago, Ill.)

Dixmier, Jacques 3895

Quasi-dual d'un idéal dans une C^*-algèbre.

Bull. Sci. Math. (2) **87** (1963), *1ière partie*, 7–11.

Let A denote a C^* algebra. The "dual object" $\hat{A}$ which consists of the unitary equivalence classes of irreducible representations appears to be of significance only for A of type I, and for this reason J. A. Ernest [Trans. Amer. Math. Soc. **104** (1962), 252–277; MR **25** #3383] has introduced the "quasi-dual" $A^\sim$ which consists of "primary classes" of representations. A representation in a "primary class" yields a "factor" M for its von Neumann algebra of operators, and while two different representations in the same class will have the same M when regarded as a C^* algebra, they may not be unitarily equivalent since the M' may be different. The concept of σ-ring and Borel set can be introduced into $A^\sim$ in a manner similar to the corresponding discussion for $\hat{A}$. Let J be a closed, two-sided ideal in A. A representation of A/J can be readily extended to a representation of A which is zero on J and thus $(A/J)^\sim$ can be mapped onto a subset E of $A^\sim$. It is shown that $A - E$ can be mapped on $J^\sim$ by simply restricting the domain of representations in $A - E$ to J. It is established that these mappings preserve type and normalcy, and that E and $A - E$ are Borel sets.

F. J. Murray (Durham, N.C.)

Lumer, G. 3896

Remarks on n-th roots of operators.

Acta Sci. Math. (*Szeged*) **25** (1964), 72–74.

This paper contains a quantitative refinement of the results of Halmos and Lumer [Proc. Amer. Math. Soc. **5** (1954), 589–595; MR **16**, 48] concerning operators on a Hilbert space that are invertible and lack a (square) root. Using the same method involving spectral multiplicity at points of the "compression spectrum" of the operator, it is proved that if H is a separable Hilbert space, there exist (1) an operator A of norm 1 such that every operator in the open sphere of radius 1 centred at A lacks a root of any order; (2) for every R, $0 < R < \frac{1}{2}\sqrt{2}$, an invertible operator A_R of norm 1 such that every operator in the open sphere of radius R centred at A_R lacks a root of any order; and (3) an invertible operator A_0 of norm 1 such that every operator in the open sphere of radius $(2\sqrt{2}+1)^{-1}$ centred at A_0 is invertible and lacks a root of any order. Several remarks are made concerning the analogous problem for commutative Banach algebras, for which the problem is much simpler. The question as to whether the set of invertible operators without roots of any order on a separable Hilbert space is open in the set of all invertible operators has almost surely a negative answer, illustrated by operators of the type described by Deckard and Pearcy [ibid. **14** (1963), 445–449; MR **26** #6774].

J. J. Schäffer (Pittsburgh, Pa.)

Schue, John R. 3897

The structure of hyperreducible triangular algebras.

Proc. Amer. Math. Soc. **15** (1964), 766–772.

The author proves a kind of Wedderburn theorem for triangular algebras in the sense of Kadison and Singer [Amer. J. Math. **82** (1960), 227–259; MR **22** #12409]: If $\mathcal{T}$ is a maximal hyperreducible triangular algebra acting on a separable Hilbert space with diagonal $\mathcal{A}$, where $\mathcal{A}$ is a maximally abelian self-adjoint subalgebra of a factor $\mathcal{B}$, then $\mathcal{T}$ is the algebraic sum of $\mathcal{A}$ and $\mathcal{S}$, where $\mathcal{S}$ is the weak closure of an increasing sequence of weakly closed nilpotent ideals. The author also proves that the sum $\mathcal{A} + \mathcal{S}$ is direct if (1) $\mathcal{B}$ is of type II, or (2) $\mathcal{B}$ is of type I and $\mathcal{A}$ is generated by an hermitian operator with pure point spectrum. It is interesting to note that the author's main technique is the analysis of the spectral manifolds introduced by Halmos [cf. *Introduction to Hilbert spaces and the theory of spectral multiplicity*, Chelsea, New York, 1951; MR **13**, 563], which is applied to the inner derivation $D_X Y = XY - YX$ considered as an operator acting on $\mathcal{A}$.

M. Nakamura (Osaka)

Stecenko, V. Ja. 3898

An estimate for the spectrum of some classes of linear operators. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 1054–1057.

The purpose of this paper is to discuss bounds on positive operators A acting on a Banach space with a partial ordering induced by a normal cone [M. G. Kreĭn and M. A. Rutman, Uspehi Mat. Nauk **3** (1948), no. 1 (23), 3–95; MR **10**, 256; M. A. Krasnosel'skiĭ, *Positive solutions of operator equations* (Russian), Fizmatgiz, Moscow, 1962; MR **26** #2862] if the behavior of A on just one single element g in the interior K^0 of K is prescribed by an inequality $A^n g \leq \lambda_0 g$. An idea of Kreĭn's in the above-quoted paper is used and extended. It is asserted in Theorem 1 that, under the hypotheses just mentioned, the spectrum of A lies in the circle $|\lambda| \leq \sqrt[n]{\lambda_0}$. Theorem 2 is proved, asserting that the hypothesis that $g \in K^0$ can be weakened to obtain the same conclusion. Two other similar theorems are stated weakening the hypotheses on g and the operator A being positive, respectively.

G. K. Kalisch (Minneapolis, Minn.)

Ezajan, A. R.; Stecenko, V. Ja. 3899

Estimates of the spectrum of integral operators and infinite matrices. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 254–257.

The purpose of the note is to announce some estimates for the spectral radii of certain linear integral operators in the spaces C and L_p, $p>1$, and for certain infinite matrices. The main tool is a theorem asserting that if an operator B, in a real Banach space with a positive cone K which is normal and reproducing, is dominated by a positive linear operator A which is u_0-bounded above and such that $A^m u_0 \leq \lambda u_0$ for $u_0 \in K$, then the spectral radius of B is at most $\lambda^{1/m}$. Six estimates are given; we shall cite only one. Let Ω be a compact subset of R^n and let $K(t, s)$ $(t, s \in \Omega)$ be such that the operators B, A defined in $C(\Omega)$ by

$$Bx(t) = \int_\Omega K(t, s)x(s)\, ds, \qquad Ax(t) = \int_\Omega |K(t, s)|x(s)\, ds$$

are bounded. If $\lambda > 0$ is such that

$$\int_\Omega |K(t, s)|\left\{\lambda - \int_\Omega |K(s, r)|\, dr\right\} ds \geqq 0,$$

then it follows that the spectral radius of B is at most λ.

R. G. Bartle (Urbana, Ill.)

Greiner, Peter C. 3900

Eigenfunction expansions and scattering theory for perturbed elliptic partial differential operators.

Bull. Amer. Math. Soc. **70** (1964), 517–521.

The eigenfunction expansions are considered for partial differential operators of the form $H = P(D) + q(x)$ in R_n, where $P(D)$ is an elliptic polynomial of degree $2p > n/2$ in the n variables $D_j = i\partial/\partial x_j$ and $q(x) \in C_{2[n/2]}$, $q(x) = O(|x|^{-n-h})$ for some $h > 0$. H is self-adjoint (under suitable definition of $P(D)$) and has pure point spectrum in $(-\infty, 0)$. For the remaining part of the spectrum it is shown that, under certain additional assumptions on the kernel representing the resolvent $(H-z)^{-1}$, the part of H in the spectral interval (a, b) is unitarily equivalent to the part of $H_0 = P(D)$ in the same interval, the equivalence being given explicitly in terms of an integral involving the eigenfunctions $F(x, k)$ of H for "continuous eigenvalues" $P(k)$. These results are then applied to the special case $n = 2m+1$, $P(D) = (-\Delta)^p$, $4p > 2m+1$; it follows that the point spectrum of H in $(0, +\infty)$ is a closed, denumerable, nowhere dense set, and the unitary equivalence stated above holds true for any (a, b) containing no point spectrum.

T. Kato (Berkeley, Calif.)

Kasiev, A. 3901

A condition for the existence of an inverse operator for a differential operator in the space of generalized functions. (Russian. Uzbek summary)

Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk **1964**, *no.* 2, 23–30.

The paper is concerned with the partial differential equation

$$L\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)U = T,$$

where $L(\partial/\partial x_1, \cdots, \partial/\partial x_n)$ is a homogeneous elliptic differential operator of order $2m$ $(2m < n)$ with constant coefficients and T is a distribution. A necessary and sufficient condition on T is given for the existence of a unique solution U in a class of distributions vanishing at infinity together with some of their derivatives. A sufficient condition was given by the author in a previous paper [same Izv. **1962**, no. 6, 27–31; MR **27** #5031].

Z. Zieleźny (Wrocław)

Carroll, Robert 3902

Problems in linked operators. I.

Math. Ann. **151** (1963), 272–282.

This is the first of a series of papers, announced by the author, on the subject of what he calls linked operators. The aim is to get general results on abstract initial-value problems and, particularly, on uniqueness of the solutions. In such problems, one usually deals with differential operators $S = D_t + A$, where $D_t = d/dt$ and A is some linear operator on a space of functions or distributions in the "space variables", e.g., on the space L^2 of an open subset of R^n, Ω. When A is independent of t, S can be written $D_t \otimes I + I \otimes A$, and viewed as operating (at first) on subsets of $L_t^2 \otimes L_x^2(\Omega)$. One seeks a solution in suitable subspaces of the space of L^2 functions of t with values in the Hilbert space $L_x^2(\Omega)$; of course, this is simply $L^2_{t,x}(R_+^1 \times \Omega)$. But it can also be interpreted as the completion of the above tensor product for the $L^2_{t,x}$ norm. Thus it is natural to raise a more general question. Let E, H be two Hilbert spaces, $E \hat{\otimes}_\alpha H$ the completion of $E \otimes H$ for some "cross-norm" α, L, A two linear operators respectively defined in dense subspaces of E, H and with values in E, H. When are there extensions $\hat{S}$ of $S = L \otimes I + I \otimes A$ in $E \hat{\otimes}_\alpha H$ with the following three properties: $\hat{S}$ is 1-1; $\hat{S}$ is onto; $\hat{S}^{-1}$ is continuous? In the paper, sufficient conditions for the existence of such extensions are proved; the adjoint S^* is studied; simple examples are given.

F. Treves (New York)

Kesel'man, G. M. 3903

On the unconditional convergence of eigenfunction expansions of certain differential operators. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 2 (39), 82–93.

The author studies a regular boundary-value problem H [see M. A. Naimark, *Linear differential operators* (Russian), GITTL, Moscow, 1954; MR **16**, 702]. Main result: Under certain conditions the eigenfunctions of H form an absolute basis in $L_2(0, 1)$. Recall that a basis $\{e_n\}$ is absolute if $\{e_{\alpha_n}\}$ is also a basis for each one-to-one map of the indices. In the case under consideration a basis is absolute if and only if it is a Riesz basis [N. K. Bari, Moskov. Gos. Univ. Učen. Zap. **148** (1951), Mat. **4**, 69–107; MR 14, 289; I. M. Gel'fand, ibid. **148** (1951), Mat. **4**, 224–225; MR 14, 289].

An analogous result was obtained independently by V. P. Mihaĭlov [Dokl. Akad. Nauk SSSR **144** (1962), 981–984; MR **26** #6824].

Z. Zieleźny (Wrocław)

Kovtun, I. I. 3904

On the solution of an operator differential equation. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 322–324.

The author presents a method for the solution of the equation

$$\frac{dx}{dt} = [A + \varepsilon F(t)]x \tag{1}$$

in a Hilbert space H with a given initial condition $x(0)$, where x is a vector, A is a bounded operator in H, and $F(t)$ is some operator in H depending on the parameter t, which is the sum of a convergent series of operators. The method consists of a combination of projection and asymptotic methods. Thus, if $\{H_n\}$ denotes a sequence of finite-dimensional subspaces of H and P_n is a projection operator of H onto H_n, then, together with (1), the author considers the approximate equation

$$\frac{dx_n}{dt} = [A_n + \varepsilon F(t)]x_n, \tag{2}$$

where x_n is in H_n and $A_n = P_n A$ is the operator constructed by some projection method which has the property that $\|A_n - A\| \to 0$ as $n \to \infty$. Using the asymptotic method the author obtains a formal solution x_n^* of (2) and then establishes the asymptotic estimate for the error $\|x - x_n^*\|$, where x is the solution of (1).

W. V. Petryshyn (Chicago, Ill.)

Lions, Jacques-Louis; Strauss, Walter A. 3905

Sur certains problèmes hyperboliques non linéaires.

C. R. Acad. Sci. Paris **257** (1963), 3267–3270.

Gli autori risolvono il problema misto (con condizioni nulle iniziali e nulla frontiera) per l'equazione non lineare $-\Delta u + u'' + \beta(u') = f$, dove Δ indica l'operatore di Laplace, $u' = \partial u/\partial t$, $u'' = \partial^2 u/\partial t^2$ ed è $\beta(u') = |u'|^{\rho-1}u'$ $(\rho > 1)$.

Precisamente, si dimostra che se $f \in L^2(0, T; L^2(\Omega))$ (qui Ω è un qualunque aperto di R^n) esiste una ed una sola soluzione (generalizzata) $u(t)$ tale che

$$u(0) = u'(0) = 0, \qquad u \in L^2(0, T; H_0^1(\Omega)),$$

$$u' \in L^\infty(0, T; L^2(\Omega)) \cap L^{\rho+1}(0, T; L^{\rho+1}(\Omega)).$$

Il procedimento è quello di Galerkin; si costruisce una successione approssimante u_n tale che u_n' sia debolmente convergente in $L^{\rho+1}(0, T; L^{\rho+1}(\Omega))$ verso u' e che $\beta(u_n')$ sia debolmente convergente in $L^{\rho+1/\rho}(0, T; L^{\rho+1/\rho}(\Omega))$ verso g; si dimostra che $g = \beta(u')$, seguendo un'idea di G. Minty.

Vengono poi dati risultati di regolarizzazione, in dipendenza da ulteriori proprietà di regolarità di f.

G. Prodi (Pisa)

Ljance, V. È. 3906

Extension of the Fourier L-transform to locally square integrable functions. (Russian)

Dokl. Akad. Nauk SSSR **158** (1964), 1026–1029.

If L is the operator in $L^2(0, \infty)$ generated by the differential operator $ly = -y'' + p(x)y$ with boundary conditions $y'(0) - \theta y(0) = 0$, where p and θ are such that L is not selfadjoint but has a finite set of spectral singularities, then the Fourier L-transform of $f \in L^2(0, \infty)$ is defined by the function $\omega(f, \lambda) = \int f(x)\omega(x, \lambda)\,dx$, where $\omega(x, \lambda)$ are solutions of $ly = \lambda y$ such that $\omega(0, \lambda) = 1$, $\omega_x'(0, \lambda) = \theta$, and its derivatives at spectral singularities of L. An extension of this definition for functions of exponential growth was given in the author's article [same Dokl. **152** (1963), 816–819; MR **28** #2276]. An extension to all functions locally square summable and valid for a wider class of p is given here, based on a Parseval equation for the L-transform defined above, and questions of its non-uniqueness and inversion are studied. Applications to partial differential equations are anticipated.

J. L. B. Cooper (Pasadena, Calif.)

Matsuura, Shigetake 3907

Factorization of differential operators and decomposition of solutions of homogeneous equations.

Osaka Math. J. **15** (1963), 213–231.

Let Ω be an open set in n-dimensional Euclidean space. Let $L(X)$ be a polynomial in n variables $X = (X_1, \cdots, X_n)$. Suppose $D = (D_1, \cdots, D_n)$, where $D_k = (1/i)\partial/\partial x_k$. Problem: If $L(X) = P(X)Q(X)$, where P and Q are polynomials, when can a solution u of $L(D)u = 0$ in Ω be expressed in the form $u = u_1 + u_2$, where $P(D)u_1 = 0$ and $Q(D)u_2 = 0$ in Ω. When $\Omega = R^n$, V. P. Palamodov [Dokl. Akad. Nauk SSSR **142** (1962), 1278–1281; MR **26** #1598] solved the problem for various kinds of spaces of ordinary and generalized functions by giving an explicit form of a general solution of $L(D)u = 0$. In the present article the same problem is solved for various regions Ω and for various spaces of solutions, to wit, spaces of distributions which are stable under the operations of partial differentiations, spaces of C^∞ functions, spaces of polynomials, and spaces of complex-valued functions whose real and imaginary parts are analytic.

R. C. Gilbert (Fullerton, Calif.)

Mikolás, Miklós 3908

Sur la propriété principale des opérateurs différentiels généralisés.

C. R. Acad. Sci. Paris **258** (1964), 5315–5317.

Define ${}_aD^\mu f = {}_aD^\mu f|x = \lim_{n\to\infty}((x-a)/n)^{-\mu}{}_a\Delta_n^\mu f$, where ${}_a\Delta_n^\mu f = {}_a\Delta_n^\mu f|x = \sum_{p=0}^{n-1}(-1)^p\binom{\mu}{p} f(x - p(x-a)/n)$. Then ${}_aD^\nu f = f^{(\nu)}(x)$ $(\nu = 0, 1, 2, \cdots)$ if the right-hand term exists, and, for $\mu < 0$, ${}_aD^\mu f = {}_aI^{-\mu}f = \Gamma(-\mu)^{-1} \times \int_a^x f(t)(x-t)^{-\mu-1}\,dt$. Thus ${}_aD^\mu f$ is an "integro-derivative" operator. Assume the existence of either one of ${}_aD^{\mu_1}({}_aD^{\mu_2}f)$ or ${}_aD^{\mu_1+\mu_2}f$. Then it is proved that ${}_aD^{\mu_1}({}_aD^{\mu_2}f) = {}_aD^{\mu_1+\mu_2}f$ if and only if $\lim(n^{\mu_1}\cdot{}_a\Delta_n^{\mu_1}\eta_N|x) = 0$, where $\eta_m(t) = ((t-a)/m)^{-\mu_2}{}_a\Delta_m^{\mu_2}f|t - {}_aD^{\mu_2}f|t$ $(m = 1, 2, \cdots)$ and $N = N(t, x) = [n(t-a)/(x-a)]$.

K. Yosida (Tokyo)

Tanabe, Hiroki 3909

Evolution equations. (Japanese)

Sûgaku **14** (1962/63), 137–152.

This is an expository paper on evolution equations: $du/dt = A(t)u + f(t)$, where $A(t)$ is a closed and additive operator which is defined on a subspace of a Banach space X and whose values are in X, $f(t) \in X$, and $0 \leq t \leq T$. The quantity u is an unknown function of t whose values are in X. The problem treated in this paper is the existence and uniqueness of the initial-value problem for this equation. The author classifies methods roughly into two kinds: one based on the theory of one-parameter semi-groups and the other based on the theory of representations

of linear functionals in Hilbert spaces by bilinear forms. This paper is mostly concerned with the methods based on the theory of one-parameter semigroups. The main part of the paper is devoted to the explanation of the results in the following two papers: T. Kato [Nagoya Math. J. **19** (1961), 93–125; MR **26** #631] and T. Kato and H. Tanabe [Osaka Math. J. **14** (1962), 107–133; MR **25** #4367]. The author also briefly mentions an existence and uniqueness theorem due to J. L. Lions [*Équations différentielles opérationnelles et problèmes aux limites*, p. 127, Springer, Berlin, 1961; MR **27** #3935].

Y. Sibuya (Minneapolis, Minn.)

Zaidman, Samuel 3910

Soluzioni quasi-periodiche per alcune equazioni differenziali in spazi hilbertiani.

Ricerche Mat. **13** (1964), 118–134.

Let V and H be two Hilbert spaces with $V \subset H$, the canonical injection being continuous. Assume also that V with the topology induced by H is dense in H. Let $A(\ ,\ )$ be a continuous, Hermitian-symmetric sesquilinear form on V such that for each $\beta > 0$ there exists an $\alpha > 0$ such that

$$a(u, u) + \beta(u)_H{}^2 \geqq \alpha(u)_V{}^2, \qquad v \in V.$$

Here $|\cdot|_H$, $|\cdot|_V$ are the norms in V and H, respectively. Let A be the self-adjoint operator in H associated with the form a.

The author considers the differential equation $u''(t) - Au(t) = f(t)$, $-\infty < t < \infty$, and proves regularity theorems for weak solutions of this equation. In particular, he shows that if $f(t)$ is (Bochner) almost periodic in H and u is a bounded (in H) solution of $u'' - Au = f$, then u is almost periodic in V and u' is almost periodic in H.

R. S. Freeman (College Park, Md.)

Nashed, M. Z. 3911

The convergence of the method of steepest descents for nonlinear equations with variational or quasi-variational operators.

J. Math. Mech. **13** (1964), 765–794.

Zur Lösung der nichtlinearen Operatorgleichung $G(x) = \Theta$ in einem Hilbertraum werden Verfahren der Gestalt $x_{n+1} = x_n - \alpha_n T_n G(x_n)$ betrachtet. Seien $G(x) = x + F(x)$, der Operator F vollstetig, stetig differenzierbar im Fréchetschen Sinne und die Ableitung F_x' symmetrisch, weiterhin G_x' gleichmäßig bezüglich x positiv definit, die Operatoren T_n gleichmäßig bezüglich n beschränkt und positiv definit und T_n und G_{x_n}' miteinander vertauschbar, so gibt es Zahlen α_n, festgelegt durch gewisse Ungleichungen, für welche die Elemente x_n gegen eine Lösung konvergieren. Die Konvergenzgeschwindigkeit ist linear. Beim Beweis wird ausgenutzt, daß G unter den Voraussetzungen nach einem Satz von Kerner der Gradient eines Funktionals ist. Es folgen Erweiterungen auf bestimmte Operatoren, welche sich nicht als Gradient eines Funktionals darstellen lassen, sowie Anwendungen der Theorie auf das Dirichletsche Problem bei quasilinearen Differentialgleichungen und auf nichtlineare Integralgleichungen vom Hammersteinschen Typ.

J. W. Schmidt (Dresden)

Rall, L. B. 3912

Variational methods for nonlinear integral equations.

Nonlinear Integral Equations (Proc. Advanced Seminar Conducted by Math. Research Center, U.S. Army, Univ. Wisconsin, Madison, Wis., 1963), pp. 155–189. *Univ. Wisconsin Press, Madison, Wis.*, 1964.

A survey of the variational methods in Banach spaces, as applied to the study of non-linear equations. It is an exposition of the recent results of the Soviet school (as summarized in Vaĭnberg's book [*Variational methods for investigation of non-linear operators* (Russian), GITTL, Moscow, 1956; MR **19**, 567]), which are not available in English. Written very clearly, supplemented with an extensive bibliography. No proofs.

A. Alexiewicz (Poznań)

CALCULUS OF VARIATIONS

See also 3519, 3776, 4565, 4635, 4636.

Bellman, Richard 3913

On a variational problem of Miele.

Astronaut. Acta **9** (1963), 199–200.

The problem of Miele and Saaris [same Acta **9** (1963), 184–198; MR **28** #4746] referred to in the title of this paper is that of minimizing the integral

$$\int_0^T \left[\frac{\rho^6}{\rho^2+\dot\rho^2} + 2(\rho^2+\dot\rho^2)^{1/2}\right] d\theta$$

subject to the constraint $\int_0^T \rho^2\, d\theta = a$. The author now shows how the theory of inequalities, specifically Young's inequality, can be used in giving simple alternative derivations of the fundamental properties of the solutions of this problem. Thus, from the observation that, by Young's inequality, $\int_0^T u^3\, dt + 2\int_0^T v^{3/2}\, dt \geq 3\int_0^T uv\, dt$ for any non-negative functions u and v, it follows by a simple substitution that

$$\int_0^T \left[\frac{\rho^6}{\rho^2+\dot\rho^2} + 2(\rho^2+\dot\rho^2)^{1/2}\right] d\theta \geq 3\int_0^T \rho^2\, d\theta,$$

with equality if and only if $\rho^2 + \dot\rho^2 = \rho^4$.

E. F. Beckenbach (Los Angeles, Calif.)

Cinquini, Silvio 3914

A proposito dell'esistenza dell'estremo assoluto per gli integrali estesi a intervalli infiniti.

Math. Notae **19** (1964), 161–170.

This note is devoted to a theorem giving sufficient conditions for the existence of an absolute minimum for an integral of the type

$$\int_a^\infty f(x, y(x), y'(x), \cdots, y^{(n)}(x))\, dx$$

over a given class of functions. The proof is made by showing that the hypotheses imply those of an earlier theorem by the same author [Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **32** (1962), 845–851; MR **27** #4104]. However, an example is given to show that in certain cases the present theorem may be more easily applied than the earlier one.

W. R. Transue (Gambier, Ohio)

Lebesgue, Henri 3915
En marge du calcul des variations.
Préface par L.-C. Young.
Enseignement Math. (2) **9** (1963), 209–326.

As indicated by its title, this is a collection of interesting remarks by Lebesgue on various aspects of variational theory. The titles of the chapters are as follows. I: Sur une question de minimum. II: Sur le problème des isopérimètres. III: Sur quelques questions de minimum relatives aux courbes orbiformes et sur leurs rapports avec le calcul des variations. IV: Sur la plus courte distance entre deux points d'une surface développable. V: La méthode classique du calcul des variations. VI: La méthode directe du calcul des variations. Chapters I, II, III and the major part of Chapter VI were previously published in various journals. Chapter V uses a simple two-parameter family of variations to derive the Euler equation, the existence of the second derivative on the minimizing curve wherever $f_{y'y'} \neq 0$, and the concavity of $f(x, y, y')$ as a function of y'. Chapter VI is largely concerned with existence proofs for the Dirichlet problem by variational methods, and with errors in a related 1877 memoir of Carl Neumann [*Untersuchungen über das logarithmische und Newton'sche Potential*, Leipzig, 1877].

L. M. Graves (Chicago, Ill.)

Morrey, Charles B., Jr. 3916
Quelques résultats récents du calcul des variations.
Les Équations aux Dérivées Partielles (*Paris*, 1962), *pp.* 129–149. *Éditions du Centre National de la Recherche Scientifique, Paris*, 1963.

Considerati gli integrali $I(z, G) = \int_G f[x, z(x), \nabla z(x)]\,dx$, ove $x = (x^1, \cdots, x^\nu)$, $z = (z^1, \cdots, z^N)$, $\nabla(z) = \{\partial z^i/\partial x^\alpha\}$, l'autore espone i risultati, relativi alla differenziabilità delle estremali degli integrali $I(z, G)$, stabiliti in altro lavoro [*Partial differential equations and continuum mechanics*, pp. 241–270, Univ. Wisconsin Press, Madison, Wis., 1961; MR **22** #12424], arrecando alcune semplificazioni alle proprie precedenti dimostrazioni.

S. Cinquini (Pavia)

Picone, Mauro 3917
Criteri sufficienti in generali problemi di calcolo delle variazioni riguardanti integrali pluridimensionali d'ordine qualsivoglia nel vettore minimante a più componenti.
Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez. I (8) **7** (1963), 33–58.

This memoir lives up to its title, and also includes a number of sufficiency theorems for special cases. Necessary conditions for a minimum are briefly treated in Section 1.

L. M. Graves (Chicago, Ill.)

Picone, M. 3918
Sur la condition de Weierstrass pour le minimum d'une intégrale à plusieurs dimensions.
Ann. Polon. Math. **15** (1964), 153–158.

Rund, Hanno 3919
Canonical formalism for parameter-invariant integrals in the calculus of variations whose Lagrange functions involve second order derivatives.
Ann. Mat. Pura Appl. (4) **64** (1964), 99–107.

Dans une publication précédente [mêmes Ann. (4) **55** (1961), 77–104; MR **25** #1469], l'auteur avait développé une théorie des intégrales, invariantes par changement de paramètre, dont le lagrangien contient des dérivées d'ordre supérieur; cette théorie s'était révélée insuffisante pour les applications, parce que l'hamiltonien ne dépendait pas explicitement des dérivées premières des coordonnées de position. On définit ici un nouvel hamiltonien qui donne lieu à un formalisme moins compliqué que le précédent et plus immédiatement applicable à des problèmes spéciaux. Se généralisent au cas présent: la notion de champ géodésique (de Carathéodory), l'équation de Hamilton-Jacobi, les parenthèses de Poisson. La méthode est illustrée par un exemple d'un caractère assez général.

M. Janet (Paris)

Vinti, Calogero 3920
L'integrale di Weierstrass e l'integrale del Calcolo delle Variazioni in forma parametrica.
Ist. Lombardo Accad. Sci. Lett. Rend. A **97** (1963), 101–114.

A definition for Weierstrass integrals of the calculus of variations previously given by the author is here adapted to integrals W_C relative to the integrand $\Phi(x, y; x', y', \theta')$, where θ is the slope of tangent to $C \equiv [x(t), y(t)]$, even when $x'(t)$, $y'(t)$ are of bounded variation. Invariance of W_C with respect to parametrizations is obtained, and when the length L_C of C is finite, a theorem of approximation expressed by

$$\lim_{h\to 0} \int_0^{L_C} \Phi\left[x(s), y(s), x'(s), y'(s), \frac{\theta(s+h)-\theta(s)}{h}\right] ds = W_C$$

(s = arc length) is proved. The Lebesgue integral I_C is $\leq W_C$, as expected, and if use is made of a result of S. Cinquini [Atti Accad. Naz. Lincei Rend. Cl. Fis. Mat. Natur. (8) **29** (1960), 515–520; MR **25** #472], equality holds if and only if x', y' are absolutely continuous. Extensions to integrands with higher derivatives are sketched.

E. Baiada (Modena)

GEOMETRY

See also 3390, 3479, 3513, 3545, 3549, 3619, 3956, 3976–3978.

Chilton, B. L. 3921
On the projection of the regular polytope {5, 3, 3} into a regular triacontagon.
Canad. Math. Bull. **7** (1964), 385–398.

The polytope {5, 3, 3} is one of the six convex regular 4-dimensional polytopes, and in some ways is the most complicated, being bounded by 120 dodecahedra and therefore called a 120-cell. The author uses two distinct methods to show the possibility of its projection, onto a plane, to be bounded by a regular 30-gon such that its symmetry group will be that of the regular triacontagon, viz., D_{30}, the dihedral group of order 60. In the projection, the 600 vertices can be classified in 12 sets, each consisting of 30 or 60 vertices which lie on a circle concentric with the outer triacontagon. A beautiful picture of the projection has been drawn as published and acknowledged earlier by Coxeter [*Introduction to geometry*, p. 403, Wiley, New York, 1961; MR **23** #A1251] showing the four rings of 30 regular

pentagons of four different sizes with the smallest forming the outermost and the largest the innermost ring. To realize and visualize this projection, the properties of the triacontagonal projection of the reciprocal polytope {3, 3, 5}, called the 600-cell [Coxeter, *Regular polytopes*, Macmillan, New York, 1963; MR 27 #1856], have been used in both methods. *S. R. Mandan* (Kharagpur)

Blum, R.; Guinand, A. P. 3922

A quartic with 28 real bitangents.

Canad. Math. Bull. 7 (1964), 399–404.

The authors show that the locus of the centres of ellipses which have given semi-axes a, b and touch two fixed lines is a quartic having 28 real bitangents if $\sqrt{(b/a)} < \tan\phi < \sqrt{(a/b)}$, 2ϕ being the angle between the fixed lines. They also draw the figure showing its practical construction by ruler and compass only. Its symmetry gives it the look of a beautiful pattern. *S. R. Mandan* (Kharagpur)

Fucke, Rudolf; Kirch, Konrad; Nickel, Heinz 3923

★Darstellende Geometrie.

Zweite, verbesserte Auflage. Lehrbücher der Mathematik.

VEB Fachbuchverlag, Leipzig, 1963. 293 pp. DM 12.80.

A general text for mathematics and engineering students commissioned by the Zentrale Fachkommission für Mathematik beim Staatssekretariat für das Hoch- und Fachschulwesen. Table of contents: (1) Orthogonale Mehrtafelprojektion; (2) Eintafelprojektion; (3) Axonometrie; (4) Zentralprojektion.

Zwikker, C. 3924

★The advanced geometry of plane curves and their applications.

(Formerly titled: *Advanced plane geometry*)

Dover Publications, Inc., New York, 1963. xii + 299 pp. $2.00.

This edition is an unabridged and unaltered republication of the work first published in 1950 under the title *Advanced plane geometry* [North-Holland, Amsterdam, 1950]. The geometry of special plane curves is treated by complex variable methods. *P. J. Davis* (Providence, R.I.)

Amir-Moéz, Ali R. 3925

Les sommets d'une surface.

Enseignement Math. (2) 10 (1964), 255–260.

Un point P d'une surface est un sommet quand P est un sommet de la quadrique osculatrice à P. L'auteur donne un critère pour que P soit un sommet au moyen d'une égalité entre deux matrices. *O. Bottema* (Delft)

Amir-Moéz, Ali R. 3926

Vertex points of functions.

Enseignement Math. (2) 10 (1964), 261–266.

Generalization of #3925 above. The author considers here functions of n complex variables. *O. Bottema* (Delft)

Schwerdtfeger, Hans 3927

Projective geometry in the one-dimensional affine group.

Canad. J. Math. 16 (1964), 683–700.

The elements of the affine group G of one variable $x' = \xi x + \eta$ can be represented by the points in the (ξ, η)-plane for which $\xi \neq 0$. In a projective interpretation, this means that the elements of the group are mapped onto the points of a projective plane from which two lines, L_∞ and L_0 (here $\xi = 0$), have been lifted. This gives rise to two kinds of parallelism. Of importance are the points $U = L_0 \cap L_\infty$, $0 \in L_0$, a point I in the plane representing the unit of the group, and $\infty = 0L \cap L_\infty$. The lines through I, except IV, are the normalizers of elements of G, IV is the commutator subgroup. The 0-parallel lines are left cosets, the ∞-parallels right cosets of the groups represented by lines through I. A projectively invariant construction is given for product and inverse. This allows a very detailed study of the lattice generated by normalizers, commutators, and their cosets by translation of statements of projective geometry. Some examples are given, and the treatment of the affine group over an arbitrary field F is indicated. *H. W. Guggenheimer* (Minneapolis, Minn.)

Bouligand, Georges; Flocon, Albert; Barre, André 3928

Étude comparée de différentes méthodes de perspective, une perspective curviligne.

Acad. Roy. Belg. Bull. Cl. Sci. (5) 50 (1964), 294–308.

Authors' summary: "Avis liminaire. Les bases de la perspective classique. L'axiome fondamental. Une nouvelle perspective. Retour à la demande fondamentale; les composants. Étude comparée des solutions. Le choix. Un point d'histoire."

Nguyễn Cảnh Toàn [Nguen, Kan Toan] 3929

Decomposition d'une collineation de l'espace P_n en produit de perspectives ou en produit d'homologies centrales application aux matrices.

Publ. Math. Debrecen 10 (1963), 1–9.

The author proves some results on collineations of finite-dimensional projective spaces. These are not all new [cf. Baer, *Linear algebra and projective geometry*, Chapter III.2, Academic Press, New York, 1952; MR 14, 675]. *P. Dembowski* (Frankfurt a.M.)

Mammana, Carmelo 3930

Sulle corrispondenze cremoniane piane la cui jacobiana ha una sola componente irriducibile.

Matematiche (Catania) 18 (1963), 164–186.

Es sei n der Grad der Transformation, μ der Grad der Jacobischen Kurve. Für $\mu = 1$ erhält man die ganzen Cremona-Transformationen. Es ist $\mu^2 | n-1$ oder $3|\mu$ $9 \times \mu$, $9|n+1$. Für die Zahlen n und μ mit dieser Nebenbedingung gibt es Transformationen. Die F-Punkte sind alle Nachbarpunkte eines von ihnen. Sie folgen einander auf einem Zweig z. Ihre Vielfachheit ist durch μ teilbar.

Ist der Zweig z linear, so haben alle F-Punkte die Vielfachheit μ. Ihre Anzahl sei t. Folgende Fälle, und nur sie, sind möglich:

μ	n	t
1	2	3
2	5	6
3	8	7
6	17	8

Ist der Zweig z von zweiter Ordnung, so sind 8 Fälle möglich, von denen 5 durchgerechnet werden.

O.-H. Keller (Halle)

Mammana, Carmelo 3931

Gruppi di corrispondenze cremoniane di S_r definiti da proprietà delle jacobiane.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **35** (1963), 27–34.

Sei H die Gruppe aller nichtausgearteten Homographien eines projektiven komplexen Raumes S_r der Dimension $r \geq 2$, G die Gruppe aller Cremona-Transformationen von S_r und $\{F\}$ eine Menge von irreduziblen Hyperflächen in S_r. Gewisse Untermengen von G, deren Elemente durch die Jacobi'sche Form der Cremona-Transformationen und ihrer Inversen in Bezug auf die Hyperflächen aus $\{F\}$ charakterisiert sind, werden als Untergruppen von G nachgewiesen. Dabei verwendet der Verfasser eine Arbeit von G. Dantoni [Ann. Mat. Pura Appl. (4) **29** (1949), 243–246; MR **11**, 738]. In G existieren stets Cremona-Transformationen T derart, daß die Jacobi'schen Formen von T und T^{-1} einen gegebenen irreduziblen Kegel des S_r als einzige irreduzible Komponente besitzen. Als Anwendung werden im Falle $r \geq 3$ bzw. $r \geq 2$ und unter zusätzlichen Voraussetzungen über $\{F\}$ weitere Untergruppen von G angegeben. Besonderes Interesse beansprucht der Fall, daß $\{F\}$ aus einer Hyperebene des S_r bzw. aus einer ebenen Kurve vom Geschlecht $p \geq 0$ besteht. Abschließend definiert der Verfasser den Begriff der Klasse von Erzeugenden von G und leitet ein Theorem ab, das den bekannten Unterschied in der Faktorenzerlegung von G in den Fällen $r=2$ und $r \geq 3$ beleuchtet.

R. Moufang (Frankfurt a.M.)

Benedicty, Mario 3932

On plurilinearities among projective spaces.

Rend. Sem. Mat. Univ. Padova **34** (1964), 110–134.

The author investigates graphic plurilinearities, which are simply pluricorrespondences among projective spaces which generalize the concept of collineations between two linear spaces. The plurilinearities among three projective lines are classified.

D. Pedoe (Lafayette, Ind.)

Freudenthal, Ernst; Heinrich, Werner 3933

★Neue Behandlung der Kurven zweiter Ordnung durch Invarianten.

Mathematische Arbeitshefte, 18.

Ernst Klett Verlag, Stuttgart, 1963. 84 pp. *DM* 3.00.

From the authors' preface: "Dieser Lehrgang arbeitet ausschliesslich mit den erstmalig in der Ebene gedeuteten Bewegungsinvarianten der Kurven zweiter Ordnung (K_2) und zieht daher die Koordinatentranslations- und Koordinatendrehungsformeln nicht mehr heran. Er benutzt keine projektiven Koordinaten, keine Matrizen und kaum die Vektorrechnung. Der Lehrgang ist mehrfach in der Sachsenwaldschule in Reinbek (Bezirk Hamburg) und der Johann Heinrich Voss-Schule in Eutin (Holstein) erprobt worden."

This book will be of no use to intending scientists looking for a quick introduction to important properties of conics, but it is a very effective introduction for a young mathematician to coordinate geometry. It underlines far more effectively than most texts the essential distinction between intrinsic properties of a geometrical configuration and the relationship of such a configuration to a particular choice of axes. It treats all the basic properties of conics and systems of conics.

The chapter headings are: (I) Die fokalerzeugten Kurven zweiter Ordnung; (II) Fokalerzeugte Kurven zweiter Ordnung in allgemeiner; (III) Klassifikation und nahere invariante Beschreibung der Kurven zweiter Ordnung; (IV) Mannigfaltigkeiten von Kurven zweiter Ordnung.

D. B. Scott (Brighton)

Ellers, Erich; Karzel, Helmut 3934

Endliche Inzidenzgruppen.

Abh. Math. Sem. Univ. Hamburg **27** (1964), 250–264.

Ein projektiver Raum einer Dimension >1 zusammen mit einer auf der Menge seiner Punkte erklärten Gruppenverknüpfung heißt Inzidenzgruppe, wenn die Linkstranslationen der Gruppe Kollineationen des Raumes sind. Als Gruppen betrachtet sind die Inzidenzgruppen (bis auf Isomorphie) gerade die (auf den Punkten) scharf einfach transitiven Gruppen von Kollineationen projektiver Räume (mit Dimension >1). Die desarguesschen Inzidenzgruppen lassen sich durch Paare (F, K) darstellen, wobei F ein Fastkörper und K ein Unterschiefkörper von F ist, dessen multiplikative Gruppe Normalteiler in der von F ist und über dem F einen Rang >2 hat. Mittels der Zassenhausschen Ergebnisse über die Struktur der Fastkörper werden so die sämtlichen endlichen desarguesschen Inzidenzgruppen angegeben und daraufhin untersucht, wenn die Linkstranslationen als Kollineationen sämtlich linear sind.

G. Pickert (Giessen)

Srinivasan, B. R. 3935

On the net of conics with a common self-polar triangle. II.

J. Madras Univ. B **32** (1962), 71–101.

In this second section of the author's Madras thesis of 1959 [for the first section see J. Madras Univ. Sect. B **31** (1961), 75–95; MR **27** #1864], Chapter IV deals with the one-to-one mapping of the conics S of the net $ax^2+by^2+cz^2=0$ on the point $S'(a,b,c)$ of the projective plane. Studied are the "F-curve" of two net conics S and S', i.e., the conic through the points of contact of the common tangents to S and S', the reciprocation of the conics of the net with respect to a net conic, considered as a quadratic involution, in-, out- and apolarity of conics (S is out-polar to S' if the polar conic of point S with respect to the fundamental triangle passes through S') and conics which are harmonically separated by two of the degenerate conics in their pencil. Chapter V brings in a metrical specialization, Chapter VI a dual representation, so that instead of point loci in the projective plane we have envelopes of lines.

D. J. Struik (Belmont, Mass.)

Baldus, Richard 3936

★Nichteuklidische Geometrie. Hyperbolische Geometrie der Ebene.

Vierte Auflage. Bearbeitet und ergänzt von Frank Löbell. Sammlung Göschen, Bd. 970/970a.

Walter de Gruyter & Co., Berlin, 1964. 158 pp. *DM* 5.80.

There are very few changes compared with the pre-war editions. Trigonometry and analytic geometry have been expanded, and an appendix on Clifford-Klein surfaces has been added. It remains one of the outstanding books on non-euclidean geometry. (The careful treatment of area has been taken over unchanged from the second edition (1942).) *H. W. Guggenheimer* (Minneapolis, Minn.)

Marchionna, Ermanno 3937

Varietà di prima specie ed ipersuperficie di aggiunzione.
Univ. Nac. Tucumán Rev. Ser. A **14** (1962), 105–125.

Soit dans un espace projectif S_r une variété algébrique définie dans le corps complexe pure, c'est-à-dire dont toutes les composantes irréductibles aient la même dimension d; elle est localement normale si à partir de l assez grand le système linéaire coupé par la totalité des formes d'ordre l est complet. L'auteur démontre que si l'on considère deux variétés V_d et V_d', V_d irréductible localement normale, V_d' pure sans parties multiples telles que V_d+V_d' soit de la première espèce au sens de Dubreil, que leur intersection D soit aussi pure sans parties multiples, si E est la section hyperplane générique de V_d, le système linéaire complet $|lE-D|$ qui existe effectivement est coupé sur V_d en dehors de D par la totalité des formes d'ordre l passant par V_d'; généralisation du Restsatz de Severi. Si l'on note $P(W_d, l)$ la postulation de W_d pour les formes d'ordre l, la relation $P(V_d, l)+P(V_d', l)=P(V_d+V_d', l)+P(D, l)$ exprime que V et V' sont "régulièrement enchainées". Si V_d et V_d' satisfont aux conditions du théorème, une condition suffisante pour qu'elles soient toutes deux de la première espèce est que D le soit. Si de plus une section hyperplane générale de $V+V'$ est de la première espèce, la condition est nécessaire. Sur une V_d irréductible, privée de variétés multiples de dimension $d-1$, de section hyperplane E, une variété $(d-1)$-dimensionnelle A pure sans parties multiples est hypersurface d'adjonction si les formes d'ordre quelconque l, passant par A, coupent sur V_d en dehors de A le système complet $|lE-A|$. Étude de quelques propriétés de ces hypersurfaces d'adjonction, une V_d localement normale qui contient une hypersurface A de la première espèce l'est aussi si, sur elle, A est d'adjonction. Lien avec de nombreuses propriétés des variétés arithmétiquement normales, conduisant à de nouveaux résultats sur les multiples de $|E|$ pour une variété irréductible. *B. d'Orgeval* (Dijon)

Etayo, José Javier 3938

Extension of the notion of linearity in algebraic geometry and a concept of algebraicity. (Spanish)
Rev. Mat. Hisp.-Amer. (4) **22** (1962), 215–238.

Permutti, Rodolfo 3939

Sulla varietà delle corde di una varietà algebrica irriducibile non singolare. (English summary)
Rend. Accad. Sci. Fis. Mat. Napoli (4) **30** (1963), 275–280.

Author's summary: "The author studies in which cases the manifold of chords of an algebraic irreducible non-singular manifold V_k of S_{2k+1}, meeting a general S_l, is reducible and gives a simple proof of a known theorem concerning the double manifold of the projection of a V_k of S_{2k+1} from an S_{k-1} into an S_{k+1}."

Godeaux, Lucien 3940

Construction de quelques surfaces projectivement canoniques. I.
Acad. Roy. Belg. Bull. Cl. Sci. (5) **50** (1964), 227–234.

Author's summary: "Construction, au moyen de la théorie des involutions, de quatre surfaces projectivement canoniques, c'est-à-dire dont le système canonique coïncide avec le système des sections hyperplanes."

Longo, C. 3941

Classificazione di trivettori o di complessi lineari di piani.
Univ. e Politec. Torino Rend. Sem. Mat. **22** (1962/63), 19–38 (1964).

Eingangs werden die p-Vektoren eines $(n+1)$-dimensionalen Vektorraumes E_{n+1} und der duale Begriff der p-Formen erklärt. Dem entsprechend folgt die Definition eines linearen Komplexes $L_n^{(p-1)}$ von projektiven Räumen S_{p-1} im projektiven Raum S_n der durch die (kontravarianten) Vektoren von E_{n+1} bestimmten Richtungen mittels einer linearen Gleichung in den Grassmannschen Koordinaten der S_{p-1}. Das Problem der Kennzeichnung der p-Vektoren besteht nun darin, die kleinste Zahl von unzerlegbaren (unabhängigen) p-Vektoren zu bestimmen, mittels derer ein gegebener p-Vektor als Linearkombination darstellbar ist. Eine entsprechende Formulierung dieser Aufgabe wird auf dem Begriff der linearen Komplexe $L_n^{(q)}$ basiert.

Zweck der Abhandlung ist, die verschiedenen (bekannten) Methoden zu untersuchen und vergleichen, die im Fall $p=3$ (Trivektoren) und für $n\leq 7$ und teilweise auch für $n=8$ zu dieser Klassifikation führen. *H. R. Müller* (Braunschweig)

Tazzi Cantalupi, Gabriella 3942

Alcuni teoremi esistenziali sui fasci reali massimali di curve algebriche.
Ist. Lombardo Accad. Sci. Lett. Rend. A **97** (1963), 763–772.

This is a contribution to the topology of curves and surfaces in algebraic geometry over the real number field, in particular, of a surface Σ on which is a pencil of curves Φ of genus p, without singular members, each Φ having the maximum number $p+1$ of disjoint circuits. The main result is that for every integer $m>0$, there exist such surfaces for $p=(m-1)(2m-1)$, consisting of m sheets which are topological spheres, each containing one circuit of each curve Φ and two base points of the pencil, and $(m-1)^2$ sheets which are topological tori, each containing two circuits of each curve of the pencil and no base points. This is constructed as a surface of revolution of order $2m$, the curves Φ being the meridian plane sections. Proof of existence is by induction on m. *P. Du Val* (London)

Gallarati, Dionisio 3943

Sul contatto di ipersuperficie algebriche di S_r lungo varietà $(r-2)$-dimensionali.
Univ. e Politec. Torino Rend. Sem. Mat. **22** (1962/63), 209–214 (1964).

La condition nécessaire et suffisante pour que deux hypersurfaces algébriques F et G de l'espace S_r aient le long de

n variétés à $r-2$ dimensions $C_1, C_2, \cdots, C_n$ des contacts respectivement d'ordre q_i-1 (≥ 0), $i=1, 2, \cdots, n$, les C_i étant simples pour F et G et $\sum_1^n q_i C_i$ étant leur intersection complète, est que l'on puisse associer à tout ensemble de n hypersurfaces $H_1, H_2, \cdots, H_n$ passant génériquement par la C_i de même indice, deux formes A et B, telles que la variété $A=B=F=G=0$ soit de dimension au plus $r-3$, et donnant l'identité: $AF+BG=H_1^{q_1}H_2^{q_2}\cdots H_n^{q_n}$.
B. d'Orgeval (Dijon)

Speranza, Francesco 3944

Le trasformazioni d'un piano in sè approssimabili con una trasformazione quadratica dotata d'una conica di punti uniti. (English summary)

Boll. Un. Mat. Ital. (3) **19** (1964), 193–205.

Author's summary: "Conditions are given for a point-transformation between two superposed planes to have, in a fixed point or in a general pair of corresponding points, at least an osculating, or semi-osculating, quadratic transformation with a conic of fixed points."
H. R. Müller (Braunschweig)

Subramanyam, S. S. 3945

On the general isotomic line-involution.

Math. Student **31** (1963), 47–50 (1964).

When two lines in a plane are conjugate with respect to all conics touching four given lines d forming a proper quadrilateral, then they form what is called a general isotomic involution; it can be considered as a dual of a general isogonal quadratic Cremona transformation. When one of the lines d is at infinity, the other lines d form the medial triangle of the diagonal triangle of the degenerate quadrilateral, and we speak of isotomic conjugate lines [the author, Math. Student **28** (1960), 70–73; MR **26** #659]. Now the line transformation is considered for the case of the net of conics S for which a given triangle is self-polar. Then the line transformation between pairs of asymptotes of the conics of the net is a one-to-one involution, hence a Cremona line involution. One of the results proved is that the asymptotes of a conic S are also the asymptotes of a definite conic which circumscribes the anti-medial triangle of any triangle self-polar to S.
D. J. Struik (Belmont, Mass.)

Demaria, Davide Carlo 3946

Sui caratteri dei sistemi lineari d'ipersuperficie segati sopra due varietà complementari dalle forme dello spazio ambiente.

Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **98** (1963/64), 147–174.

The ideas and results of this work are closely related to those of a paper by Marchionna [Ann. Mat. Pura Appl. (4) **54** (1961), 159–199; MR **24** #A3547]. A leading part is played by the indices of irregularity σ^h ($h=1, 2, \cdots, d-1$) of an algebraic variety V_d.

The author considers two non-singular varieties V_d, V_d' of a complex projective space S_r which make up the complete intersection of $(r-d)$-forms of S_r having respective orders $n_1, n_2, \cdots, n_{r-d}$. It is further assumed that the variety $D=V_d\cap V_d'$ is non-singular and of dimension $d-1$.

With the usual notation, let (E_m), (E_m'), (E_m^*) denote the linear systems cut by the forms of order m on V_d, V_d' and D, respectively; let $\Delta_m, \Delta_m', \Delta_m^*$ be the relative deficiencies of these systems, and $\sigma^h(E_m)$, $\sigma^h(E_m')$, $\sigma^h(E_m^*)$ the corresponding indices of irregularity. According to convention, $\Delta_m=0$ for $m\leq 0$; also $\sigma^h(E_m)=0$ for $m<0$.

Writing $\rho=n_1+n_2+\cdots+n_{r-d}-r-1$, we first recall that, for $d=1$, $\Delta_m=\Delta_m'$ (Severi and Gaeta). The author establishes the following generalisation of this result:

$$\Delta_m=\sigma^{d-1}(E'_{\rho-m}),\qquad \sigma^1(E_m)=\sigma^{d-2}(E'_{\rho-m}),$$
$$\sigma^2(E_m)=\sigma^{d-3}(E'_{\rho-m}),\ \cdots.$$

It now readily follows that for $m>\rho$ the linear systems (E_m), (E_m') are complete and regular, a theorem previously obtained by other authors using different methods. Further, denoting by $g_k(W)$ the number of differential forms of the first kind of degree k attached to a variety W, the author shows that

$$\Delta_\rho'=g_{d-1}(V_d),\qquad \sigma^{d-h}(E_\rho')=g_{h-1}(V_d)$$
$$(h=2, 3, \cdots, d-1).$$

Finally, he obtains the relations

$$\Delta_m^*=\Delta_m+\Delta_m',\qquad \sigma^h(E_m^*)=\sigma^h(E_m)+\sigma^h(E_m'),$$
$$g_h(D)=g_h(V_d)+g_h(V_d')\qquad (h=1, 2, \cdots, d-2).$$
L. Roth (London)

Demaria, Davide Carlo 3947

Sui piani grafici esagonali.

Univ. e Politec. Torino Rend. Sem. Mat. **22** (1962/63), 115–158 (1964).

Nella prima parte di questa memoria (in parte monografica) l'autore espone con chiarezza e concisione quelle premesse e quelle definizioni (4-arco, reticolato intero, proprietà esagonale) sui piani grafici e sugli anelli ternari di M. Hall associati ai loro sistemi normali di coordinate affini, che occorrono al lettore per la comprensione delle parti successive.

Nella seconda parte si dimostra che ogni piano grafico esagonale π è localmente desarguesiano. Ciò significa che ogni 4-arco (quaterna di punti a 3 a 3 non allineati) di π ne genera un subpiano isomorfo ad un piano lineare su di un corpo primo di caratteristica $p\geq 0$ (questo teorema era già stato dimostrato dall'autore per $p>0$ [stessi Rend. **18** (1958/59), 43–52; MR **22** #12443]).

Nella terza parte l'autore considera i piani grafici esagonali per cui valgono anche i postulati dell'ordine e di Archimede. In particolare dimostra che condizione necessaria e sufficiente perchè un piano archimedeo sia desarguesiano è che esso contenga un subpiano esagonale.

Nella quarta parte si fa vedere come nei postulati che definiscono secondo D. Hilbert il piano proiettivo reale si possa sostituire l'ipotesi della validità della proprietà esagonale in un subpiano del piano stesso a quella della validità universale del teorema di Desargues.
E. Morgantini (Padova)

Salzmann, Helmut 3948

Zur Klassifikation topologischer Ebenen. II.

Abh. Math. Sem. Univ. Hamburg **27** (1964), 145–166.

The author continues his investigation of "flat" projective

planes **P** (topological planes whose point space is homeomorphic to that of the real projective plane) whose collineation group Γ is not too small. In his earlier papers [Math. Ann. **145** (1961/62), 401–428; MR **25** #2509; Arch. Math. **13** (1962), 98–109; MR **26** #666; Math. Ann. **150** (1963), 226–241; MR **27** #1922] it was shown (i) that **P** is desarguesian (and hence isomorphic to the real projective plane) if $\dim \Gamma > 4$, (ii) that **P** is a Moulton plane [Moulton, Trans. Amer. Math. Soc. **3** (1902), 192–195] if $\dim \Gamma = 4$, and (iii) all **P** with $\dim \Gamma \geq 3$ and Γ simple were completely determined. In the present paper the case $\dim \Gamma = 3$ and Γ non-simple is investigated, with topological group theory the main tool in the proofs. There are essentially five different possibilities for the configuration of the fixed elements of a closed connected three-dimensional subgroup Δ of Γ. In four of these cases the structure of **P** is completely determined, and the last case (where the fixed elements are an incident point-line pair) is postponed to a later publication. Except in the author's case (2), where there are two fixed points and two fixed lines of Δ, the plane **P** must again be desarguesian or moultonian. In case (2), however, there are other possibilities: **P** can then be coordinatized by a cartesian group [cf. Pickert, *Projektive Ebenen*, p. 90, Springer, Berlin, 1955; MR **17**, 399] whose additive group is that of the real numbers and whose multiplication depends on three real parameters.

P. Dembowski (Frankfurt a.M.)

André, Johannes 3949

Bemerkung zu meiner Arbeit "Über verallgemeinerte Moulton-Ebenen".

Arch. Math. **14** (1963), 359–360.

In Verbesserung einer früheren Arbeit [dasselbe Arch. **13** (1962), 290–301; MR **26** #667] wird gezeigt, daß die Bedingung III ($(as)b = a(sb)$, falls $s > 0$) bei der Kennzeichnung der verallgemeinerten Moulton-Ebenen (v.M.E.) fortzulassen ist, da sie nicht bei allen v.M.E. gilt und bei den Beweisen entbehrt werden kann. Ferner wird darauf hingewiesen, daß in Satz 4.3 (der genannten Arbeit) für die Umkehrung die Zugehörigkeit von u, U zur Unterebene benötigt wird.

G. Pickert (Giessen)

Pierce, William A. 3950

Collineations of projective Moulton planes.

Canad. J. Math. **16** (1964), 637–656.

Als affine Moulton-Ebene über dem Körper K wird eine affine Ebene bezeichnet, die sich (mit ϵ als Inzidenz) so bilden läßt: Punkte sind die Paare (x, y) aus $K \times K$, Geraden die Mengen $\{(x, y);\ x = c\}$, $\{(x, y);\ y = a \circ x + b\}$ $(a, b, c \in K)$; dabei wird die Verknüpfung $\circ$ mittels einer Untergruppe P vom Index 2 in der multiplikativen Gruppe von K und einer nichtidentischen umkehrbaren Abbildung ϕ von K auf sich folgendermaßen erklärt: $a \circ x = ax$ für $x \in P$, $a \circ x = \phi(a)x$ sonst. Damit nach diesem Verfahren überhaupt eine affine Ebene entsteht, muß ϕ gewissen Bedingungen genügen; ohne Einschränkung der Allgemeinheit wird darüber hinaus $\phi(0) = 0, \phi(1) = 1$ angenommen. Die projektive Abschließung einer affinen Moulton-Ebene wird als projektive Moulton-Ebene (im folgenden kurz: M.E.) bezeichnet; l_∞ ist ihre uneigentliche Gerade und Y_∞ der uneigentliche Punkt der Geraden $[0] =_{\mathrm{def}} \{(x, y);\ x = 0\}$. Es werden nun die Kollineationen einer M.E. auf eine M.E. bestimmt, zugleich mit den Bedingungen, wann zwei M.E. isomorph sind (d.h. eine Kollineation der einer auf die andere existiert). Ein Ausnahmefall liegt vor, wenn K die kleinstmögliche Ordnung, nämlich 9, besitzt. Nur in diesem Fall gilt nicht, daß jede Kollineation einer M.E. mit Y_∞ als Fixpunkt die Geraden $[0]$, l_∞ vertauscht oder festläßt. Besonders bemerkenswert ist das folgende Ergebnis: Genau dann, wenn eine M.E. (über K) eine den Punkt Y_∞ nicht festlassende Kollineation besitzt, ist K ein angeordneter Körper und die M.E. isomorph zu einer M.E. über K, bei der ϕ durch ein Element $q > 0$, $\neq 1$ von K bestimmt wird gemäß: $\phi(a) = a$ für $a > 0$, $\phi(a) = aq$ sonst.

G. Pickert (Giessen)

Lee, Boon-Phiew 3951

Plane configurations.

Bull. Math. Soc. Nanyang Univ. **1963**, 47–50.

A point and a line are incident if either the point lies on the line or the line passes through the point. A plane configuration is composed of p points and q lines such that every point is incident with a fixed number α of q lines and every line is incident with a fixed number β of p points. The symbol (p_α, q_β) denotes such a plane configuration whose dual is (q_β, p_α). If $\alpha = \beta$, the configuration is represented by (p_α) (self-dual configuration). (k_3) does not exist for $k = 7$. Whatever be the configuration, $p\alpha = q\beta$. The symbol $\{[q(q-1)/2]_2, q_{q-1}\}$ represents the only configuration for which every point of intersection of configuration lines is also a configuration point. The dual statement is satisfied by $\{p_{p-1}, [p(p-1)/2]_2\}$. If, finally, this is true for a self-dual configuration, the corresponding configuration is the triangle (3_2).

F. Şemin (Istanbul)

CONVEX SETS AND GEOMETRIC INEQUALITIES

See also 3443, 3444, 3852.

Alemany, R. E. 3952

Numerical values of certain constants related to mixed volumes of convex bodies. (Spanish)

Rev. Un. Mat. Argentina **21**, 113–118 (1963).

I. Fary [Illinois J. Math. **5** (1961), 425–430; MR **24** #A811] has obtained linear expressions for integrals of certain continuous, additive, translation-invariant functionals on the class of convex bodies in euclidean n-space, and in particular for certain "mixed volume" functionals. In the present paper the coefficients in these linear expressions are evaluated and used to prove a special case of a basic formula from integral geometry.

T. A. Botts (Charlottesville, Va.)

Shephard, G. C. 3953

Convex bodies and convexity on Grassmann cones. VIII. Projection functions of vector sums of convex sets.

J. London Math. Soc. **39** (1964), 417–423.

This paper uses the notation and terminology of the earlier papers in the same series (joint work of the author, H. Busemann and G. Ewald [Busemann, Ewald and Shephard, Math. Ann. **151** (1963), 1–41; MR **28** #522a; Arch. Math. **13** (1962), 512–526; MR **28** #522b; Shephard, J. London Math. Soc. **39** (1964), 307–319; MR **29** #520]).

Since it had been shown in an earlier paper in the series that two convex bodies can have the property that each has a convex projection function while their vector sum does not, it is natural to look for conditions under which it is possible to assert the convexity of a vector sum of convex sets about which we know nothing except their dimensions. A comprehensive result in this direction, too complicated to summarize here, is obtained in Theorem (10). An important tool in this work is the mixed projection function, introduced and studied at the beginning of the paper. This is the Minkowski mixed volume of the orthogonal projections of a number of convex sets on an r-flat, and its properties are of independent interest for their own sake. *A. M. Macbeath* (Birmingham)

Fejes Tóth, L. 3954

On the isoperimetric property of the regular hyperbolic tetrahedra. (Russian summary)

Magyar Tud. Akad. Mat. Kutató Int. Közl. **8** (1963), 53–57.

The isoperimetric property of the regular tetrahedron is the fact that, of all tetrahedra having a given surface area, the regular one has the greatest volume. Two simple proofs are known for Euclidean space, but neither of them extends in any natural way to non-Euclidean spaces, and the case of elliptic space still presents an unsolved problem. For his ingenious proof of the isometric property in hyperbolic space, the author uses the following "moment lemma" [cf. the reviewer and the author, Quart. J. Math. Oxford Ser. (2) **14** (1963), 273–284, p. 277; MR **28** #1532]: Of all convex n-gons D (in the absolute plane) having a given area, the regular n-gon with center O maximizes the "moment" $\int_D f(r)\,dp$, where $f(r)$ is a strictly decreasing function of the distance r from a fixed point O to a variable point P of D, and dp is the element of area at P.

H. S. M. Coxeter (Hanover, N.H.)

Klee, Victor 3955

On the number of vertices of a convex polytope.

Canad. J. Math. **16** (1964), 701–720.

Let P be a d-dimensional convex polytope in R^d, and let f_s denote the number of s-faces of P. It has been conjectured that

$$(1)\qquad f_0 \le \binom{f_{d-1}-[\frac{1}{2}(d+1)]}{f_{d-1}-d}+\binom{f_{d-1}-[\frac{1}{2}(d+2)]}{f_{d-1}-d},$$

$$(1^*)\qquad f_{d-1} \le \binom{f_0-[\frac{1}{2}(d+1)]}{f_0-d}+\binom{f_0-[\frac{1}{2}(d+2)]}{f_0-d},$$

the upper bound in (1*) being attained by the neighborly polytopes. Special cases of this conjecture have been established by various authors; e.g., Gale proved (1*) in case $f_{d-1}=d+2$ or $d+3$ [same J. **16** (1964), 12–17; MR **28** #516] (the present paper contains a history of known results). The author establishes (1*), in case $f_0 \ge (\frac{1}{2}d)^2-1$, not only for d-polytopes, but also for arbitrary Eulerian $(d-1)$-manifolds of Euler characteristic $1-(-1)^d$. For d-polytopes with $f_{d-1}\ge(\frac{1}{2}d)^2-1$, (1) follows by duality. Further, it is proved that for f_0 sufficiently large, among all d-polytopes with f_0 vertices, the neighborly polytopes maximize not only f_{d-1} but also f_s, $1\le s\le d-2$. Another result is the sharp inequality, $f_s \ge \binom{d+1}{s+1}+\binom{d-1}{s}$, $0\le s\le d-1$, if P is not a d-simplex. The author also proves some results pertaining to a closely related problem of Dantzig (to determine those convex sets with a maximum number of extreme points among all sets determined by m linear equations in n non-negative variables).

G. D. Chakerian (Davis, Calif.)

Bollobás, Béla 3956

Filling the plane with congruent convex hexagons without overlapping.

Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **6** (1963), 117–123.

The problem of filling and covering the Euclidean plane with congruent (but not necessarily "equivalent") polygons was discussed by Kepler, Hilbert, and especially H. Heesch and O. Kienzle [*Flächenschluss*, Springer, Berlin, 1963; MR **27** #6185]. The author obtains some significant restrictions on the possible kinds of convex hexagons that can serve as a tile. In particular, he proves that if the plane can be filled with congruent hexagons, it can be filled with the same hexagons in such a way that each vertex belongs to exactly three of them. On the other hand, vertices of higher valency can easily occur; for instance, the author has drawn a pattern of irregular (but equilateral) hexagons having the symmetry of a regular pentagon. *H. S. M. Coxeter* (Hanover, N.H.)

Heppes, Aladár 3957

Filling of a domain by discs. (Russian summary)

Magyar Tud. Akad. Mat. Kutató Int. Közl. **8** (1963), 363–371 (1964).

L. Fejes Tóth [Publ. Math. Debrecen **5** (1957), 119–127; MR **19**, 763] proved a theorem on the average area of isoperimetric convex discs packed in a convex hexagon. The proof is valid for the more general Theorem A stated below. Let $a_6(p)$ be the maximum possible area of a disc of perimeter $\le p$ lying in a regular hexagon of unit area. Then Theorem A states that if $D_1, \cdots, D_n$ are n convex discs of perimeter $\le p$ packed in a convex hexagon H, their average area does not exceed $a_6(p)$. The author proves that Theorem A is valid if (A*) the discs D_i are not necessarily convex, and (A**) the D_i are not necessarily convex and H is replaced by the union $\bigcup$ of n faces of a tessellation consisting of regular hexagons of unit area. In both cases the case of non-convex D_i is reduced to that of convex discs as follows. The D_i are first replaced by their convex hulls C_i, which are shown to intersect simply [in the notation of the reviewer and C. A. Rogers, J. London Math. Soc. **27** (1952), 304–314; MR **13**, 971]. Then, using the method of the paper by Rogers and the reviewer, the C_i are replaced by non-overlapping discs E_i such that $\bigcup C_i=\bigcup E_i$. Theorem A* follows at once from Theorem A, while A** is proved by replacing the E_i by non-overlapping polygonal discs whose average number of sides does not exceed six.

R. P. Bambah (Columbus, Ohio)

Strohmajer, J. 3958

Über die Verteilung von Punkten auf der Kugel.

Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **6** (1963), 49–53.

The problem of arranging n points on the surface of a

sphere such that the minimum distance between them is maximum has attracted a lot of attention since its formulation by P. M. L. Tammes in 1930. The solution is known for $n \leq 12$ and $n = 24$; various cases having been disposed of by L. Fejes Tóth, P. M. L. Tammes, B. L. van der Waerden, K. Schütte, L. Danzer and R. M. Robinson. Good configurations for $n = 13$, 14, 15, 16, 17, 20, 25, 32, 33, 42 have been given by different authors. In this note good configurations for $n = 18$, 21, 22, 26, 30, 31, 52 due to the author, D. Kólya, J. Györffy and K. Böröczky are described. The density D_n of the corresponding packing of circles is compared with the upper bound Δ_n due to R. M. Robinson [Math. Ann. **144** (1961), 17–48; MR **24** #A3565]. *R. P. Bambah* (Columbus, Ohio)

Bambah, R. P.; Rogers, C. A.; Zassenhaus, H. 3959

On coverings with convex domains.

Acta Arith. **9** (1964), 191–207.

Let K be a closed, bounded, strictly convex region of area $a(K)$ in the real affine plane, $t(K)$ the area of the largest triangle contained in K, $\theta^*(K)$ the density of the best covering of the plane by translated replicas of K, and $\theta(K)$ the same with the restriction that the translations form a group. The authors prove that there exists a triangulation of the plane, with vertices at corresponding points in the replicas of K, such that each triangle has area $\leq t(K)$. They deduce that

$$\theta^*(K) \geq a(K)/\{2t(K)\}$$

and that, if K is centrally symmetric,

$$\theta^*(K) = a(K)/\{2t(K)\} = \theta(K).$$

They remark that the above inequality is sometimes (but not always) stronger than a similar inequality (using a hexagon instead of a triangle) due to Fejes Tóth [*Regular figures*, p. 167, Macmillan, New York, 1964; MR **29** #2705]. By suitably triangulating a polygonal region (their Theorem 5), they obtain a reasonably concise proof for a theorem which had been announced by Bambah and Rogers [J. London Math. Soc. **27** (1952), 304–314; MR **13**, 971]; their original proof was too complicated for publication. *H. S. M. Coxeter* (Hanover, N.H.)

Rogers, C. A. 3960

Covering a sphere with spheres.

Mathematika **10** (1963), 157–164.

Let N be the least number of its spherical caps of chord 2 required to cover the surface of a sphere of radius R, and let N^* be the least number of spheres of radius 1 required to cover the solid sphere of radius R. Upper bounds for N, N^* are obtained for $R > 1$, $n \geq 9$. The upper bound of N is of the type $N \leq f(n, R)\theta_R^{-1} + 1 = N_1$, where θ_R is the proportion of the surface of the sphere covered by one of the caps. This is used to obtain an estimate for N^* of the same type. An earlier result of the author [Mathematika **4** (1957), 1–6; MR **19**, 877] is used to obtain an estimate for N^* for $R \geq n/\log n$. This is also valid for the number of convex bodies K with centre O covering RK. Combined with the other estimate, this gives $N^* = O(n \log n R^n)$ if $R \geq n$, and $O(n^{5/2}R^n)$ if $R < n$.

To obtain the estimate of N above, the author proves (as in his earlier paper referred to above) that on the average $[N_1]$ caps of chord ξ (a certain length less than 2) cover a good proportion of the surface. Taking a configuration at least as good as the average and enlarging the caps to those of chord 2, he gets a covering of the surface by $[N_1]$ such caps. The problem of getting an estimate for N with $f(n, R)$ independent of R is open. *R. P. Bambah* (Columbus, Ohio)

Molnár, József 3961

Estensione del teorema di Segre-Mahler allo spazio.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **35** (1963), 166–168.

The author defines a Segre-Mahler polytope to be one whose dihedral angles are $\leq 120°$. He makes the conjecture that, when equal spheres of radius r are packed in such a region of n-space, the density is $\leq d_n(r)$, where $d_n(r)$ is the density, inside a regular simplex of edge $2r$, of the packing of $n+1$ spheres of radius r with their centers at the vertices. In the notation of Schläfli [*Gesammelte mathematische Abhandlungen*, Band II, p. 177, Birkhäuser, Basel, 1953; MR **14**, 833] and the reviewer [Proc. Sympos. Pure Math., Vol. VII, pp. 53–71, esp. pp. 56–64, Amer. Math. Soc., Providence, R.I., 1963; MR **29** #1581], if the space is spherical or elliptic,

$$d_n(r) = \frac{\int_0^r \sin^{n-1}\rho\, d\rho}{\int_0^{\pi/2} \sin^{n-1}\rho\, d\rho}\,\frac{F_n(\alpha)}{F_{n+1}(\alpha)}, \quad \sec 2\alpha = n - 1 + \sec 2r;$$

if it is hyperbolic, there is a somewhat similar formula; and the Euclidean case can be derived by making r tend to zero.

This conjecture, which has already been established in the Euclidean case by Rogers [Proc. London Math. Soc. (3) **8** (1958), 609–620; MR **21** #847], is now proved for non-Euclidean 3-spaces. The author remarks that, in the spherical (or elliptic) case with $r = \pi/10$, the upper bound

$$d_3\left(\frac{\pi}{10}\right) = 12\left(1 - \frac{5}{\pi}\sin\frac{\pi}{5}\right) = 0.774\cdots$$

is attained by the "packing" of a single sphere of radius $\pi/10$ in one of the cells of the spherical honeycomb $\{5, 3, 3\}$ [the reviewer, Acta Math. Acad. Sci. Hungar. **5** (1954), 263–274, esp. p. 266; MR **17**, 523].

In the hyperbolic case, he mentions Florian's result that $d_3(r)$ is monotonic [Böröczky and Florian, ibid. **15** (1964), 237–245; MR **28** #3369] and deduces that, when equal spheres of radius r are packed in a Segre-Mahler polyhedron, the density is always less than

$$d_3(\infty) = \left(1 + \frac{1}{2^2} - \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} - \cdots\right)^{-1} = 0.853\cdots.$$

H. S. M. Coxeter (Hanover, N.H.)

DIFFERENTIAL GEOMETRY

See also 3484, 3522, 3523, 3940–3942, 3944, 4069, 4071, 4467, 4572.

Belov, K. M. 3962

On surfaces of constant mean curvature. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 746–749.

Let A be a regular surface of constant mean curvature in three-dimensional Euclidean space and x, y isometric parameters of the surface A, so that

$$(*)\qquad ds^2 = \Lambda(x, y)(dx^2+dy^2).$$

The author examines by means of analytic functions the distribution of umbilical points and the lines of curvature of the surface A, formulates necessary and sufficient conditions for the realization of the metric (*) on the surface A and studies the deformation of the surface A.

Z. Nádeník (Prague)

Santaló, L. A. 3963

On some characteristic properties of the sphere. (Spanish. English summary)

Univ. Nac. Tucumán Rev. Ser. A **14** (1962), 287–297.

Author's summary: "Theorem: (a) Let k be the curvature and τ the torsion of a closed curve C. Let s be the arc length of C and $f(k, \tau)$ a function of k and τ. If the relation

$$(1)\qquad \int_C f(k, \tau)\,ds = 0$$

holds for every closed spherical curve C, then $f=\varphi(k)\tau$, where $\varphi(k)$ is an arbitrary function of k. Reciprocally, if the function $f(k, \tau)$ has the form $f=\varphi(k)\tau$, then (1) holds for every closed spherical curve such that $k^{-1}\neq 0$ at every point. (b) Given a function $\varphi(k)$, if the relation

$$(2)\qquad \int_C \varphi(k)\tau\,ds = 0$$

holds for every closed curve C on a given surface Σ, then Σ is either a plane or a sphere. The curve C and the surface Σ are assumed of class C^3. This theorem generalizes previous results, see, e.g., G. Saban [Atti Accad. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **25** (1958), 457–464; MR **21** #5967].

H. W. Guggenheimer (Minneapolis, Minn.)

Bereis, Rudolf 3964

Über das Raumbild eines ebenen Zwanglaufes (kinematische Abbildung von Blaschke und Grünwald).

Wiss. Z. Techn. Univ. Dresden **13** (1964), 7–16.

Die Arbeit bringt zunächst eine möglichst elementar gehaltene Einführung in die aus dem Jahre 1911 stammende kinematische Abbildung von W. Blaschke und J. Grünwald, bei der die Bewegungen der Ebene auf die Punkte eines quasielliptischen Parameterraumes übertragen werden. Hauptsächliches Hilfsmittel der Darstellung sind dabei geometrisch die Drehnetze der Cliffordschen Parallelen des quasielliptischen Raumes und die mit ihnen verbundenen Netzprojektionen, die sich, ebenso wie die Bewegungen, analytisch in einfacher Weise mittels komplexer Zahlen beschreiben lassen. Den Geraden g des Raumes sind dabei im kinematischen Bilde zwei Bildpunkte g_l, g_r zugeordnet und schneidenden Geraden g, g' entsprechen äquidistante Bildpaare ($g_lg_l'=g_rg_r'$).

Jeder stetig differenzierbare ebene Bewegungsvorgang ("Zwanglauf") besitzt dabei im quasielliptischen Bildraum als kinematisches Bild eine bestimmte Raumkurve, wobei dem System der Tangenten t der Raumkurve in der Bildebene die beiden Polkurven (t_l) und (t_r) der Bewegung entsprechen. Die Arbeit entwickelt die Bildkurven verschiedener wichtiger Bewegungsvorgänge, nämlich der elliptischen Bewegung und der symmetrischen Rollungen, insbesondere der verschiedenen Arten symmetrischer Kegelschnittsrollung, und geht zum Schluß auf die Konstruktion des kinematischen Raumbildes einer ebenen Bewegung ein, die durch zwei vorgegebene Punktbahnen festgelegt ist.

K. Strubecker (Karlsruhe)

Geise, Gerhard 3965

Über den Zusammenhang von Netzprojektion und kinematischer Abbildung.

Wiss. Z. Techn. Univ. Dresden **13** (1964), 17–18.

Im Anschluß an die vorausgehende Arbeit von R. Bereis [#3964] wird auf elementarem Wege die kinematische Abbildung von W. Blaschke und J. Grünwald und eine naheliegende Verallgemeinerung davon aus der Netzprojektion hergeleitet. Während bei der üblichen kinematischen Abbildung den eigentlichen Punkten des Bildraumes die Drehungen der Ebene entsprechen, treten bei der genannten Verallgemeinerung an die Stelle der Drehungen gewisse Drehstreckungen. Den Fernpunkten des Bildraumes entsprechen in beiden Fällen die Translationen der Ebene.

K. Strubecker (Karlsruhe)

Bottema, O. 3966

On the determination of the Burmester points for five positions of a moving plane.

Nederl. Akad. Wetensch. Proc. Ser. A **67** = *Indag. Math.* **26** (1964), 310–318.

Referring to the paper by F. Freudenstein and G. N. Sandor [J. Appl. Mech. **28** (1961), 41–49; MR **22** #10239], the author offers a direct analytical method to determine Burmester pairs, i.e., the combinations of a Burmester point with the center of its corresponding circle. The Burmester points are found as the intersections of two conics. In general, the configuration of the Burmester points is affinely related to that of the corresponding centers. If one Burmester point has its corresponding center at infinity, then the other Burmester points are collinear. Attention is paid to the case in which three of the five positions of the moving plane are parallel and to the cases in which all or a part of these positions are infinitesimally separated.

D. J. Struik (Belmont, Mass.)

Finn, R.; Osserman, R. 3967

On the Gauss curvature of non-parametric minimal surfaces.

J. Analyse Math. **12** (1964), 351–364.

For a minimal surface S defined by $z=\phi(x, y)$, $x^2+y^2<R^2$, $\phi(0, 0)=0$, the authors continue the investigation of the value K of the Gauss curvature of S at the origin.

It is a result of E. Heinz [Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. Math.-Phys.-Chem. Abt. **1952**, 51–56; MR **14**, 885] that there is a universal constant c_0 such that $|K|\leq c_0/R^2$, and of E. Hopf [J. Rational Mech. Anal. **2** (1953), 519–522; MR **14**, 1119] that there is a universal constant c_1 such that $|K|\leq c_1/(R^2W^2)$, where W^2 is the value of $1+\phi_x^2+\phi_y^2$ at the origin. Further, R. Osserman [Trans. Amer. Math. Soc. **96** (1960), 115–128; MR **22** #12457] has shown that there is a universal constant c_2

such that $|K| \leq c_2/(d^2W^2)$, where d is the distance on S from the origin to the boundary of S.

It is now established that $|K| < g(W)/R^2$, where

$$g(W) = \frac{1}{2}\left(1+\frac{1}{W^2}\right)^2\left(\frac{\pi}{2}+\tan^{-1}\sqrt{\left(\frac{W^2-1}{2}\right)}\right)^2$$

and that in case $W=1$, i.e., in case S has a horizontal tangent plane at the origin, the value $g(1)=\pi^2/2$ is best possible.

As for best values for the constants c_0, c_1, c_2, it was previously known only that $c_0 \leq c_1 \leq c_2$ and that $7\frac{1}{8} \leq c_2 < 8$. The present result yields $\pi^2/2 \leq c_0 < (\pi+1/\pi)^2 < 6$, so that, in particular, $c_0 < c_2$.

E. F. Beckenbach (Los Angeles, Calif.)

Ruscior, Ştefania 3968

Sur une classification affine des hypersurfaces réglées dans un E_4.

Acad. Roy. Belg. Bull. Cl. Sci. (5) **50** (1964), 309–314.

Author's summary: "Les résultats acquis dans ce travail, consacré à la classification affine des hypersurfaces réglées dans un E_4, se trouvent synthétisés dans un tableau inséré à la fin de cet article. A la base de cette classification nous avons choisi la section de l'hypersurface réglée par l'hyperplan impropre et la nature de l'enveloppe des hyperplans tangents, menés le long d'une génératrice."

Berezina, L. Ja. 3969

On the theory of the two-dimensional surface in E_4. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 4 (41), 12–18.

On considère un espace pseudoeuclidien E_4 de dimension 4, c'est-à-dire un espace affine dans lequel le produit scalaire des vecteurs $a(a_0, a_1, a_2, a_3)$ et $b(b_0, b_1, b_2, b_3)$ est $ab = a_1b_1 + a_2b_2 + a_3b_3 - a_0b_0$. La géométrie des vecteurs unitaires de E_4 coïncide avec celle de l'espace de Lobatchevski L_3 de dimension 3. Dans ce travail, on considère simultanément les espaces E_4 et L_3 ce qui permet d'utiliser les résultats obtenus dans la théorie des congruences de droites dans L_3 pour l'étude des surfaces de dimension 2 dans E_4.

M. Decuyper (Lille)

Doľkal, Ljerka 3970

Kongruenz der Gemeinlote von Erzeugenden einer rationalen windschiefen Regelfläche n-ten Grades. (Serbo-Croatian summary)

Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hrvatske Ser. II **17** (1962), 205–222 (1963).

Die Arbeit entwickelt zuerst Formeln für die Ordnung N und die Klasse M der Linienkongruenz, die von den Gemeinloten der Erzeugendenpaare einer rationalen windschiefen Regelfläche n-ten Grades gebildet wird. Man findet im allgemeinen Falle $N=(2n-1)(n-1)$ und $M=\frac{1}{2}(n-1)(3n-2)$. Die Gemeinlote der Erzeugendenpaare einer Regelschar des einschaligen Hyperboloids ($n=2$) bilden danach eine Linienkongruenz der Ordnung $N=3$ und Klasse $M=2$.

Im folgenden wird der Sonderfall der Kongruenz der Gemeinlote aller Erzeugendenpaare einer Regelschar eines einschaligen Drehhyperboloids betrachtet. Die Verfasserin geht dabei synthetisch vor und verwendet die konstruktiven Hilfsmittel der Darstellenden Geometrie, insbesondere der Perspektive. Wird eine Erzeugende e festgehalten, so bilden die sie treffenden Gemeinlote ein Plückersches Konoid, welches das Hyperboloid außer in seiner doppelten Leitlinie e und zwei Erzeugenden noch in einer Ellipse schneidet. Aus den Erzeugenden dieses Plückerschen Konoids kann die Kongruenz durch Drehung um die Hyperboloidachse gewonnen werden. Es wird noch eine weitere Zergliederung der Kongruenz in ein Büschel von Plückerschen Konoiden K, studiert und konstruktiv sowie figürlich erläutert.

Mit der Kongruenz der Gemeinlote der Erzeugendenschar eines (beliebigen) einschaligen Hyperboloids hat sich, wie der Autorin offenbar entgangen ist, schon W. Waelsch [S.-B. Math.-Natur. Cl. Akad. Wiss. Wien **96** (1887), II. Abt., 781–801] befaßt. Die große Bedeutung dieser Kongruenz für die Theorie der linearen Gewindemannigfaltigkeiten und ihre wichtigsten Eigenschaften findet man vor allem in E. Study [*Geometrie der Dynamen*, Teubner, Leipzig, 1902] entwickelt.

K. Strubecker (Karlsruhe)

Dadažanov, N. 3971

Cyclic pairs of T_n complexes. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 793–803.

The author studies cyclic pairs of T_n complexes, which are generalizations of the well-known pairs of T complexes. They are introduced as follows.

Let there be given a one-to-one correspondence between the lines of two complexes G, G'. The plane π which corresponds to the point $M \in l$ in the main (normal) correlation of the G on the line l intersects the corresponding line l' in a point M'. To this point we can construct analogously a point M^* on the line l. The correspondence $\pi: M \to M^*$ is a projective transformation.

If π is the identity, the complexes G, G' form a T-pair. Cyclic pairs of T_n complexes are defined by the condition that π is a cyclic projective transformation of order $n \geq 2$.

A. Urban (Prague)

Korovin, V. I. 3972

Stratification of two-parameter families of lines in a higher-dimensional projective space. (Russian)

Mat. Sb. (*N.S.*) **64** (**106**) (1964), 378–395.

L'ensemble formé de trois systèmes (l_i) ($i=1, 2, 3$) à deux paramètres de droites dans un espace projectif S_5 s'appelle cycle stratifiable si l'on peut attacher, à chaque système (l_i), une couche de surfaces de telle façon que les plans tangents, au points d'une droite l_i quelconque, passent par la droite l_{i+1} ($l_4 = l_1$) qui correspond à l_i dans une correspondance définie entre (l_i) et (l_{i+1}). Les cycles en question dépendent de 10 fonctions d'une variable. On démontre que les systèmes formant un cycle stratifiable sont des congruences R de droites. Les cycles qui renferment une congruence R donnée dépendent de 13 constantes arbitraires. On étudie les cycles qui admettent un système linéaire de droites ayant avec les congruences du cycle un contact du second ordre. Ces cycles dépendent de deux fonctions d'une variable. La généralité des cycles contenus dans un système linéaire en question dépendent d'une fonction d'une variable. Les transformations de Laplace des congruences d'un cycle stratifiable forment aussi des cycles stratifiables. Les cycles particuliers, définis par

l'égalité des invariants de Darboux des surfaces focales des congruences du cycle, dépendent de 26 constantes arbitraires et ils sont caractérisés par la propriété que les systèmes à deux paramètres des plans, déterminés par les triples de foyers correspondants des congruences du cycle, forment un couple stratifiable. Les recherches ultérieures sont consacrées à l'étude des transformations des systèmes à deux paramètres de droites moyennant des systèmes à un paramètre de droites situées sur une hyperquadrique. On arrive ainsi à une transformation T des congruences de droites dans S_3. A la fin du travail l'auteur donne un bref résumé des résultats relatifs à la stratification des systèmes à deux paramètres de droites dans S_{2n+1}. Quelques problèmes de ce mémoire ont été considérés dans un travail du rapporteur [Ann. Mat. Pura Appl. (4) **57** (1962), 239–255; MR **26** #4274]. *K. Svoboda* (Brno)

Stanilov, G. 3973

Classification of complexes in a biaxial space. (Russian)

C. R. Acad. Bulgare Sci. **17** (1964), 221–222.

The author presents the classification of line complexes in a projective 3-space, the groups of its automorphisms consisting of the collineations preserving two lines.

A. Švec (Waltham, Mass.)

Porcu, Livio 3974

Sugli invarianti proiettivi di coppie di calotte del 2° ordine.

Ist. Lombardo Accad. Sci. Lett. Rend. A **97** (1963), 603–630.

Two (possibly coincident) regular $(r-1)$-dimensional differential caps are considered in a projective r-dimensional space $(r \geq 3)$, and the homographies transforming one of them into the other are studied. Particular attention is paid to the case $r=3$ and to the case when the two caps have the origin in common, the tangent prime and possibly the asymptotic tangent cone at this point. Applications are made concerning some of the well-known differential invariants of the two caps.

B. Segre (Rome)

Backes, F. 3975

Sur les paramètres de distribution de diverses surfaces réglées.

Acad. Roy. Belg. Bull. Cl. Sci. (5) **50** (1964), 11–18.

Author's summary: "L'étude des paramètres de distribution des génératrices de certaines surfaces réglées admettant une courbe donnée pour ligne de striction, suggère une étude analogue relative aux surfaces réglées engendrées par les droites d'un corps solide dans son mouvement le plus général."

Godeaux, Lucien 3976

Sur un théorème de Darboux concernant les congruences W.

Acad. Roy. Belg. Bull. Cl. Sci. (5) **49** (1963), 758–759.

Author's summary: "Démonstration géométrique du théorème de Darboux suivant lequel les coordonnées d'une droite engendrant une congruence W satisfont à une équation de Laplace."

Godeaux, Lucien 3977

Un cas de fermeture d'une suite de Laplace.

Acad. Roy. Belg. Bull. Cl. Sci. (5) **49** (1963), 851–855.

Author's summary: "Examen du cas où la suite de Laplace attachée dans un espace à cinq dimensions à une surface de l'espace ordinaire se termine dans un sens en présentant le cas de Laplace, le point de fermeture décrivant une courbe qui appartient à un espace à trois dimensions."

Godeaux, Lucien 3978

Remarque sur la suite de Laplace associée à une surface.

Acad. Roy. Belg. Bull. Cl. Sci. (5) **50** (1964), 8–10.

Author's summary: "Construction de relations linéaires entre sept points consécutifs de la suite de Laplace de l'espace à cinq dimensions associée à une surface."

Godeaux, Lucien 3979

Sur les directrices de Wilczynski des surfaces ayant mêmes quadrilatères de Demoulin.

Acad. Roy. Belg. Bull. Cl. Sci. (5) **50** (1964), 48–55.

Author's summary: "Étude des relations entre les directrices de Wilczynski de deux surfaces ayant mêmes quadrilatères de Demoulin et de questions connexes."

Picasso, E. 3980

Su certi sistemi ∞^2 di rette associati ai punti di una superficie differenziabile di un iperspazio.

Rend. Circ. Mat. Palermo (2) **12** (1963), 114–128.

The author extends in various ways Blaschke's results concerning a surface in a projective space P^4 and a hyperplane (or primal) P^3 in it; the tangent planes to the surface intersect P^3 in a congruence of lines; special properties of this congruence are projective properties of the surface and the hyperplane.

The configurations examined by the author are the following. (1) a surface V_2 in P^4 and a P^3: the congruences of the osculating planes to the curves of the conjugate system of V_2; (2) a V_2 in a P^5 and the intersections of the tangents to its parametric lines with a fixed P_4; (3) a V_2 with a double conjugate system in $P^{2\nu}$ and a fixed $P^{2\nu-1}$.

E. Bompiani (Rome)

Sorace, Orazio 3981

Superficie Φ e congruenze W.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **34** (1963), 620–627.

Erfüllen die inhomogenen Koordinaten $z^i = z^i(u, v)$ (mit $i = 1, 2, \cdots, n+2$) einer Fläche F des projektiven Raumes S_{n+2} $(n \geq 2)$ eine Laplacesche Differentialgleichung $z_{uv} = Pz_u + Qz_v$, so heißt die Fläche F vom (nicht parabolischen) Typus ϕ. Die Koordinatenlinien $u =$ konst. und $v =$ konst. von F bilden dann das charakteristische Netz obiger Differentialgleichung. Die Schmieg S_n, die die Linien $u =$ konst. von n-ter Ordnung berühren, schneiden einen festen S_3 allgemeiner Lage in den Strahlen einer Strahlkongruenz. Der Verfasser sucht die notwendigen und hinreichenden Bedingungen, denen die Fläche F und der S_3 genügen müssen, damit so eine W-Kongruenz erzeugt wird. Hierbei sind die Fälle n gerade oder n ungerade zu unterscheiden. *H. R. Müller* (Braunschweig)

Krasnodębski, R. 3982

A natural parameter of a curve in the symplectic space.
Ann. Polon. Math. 15 (1964), 189–194.

Nomizu's technique [Tôhoku Math. J. (2) **11** (1959), 106–112; MR **21** #5966] is used to define a natural parameter in a geometry given by an invariant form $\Omega = \sum_1^n x^i \wedge x^{n+i}$ of maximal rank n. The rank of a curve is the maximum dimension p such that the osculating spaces of dimension $k-1$ and k are in involution relative to Ω for $k \leq p$. This notion obviously is invariant in diffeomorphisms of high enough class. If the rank is constant on an arc, there is only one integral invariant θ such that

$$\Omega\left(\frac{d^p x}{d\theta^p}, \frac{d^{p+1}x}{d\theta^{p+1}}\right) = 1$$

for curves of class $\geqq 2p+1$.

H. W. Guggenheimer (Minneapolis, Minn.)

Dolci, Alba 3983

Brevi appunti di calcolo vettoriale.
Rend. Sem. Fac. Sci. Univ. Cagliari **33** (1963), 465–514.

An expository article based on lectures of L. Castoldi.

Botella Raduán, F. 3984

On the proper differential of a topological space. (Spanish)
Rev. Mat. Hisp.-Amer. (4) **23** (1963), 75–83.

For the author a differential of a topological space is a sheaf of groups arising from a presheaf of local homeomorphism groups of the space. A proper differential can only occur on some special topological spaces, those which have a certain amount of local homogeneity. Proper differentials are associated with proper coordinate bundles, meaning those which have fibres homeomorphic to neighborhoods in the base space. This paper is concerned with the question of whether the sheaves depend finely on some neighborhood. However, this reviewer was unable to decipher the definition of this concept either through its statement or through the proofs. The definition as stated seems to be concerned with whether or not the part of a fibre bundle given by a coordinate function is a product bundle, but the reviewer's interpretation of a coordinate function is that it is always a representation of the part of the bundle as a product.

R. L. Bishop (Urbana, Ill.)

Botella Raduán, F. 3985

Differential connexion on a topological space and on a generalized manifold. (Spanish)
Rev. Mat. Hisp.-Amer. (4) **23** (1963), 181–188.

A presheaf of groups of homeomorphisms of neighborhoods of a topological space X prolongs to a coordinate bundle and gives a sheaf over X which is called a differential connexion with respect to the coordinate bundle. Parallel displacement in this bundle along a curve C in X is defined as being the collection of all those homeomorphisms between the fibres at the ends of C which are given by piecewise lifts of C with respect to given coordinate functions and their distortions by the local groups.

The definitions require that X be locally strongly ("fuertamente", abbreviated f) homogeneous. The definition of this concept is given in the author's paper [same Rev. (4) **22** (1962), 123–140], and here a characterization of it is given in terms of the existence of differential connexions and parallel displacement for them. In particular, a generalized manifold-α, defined in the next paper, satisfies the conditions.

R. L. Bishop (Urbana, Ill.)

Botella Raduán, F. 3986

A generalized manifold. (Spanish)
Rev. Mat. Hisp.-Amer. (4) **23** (1963), 189–193.

A generalized manifold is a separable locally strongly homogeneous topological space, which has been defined previously to be a space with a system of neighborhoods, each of which is homeomorphic to a homogeneous space A.

This paper is concerned with defining analogues to the tangent bundle, called a proper coordinate bundle, with the existence of differentials (which are certain sheaves of groups) and the existence of differentiable functions (those which induce mappings between differentials, e.g., all continuous functions). A more restrictive concept of generalized manifolds-α is defined by requiring A to be a topological ball, which means that all bundles over A are trivial, and it is pointed out that manifolds in the usual sense are generalized manifolds-α.

R. L. Bishop (Urbana, Ill.)

Liber, A. E. 3987

Quasilinears as characteristic objects of subgroups of the linear group. (Russian)
Trudy Sem. Vektor. Tenzor. Anal. **12** (1963), 63–70.

Let G be a subgroup of the full linear group $GL(n)$. Then G is isomorphic to a subgroup G of the group of all automorphisms of the centred affine n-space E_n. Hence there exists a geometric object in E_n that is the characteristic object of G (i.e., G is the group of all automorphisms of E_n preserving this object). Characteristic objects of subgroups of $GL(n)$ have been found till now in very special cases only [see, e.g., P. K. Raševskiĭ, same Trudy **10** (1956), 105–117; MR **18**, 907]. The author finds such objects for almost all one-parameter subgroups.

Let G be a one-parameter subgroup of $GL(n)$. If a coordinate system in E_n is fixed, then G is isomorphic to the group of transformations ${}^*x^\alpha = P_\beta{}^\alpha(t)x^\beta$, where x^β are coordinates of a point in E_n and t is a canonical parameter. The author introduces a special class of geometric objects in E_n—quasilinears—and proves that if the real parts of the eigenvalues of the matrix $\|dP_\beta{}^\alpha(0)/dt\|$ are not all equal to 0, then G has a quasilinear as its characteristic object. There are also found different properties of quasilinears (e.g., their Lie derivatives).

B. M. Schein (*Šaĭn*) (Saratov)

Hadžiivanov, Nikolai; Gavrilov, Mihail 3988

On algebraic exterior forms. I. (Bulgarian. French summary)
Annuaire Univ. Sofia Fac. Sci. Phys. Math. Livre 1 *Math.* **56** (1961/62), 139–160 (1963).

Authors' summary: "Dans ce travail nous donnons un exposé axiomatique de la théorie des formes algébriques extérieures. Au commencement du travail nous donnons une axiomatique de la théorie des formes algébriques extérieures. Sur la base de cette axiomatique on démontre

une multitude des affirmations de cette théorie. On définit la valeur d'une forme algébrique extérieure. Nous démontrons que la théorie des formes algébriques extérieures de degré qui ne surpasse pas n est une théorie mathématique non-contradictoire et catégorique."

A. Švec (Waltham, Mass.)

Hadžiivanov, Nikolai; Petkov, Pet'o 3989

On algebraic exterior forms. II. (Bulgarian. French summary)

Annuaire Univ. Sofia Fac. Sci. Phys. Math. Livre 1 *Math.* **56** (1961/62), 161–183 (1963).

Authors' summary: "Dans un travail précédent [#3988], on a donné un système d'axiomes pour la théorie des formes algébriques extérieures. Dans ce travail on construit deux modèles de cette théorie et on démontre quelques théorèmes qui se rapportent au rang d'un ensemble de formes extérieures."

A. Švec (Waltham, Mass.)

Kotō, Satoshi 3990

Infinitesimal transformations of a manifold with f-structure.

Kōdai Math. Sem. Rep. **16** (1964), 116–126.

An f-structure on a differentiable manifold is determined by a tensor field f of type $(1, 1)$ such that $f^3+f=0$. This paper studies infinitesimal transformations v related to f-structures by the condition that the Lie derivative of f with respect to v is zero. *T. J. Willmore* (Liverpool)

Ishihara, Shigeru; Yano, Kentaro 3991

On integrability conditions of a structure f satisfying $f^3+f=0$.

Quart. J. Math. Oxford Ser. (2) **15** (1964), 217–222.

Let M be an n-dimensional manifold, f a non-trivial tensor field of type $(1, 1)$ satisfying $f^3+f=0$; so f has the form

$$\begin{pmatrix} 0 & -I_m & 0 \\ I_m & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$2m=\operatorname{rank} f$, with respect to suitable bases. Then f is integrable if the Nijenhuis tensor vanishes. This is proved by considering integrability of the distributions associated with the projections $-f^2$ and f^2+1.

R. J. Crittenden (Providence, R.I.)

Fujimoto, Atsuo 3992

Conformal geometry of G-structures. I, II.

Tensor (*N.S.*) **15** (1964), 181–190; *ibid.* (*N.S.*) **15** (1964), 191–199.

Let M be a connected, paracompact n-dimensional manifold, G a Lie subgroup of $\mathrm{Gl}(n, R)$ with trivial intersection with R_+, where R is realized as the scalars in $\mathrm{Gl}(n, R)$, $H_s(M, G)$ a G-structure. Then the bundle $\hat{H}_s(M, \hat{G})$ obtained by enlarging the group of $H_s(M, G)$ to $\hat{G}=R_+\cdot G$ is the conformal extension of $H_s(M, G)$. G-structures $H_s(M, G)$ and $H_{s'}(M, G)$ are conformally equivalent if they have the same conformal extension, i.e., they differ by a positive C^∞ function on M. They are homothetically equivalent if the function is constant. A C^∞ transformation F of M is a conformal [homothetic] transformation of $H_s(M, G)$ if $\tilde{f}H_s(M, G)$ is conformally [homothetically] equivalent to $H_s(M, G)$, where $\tilde{f}$ is the prolongation of f to the bundle of bases ("frames" in the paper). An infinitesimal conformal [homothetic] transformation of $H_s(M, G)$ is a vector field X on M whose local one-parameter group of local diffeomorphisms consists of conformal [homothetic] transformations of $H_s(M, G)$.

A number of fairly simple results are established, including equivalent formulations of the above definitions, relations with structure tensors and connexions, and the fact that the bracket of infinitesimal homothetic transformations is an infinitesimal automorphism of the G-structure. *R. J. Crittenden* (Providence, R.I.)

de Barros, Constantino M. 3993

Espaces infinitésimaux; algèbre de Lie graduée associée à un espace infinitésimal de Cartan.

C. R. Acad. Sci. Paris **258** (1964), 3956–3959.

As the second in a series, this paper continues the algebraic construction of systems of modules, algebras, etc., derived by formal analogy from tangent vector fields, the algebra of C^∞ functions and the exterior algebra of a C^∞ manifold. The first part [same C. R. **258** (1964), 3624–3627; MR **28** #5396] deals with the more classical differential geometric structures; this second part sets up a system analogous to the theory of vector forms and derivations or differential forms of Frölicher and the reviewer [Nederl. Akad. Wetensch. Proc. Ser. A **59** (1956), 338–359; MR 18, 569].

A. Nijenhuis (Philadelphia, Pa.)

de Barros, Constantino M. 3994

Espaces infinitésimaux; dérivée absolue.

C. R. Acad. Sci. Paris **258** (1964), 5330–5333.

In previous papers [same C. R. **258** (1964), 3624–3627; MR **28** #5396; see also #3993 above] the author developed a highly algebraic axiomatic theory of infinitesimal spaces. Here the notion of graduated infinitesimal translation is introduced, and a theory of linear connections and absolute derivation is developed in terms of the theory of infinitesimal spaces. *A. Goetz* (Wrocław)

Nguyen-Van-Hai [Nguyen Van Hieu] 3995

Conditions nécessaires et suffisantes pour qu'un espace homogène admette une connexion linéaire invariante.

C. R. Acad. Sci. Paris **259** (1964), 49–52.

The author gives a necessary and sufficient condition for a homogeneous space G/H to admit an invariant affine connection (Propositions 1 and 2). The condition is essentially the same as the one discovered by Wang [Nagoya Math. J. **13** (1958), 1–19; MR **21** #6001; see also Kobayashi and Nomizu, *Foundations of differential geometry*, Vol. I, p. 106, Interscience, New York, 1963; MR **27** #2945]. While Wang treats the general case where a Lie group is acting fibre-transitively on a principal bundle (with group K) and expresses the condition in terms of Lie algebras, the author treats the case where a Lie group G is acting fibre-transitively on the bundle of linear frames and expresses the condition in terms of Lie algebras and covariant differentiation. In Proposition 3,

the author gives a necessary and sufficient condition for G/H to admit an invariant torsion-free connection in terms of Lie algebras. *S. Kobayashi* (Berkeley, Calif.)

Kantor, I. L. 3996

A generalization of reductive homogeneous spaces. (Russian)

Dokl. Akad. Nauk SSSR **151** (1963), 1268–1270.

Let H be a subalgebra of the Lie algebra G. The homogeneous space corresponding to this pair (denoted by G/H) is said to be ν-reductive if there exists a direct sum decomposition $G=H_0+H_1+\cdots+H_\nu$ with $H_1+\cdots+H_\nu=H$ and $[H_i, H_0]\subset H_{i-1}$, for $i=1,2,\cdots,\nu$. (There seems to be a hypothesis missing, since Lemma 1, that $[H_i, H_k]\subset H_{i+k-1}$, does not always follow from the case $k=0$ when, e.g., H_0 is a direct summand of G and the H_i are renumbered.) If $\nu=1$, we have the usual reductive case, but if $\nu>1$ the bundle carries no invariant affine connection. Examples are discussed which arise from subgroups of differential groups of finite order. By "abelianizing" H_0, the author constructs an associated ν-reductive space M' and a fiber space U with base G/H and fiber M'. A canonical connection is defined in U and (following H. C. Wang [Nagoya Math. J. **13** (1958), 1–19; MR **21** #6001]) shown to be G-invariant. Projective and conformal connections are characterized among the 2-reductive canonical connections.

L. W. Green (Minneapolis, Minn.)

Libermann, Paulette 3997

Surconnexions. Propriétés générales.

C. R. Acad. Sci. Paris **258** (1964), 6327–6330.

Using the non-holonomic prolongations of the tangent bundle, the author introduces the notion of "semi-holonomic 'surconnexion' of order q" which reduces to an ordinary affine connexion if $q=1$. She also introduces the notion of geodesic of order q.

S. Kobayashi (Berkeley, Calif.)

Vanstone, J. R. 3998

Connections satisfying a generalized Ricci lemma.

Canad. J. Math. **16** (1964), 549–560.

Let $a_{ij}(x)$ be components of a tensor field of class C^1; let the symmetric and skew-symmetric parts of a_{ij} be denoted by s_{ij} and p_{ij}; let (s_{ij}) be non-singular and denote its reciprocal by (s^{ij}); define $(b^i{}_j)$ by the relation $b^i{}_j=s^{ir}p_{rj}$. The principal theorem proved is the following: There exists a connection $(\Gamma_j{}^i{}_k)$ such that the corresponding covariant derivative of (a_{ij}) vanishes if and only if the Jordan form of $(b^i{}_j)$ is constant.

T. J. Willmore (Liverpool)

Takano, Kazuo 3999

On Y. C. Wong's conjecture.

Tensor (N.S.) **15** (1964), 175–180.

An n-dimensional affinely connected space A_n for which $\nabla_m B^i{}_{jkl}=W_m B^i{}_{jkl}$ $(W_m\neq 0)$, where ∇_m denotes covariant differentiation with respect to the (symmetric) connection and $B^i{}_{jkl}$ is the curvature tensor, is called an A_n with recurrent curvature. Wong [Trans. Amer. Math. Soc. **102** (1962), 471–506; MR **24** #A3601] has conjectured that in an A_n with recurrent curvature, the tensor $\nabla_j W_k$ is symmetric if the Ricci tensor is symmetric. In the present paper, the author proves that in an A_n with recurrent curvature and symmetric Ricci tensor, $\nabla_j W_k$ is symmetric if there exists a vector ξ^i satisfying $W_i\xi^i=1$ and $W_h B^h{}_{ijk}\xi^i=0$. The proof depends upon use of the Bianchi identities and the integrability conditions of $\nabla_m B^i{}_{jkl}=W_m B^i{}_{jkl}$. *A. Fialkow* (Brooklyn, N.Y.)

Fava, Franco 4000

Connessioni composte e movimenti di ordine superiore.

Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **98** (1963/64), 707–749.

A composed connection of order h is a system of operators, defined at each point x of a differentiable manifold V_n and associating with every ordered finite set $v_1,\cdots,v_p, v$ $(p\leq h)$ of tangent vectors at x a new tangent vector at x, denoted by $D_{v_1\cdots v_p}(v)$, which has the form

$$D_{v_1\cdots v_p}(v)^i = L^i_{jr_1\cdots r_p}v^jv^{r_1}\cdots v^{r_p},$$

where higher indices denote the components of a vector with respect to the canonical frame defined by a local coordinate system in the linear tangent space at x. Moreover, the functions L are subject to the transformation laws:

$$L^q_{mr_1}\theta^{q'}_q = \theta^{q'}_{mr_1}+L^{q'}_{m'r_1'}\theta^{m'}_m\theta^{r_1'}_{r_1}$$

$$\left(\theta^{q'}_q = \frac{\partial x^{q'}}{\partial x^q},\ \theta^{q'}_{mr} = \frac{\partial^2 x^{q'}}{\partial x^m\partial x^r},\cdots\right),$$

$$L^q_{mr_1r_2}\theta^{q'}_q = \theta^{q'}_{mr_1r_2}+L^{q'}_{m'r_1'}(\theta^{m'}_{mr_2}\theta^{r_1'}_{r_1}+\theta^{m'}_m\theta^{r_1'}_{r_1r_2})$$
$$+L^{q'}_{m'r_2'}\theta^{m'}_{mr_1}\theta^{r_2'}_{r_2}+L^{q'}_{m'r_1'r_2'}\theta^{m'}_m\theta^{r_1'}_{r_1}\theta^{r_2'}_{r_2}$$

when a coordinate transformation $x^{q'}=x^{q'}(x^1,\cdots,x^n)$ is performed.

With a composed connection certain curvature tensors are associated. The first of these is defined by

$$R^q_{mr_1r_2} = L^q_{mr_1r_2}-\partial_{r_2}L^q_{mr_1}-L^p_{mr_1}L^q_{pr_2},$$

and this tensor vanishes if and only if $D_{v_1v_2}$ coincides with the second-order absolute derivative defined by the connection $(L^q_{mr_1})$. The second curvature tensor is

$$R^q_{mr_1r_2r_3} = L^q_{mr_1r_2r_3}-\partial_{r_3}L^q_{mr_1r_2}-L^p_{mr_1r_2}L^q_{pr_3}$$
$$-L^p_{mr_1}R^q_{pr_2r_3}-L^p_{mr_2}R^q_{pr_1r_3}-L^p_{r_1r_2}R^q_{mpr_3}$$

and its vanishing is a further necessary and sufficient condition in order that $D_{v_1v_2v_3}$ also be defined by the connection $(L^q_{mr_1})$ by iteration of absolute derivation.

A composed connection defines a parallelism of tangent vectors, and a condition is given in order that the space admit a field of parallel vectors for a second-order composed connection $(L^q_{mr_1}, L^q_{mr_1r_2})$.

A family of paths is also associated with a composed connection. The paper concludes with a study of the motions of a composed connection of the second order, and it is proved that this group has a dimension at most equal to n^2+n. The projective motions are also defined.

G. Vranceanu (Bucharest)

Fava, Franco 4001

Connessioni tensoriali composte.

Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **98** (1963/64), 773–789.

A composed tensor connection is introduced as a generalisation of the composed connections defined in the paper reviewed above [#4000]. When second-order tensors are considered, the author's generalisation is given by a system of two linear operators acting in the vector spaces of tensor squares $T_x \otimes T_x$ of the tangent vector spaces T_x of a differentiable manifold V_n, with respect to a local coordinate system. These operators are of the form

$$D_{v_1}(t)^{q_1 q_2} = L^{q_1 q_2}_{m_1 m_2 r_1} v_1^{r_1} t^{m_1 m_2},$$

$$D_{t_1}(t)^{q_1 q_2} = L^{q_1 q_2}_{m_1 m_2 r_1 r_2} t_1^{r_1 r_2} t^{m_1 m_2},$$

where $v_1 \in T_x$ and $t, t_1 \in T_x \otimes T_x$, $x \in V_n$. The functions L obey the transformation law

$$L^{q_1 q_2}_{m_1 m_2 r_1} \theta^{q_1'}_{q_1} \theta^{q_2'}_{q_2} = \partial_{r_1}(\theta^{q_1'}_{m_1} \theta^{q_2'}_{m_2}) + L^{q_1' q_2'}_{m_1' m_2' r_1'} \theta^{m_1'}_{m_1} \theta^{m_2'}_{m_2} \theta^{r_1'}_{r_1},$$

$$L^{q_1 q_2}_{m_1 m_2 r_1 r_2} = \partial_{r_1 r_2}(\theta^{q_1'}_{m_1} \theta^{q_2'}_{m_2}) + L^{q_1' q_2'}_{m_1' m_2' r_1'} \partial_{r_2}(\theta^{m_1'}_{m_1} \theta^{m_2'}_{m_2}) \theta^{r_1'}_{r_1}$$
$$+ L^{q_1' q_2'}_{m_1' m_2'} \partial_{r_1'}(\theta^{m_1'}_{m_1} \theta^{m_2'}_{m_2}) \theta^{r_2'}_{r_2}$$
$$+ L^{q_1' q_2'}_{m_1' m_2' r_2'} \theta^{m_1'}_{m_1} \theta^{m_2'}_{m_2} \theta^{r_2'}_{r_1 r_2}$$
$$+ L^{q_1' q_2'}_{m_1' m_2' r_1' r_2'} \theta^{m_1'}_{m_1} \theta^{m_2'}_{m_2} \theta^{r_1'}_{r_1} \theta^{r_2'}_{r_2}.$$

A curvature tensor is defined by

$$R^{q_1 q_2}_{m_1 m_2 r_1 r_2} = L^{q_1 q_2}_{m_1 m_2 r_1 r_2} - \partial_{r_2} L^{q_1 q_2}_{m_1 m_2 r_1} - L^{p_1 p_2}_{m_1 m_2 r_1} L^{q_1 q_2}_{p_1 p_2 r_2},$$

and also such geometrical notions are introduced as parallel transport and autoparallel curves. Similar considerations are sketched for the case when $T_x \otimes T_x$ is replaced by $T_x \otimes T_x^*$, where T_x^* is the dual vector space of T_x. *G. Vranceanu* (Bucharest)

Fava, Franco 4002

Connessioni tensoriali in spazi proiettivi curvi.

Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **97** (1962/63), 1064–1084.

Cet article est consacré à une étude des connexions tensorielles (c.t.) de E. Bompiani [Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) 1 (1946), 478–482; MR 8, 404], dans le sens de la possibilité d'extension à ces connexions de la notion de courbes autoparallèles pour une connexion (vectorielle) ordinaire. Après avoir rappelé quelques notions fondamentales relatives aux c.t., et notamment à celles de ces connexions de type (2, 0) [E. Bompiani, loc. cit.], l'auteur introduit des systèmes de courbes jouant, dans une variété V_n, pour la c.t. envisagée, le rôle d'autoparallèles. A partir du vecteur λ tangent à une courbe γ de V_n il construit le tenseur ξ, tangent à γ de type (2, 0) et deuxième puissance tensorielle de λ, et définit γ comme autoparallèle de V_n si le long de cette courbe est réalisé le transport parallèle de ξ. Contrairement à ce qui a lieu pour les connexions ordinaires, il n'existe généralement pas pour les c.t. de systèmes de courbes autoparallèles dont les courbes constituantes sont déterminées par un point et une direction de V_n. Mais, se plaçant dans un espace projectif courbe, l'auteur montre que de tels systèmes existent pour des classes particulières de c.t. (dites précisément associables à des espaces projectifs courbes) caractérisées par l'annulation d'un certain tenseur. L'étude de certaines de ces c.t. l'amène à établir quelques propriétés fondamentales pour l'édification d'une théorie projective des connexions tensorielles envisagées, et à donner quelques indications sur leur possibilité d'extension à des c.t. plus générales. *P. Vincensini* (Marseille)

Fulton, Curtis M. 4003

Clifford vectors.

Pacific J. Math. **14** (1964), 917–918.

In a Riemannian 3-space a Clifford vector field represented by the unit vector V^h is defined by $V_{i,j} = L\eta_{ijh} V^h$, where η_{ijh} is the covariant tensor whose components are $\sqrt{|g|}$ $[-\sqrt{|g|}]$ if (i, j, h) is an even [odd] permutation of (1, 2, 3), and zero otherwise, and L is a non-zero scalar. The author proves the theorem: The necessary and sufficient condition for a Riemannian 3-space to admit Clifford vector fields in all directions is that it be of constant positive curvature K with $K = L^2$.

R. Blum (Saskatoon, Sask.)

Zirilli, Francesco 4004

Ipersuperficie ricorrenti di uno spazio euclideo.

Ricerche Mat. **12** (1963), 181–194.

Given a Riemannian space V_n, this space is said to be recurrent (Schouten, Walker, Ruse) if we have the formulae

$$(1) \qquad R_{hijk,l} = k_l R_{hijk}$$

in which R_{hijk} is the curvature tensor and $R_{hijk,l}$ its covariant derivative. We see that for $k_l = 0$ we obtain the symmetric spaces of Cartan.

The aim of the paper is to show that if V_n is immersed in a Euclidean space E_{n+1}, then V_n is a cylinder (and can be considered as the orthogonal projection from a vertex E_{n-3} of a V_3). The proof given is a direct one by taking V_n to be defined by $z = z(x^1, \cdots, x^n)$ when $x^1, \cdots, x^n, z$ are orthogonal coordinates in E_{n+1}, and $z(x^1, \cdots, x^n)$ is of class C^4 and by using the relations of Bianchi for $k_l \neq 0$,

$$(2) \qquad k_l R_{hijk} + k_j R_{hikl} + k_k R_{hilj} = 0.$$

G. Vranceanu (Bucharest)

Nirmala, K. 4005

Curves and invariants associated with a vector field of a Riemannian V_n in relation to a curve C in a subspace V_m.

Proc. Nat. Inst. Sci. India Part A **29** (1963), 394–406.

Generalizing familiar notions of Riemannian geometry, Mishra and Krishna [Tensor (N.S.) 6 (1956), 125–131; MR 18, 761] defined the absolute curvature and normal curvature of a congruence of curves in a Riemannian manifold V_n with respect to a curve C in a subspace V_m. The definitions were given in terms of the derived vector of λ^a along C and its normal component in V_m, where λ^a is a unit tangent to a curve of the congruence. The present author considers the tangential component of this derived vector, leading to a definition of the geodesic curvature of the congruence λ^a with respect to C. Two possible generalizations of lines of curvature, principal curvatures and asymptotic lines, as well as their properties, are discussed. Finally, a general expression for the tendency of λ^a with respect to a curve C in V_m is obtained, and a number of its properties are given.

A. Fialkow (Brooklyn, N.Y.)

Nirmala, K. 4006

Differential invariants in Riemannian space.

Tensor (N.S.) **15** (1964), 282–291.

Let V_n be imbedded as a hypersurface of a Riemann space

V_{n+1} by the equations $y^\alpha = y^\alpha(x^i)$, and let $g_{ij}\,dx^i dx^j$ and $\Omega_{ij}\,dx^i dx^j$ be the first and second fundamental forms of V_n. Let (λ) be a congruence such that one curve of (λ) passes through each point of V_n, and let C be a curve in V_n. The author shows that various geometric quantities (e.g., the normal curvature and geodesic curvature of (λ) with respect to C) may be expressed by means of the two differential operators

$$\nabla = \frac{\partial y^\alpha}{\partial x^i} g^{ij} \frac{\delta}{\delta x^j}, \qquad \nabla^* = \frac{\partial y^\alpha}{\partial x^i} \Omega^{ij} \frac{\delta}{\delta x^j},$$

where $\delta/\delta x^j$ is the symbol of covariant differentiation. These operators also permit the author to generalize the Darboux functions associated with a surface to the case of a V_n in V_{n+1} and to discuss some of their properties.

A. Fialkow (Brooklyn, N.Y.)

Lichnerowicz, André 4007

Sur les transformations conformes d'une variété riemannienne compacte.

C. R. Acad. Sci. Paris **259** (1964), 697–700.

Let M be a compact Riemannian manifold with constant scalar curvature which is not isometric to a sphere. It is an open question whether the largest connected group of conformal transformations of M coincides with the largest connected group of isometries of M. The author gives an affirmative answer to this question under the additional assumption that $R^{\alpha\beta}R_{\alpha\beta}$ is also constant.

S. Kobayashi (Berkeley, Calif.)

Cattaneo-Gasparini, Ida 4008

Connessioni adattate a una struttura quasi prodotto.

Ann. Mat. Pura Appl. (4) **63** (1963), 133–150.

A unit tangent vector field in a Riemannian manifold gives rise to an almost product structure with perpendicular summands in each tangent space. The author chooses a special adapted connection, and she shows how covariant derivatives of vector fields with respect to this connection can be expressed very simply through the Riemannian covariant derivatives and the projection operators. Applications to relativity are anticipated.

A. Nijenhuis (Philadelphia, Pa.)

Radziszewski, K. 4009

Sur une condition de coïncidence des surfaces convexes isométriques.

Ann. Polon. Math. **15** (1964), 167–177.

Die klassische Theorie der regulären Flächen beweist den Eindeutigkeitssatz, daß zwei solche Flächen S_1 und S_2 deren erste und zweite Grundformen übereinstimmen, zueinander kongruent sind. Diese Bedingungen können durch die Forderungen abgewandelt werden, daß die beiden Flächen S_1 und S_2 zueinander isometrisch sind und in isometrisch entsprechenden Richtungen identische Normalkrümmungen haben. Bei konvexen Flächen kann die letzte Bedingung durch die Forderung ersetzt werden, daß isometrisch entsprechende Kurven gleiche Krümmungen besitzen.

In der vorliegenden Arbeit wird der obige Eindeutigkeitssatz für konvexe Flächen unter der Abänderung bewiesen, daß neben der Isometrie der Flächen die Identität der integralen Krümmungen isometrisch entsprechender Kurven gefordert wird. Der Begriff der integralen Krümmung $k(s)$ ist dabei im Sinne von A. D. Aleksandrov [Uspehi Mat. Nauk **2** (1947), no. 3 (19), 182–184] zu verstehen. Ist AB ein ebener konvexer Kurvenbogen und M einer seiner Punkte, ferner s die Länge des Bogens AM und $h(s)$ der algebraische Abstand des Punktes M von der Tangente der Kurve AB im Punkt A, schließlich $k(s)$ die integrale Krümmung des Kurvenbogens AM, so gilt dabei die Formel

$$h(s) = \int_0^s \sin k(x)\,dx.$$

Der Beweis des genannten Eindeutigkeitssatzes wird schrittweise auf 7 Hilfssätze gestützt. An verschiedenen Stellen werden dabei Ergebnisse von A. D. Aleksandrov [*Die innere Geometrie der konvexen Flächen* (russisch), OGIZ, Moscow, 1948; MR **10**, 619; deutsche Übersetzung, Akademie-Verlag, Berlin, 1955; MR **17**, 74] illustriert.

K. Strubecker (Karlsruhe)

Mizusawa, Hideo 4010

On infinitesimal transformations of K-contact and normal contact metric spaces.

Sci. Rep. Niigata Univ. Ser. A No. 1 (1964), 5–18.

A K-contact space is a contact metric space with structure (ϕ, ξ, η, g), where ξ is a Killing vector with respect to the metric g. The author proves: (1) In a normal contact metric space of constant scalar curvature $R \neq n(n-1)$, $n > 3$, an infinitesimal conformal transformation is isometric; (2) Under the hypotheses of (1), an infinitesimal projective transformation is isometric if its associated vector ρ_i has the property $\xi^i\rho_i \neq 0$; (3) In a contact metric space, a vector v is an infinitesimal ϕ-transformation, $\mathfrak{L}_v\phi^i{}_j = 0$, if and only if $\mathfrak{L}_v\eta_i = \sigma\eta_i$ and $\mathfrak{L}_v g_{ij} = \sigma(g_{ij} + \eta_j\eta_i)$, and then σ is a constant; (4) In an Einstein contact space with $R \neq 0$, or in a K-contact space of constant scalar curvature $R \neq -(n-1)$, an infinitesimal ϕ-transformation is an automorphism; (5) In a K-contact space, an infinitesimal affine contact transformation is an automorphism. These are generalizations of some results in normal contact metric spaces [M. Okumura, Tôhoku Math. J. (2) **14** (1962), 398–412; MR **26** #4294].

Y. Tashiro (Okayama)

Kosmanek, Édith 4011

Une propriété caractéristique des variétés kählériennes à courbure holomorphe constante.

C. R. Acad. Sci. Paris **259** (1964), 705–708.

Let V be a Kähler manifold with complex structure operator J. For every geodesic γ in V there is a Jacobi field ψ along γ such that $\psi(0) = 0$, $\psi'(0) = J\gamma'(0)$, where γ' is the tangent field to γ, and $\psi' = D_{\gamma'}\psi$, the covariant derivative of ψ. The Kähler manifold satisfies (CJ) if for every such ψ, $\psi = fJ\gamma'$ for some real-valued function f. The author shows that satisfying (CJ) is equivalent to having constant holomorphic curvature.

R. L. Bishop (Urbana, Ill.)

Hashimoto, Shintaro 4012

On differentiable manifold with almost quaternion contact structure.

Tensor (N.S.) **15** (1964), 258–268.

A quaternion almost complex structure on a $4m$-dimensional manifold was studied exhaustively by M. Obata and others [Obata, Japan. J. Math. **26** (1956), 43–77; MR **20** #1796a; J. Math. Soc. Japan **9** (1957), 406–416; MR **20** #1796b; Tôhoku Math. J. (2) **10** (1958), 11–18; MR **20** #1796c]. The author investigates almost quaternion structure on the $(4m+1)$-dimensional differentiable manifold, i.e., the (F, G, H, ξ, η)-structure on M^{4m+1}, where the tensor fields F_j^i, G_j^i, H_j^i, and the contravariant and covariant vector fields ξ^i and η_i over M^{4m+1} satisfy certain conditions. The manifold M^{4m+1} with the (F, G, H, ξ, η)-structure is said to have contact structure if there exists a 1-form $\eta = \eta_i dx^i$ over M^{4m+1} such that $\eta \wedge (d\eta)^{2m} \neq 0$. When a contact structure $\eta = \eta_i dx^i$ satisfies the relations $\partial_i \eta_j - \partial_j \eta_i = (1/\sqrt{3})(F_{ij} + G_{ij} + H_{ij})$, the (F, G, H, ξ, η)-structure is called a quaternion contact structure on M^{4m+1}. In this case the author considers the product manifold $M^{4m+1} \times R$, R a real line, and studies Nijenhuis tensors systematically, making use of the product manifold and pseudo-group of transformations. Thus he obtains 16 types of Nijenhuis tensors and many theorems concerning Nijenhuis tensors of this quaternion contact structure. His methods are analogous to those used by M. Obata and S. Sasaki [Sasaki and Hatakeyama, ibid. (2) **13** (1961), 281–294; MR **25** #1513].

T. Ohkubo (Kumamoto)

Hashimoto, Shintaro 4013

On differentiable manifold with almost quaternion contact structure. III.

Sci. Rep. Fac. Lit. Sci. Hirosaki Univ. **10** (1963), 5–9.

For the concept of an almost quaternionian contact structure (F, G, H, ξ, η), we refer to the author [same Rep. **8** (1961), 1–7; MR **27** #679a; ibid. 8 (1961), 79–87; MR **27** #679b]. By a method quite similar to that of M. Obata [Japan. J. Math. **26** (1956), 43–77; MR **20** #1796a], he shows that there is an affine connection, called an (F, G, H, ξ, η)-connection, with respect to which the tensors of the structure are covariant constant, and that the connection having a given torsion is unique. {Out of many misprints, F should be put ahead of $\mathring{\Gamma}$ in the first and second lines of the proof of Theorem 2.}

Y. Tashiro (Okayama)

Bonan, Edmond 4014

Tenseur de structure d'une variété presque quaternionienne.

C. R. Acad. Sci. Paris **259** (1964), 45–48.

An almost quaternionian (a.q.) structure (I, J, K) on V_m $(m = 4n)$ induces four projectors S, S', S'', S''' in the tangent space which are orthogonal to each other and whose sum is the identity. In $B = R^m \otimes R^{m*}$, $S(B)$ is identified with the Lie algebra $\underline{G}$ of $G = GL(n, H)$ and $\Gamma = S'(B) + S''(B) + S'''(B)$ is supplementary to $\underline{G}$ and invariant under G. Let $N_2 = B \otimes R^{m*}$, $N_G = \underline{G} \otimes R^{m*}$, $P = R^m \otimes \wedge^2 R^{m*}$, $A: N_2 \to P$, $V_G = A(N_G)$, $W = A(\Gamma \otimes R^{m*})$. Then W is a supplement of V_G in P. Applying results on G-structures due to D. Bernard [Ann. Inst. Fourier (Grenoble) **10** (1960), 151–270; MR **23** #A4094], the torsion tensor Σ of an a.q. connection is decomposed into $\Sigma = \Sigma_W + \Sigma_{V_G}$, and Σ_W is identified with the tensor T of the structure and $T = \frac{1}{2}([I, I] + [J, J] + [K, K])$. Moreover, on an a.q. Hermitian manifold, Σ_{V_G} is decomposed into its symmetric part Σ_U and its anti-symmetric part Σ_{V_g} in $\underline{G}$. $T + \Sigma_U$ is the tensor of the Sp(n)-structure.

Let γ be the Riemannian connection with respect to an a.q. Hermitian metric and ω the canonical a.q. connection. $\alpha = S\gamma$ is called the first canonical connection of Lichnerowicz and β, defined by $B_\alpha^\beta = \omega_\alpha^\beta + \frac{1}{2}g^{\beta\rho}\nabla g_{\rho\alpha}$, the second canonical connection. The torsion of β is equal to $T + \Sigma_U$. If, in a compact a.q. Hermitian manifold, an infinitesimal affine transformation with respect to α preserves the a.q. structure, the transformation is an isometry.

Y. Tashiro (Okayama)

Rizza, Giovanni Battista 4015

Strutture di Finsler di tipo quasi Hermitiano.

Riv. Mat. Univ. Parma (2) **4** (1963), 83–106.

The author studies Finsler metrics compatible with quasi-complex structures. Since the quasi-complex structure introduces almost a rotation group in the tangent planes, one expects the Finsler metrics to have special properties. In fact, the indicatrix of the Finsler metric is shown to be a complete circular domain in the terminology of Behnke and Thullen, hence in particular such a metric is symmetric (this is elementary from $J^2 = -1$). Also, there is a simple relation between the two kinds of cosines introduced in Rund's book [*The differential geometry of Finsler spaces*, Springer, Berlin, 1959; MR **21** #4462]. It follows that the Finsler metric and the quasi-complex tensor determine identical angular metrics in all two-dimensional characteristic elements. This leads immediately to quasi-hermitian Finsler structures. These structures are characterized by the symmetry of the usually unsymmetric cosine, and by a breaking up of the Euler relation for homogeneity of the metric form into complex and conjugate part. Finally, an averaging process due to Lichnerowicz is used to construct an invariant metric out of an arbitrary metric and a quasi-complex tensor. A non-trivial example is given which is invariant under a quasi-complex tensor field relative to a given field of elements of support but not relative to an arbitrary field of elements of support, and therefore is quasi-complex but not quasi-hermitian.

H. W. Guggenheimer (Minneapolis, Minn.)

Kawaguchi, Syun-ichi 4016

On a special Kawaguchi space of recurrent curvature.

Tensor (N.S.) **15** (1964), 145–158.

A special Kawaguchi space is an n-dimensional metric space in which the arc length of a curve $x^i = x^i(t)$ is given by

$$s = \int \{A_i(x, x')x''^i + B(x, x')\}^{1/p}\, dt.$$

A covariant derivative ∇ of tensors can be introduced in such a space and with it a curvature tensor $R_{jkl}^{\ \ \ \ i}$ and another tensor $K_{jk}^{\ \ \ i}$ defined by the identity

$$(\nabla_j \nabla_k - \nabla_k \nabla_j)v^i = -R_{jkl}^{\ \ \ \ i}v^l + K_{jk}^{\ \ \ l}\partial v^i/\partial x'^l$$

for an arbitrary contravariant vector v^i [see A. Kawaguchi, Trans. Amer. Math. Soc. **44** (1938), 153–167]. In this paper, the author obtains a few necessary conditions for a special Kawaguchi space to be of recurrent curvature, i.e., for it to satisfy the following condition:

$$\nabla_m R_{jkl}^{\ \ \ \ i} = v_m R_{jkl}^{\ \ \ \ i},$$

where v_m is some covariant vector. Among other results,

it is proved that in a special Kawaguchi space, (i) either v_m is a function of x^1 alone or $K_{jhk}{}^{\cdots h}x'^jx'^k=0$; and (ii) v_m satisfies

$$\nabla_q(\nabla_m v_n-\nabla_n v_m) = v_q(\nabla_m v_n-\nabla_n v_m),$$

$$v_q(\nabla_m v_n-\nabla_n v_m)+v_m(\nabla_n v_q-\nabla_q v_n)+v_n(\nabla_q v_m-\nabla_m v_q) = K_{qm}{}^{\cdots p}\partial v_n/\partial x'^p+K_{mn}{}^{\cdots p}\partial v_q/\partial x'^p+K_{nq}{}^{\cdots p}\partial v_m/\partial x'^p.$$

{Remark: The two identities in (ii) are analogous to those obtained by the reviewer for the case of linear connexions with zero torsion and recurrent curvature [ibid. **102** (1962), 471–506; MR **24** #A3601].}

Y.-C. Wong (Hong Kong)

Zaustinsky, Eugene M. 4017

Extremals on compact *E*-surfaces.

Trans. Amer. Math. Soc. **102** (1962), 433–445.

H. Busemann investigated systematically the theory of *G*-spaces (finitely compact metric spaces with locally unique geodesics) [*The geometry of geodesics*, Academic Press, New York, 1955; MR **17**, 779]. The author has extended many results of Busemann's study, which depend upon the symmetry of the metric, to the case without the assumption of symmetry [Mem. Amer. Math. Soc. No. 34 (1959); MR **22** #8462]. In the Riemannian case, if the genus of the surface is 1, the following results are known: (1) for a Euclidean straight line of the Poincaré model, there is a geodesic of the universal covering space which remains within a fixed distance of the given straight line; (2) a unique geodesic segment lies within a constant distance of the Euclidean straight line segment joining its endpoints. If the genus is >1, under the assumption of the Riemannian character of the metric, the following analogous results are known: (1′) for a hyperbolic straight line of the model, there is a geodesic which remains within a fixed distance of the straight line; (2′) a straight geodesic remains within a constant distance of some suitable hyperbolic straight line; (3′) the space possesses transitive geodesics.

In the present paper the author carries over the above results (1)–(2), (1′)–(3′) to *E*-spaces, i.e., non-symmetric Finsler spaces, using the notions of axis and axial motion (§§ 3, 4). He also proves that the universal covering space of a compact *E*-space with the divergence property is straight (§ 5). The last section is devoted to the question of the existence of transitive extremals on compact surface of higher genus, the distance being non-symmetric.

T. Ohkubo (Kumamoto)

GENERAL TOPOLOGY

See also 3398, 3407, 3604, 3948, 3984–3986, 4052, 4063.

Moore, Theral O. 4018

★Elementary general topology.

Prentice-Hall Mathematics Series.

Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964. xi+174 *pp.* $7.95.

An undergraduate text with unusual precision, efficiency, and very good problems. The usual material is covered, and two fine chapters on nets and Peano spaces, respectively, are added. All concepts are well-motivated and studied to the extent practicable on this level.

J. Mayer (Albuquerque, N.M.)

Hammer, Preston C. 4019

Extended topology: Additive and subadditive subfunctions of a function.

Rend. Circ. Mat. Palermo (2) **11** (1962), 262–270.

The author continues his investigation of generalized closure functions. A set function f is subadditive if $f(X\cup Y)\subseteq f(X)\cup f(Y)$; it is additive if equality holds; it is isotonic if $X\supseteq Y$ implies $f(X)\supseteq f(Y)$. A function g is a subfunction of f if always $g(X)\subseteq f(X)$. Given f, it is established that maximum subadditive, additive and isotonic subfunctions of f exist. Various relations among these properties are established and high-order additivity properties are considered.

M. L. Curtis (Houston, Tex.)

Sieber, J. L.; Pervin, W. J. 4020

Connectedness in syntopogenous spaces.

Proc. Amer. Math. Soc. **15** (1964), 590–595.

From the authors' introduction: "In his monograph [*Fondements de la topologie générale*, Gauthier-Villars, Paris, 1960; Akad. Kiadó, Budapest, 1960; MR **22** #4043], Császár introduced the notion of a syntopogenous space, which generalized the notions of a topological space, a proximity space, and a uniform space. Although Császár was able to obtain many of the usual theorems of general topology in this more general setting, the basic topological notion of connectedness was not introduced at all. Mrówka and Pervin [Proc. Amer. Math. Soc. **15** (1964), 446–449; MR **28** #4515] have discussed the concepts of $\mathscr{U}$-connectedness and δ-connectedness for uniform and proximity spaces, respectively. In this paper we shall give a definition for connectedness in syntopogenous spaces and show that it agrees with the corresponding properties in topological, proximity, and uniform spaces. Furthermore, we shall obtain some of the theorems concerning connected sets from general topology in this more general setting."

K. W. Kwun (Princeton, N.J.)

Lavallee, Lorraine D. 4021

Mosaic spaces, P_1-mappings, and property K.

Boll. Un. Mat. Ital. (3) **19** (1964), 95–97.

A collection $\{(X_a, \mathfrak{F}_a): a\in A\}$ is said to be a mosaic of topological spaces on a set X if and only if (i) each $(X_a, \mathfrak{F}_a)$ is a topological space; (ii) $X=\bigcup\{X_a: a\in A\}$; and (iii) for all subsets E of X and all $a, b\in A$, if $E\subseteq X_a$ and E is $\mathfrak{F}_a$-closed, then $E\cap X_b$ is $\mathfrak{F}_b$-closed. For a mosaic of topological spaces on X the mosaic topology $\mathfrak{F}$ is defined as follows: for all $E\subseteq X$, E is $\mathfrak{F}$-closed if and only if $E\cap X_a$ is $\mathfrak{F}_a$-closed for all $a\in A$. If each $(X_a, \mathfrak{F}_a)$ is a compact metric space, then the topological space $(X, \mathfrak{F})$ determined by the collection $\{(X_a, \mathfrak{F}_a): a\in A\}$ is called a mosaic space. An hereditary mosaic space is a mosaic space with the property that each subspace is a mosaic space.

A topological space $(X, \mathfrak{F})$ is said to have property K if for each point x and each subset E of X having x as an $\mathfrak{F}$-limit point, there exists an $\mathfrak{F}$-compact subset of $E\cup\{x\}$ which has x as an $\mathfrak{F}$-limit point.

The main theorem is that if $(X, \mathfrak{F})$ is a mosaic space, then $(X, \mathfrak{F})$ has property K if and only if $(X, \mathfrak{F})$ is hereditary. *S. Roy Schubert* (Riverside, Calif.)

Wang, Shu-tang 4022

Remarks on ω_μ-additive spaces.

Fund. Math. **55** (1964), 101–112.

A set X is called an ω_μ-additive space, where ω_μ is a regular initial ordinal number, if there is defined for every subset A of X a closure operation, $A \to \bar{A}$, satisfying the following axioms: (i) $\overline{\sum A_\xi} = \sum \bar{A}_\xi$ for every α-sequence of sets $\{A_\xi\}$, $0 \leq \xi < \alpha < \omega_\mu$; (ii) $\bar{A} = A$ for every finite subset A; (iii) $\bar{\bar{A}} = \bar{A}$ [R. Sikorski, Fund. Math. **37** (1950), 125–136; MR **12**, 727]. A topological space X is a $(U)_m$-space if its topology can be derived from a uniformity with a basis of power m, where m is the smallest possible. Definitions are also given for an ω_μ-metric and an ω_μ-metrizable topological space. The author obtains necessary and sufficient conditions for a $(U)_m$-space to be ω_μ-additive. Relationships between $(U)_m$-spaces and ω_μ-metrizable spaces are studied, and some necessary and sufficient conditions for an ω_μ-additive space to be ω_μ-metrizable are given. *E. Duda* (Coral Gables, Fla.)

Fröhlich, Otto 4023

Das Halbordnungssystem der topologischen Räume auf einer Menge.

Math. Ann. **156** (1964), 79–95.

Topologies on a set E form a subset Σ of the set $\mathfrak{B}(E)$ of sets of subsets of E. Σ is ordered ("ein Halbordnungssystem") by inclusion with the discrete topology $\mathfrak{B}(E)$ as greatest element. A maximal element of the ordered set $\Sigma - \{\mathfrak{B}(E)\}$ is called an ultraspace on E.

Let $\mathfrak{z}(a, \mathfrak{U}) = \mathfrak{B}(E - \{a\}) \cup \mathfrak{U}$ for a point a of E and an ultrafilter $\mathfrak{U}$ on E. If $\mathfrak{U} \neq \mathfrak{U}(a)$ (i.e., $\{a\} \notin \mathfrak{U}$), then $\mathfrak{z}(a, \mathfrak{U})$ is an ultraspace. All ultraspaces are obtained in this way, and a and $\mathfrak{U}$ are determined uniquely by $\mathfrak{z}(a, \mathfrak{U})$. The ultraspace $\mathfrak{z}(a, \mathfrak{U})$ is called principal if $\mathfrak{U} = \mathfrak{U}(b)$ for a point $b \neq a$ of E.

An automorphism of the ordered set Σ is determined by its action on ultraspaces. Any permutation of E induces an automorphism of Σ. If E is infinite, all automorphisms of Σ are obtained in this way. If E is finite with more than two elements, this is false.

Two ultraspaces $\mathfrak{z}(a, \mathfrak{U})$ and $\mathfrak{z}(b, \mathfrak{B})$ are homeomorphic if and only if there is a permutation of E which maps $\mathfrak{U}$ onto $\mathfrak{B}$. Homeomorphism classes of ultraspaces, and other classifications of ultraspaces, are discussed in detail; the results of this study cannot be reproduced here.

O. Wyler (Albuquerque, N.M.)

Hayashi, Eiichi 4024

Topologies defined by local properties.

Math. Ann. **156** (1964), 205–215.

Guided by certain properties of the first category sets of a topological space the author considers a family of sets $\mathfrak{p}$ of a topological space which contains the unit sets, is hereditary and finitely additive, and a set X is in $\mathfrak{p}$ if every point of X has a neighborhood U such that $U \cap X$ is in $\mathfrak{p}$. If (R, τ) is a T_1-space and $\mathfrak{p}$ is a family of subsets of R satisfying the conditions above, then, for each subset X of R, the set X^* consists of the points of R for which every neighborhood intersects X in a set not in $\mathfrak{p}$. Following a familiar procedure, the author defines a new topology τ^*, obviously containing τ, on R by taking as closed sets those sets X such that $X = X \cup X^*$.

A variant of the Cantor-Bendixson theorem is proved. Also, it is shown that, when (R, τ^*) is dense in itself, (R, τ) is connected, Hausdorff, and lightly compact if and only if (R, τ^*) satisfies the corresponding condition; and, if (R, τ^*) is regular, then $\tau = \tau^*$.

R. W. Bagley (Coral Gables, Fla.)

Levine, Norman 4025

Semi-open sets and semi-continuity in topological spaces.

Amer. Math. Monthly **70** (1963), 36–41.

A set which is a subset of the closure of its interior is defined to be semi-open. Several quite elementary properties of such sets are established (e.g., the image of a semi-open set under an open continuous map is semi-open). A function from a topological space to a topological space is defined to be semi-continuous if the inverse of each open set is semi-open. Again, some elementary properties are established (e.g., the set of points of discontinuity of a semi-continuous function from a topological space to a perfectly separable topological space is of the first category).

The choice of terminology ("semi-continuous") is unfortunate in that neither upper semi-continuous functions nor lower semi-continuous functions need be "semi-continuous". Also the term "semi-continuous" has been previously used as a property of binary relations [A. D. Wallace, Bull. Amer. Math. Soc. **51** (1945), 413–416; MR **6**, 278]. *F. Burton Jones* (Riverside, Calif.)

Kowalsky, Hans-Joachim 4026

Bemerkungen zum Brückensatz.

Bayer. Akad. Wiss. Math.-Natur. Kl. S.-B. **1963**, *Abt. II*, 71–81 (1964).

Let X be a connected topological space such that X is normal and compact (i.e., bicompact). The author proves two "bridge" theorems: (I) If H and K are non-intersecting closed point sets (non-void), then some component of $X - (H + K)$ has a limit point in each of H and K, and (II) If H and K are non-intersecting closed point sets (non-void) and Q is a closed point set containing $H + K$ such that every limit point of a component of $X - Q$ belonging to Q must belong to $H + K$, then either $Q - (H + K)$ contains a connected point set having a limit point in each of H and K or $X - Q$ contains such a connected point set.

{The reviewer is somewhat puzzled by the interest in (II) as a "generalization" of (I). For if Q contains a continuum intersecting both H and K, then (II) follows from (I). If Q contains no such continuum, then Q is the sum of two non-intersecting closed point sets Q_H and Q_K containing H and K, respectively, and (II) again follows from (I). (Both theorems are immediate consequences of Theorems 43, 44 and 47 of Chapter I of R. L. Moore's *Foundations of point set theory* [revised edition, Amer. Math. Soc., Providence, R.I., 1962; MR **27** #709]. Moore's arguments apply without change.)}

The author also proves (I) and (II) when X is "locally

connected" instead of "normal and compact". (See the arguments for Theorems 2, 4 and 26 of Chapter II of Moore's book [loc. cit.].)

F. Burton Jones (Riverside, Calif.)

Dwinger, Ph. 4027

Amalgamation of Boolean spaces.

Nieuw Arch. Wisk. (3) **12** (1964), 25–31.

Let Y_γ, $\gamma \in \Gamma$, be a family of Boolean spaces, and let X be a Boolean space such that for every $\gamma \in \Gamma$ there exists an imbedding $h_\gamma : X \to Y_\gamma$. The author defines the "generalized Boolean topological sum of this family with the amalgamated Boolean subspace X" as a Boolean space Z such that (i) for every $\gamma \in \Gamma$ there exists an imbedding $f_\gamma : Y_\gamma \to Z$ such that for every pair $\gamma', \gamma'' \in \Gamma$, $f_{\gamma'} h_{\gamma'} = f_{\gamma''} h_{\gamma''}$, and (ii) if V is an arbitrary Boolean space and if g_γ, $\gamma \in \Gamma$, is a family of continuous maps, $g_\gamma : Y_\gamma \to V$, such that for every pair $\gamma', \gamma'' \in \Gamma$, $g_{\gamma'} h_{\gamma'} = g_{\gamma''} h_{\gamma''}$, then there exists a unique continuous map $g : Z \to V$ such that $gf_\gamma = g_\gamma$ for every $\gamma \in \Gamma$. He then proves that such a space Z always exists and is unique, as follows. Let Y be the ordinary topological sum of the Y_γ, and let Y^* be the quotient space of Y obtained by identifying those points of Y that are the images of the same point of X under the mappings h_γ. Then Z is the largest zero-dimensional Hausdorff compactification of Y^*.

L. Gillman (Rochester, N.Y.)

Frink, Orrin 4028

Compactifications and semi-normal spaces.

Amer. J. Math. **86** (1964), 602–607.

A base Z for the closed sets of a T_1 space X is called a normal base if: (i) Z is a ring of sets, (ii) for every $F = \bar{F} \subset X$ and $x \notin F$ there exists $A \in Z$ such that $x \in A$ and $A \cap F = 0$, (iii) for any two disjoint $A, B \in Z$ there exists $C, D \in Z$ satisfying $A \subset C'$, $B \subset D'$ and $C' \cap D' = \varnothing$.

Theorem 1: A T_1 space X is completely regular if and only if there exists a normal base for the closed sets of X. Proof: In the completely regular T_1 space X the family of all zero-sets is a normal base for the closed sets of X. If Z is a normal base for the closed sets of a T_1 space X, then the family $w(Z)$ of all ultrafilters of Z with the Wallman topology [H. Wallman, Ann. of Math. (2) **39** (1938), 112–126] is a compactification of the space X. The author gives also a condition for the possibility of extending a real-valued function (and a continuous map into a compact space) from X to $w(Z)$. It is not known whether every compactification of the space X is of the form $w(Z)$.

R. Engelking (Warsaw)

Moors, R. 4029

Extensions d'un espace topologique associées à une famille de tamis; compactifications d'un espace topologique.

Bull. Soc. Roy. Sci. Liège **33** (1964), 59–81.

A sieve ("tamis") on a topological space $(E, \mathcal{T})$ is a nonempty subset $\mathcal{U}$ of $\mathcal{T}$, closed under finite intersections, such that $U \cup T \in \mathcal{U}$ whenever $U \in \mathcal{U}$ and $T \in \mathcal{T}$. A family $\{\mathcal{U}_i\}_{i \in I}$ of sieves on $(E, \mathcal{T})$ defines various "extensions" (K, θ) of $(E, \mathcal{T})$. In each case $K = E \cup I$, and (sample result, Theorem 2.5.2) the minimal extension has as base for θ all sets of the form $B \cup \{i : B \in \mathcal{U}_i\}$ $(B \in \mathcal{T})$.

Let $\mathcal{F}$ be a family of finite open covers for E, hereditary in a sense made precise by the author (p. 69). Each open cover i which is maximal with respect to the property $(\mathcal{V} \subset i \Rightarrow \mathcal{V} \notin \mathcal{F})$ gives rise naturally to a sieve $\mathcal{U}_i$ on E, and the minimal extension (K, θ) defined by the family $\{\mathcal{U}_i\}$ is a compactification of $(E, \mathcal{T})$. In case E admits a Wallman compactification, or a Stone-Čech compactification, these can be obtained by judicious choice of $\mathcal{F}$.

W. W. Comfort (Rochester, N.Y.)

Njåstad, Olav 4030

On real-valued proximity mappings.

Math. Ann. **154** (1964), 413–419.

The author defines and studies "realcompleteness" of a proximity space in analogy with realcompactness of a topological space.

L. Gillman (Rochester, N.Y.)

Dolčinov, D. 4031

A method of introducing the concept of proximity. (Russian)

C. R. Acad. Bulgare Sci. **17** (1964), 349–351.

If δ is a proximity relation on X, then a cover u of X is a δ-cover if $A\delta B$ implies that $A \cap C \neq \varnothing$ and $B \cap C \neq \varnothing$ for some $C \in u$. This note shows how the theory of proximity spaces may be axiomatized using the set of δ-covers instead of the proximity relation itself as the primitive notion.

D. Bushaw (Pullman, Wash.)

Heath, Robert W. 4032

Arc-wise connectedness in semi-metric spaces.

Pacific J. Math. **12** (1962), 1301–1319.

A semi-metric space differs from a metric space in that it need not satisfy the triangle inequality. The author studies in detail possible ways of extending to semi-metric spaces the arc theorem: A connected and locally connected Cauchy complete metric space is arc-wise connected. He shows that it cannot be generalized by replacing "metric" by "regular semi-metric" even in the presence of such properties as possessing a uniformity and being compactly connected, but can be extended to a class of spaces satisfying a more general completeness condition. In addition, topological characterizations are given for semi-metric, developable, and metric spaces. The proofs are for the most part related to R. L. Moore's proof of the arc theorem, and most results are related to their equivalent statements in terms of Moore's axioms.

L. K. Barrett (Knoxville, Tenn.)

Heath, R. W. 4033

Screenability, pointwise paracompactness, and metrization of Moore spaces.

Canad. J. Math. **16** (1964), 763–770.

Several relationships between the terms of the title are obtained, the first theorem being related to the assumption that $2^{\aleph_0} < 2^{\aleph_1}$. Some of the results proved are the following. (1) Every separable normal Moore space is metrizable if and only if every uncountable subspace M of E^1 contains a subset which is not an F_σ in M. (2) Let S be a topological space in which every closed set is a G_δ. If S is screenable, then S is pointwise paracompact. (3) A T_3-space S is a pointwise paracompact Moore space if and only if S has

a uniform base (in the sense of Aleksandrov). (4) A necessary and sufficient condition that a space S be screenable is that every open covering of S have a σ-star-countable refinement which is an open covering of S.

R. W. Bagley (Coral Gables, Fla.)

Morita, Kiiti 4034

Products of normal spaces with metric spaces. II.

Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **8**, 87–92 (1964).

Part I appeared in Math. Ann. **154** (1964), 365–382 [MR **29** #2773]. The author introduces so-called basic coverings of a product $X \times Y$, and so-called special refinements of such coverings; the definitions of these concepts are too complicated to reproduce here. Some results relating these concepts to the normality of $X \times Y$ are obtained, and they have the following corollaries in whose statements the above concepts do not appear: (1) If Y is metrizable, then $X \times Y$ is paracompact and normal if and only if X is paracompact and normal and $X \times Y$ is countably paracompact and normal; (2) If Y is separable metrizable, then $X \times Y$ is regular Lindelöf if and only if X is regular Lindelöf and $X \times Y$ is countably paracompact and normal; (3) If Y is metrizable, then $X \times Y$ is normal for any countably paracompact normal space X if and only if Y is a countable union of locally compact subsets. {Note: In the author's terminology, neither paracompact nor normal imply Hausdorff.}

E. Michael (Seattle, Wash.)

Mioduszewski, J. 4035

Mappings of inverse limits.

Colloq. Math. **10** (1963), 39–44.

The author proves several generalizations for inverse systems of polyhedra of the following well-known theorem. If the sequence $f_1, f_2, \cdots$ maps the inverse sequence $\{X_m, \pi_n^m\}$ to the inverse sequence $\{Y_m, \sigma_n^m\}$ (i.e., for $m < n$, $f_m\pi_m^n = \sigma_n^m f_n$), then the induced transformation f between the inverse limits is continuous. The generalizations involve replacing equality of $f_m\pi_m^n(x)$ and $\sigma_m^n f_n(x)$ for all x in X_m with suitable uniform closeness conditions. Some of these more or less "technical" results were applied in the author's proof that every snakelike continuum is the continuous image of the pseudo-arc [Fund. Math. **51** (1962/63), 178–189; MR **26** #1859]. (A stronger result on the images of the pseudo-arc has appeared in papers by A. Lelek [Fund. Math. **51** (1962/63), 271–282; MR **26** #742] and L. Fearnley [Trans. Amer. Math. Soc. **111** (1964), 380–399; MR **29** #596].)

{Theorems 2′ and 4′ evidently need to have added to the hypothesis that for some integer pair m, n there is a map f in $\mathcal{F}$ from X_m to Y_n. On page 40, line 18 from bottom, m_{i+1} should be n_{i+1}.} *R. Bennett* (Galesburg, Ill.)

Pasynkov, B. A. 4036

On a class of mappings and on the dimension of normal spaces. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 356–376.

This paper contains numerous results (25 theorems) concerning the comparison of various notions of dimension (dim, ind, Ind), the behaviour of dimension under mappings of different types, and results concerning the dimension of quotient spaces of locally bicompact groups. The unifying concept in the paper is the notion of mappings $f: X \to Y$ of class (k, Ω), where $k \geq 0$ is an integer and Ω is a class of open coverings of the space X. These are maps with the property that, for every $\omega \in \Omega$ and every point $y \in Y$, there exists a neighborhood $O(y)$ such that $f^{-1}O(y)$ admits an open covering of order $\leq k+1$ which refines ω. The most important case is the case $(0, \Omega)$, which generalizes the uniformly 0-dimensional mappings of M. Katětov [Dokl. Akad. Nauk SSSR **79** (1951), 189–191; MR **15**, 145], the (U, V)-uniformly 0-dimensional mappings of K. Nagami [Proc. Japan Acad. **37** (1961), 207–211; MR **24** #A533] and the "disintegrating" mappings of A. Zarelua [Dokl. Akad. Nauk SSSR **144** (1962), 713–716; MR **26** #5541].

Here we can state only some of the results. Theorem 4: Let X be a normal space which admits a mapping $f: X \to Y$ of type $(0, \Omega)$ onto a metric space Y, Ω containing all finite coverings of X. Then $\dim X = \operatorname{Ind} X$. The proof depends on the theorem of M. Katětov and K. Morita that dim and Ind coincide for metric spaces, and on the fact that every mapping $f: X \to Y$ into a metric space Y admits a factorization through another metric space Z with $\dim Z \leq \dim X$ [the author, ibid. **150** (1963), 488–491; MR **27** #2955]. Theorem 7: Let X be a normal space having the property that $\dim F \leq \operatorname{ind} F$ for all closed subsets $F \subseteq X$. Furthermore, let X admit a mapping $f: X \to Y$ of type $(0, \Omega)$ onto a metric space Y, Ω being a "fine" system of coverings in the sense that, for every $x \in X$ and every neighborhood $O(x)$, there exists an $\omega \in \Omega$ such that the star $\mathrm{St}_\omega(x) \subseteq O(x)$. Then $\dim X = \operatorname{ind} X = \operatorname{Ind} X$.

The author also introduces and studies mappings called n-peripheral. $f: X \to Y$ is 0-peripheral provided it is of type $(0, \Omega)$, where Ω is a "fine" system of coverings of X. f is n-peripheral provided every point $x \in X$ can be surrounded by arbitrarily small neighborhoods with the property that f is $(n-1)$-peripheral on their boundaries. Theorem 15: If $f: X \to Y$ is n-peripheral, then $\operatorname{ind} X \leq \operatorname{ind} Y + n$. In the last section the author establishes several factorization theorems of the following type. Theorem 20: For paracompact spaces Z, every mapping $f: X \to Z$ of type $(0, \Omega)$ admits, for every $\omega \in \Omega$, a factorization into an ω-mapping $g: X \to Y$ and a mapping $h: Y \to Z$ with discrete counter-images $h^{-1}(z)$ of cardinality $k(h^{-1}(z)) \leq k(\omega)$.

S. Mardešić (Zagreb

Gropen, Arthur L. 403

Special homeomorphisms in the functional spac $\mathscr{C}(X, I_{2n+1})$.

Duke Math. J. **28** (1961), 629–637.

The author makes the following extensions of results o J. H. Roberts [same J. **8** (1941), 565–574; MR **3**, 138 and M. K. Fort, Jr. [Proc. Amer. Math. Soc. **7** (1956) 539–542; MR **18**, 918]. (1) Let M be a separable metri space of dimension n, n finite, and let $\{T_n^i\}_{i=1}^\infty$ be a arbitrary countable family of n-hyperplanes in I_{2n+} (the intersection of n-hyperplanes with the cube I_{2n+1} Then there exists a mapping f from M into I_{2n+1} with th following properties: (i) for any k-hyperplane T_k i I_{2n+1} $(n+1 \leq k \leq 2n+1)$, we have $\dim(f(M)^- \cap T_k)$: $k-n-1$; (ii) f is a homeomorphism from M into I_{2n+1} (iii) for every $i = 1, 2, \cdots$, $f(M)^- \cap T_n^i$ is empty. Furthe the set of functions having properties (i)–(iii) contains

dense G_δ in the functional space $\mathscr{C}(M, I_{2n+1})$ (uniform topology). (2) Let X be separable metric and A a closed subset of X such that $\dim(X-A) = m \geq n$, m finite. Then, if f is a mapping from A into I_n, there is an extension F of f over X such that $\dim(X-A) \cap F^{-1}(y) \leq m-n$ for all y. Further, the set of such extensions is dense in the space of all extensions of f over X (with uniform topology).

R. W. Bagley (Coral Gables, Fla.)

Anderson, R. D. 4038

Homeomorphisms of 2-dimensional continua.

General Topology and its Relations to Modern Analysis and Algebra (Proc. Sympos., Prague, 1961), *pp.* 55–58. *Academic Press, New York; Publ. House Czech. Acad. Sci., Prague,* 1962.

A technique for the construction of homeomorphisms between spaces is described. The ideas are then used to characterize two classes of two-dimensional homogeneous continua. The method of description is suggested by a refinement sequence of finite, closed, non-overlapping, connected partitions of the space with homeomorphic elements. The basic definitions used are the following. A triple of sequences $(\{F_i\}, \{\phi_i\}, \{\alpha_i\})$ is an inverse incidence system provided that, for each i, (1) F_i is a finite set; (2) ϕ_i is a map of F_{i+1} onto F_i; (3) α_i is a reflexive and symmetric binary (incidence) relation of F_i; and (4) if $(a, b), (b, c) \in \alpha_{i+1}$ then $(\phi_i(a), \phi_i(c)) \in \alpha_i$. The pair $(\{F_i\}, \{\phi_i\})$ is an inverse system whose inverse limit L is a zero-dimensional compact metric space.

Let R be a binary relation on L defined by $(\{f_i\}, \{f_i^0\}) \in R$ provided that, for each i, $(f_i, f_i^0) \in \alpha_i$. Using condition (4) of the definition above, it follows that R is an equivalence relation and that the set L of equivalence classes defined by R is an upper semicontinuous decomposition $\hat{L}$ of L. The collection $\hat{L}$ (topologized) is called the inverse incidence limit of (F_i, ϕ_i, α_i). It is interesting to note that only binary incidence relations are needed for this structure.

The basic problem discussed in this paper is to determine conditions on two sequences under which their inverse incidence limits are homeomorphic.

L. K. Barrett (Knoxville, Tenn.)

Henderson, George W. 4039

The pseudo-arc as an inverse limit with one binding map.

Duke Math. J. **31** (1964), 421–425.

A continuous function f from $[0, 1]$ onto $[0, 1]$ is constructed such that the inverse limit with f as the only binding map is a pseudo-arc. The construction is rather complex. The author states that this unusual representation may help in solving some of the open questions concerning the pseudo-arc. The assertion is made, without proof, that the construction may be modified in such a way that the map is "infinitely differential".

{There are several misprints. The statement of Lemma 1 should begin "There is a map f". On line 6 from the bottom, page 424, V should be v. It is not clear to the reviewer that the hypotheses of Lemma 2 are sufficient to imply the limit function is continuous at 1, as is stated after Lemma 2; the construction as given is sufficient, however. In the construction of $f_1, f_2, \cdots$ it appears that some further condition like $f_k^{-1}(l_k) < L_{k+1}$ needs to be imposed in choosing L_{k+1}.} *R. Bennett* (Galesburg, Ill.)

Hedrlín, Z. 4040

Remark on partial mappings.

Comment. Math. Univ. Carolinae 4 (1963), 93–97.

Let F be a semigroup of mappings under composition. For $f \in F$ and a set Y, let $f \| Y = f | (Y \cap f^{-1}(Y))$. If h is a mapping such that $h(x_1) = h(x_2)$ implies $h \circ f(x_1) = h \circ f(x_2)$ for $x_1, x_2 \in \operatorname{dom} f$, define $(h \times f)(x) = h \circ f(y)$, where $h(y) = x$. The author gives elementary conditions under which $f \to f | Y$, $f \to f \| Y$, and $f \to h \times f$ are homomorphisms.

K. A. Ross (New Haven, Conn.)

Hedrlín, Z.; Pultr, A. 4041

Remark on topological spaces with given semigroups.

Comment. Math. Univ. Carolinae 4 (1963), 161–163.

A semigroup with identity that has cardinality less than the first inaccessible cardinal is isomorphic with the semigroup of all local homeomorphisms of some T_0 topological space into itself. *K. A. Ross* (New Haven, Conn.)

Baayen, P. C.; Hedrlín, Z. 4042

Commutative polynomial semigroups on a segment.

Comment. Math. Univ. Carolinae 4 (1963), 173–179.

If X is a subset of the real line R, an X-cps is, by definition, a commutative semigroup S of mappings of X into itself, all elements of which are restrictions to X of real polynomials on R. If S contains every continuous map that commutes with all $f \in S$, then S is called maximal; S is entire if it contains restrictions to X of polynomials of every non-negative degree. Semigroups S_i on X_i $(i = 1, 2)$ are equivalent if there is a homeomorphism τ of X_1 onto X_2 such that $S_2 = \{\tau \circ f \circ \tau^{-1} : f \in S_1\}$. The authors determine, up to equivalence, all entire $[0, 1]$-cps and establish which of these is maximal. Results on $[-1, 1]$-cps and R-cps are also given. *K. A. Ross* (New Haven, Conn.)

Ogurcov, N. I. 4043

A class of topological semigroups on a direct product. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 887–890.

Let $S = \prod X_\alpha$ be a product of topological Hausdorff semigroups. The author states some elementary facts relating subsemigroups, ideals, etc., in S with these objects in the factors X_α. Theorem: A topological semigroup T with underlying space $S = \prod X_\alpha$ is topologically isomorphic with a product semigroup on S if there exist continuous homomorphisms ϕ_α of T onto topological semigroups on each factor X_α that are independent. As an example, consider the manifold $A = S^{n-1} \times I$ with boundary B, where S^{n-1} is the closed unit cube in Euclidean n-space. If T is a topological semigroup with identity on A and B is a subsemigroup, then T is topologically isomorphic with a product semigroup on A.

K. A. Ross (New Haven, Conn.)

Gottschalk, W. H. 4044

A survey of minimal sets.

Ann. Inst. Fourier (Grenoble) **14** (1964), *fasc.* 1, 53–60.

This is an expository article without proofs. A number of concepts which have arisen in the study of minimal sets—for example, universal minimal sets, the structure group, total minimality, and reversibility—are briefly discussed.

Two classes of minimal sets—coset transformation groups and symbolic flows—are discussed in some detail, and several examples of the latter are presented.

J. Auslander (New Haven, Conn.)

ALGEBRAIC TOPOLOGY

Saaty, Thomas L. 4045

The minimum number of intersections in complete graphs.

Proc. Nat. Acad. Sci. U.S.A. **52** (1964), 688–690.

Let G_n denote the complete graph on n vertices. Let I_n denote the minimum number of edge intersections when G_n is drawn in the plane (in such a way that at most two edges intersect at any point other than a vertex of G_n). The author exhibits a drawing of G_n which shows that an upper bound of I_n is given by

$$M_n = n(n-2)^2(n-4)/48, \qquad n \text{ even},$$
$$= (n-1)(n-3)(n^2-4n+1)/48, \qquad n \text{ odd}.$$

He states without proof:

$$M_n' = n(n-2)^2(n-4)/64, \qquad n \text{ even},$$
$$= (n-1)^2(n-3)^2/64, \qquad n \text{ odd},$$

is a (better) upper bound; furthermore, if there exists a minimal-intersection representation of G_n which has the property that it contains a minimal-intersection representation of G_{n-k} for each even $k<n$, then $I_n = M_n'$.

W. Moser (Montreal, Que.)

Erdős, P.; Moser, L. 4046

A problem on tournaments.

Canad. Math. Bull. **7** (1964), 351–356.

The authors establish some existence theorems for tournaments. A typical result runs as follows. "Let $n > (\log 2+\varepsilon)k^2 2^k$. Then there exists a positive $\alpha = \alpha(\varepsilon)$ so that for each $l \leq k$ and every choice of l players $x_1, x_2, \cdots, x_l$, each of the 2^l classes into which the remaining $n-l$ players are divided (two players are in the same class if they perform in an identical way against the players $x_1, x_2, \cdots, x_l$) contains more than $\alpha n/2^l$ players, for all but $o(2^{n(n-1)/2})$ of the tournaments."

W. T. Tutte (Waterloo, Ont.)

Beatty, J. C.; Miller, R. E. 4047

On equi-cardinal restrictions of a graph.

Canad. Math. Bull. **7** (1964), 369–375.

The authors define a "k-equi-cardinal restriction" of a graph G as a subgraph of G which includes all the vertices, and in which each component has exactly k vertices. Their problem is to find a minimum integer d such that every regular graph of $n = mk$ vertices and degree (valency) $\geq d$ has a k-equi-cardinal restriction.

The authors obtain the following results. If m is even, then $d = (n/2)-1$. If m is odd and k is even, then $d = n/2$. If m and k are odd and $(n+1)/2$ is even, then $d = (n-1)/2$. In the remaining case $d = (n-3)/2$. They remark that "little is known of the problem if one adds the hypothesis that the graph be connected".

W. T. Tutte (Waterloo, Ont.)

Harary, F. 4048

Recent results in topological graph theory.

Acta Math. Acad. Sci. Hungar. **15** (1964), 405–411.

This is an article reviewing some recent work (mostly by the author and collaborators) concerning the embeddability of graphs in surfaces. The main problems dealt with are the determination of the genus or thickness of complete graphs, complete bicolored graphs, and cubes. The results range in calibre from trivial (such as various consequences of Euler's polyhedral formula) to impressive (e.g., the theorem that the genus of a graph is the sum of the genera of its blocks). There is an abundance of interesting open questions, such as the determination of the genus of the complete graph K_n for $n \equiv 1, 2, 6, 8, 9, 11 \pmod{12}$. The bibliography lists nearly all recent work on the subject.

G. Sabidussi (Hamilton, Ont.)

Harary, Frank; Prins, Geert; Tutte, W. T. 4049

The number of plane trees.

Nederl. Akad. Wetensch. Proc. Ser. A **67** = *Indag. Math.* **26** (1964), 319–329.

A (planted) plane tree is a tree T embedded in the plane (with an end vertex of T distinguished as root). Two (planted) plane trees are considered to be isomorphic if there is an orientation-preserving homeomorphism of the plane onto itself which maps one tree onto the other (and root onto root). In the first part of the paper an explicit formula is obtained for the number of non-isomorphic planted plane trees with n vertices. This is simply another in a long series of applications of Pólya's well-known enumeration method. Contrary to the belief of the authors (if their choice of title is any indication) this reviewer feels that the second part of the paper is by far the more important. For $n \geq 2$ denote by P_n the set of all non-isomorphic planted plane trees with n vertices, and by T_n the set of all non-isomorphic planted trivalent plane trees (i.e., trees in which every vertex has degree 1 or 3). It is shown that there is a natural one-one correspondence f_n between P_n and T_n for all n. Both the construction of f_n and the proof that it is one-one and onto are too involved to be described here.

G. Sabidussi (Hamilton, Ont.)

Gallai, T. 4050

Neuer Beweis eines Tutte'schen Satzes. (Russian summary)

Magyar Tud. Akad. Mat. Kutató Int. Közl. **8** (1963), 135–139.

The theorem mentioned in the title gives a necessary and sufficient condition for the existence of a 1-factor in a given finite graph G [the reviewer, J. London Math. Soc. **22** (1947), 107–111; MR **9**, 297]. A simpler condition can be given for the case in which G is bipartite [ibid. **10** (1935), 26–30].

The author shows that the general condition can be deduced from that for the bipartite case with the help of a simple auxiliary theorem about "critical graphs". A critical graph can be characterized as a graph which has

no 1-factor, but which acquires one when any vertex and its incident edges are removed.

W. T. Tutte (Waterloo, Ont.)

Palásti, Ilona 4051

On the connectedness of bichromatic random graphs. (Russian summary)

Magyar Tud. Akad. Mat. Kutató Int. Közl. **8** (1963), 431–441 (1964).

The author examines the asymptotic behavior of the number of connected bipartite graphs having n and m vertices in the two vertex sets. The edges are chosen at random with one endpoint in each set. When $n=m$, the probability

$$P(n, n, N_c), \qquad N_c = [n \log n + cn]$$

that the graph with N_c edges is connected satisfies

$$\lim_{n\to\infty} P(n, n, N_c) = \exp(-2e^{-c}).$$

An analogous result is obtained when $n \neq m$.

O. Ore (New Haven, Conn.)

Tokuda, Hirotarō 4052

Singularities of n-spheres in $(n+2)$-space.

Yokohama Math. J. **11** (1963), 23–39.

This paper generalizes work of Fox and Milnor [Bull. Amer. Math. Soc. **63** (1957), 406] to obtain the following theorem. Theorem 2: If a collection $\{k_1, \cdots, k_m\}$ of m $(n-1)$-knot types can occur as the collection of singularities of a 1-flat n-sphere in E^{n+2}, then the product $\prod_{i=1}^m k_i$ is cobordant to the trivial knot. Conversely, Theorem B: If $\prod_{i=1}^m k_i$ is cobordant to the trivial knot, then $\{k_1, \cdots, k_m\}$ occurs as the set of singularities of some S^n in E^{n+2}.

L. Neuwirth (Princeton, N.J.)

Schubert, Horst; Soltsien, Kay 4053

Isotopie von Flächen in einfachen Knoten.

Abh. Math. Sem. Univ. Hamburg **27** (1964), 116–123.

Let k be a simple closed polygon in 3-space which is such that the knot type represented is not a companion [cf. H. Schubert, Acta Math. **90** (1953), 131–286; MR **17**, 291], and let g denote its genus. Then among the polyhedral orientable surfaces of genus g that span k there can be found a finite number such that any polyhedral orientable surface of genus g that spans k is isotopic to one of them by a semilinear isotopy that does not move k. It is conjectured that the condition that k not be a companion is necessary (because of the existence of cable-braids). The proof of the theorem is accomplished by means of the Haken-Schubert machine [W. Haken, ibid. **105** (1961), 245–375; MR **25** #4519a; H. Schubert, Math. Z. **76** (1961), 116–148; MR **25** #4519b].

R. H. Fox (Princeton, N.J.)

Guerra, Juan 4054

Čech homology. (Spanish)

Univ. Nac. Ingen. Inst. Mat. Puras Apl. Notas Mat. **2** (1964), 1–7.

Exposé de l'homologie et de la cohomologie de Čech d'un espace compact en insistant sur l'aspect fonctoriel de la théorie.

P. Dedecker (Rhode-St.-Genèse)

Brown, R. 4055

On Künneth suspensions.

Proc. Cambridge Philos. Soc. **60** (1964), 713–720.

Let A, A', B, B' be FD-complexes and Y a css-complex. Let N denote a normalization functor. The author shows (Theorem 1) that a css-homotopy equivalence $\lambda_1: A^Y \to A'$ is determined by a Künneth isomorphism of type $(Y, NA; NA')$. Let $\lambda_2: B^Y \to B'$ be that for the Künneth isomorphism of type $(Y, NB; NB')$. Denoting the set of cohomology operations of type (NA, NB) by $\mathrm{Op}(NA, NB)$, he defined a Künneth suspension κ: $\mathrm{Op}(NA, NB) \to \mathrm{Op}(NA', NB')$ [Proc. London Math. Soc. (3) **14** (1964), 545–565; MR **29** #604]. The main theorem (Theorem 4) says that a natural function of css-homotopy classes $\beta: [A, B] \to [A^Y, B^Y]$ is reduced to κ by canonical isomorphisms $\Phi: [A, B] \to \mathrm{Op}(NA, NB)$ and $\Phi \cdot (\lambda_1^*)^{-1} \cdot \lambda_2^*$: $[A^Y, B^Y] \to \mathrm{Op}(NA', NB')$. When Y is finite-dimensional and X is a connected css-complex, the results are used to determine the homotopy type of X^Y by induction on the Postnikov system of X. As examples, he obtains another proof for some of M. G. Barratt's results [ibid. (3) **5** (1955), 71–106; MR **17**, 290; ibid. (3) **5** (1955), 285–329; MR **17**, 395]: $\pi_{r-1}(X^Y) = Z_2$, $\pi_r(X^Y) \approx Z_4$, where $X = S^p \cup_2 e^{p+1}$ $(p>2)$, $Y = S^{p-r} \cup_2 e^{p-r+1}$.

H. Suzuki (Fukuoka)

Rutter, J. W. 4056

Relative cohomology operations.

Quart. J. Math. Oxford Ser. (2) **15** (1964), 77–88.

A relative cohomology functor Γ is a functor from the category of c.s.s. pairs with base point and c.s.s. maps to the category of Abelian groups and set-theoretic maps, having the following form: Let $\{r_i, G_i\}$, $\{s_j, G_j'\}$, $\{t_k, G_k''\}$ be finite sets of pairs, each pair consisting of a positive integer and an Abelian group. Then

$$\Gamma(X, A) = \prod H^{r_i}(X, G_i) \times \prod H^{s_j}(X, A, G_j') \times \prod H^{t_k}(A, G_k'').$$

A relative cohomology operation is a natural transformation of relative cohomology functors.

The author proves that every relative cohomology operation can be expressed in terms of absolute operations plus a few simple operations which are intrinsically relative, such as the coboundary in the sequence of a pair. The proof proceeds via Eilenberg-MacLane spaces.

W. D. Barcus (Stony Brook, N.Y.)

Hardie, K. A. 4057

On the Hopf-Toda invariant.

Trans. Amer. Math. Soc. **112** (1964), 43–54.

Let S_r^n denote the rn-skeleton of the reduced product complex S_∞^n of the n-sphere S^n. The author discusses a homomorphism

$$H: \pi_i(S_{r-1}^n, S_{r-2}^n) \to \pi_{i+1}(S^{rn})$$

which is equivalent to a special case of the relative "Hopf homomorphism" of H. Toda [J. Inst. Polytech. Osaka City Univ. Ser. A **7** (1956), 103–145; MR **19**, 1188], and uses it to establish various interesting facts about the unstable homotopy groups of the spheres.

These computations independently confirm and slightly extend the results of Toda on the subject [see *Composition methods in homotopy groups of spheres*, Princeton Univ.

Press, Princeton, N.J., 1962; MR 26 #777]. The explicit computations are as follows: let p be an odd prime, and write $[r, n]$ for the p primary component of $\pi_r(S^n)$. Then

(a) $[2t(p-1)+2m-s, 2m+1] = 0$

$$(m \geqq 1, 1 \leqq t \leqq p, 2 \leqq s \leqq 2p-3);$$

(b) (i) $[2t(p-1)+2m-1, 2m+1] = Z_p$

$$(1 \leqq t \leqq p-1, 1 \leqq m \leqq t-1);$$

(ii) $[2t(p-1)+2m-1, 2m+1] = 0$

$$(1 \leqq t \leqq p-1, m \geqq t);$$

(iii) $[2p(p-1)+1, 3] = Z_p$;

(iv) $[2p(p-1)+2m-1, 2m+1] = Z_{p^2}$ or Z_p+Z_p

$$(2 \leqq m \leqq p-1)$$

$$= Z_p \quad (m \geqq p);$$

(c) (i) $[2t(p-1)+2m, 2m+1] = Z_p$

$$(1 \leqq t \leqq p-1, m \geqq 1);$$

(ii) $[2p(p-1)+2m, 2m+1] = Z_{p^2} \quad (m \geqq 2)$;

(iii) $[2p(p-1)+2, 3] = Z_p$.

R. Bott (Cambridge, Mass.)

Sasao, Seiya 4058

On homotopy groups $\pi_{2n}(K_m^n, S^n)$.

Proc. Japan Acad. **39** (1963), 557–558.

The author obtains expressions for the relative homotopy groups in the title, where K_m^n is the complex obtained by attaching an $(n+1)$-cell to an n-sphere by a map of degree m. *I. M. James* (Oxford)

Adams, J. F.; Walker, G. 4059

An example in homotopy theory.

Proc. Cambridge Philos. Soc. **60** (1964), 699–700.

The example consists of an essential map $f: X \to Y$ of CW-complexes such that $f|X^n$ is inessential for all n, and Y is finite-dimensional. X is the suspension of infinite complex projective space, while Y has the homotopy type of a countable wedge of 4-spheres.

W. D. Barcus (Stony Brook, N.Y.)

Liulevicius, Arunas 4060

Notes on homotopy of Thom spectra.

Amer. J. Math. **86** (1964), 1–16.

This paper computes the two primary torsion groups which occur in the bordism theories associated to the classical groups of type O, U, SU, SO, and Sp. This systematic account is a welcome addition to the rather spotty literature on the subject.

The main tool of the paper is the Adams spectral sequence for the prime 2. For the Thom spectra MO, MU, and MSO, the differentials of this sequence are shown to vanish. In the other two cases the author shows that this is not so. *R. Bott* (Cambridge, Mass.)

Suzuki, Haruo 4061

Remarks on the multiplications in Postnikov systems.

Mem. Fac. Sci. Kyushu Univ. Ser. A **17** (1963), 200–201.

Let X be $(m-1)$-connected with Postnikov decomposition $\{X^{(n)}\}$. A condition is given under which the existence of an H-space structure in $X^{(n-1)}$ implies the existence of such a structure in $X^{(n)}$, provided n is in the stable range, $n < 2m-1$. *W. D. Barcus* (Stony Brook, N.Y.)

Cairns, Stewart S. 4062

The Schoenflies theorem for polyhedra.

Proc. Nat. Acad. Sci. U.S.A. **47** (1961), 328–330.

Outline of a proof of a Schoenflies extension theorem for a polyhedron P^{n-1} in euclidean E^n, where P^{n-1} is combinatorially equivalent to the boundary complex of an n-simplex. One certainly hopes to see the full proof published in the near future.

H. Guggenheimer (Zbl **107**, 402)

Hudson, J. F. P.; Zeeman, E. C. 4063

On regular neighbourhoods.

Proc. London Math. Soc. (3) **14** (1964), 719–745.

This paper presents more general and streamlined versions of many of J. H. C. Whitehead's theorems [same Proc. (2) **45** (1939), 243–327]. Where Whitehead assumed a space with a specific triangulation, the authors here assume a family of piecewise related triangulations, where Whitehead proved uniqueness of regular neighborhoods up to a piecewise linear homeomorphism, the present authors prove, with appropriate hypotheses, uniqueness up to an ambient isotopy, and where Whitehead considered absolute neighborhoods, these authors consider a relative version of a regular neighborhood. The definition of the relative neighborhood is as follows. Assume X, Y, N compact subpolyhedra of a polyhedral m-manifold M. N is a regular neighborhood of X mod Y in M if (1) N is an m-manifold, (2) N is a (topological) neighborhood of $X-Y$, (3) $N \cap Y = \dot N \cap Y = \overline{X-Y} \cap Y$, (4) N collapses to $\overline{X-Y}$.

The main theorems proved are an existence theorem for relative neighborhoods and two uniqueness theorems, the second referring to neighborhoods which meet the boundary of M in a required manner.

There are eleven very interesting corollaries proved, not all of which require the relative neighborhood theorem. For example (Corollary 4), let X be a spine of M, let N be a regular neighborhood of X in $\mathring M$; then $\overline{M-N} \cong \dot M \times I$.

Using their stronger form of Whitehead's uniqueness theorem the authors prove Corollary 7: Suppose M is without boundary, then two compact polyhedra in M are of the same type if and only if their regular neighborhoods are ambient isotopic.

The authors ask several questions, some of which are well known, some of which have been partially or fully answered by now, and some of which are novel. Question 3 in particular has been answered in the affirmative by the reviewer and J. Levine independently, provided the boundary sphere pair is unknotted. One proof of an affirmative answer to question 3 in the classical case is attributed to B. Mazur. This is incorrect: the proof in question is due to R. Fox. Question 4 has been answered negatively by Hirsch and the reviewer, and J. Stallings and J. Levine. *L. Neuwirth* (Princeton, N.J.)

TOPOLOGY AND GEOMETRY OF DIFFERENTIABLE MANIFOLDS

See also 3519, 3575, 3580, 3581, 3672, 3725, 3730, 3879, 4060.

Milnor, John 4064

Differentiable manifolds with boundary. **(Spanish)**

An. Inst. Mat. Univ. Nac. Autónoma México **1** (1961), 82–116.

Expository lecture.

Suzuki, Haruo 4065

Correction to: "An approximation of convex polyhedra by C^∞-manifolds in a Euclidean space R^n".

Mem. Fac. Sci. Kyushu Univ. Ser. A **18** (1964), 118–119.

Correction to an earlier paper [same Mem. **16** (1962), 94–100; MR **26** #5587].

Newstead, P. E.; Schwarzenberger, R. L. E. 4066

Reducible vector bundles on a quadric surface.

Proc. Cambridge Philos. Soc. **60** (1964), 421–424.

Corrections and complements to an earlier paper by the second author [same Proc. **58** (1962), 209–216; MR **25** #2611]. Study of the vector bundles over the product of two projective lines which are extensions of one line bundle by another, mainly those which can be so represented in two different ways. *P. Cartier* (Strasbourg)

Sacksteder, Richard 4067

Some properties of foliations.

Ann. Inst. Fourier (Grenoble) **14** (1964), *fasc.* 1, 31–35.

Let M be a manifold with a C^2 foliation of co-dimension 1. A nonempty compact subset C of M is said to be minimal if, for any $x \in C$, the closure of the leaf L_x containing x is C itself. A. J. Schwartz's theorem on minimal sets for the case of dim $M = 2$ [Amer. J. Math. **85** (1963), 453–458; errata, ibid. **85** (1963), 753; MR **27** #5003] is generalized as follows: If C is minimal and if, for every $x \in C$, the first Betti number of L_x is zero, then either $C = M$ or C is a compact leaf. The author also gives another result. Let $M \to B$ be a C^2 fiber space with fiber S^1, where B is an $(n-1)$-manifold with abelian $\pi_1(B)$. If the foliation of M is transversal to the fibers, then any minimal set is either M or a compact leaf. *K.-T. Chen* (New Brunswick, N.J.)

Helgason, Sigurdur [Helgason, Sigurđur] 4068

A duality in integral geometry; some generalizations of the Radon transform.

Bull. Amer. Math. Soc. **70** (1964), 435–446.

The author begins with the following reciprocity formula due to F. John. Let f be a compactly supported C^∞ function on $\mathbf{R}^{2n+1}$. Then

$$(1) \qquad f(x) = \frac{1}{2}(2\pi i)^{-2n}\Delta^n\left(\int_\Omega J(\omega, \langle\omega, x\rangle)\, d\omega\right),$$

where $\Delta = \sum_{i=1}^{2n+1} \partial^2/\partial x_i^2$, Ω is the $2n$-sphere with surface element $d\omega$, and $J\langle\omega, r\rangle = \int_{\{y:\langle y,\omega\rangle = r\}} f(y)\, dy$. With the duality aspect of (1) in mind, the following situation is considered. Let X be a manifold, G a transitive Lie group acting on X, and let Ξ be a family of subsets of X which G permutes and which has the structure of a manifold (above, $X = \mathbf{R}^{2n+1}$, G is the Euclidean group, and Ξ is the set of hyperplanes in $\mathbf{R}^{2n+1}$). The space Ξ is called the dual of X. Given $x \in X$, set $\check{x} = \{\xi \in \Xi : x \in \xi\}$. It is assumed that each ξ and $\check{x}$ have measures μ and ν, both preserved by G; denote by $D(X)$, $D(\Xi)$ the algebra of G-invariant differential operators on X, Ξ, respectively. Define $\hat{f}(\xi) = \int_{x\in\xi} f(x)\, d\mu(x)$, $\check{g}(x) = \int_{\xi\in\check{x}} g(\xi)\, d\nu(\xi)$. The author considers the following two problems: (A) Describe the mappings $f \to \hat{f}$, $g \to \check{g}$, $f \to (\hat{f})^\vee$, etc.; (B) Do there exist maps $D \to \hat{D}$ of $D(X)$ to $D(\Xi)$ and $E \to \check{E}$ of $D(\Xi)$ to $D(X)$ such that $(Df)^\wedge = \hat{D}\hat{f}$, $(Eg)^\vee = \check{E}\check{g}$?

Example: Let X be a two-point homogeneous space (that is, a compact Riemannian manifold with group of isometries G which permutes pairs of points at equal distance). Denote by δ the diameter of X and, for $x \in X$, let $A_x = \{y \in X:$ distance $(x, y) = \delta\}$ be the antipodal manifold. Let A_x be the family of antipodal manifolds ($x \neq y$ implies $A_x \neq A_y$) with the Riemannian structure such that $x \to A_x$ is an isometry. Then $f \to \hat{f}$ is one-to-one and $(\Delta f)^\wedge = \hat{\Delta}\hat{f}$, where Δ, $\hat{\Delta}$ are the Laplacians on X, Ξ. Furthermore, with one exception, $f = P(\Delta)(\hat{f})^\vee$ for a suitable polynomial P.

This article is valuable to the non-expert, as it is expository and clearly written. *Phillip A. Griffiths* (Berkeley, Calif.)

Schwartz, Jacob T. 4069

Generalizing the Lusternik-Schnirelman theory of critical points.

Comm. Pure Appl. Math. **17** (1964), 307–315.

Results in the calculus of variations in the large are, generally speaking, of two sorts. On the one hand there are results of the Morse theory type which make strong a priori assumptions on the nature of the critical locus of a smooth function f on a manifold M (i.e., non-degeneracy of critical points or of critical submanifolds) and conclude rather precise facts (e.g., the Morse inequalities) about the relation of the critical locus of f and the homology, homotopy, or even diffeomorphism type of M. On the other hand there are results of the Lusternik-Schnirelman type, which make no a priori assumption on the nature of the critical locus of f and conclude relations between the critical locus of f and the category of M (i.e., the least integer k for which there exists a covering of M by k closed sets, each contractible in M). The reviewer [Topology **2** (1963), 299–340; MR **28** #1633] and the reviewer and S. Smale [Bull. Amer. Math. Soc. **70** (1964), 165–172; MR **28** #1634] indicated how to extend Morse theory from the classical case, where f was a proper function on a finite-dimensional manifold, to the case where M is a C^2 complete Riemannian manifold modeled on a separable Hilbert space and f is a C^2 function satisfying the following condition: If S is a closed subset of M on which f is bounded, but on which $\|\nabla f\|$ is not bounded away from zero, then there is a critical point of f in the closure of S. In the present paper the author develops a Lusternik-Schnirelman theory under the same hypotheses. In particular, for example, he proves that if such an f is bounded below, then it has at least cat(M) critical points. In addition, the author gives as an application the following theorem which he attributes to Serre: If N is a compact simply connected Riemannian manifold and p and q are

points of M, then p and q are connected by geodesics of infinitely many distinct lengths.

R. S. Palais (Waltham, Mass.)

Andreotti, A.; Vesentini, E. 4070

Un teorema d'annullamento della coomologia.

Univ. e Politec. Torino Rend. Sem. Mat. **20** (1960/61), 109–114.

Article déjà publié dans Atti Convegno Internaz. Geometria Algebrica (Torino, 1961), pp. 57–62, Rattero, Turin, 1962 [MR 27 #775]. *P. Dolbeault* (Poitiers)

Peters, Klaus 4071

Über holomorphe und meromorphe Abbildungen gewisser kompakter komplexer Mannigfaltigkeiten.

Arch. Math. **15** (1964), 222–231.

The author proves the following. Let X_1 and X_2 be compact Kähler manifolds with negative first Chern class, and let $\alpha_1: X_1 \to X_2$ and $\alpha_2: X_2 \to X_1$ be meromorphic maps. Then α_1 and α_2 are in fact biholomorphic maps. This theorem implies that every meromorphic map of the complete intersection $V_n^{(a_1,\cdots,a_r)}$ onto itself is biholomorphic, provided $\sum a_i > n+r+1$.

Several more conditions are stated in order that a meromorphic or a surjective holomorphic map be biholomorphic. *H. Röhrl* (La Jolla, Calif.)

Narasimhan, M. S.; Seshadri, C. S. 4072

Holomorphic vector bundles on a compact Riemann surface.

Math. Ann. **155** (1964), 69–80.

The authors prove that the set M of equivalence classes of holomorphic vector bundles over a compact Riemann surface X of genus $g \geq 2$ that arises from an n-dimensional irreducible unitary representation of $\pi_1(X)$ forms a complex manifold. For this purpose, consider the kernel K of the homomorphism $U(n)^{2g} \to SU(n)$ that sends $(A_1, B_1, \cdots, A_g, B_g)$ into $A_1B_1A_1^{-1}B_1^{-1}\cdots A_gB_gA_g^{-1}B_g^{-1}$. The set of regular points of this real-analytic variety K corresponds precisely to the set M_0 of irreducible unitary representations of $\pi_1(X)$, whence M_0 can be given the structure of a real-analytic manifold. Since two holomorphic vector bundles over a compact complex manifold X that arise from irreducible unitary representations of $\pi_1(X)$ are isomorphic if and only if the representations are equivalent, M can be written as the orbit space of M_0 under the obvious free action of PU(n). Hence M itself can be given the structure of a real-analytic manifold. The tangent space of M at $m \in M$ can be identified canonically with $H^1(\pi_1(X), \mathrm{ad}\,\rho)$, where ρ is a unitary representations in the class m. Since for $P(\rho)$, the holomorphic principal bundle determined by ρ, the natural homomorphism $H^1(\pi_1(X), \mathrm{ad}\,\rho) \to H^1(X, \mathrm{Ad}\,P(\rho))$ is an isomorphism of real vector spaces and since $H^1(X, \mathrm{Ad}\,P(\rho))$ has a natural structure of a complex vector space, M acquires an almost complex structure. This almost complex structure in turn can be shown to be integrable.

H. Röhrl (La Jolla, Calif.)

Spencer, D. C. 4073

A type of formal exterior differentiation associated with pseudogroups.

Scripta Math. **26**, 101–106 (1963).

A concise description of the author's δ-Poincaré lemma [details are published in Ann. of Math. (2) **76** (1962), 306–398; MR **27** #6287a; ibid. (2) **76** (1962), 399–445; MR **27** #6287b], together with a new proof of the algebraic part of the lemma due to D. Mumford.

M. Kuranishi (Princeton, N.J.)

Spencer, D. C. 4074

Correction to "Deformation of structures on manifolds defined by transitive, continuous pseudogroups. I. Infinitesimal deformations of structure".

Ann. of Math. (2) **78** (1963), 204.

A correction to the paper cited in the title [same Ann. (2) **76** (1962), 306–398; MR **27** #6287a].

M. Kuranishi (Princeton, N.J.)

PROBABILITY

See also 3373, 3605, 3606, 3663, 4128, 4147, 4245, 4473, 4546.

Erdős, P.; Neveu, J.; Rényi, A. 4075

An elementary inequality between the probabilities of events.

Math. Scand. **12** (1963), 99–104.

Let n be a positive integer and let $\alpha \in (0, 1)$. Denote by $\varepsilon_n(\alpha)$ the least real number ε satisfying the inequality

$$\sum_{i=1}^{n} P(A_i) - n\alpha \leq \varepsilon$$

for any sequence of random events A_i $(i = 1, \cdots, n)$ with the condition $P(A_iA_j) \leq \alpha^2$, $1 \leq i < j \leq n$. Two equivalent definitions of $\varepsilon_n(\alpha)$ are given, and its exact value is obtained: If ν is the largest integer such that $\nu(\nu-1) \leq n(n-1)\alpha^2$, then

$$\varepsilon_n(\alpha) = \tfrac{1}{2}(1-\alpha) + (n\alpha - \nu)[(n-1)\alpha - \nu]/(2\nu).$$

The second term is zero if $n\alpha$ or $(n-1)\alpha$ is an integer: for $n \to \infty$ it is of the order $1/n$.

For any n and α there exists a sequence A_i^* $(i = 1, \cdots, n)$ such that

$$P(A_i^*) = \alpha + \varepsilon_n(\alpha)/n, \qquad P(A_i^*A_j^*) = \alpha^2 \qquad (i \neq j).$$

An example of such an extremal sequence of random events is constructed for $\alpha = \frac{1}{2}$, $n \equiv 3 \bmod 4$ by means of the theory of quadratic residues.

A particular case of this problem was investigated independently by S. Zubrzycki [Studia Math. **14** (1955), 232–242; MR **16**, 839]. *J. Kubilius* (Vilnius)

Pleszczyńska, Elżbieta 4076

Some generation methods of realizing a Poisson process. (Polish. English summary)

Algorytmy **1**, no. 2, 31–42 (1963).

Author's summary: "The paper describes the way of generating Poisson's process used on the XAM-2 computer. The so-called generation process W of pseudo-random numbers from the exponential distribution (the probability density being e^{-x}, $0 \leq x < \alpha$) is the integral part of the described process. Three various ways of designing the

process W are given, their randomness being checked. Some examples are given of the Poisson process generation used for Operations Research." *P. K. Ghosh* (Calcutta)

Cheng, Shi-hung; Mar, Fun-shi 4077
The limiting joint distribution of terms of variational series. (Chinese. English summary)
Acta Sci. Natur. Univ. Pekinensis **10** (1964), 211–226.

Authors' summary: "The following results for limiting joint distribution of variational series are obtained. (1) The limiting joint distribution of m middle terms whose limit ranks are $\lambda_1, \cdots, \lambda_m$, $0 \le \lambda_1 < \cdots < \lambda_m \le 1$, exists if and only if the limit distribution of every term exists. Therefore, by Smirnov's results, the number of types of the limiting joint distribution law of the m middle terms $\xi_{k_1}^{(n)}, \cdots, \xi_{k_m}^{(n)}$ which satisfy the regularity conditions

$$\lim_{n\to\infty} \sqrt{n}\left(\frac{k_i}{n} - \lambda_i\right) = t_i, \qquad |t_i| < \infty, \quad i = 1, \cdots, m,$$
$$0 < \lambda_1 < \cdots < \lambda_m < 1,$$

is 4^m. The fields of attraction of those types of law are obtained. (2) The limiting joint distribution of s left (or right) marginal terms of constant ranks exists if and only if the limit of every term exists. The limiting joint distribution of marginal terms of constant ranks exists if and only if: (i) The limiting distribution of one left and one right marginal term both exist; (ii) the limiting joint distribution of the m middle terms exists. If the m middle terms satisfy regularity conditions, then, by Smirnov's and Gnedenko's results, the number of types of limit law is $9\cdot 4^m$. The fields of attraction of those laws are also obtained."

Csáki, E.; Vincze, I. 4078
On some distributions connected with the arcsine law. (Russian summary)
Magyar Tud. Akad. Mat. Kutató Int. Közl. **8** (1963), 281–291 (1964).

Several results extending those of the reviewer and Feller, and generalized by E. S. Andersen, are given. The combinatorial formulae are obtained by one-to-one correspondence, from which asymptotic ones are computed. For instance, let s_i, $i = 0, 1, 2, \cdots$, be the successive sums in the coin-tossing game with stakes ± 1, $s_0 = 0$, and let $2\gamma_{2n}^{(2k)}$ denote the number of indices i ($i = 1, \cdots, 2n$) for which either $s_i > 2k$ or $s_i = 2k$ but $s_{i-1} = 2k+1$, where k is a non-negative integer; then

$$P(\gamma_{2n}^{(2k)} = g) = \frac{1}{2^{2n}}\binom{2g}{g}\binom{2n-2g}{n-g+k}, \quad g = 1, 2, \cdots, n-k,$$

$$P(\gamma_{2n}^{(2k)} = 0) = \frac{1}{2^{2n}}\sum_{j=-k}^{k}\binom{2n}{n+j}.$$

K. L. Chung (Stanford, Calif.)

Kellerer, Hans G. 4079
Linearkombinationen zufälliger Grössen und ihre gemeinsame Verteilung.
Math. Z. **84** (1964), 403–414.

If $\mathbf{x}$ is a random vector, $\mathbf{t}$ a constant vector and T a set of vectors $\mathbf{t}$ in R^n, denote by μ the probability distribution in R^n of $\mathbf{x}$ and by $\mu_{\mathbf{x}'\mathbf{t}}$ the probability distribution in R^1 of $\mathbf{x}'\mathbf{t}$. The following two problems are treated in the paper. (1) When does there correspond a distribution μ to arbitrarily given distributions $\mu_{\mathbf{x}'\mathbf{t}}$ for all $\mathbf{t} \in T$? It is shown that such a distribution μ always exists if and only if the vectors in T are linearly independent. (2) When is μ uniquely determined, given $\mu_{\mathbf{x}'\mathbf{t}}$ for all $\mathbf{t} \in T$? Cramér and Wold have proved the uniqueness of μ in the case $T = R^n$ [J. London Math. Soc. **11** (1936), 290–294]. If μ is known to belong to a certain class of distributions, this condition may be relaxed. The author studies the two cases where μ is purely discontinuous and μ has an analytic characteristic function, and gives conditions on T to ensure the uniqueness of μ. *C. G. Esseen* (Stockholm)

Krickeberg, Klaus 4080
Wahrscheinlichkeitsoperatoren von Verteilungen in Vektorräumen.
Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (*Liblice*, 1962), *pp.* 441–452. *Publ. House Czech. Acad. Sci., Prague,* 1964.

The author develops the concept of the probability operator corresponding to a probability distribution in a topological linear space. Such operators can play the role played by characteristic functions in finite-dimensional spaces, and have analogous properties, e.g., the theorems of uniqueness, continuity and convolution are valid for them. Application: a simpler proof of Prohorov's theorem on weak convergence to Brownian motion.

J. G. Wendel (Ann Arbor, Mich.)

Kubik, Lech 4081
On analogies between two classes of probability distributions: The class of infinitely decomposable distributions and the class of self-decomposable distributions.
Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (*Liblice*, 1962), *pp.* 461–465. *Publ. House Czech. Acad. Sci., Prague,* 1964.

The class $\mathscr{L}$ is the class of limit distributions of the sum $(1/B_n)\sum_1^n \xi_k - A_n$ of independent random variables ξ_k, where A_n and $B_n > 0$ are constants and ξ_k/B_n are uniformly asymptotically negligible. $\mathscr{L}$ is characterized as the class of convolutions of a finite number of distributions from the class $\mathscr{G}$ of some special distributions and their weak limits [the author, Studia Math. **21** (1961/62), 245–252; MR **26** #1916]. The author, in the present paper, announces that the class $\mathscr{G}$ is, in a certain sense, the smallest class with the property that characterizes the class $\mathscr{L}$ in the above described sense. Also he claims that the same type of relation is true between the class of Poisson distributions and the class of all infinitely divisible distributions. He also observes the isomorphism between the class of infinitely divisible distributions and its subclass $\mathscr{L}$. These results have been proved by the author [ibid. **22** (1962/63), 197–209; MR **26** #4386].

T. Kawata (Washington, D.C.)

Lukacs, Eugene 4082
A linear mapping of the space of distribution functions onto a set of bounded continuous functions.
Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **3**, 1–6 (1964).

"The great usefulness of characteristic functions in probability theory is based on three properties which are stated by the uniqueness theorem, the convolution theorem and the continuity theorem."

The author considers the linear mapping $\alpha(s; F)$ of the space of distribution functions F onto a space of continuous, uniformly bounded functions of the real variable s. It is shown that the necessary and sufficient condition for $\alpha(s; F)$ to have the three above-mentioned properties is that

$$\alpha(s; F) = \int_{-\infty}^{\infty} e^{ixA(s)}\, dF(x),$$

where $A(s)$ is a continuous real-valued function such that $|A(s)|$ assumes all non-negative real values. Thus $\alpha(s; F)$ is almost a characteristic function. This generalizes an earlier result of the author where integral transforms of distribution functions were studied [Proc. Amer. Math. Soc. **3** (1952), 508–510; MR **13**, 937]. If, in addition, $\alpha(s; F)$ maps every degenerate distribution function onto a non-negative definite function, then $A(s) = Cs$, where C is a constant.

C. G. Esseen (Stockholm)

Pfanzagl, J. 4083

On the topological structure of some ordered families of distributions.

Ann. Math. Statist. **35** (1964), 1216–1228.

A new definition of order for families of distributions, including as a special case monotone likelihood ratio families, is investigated. For such ordered families convergence in the weak sense is equivalent to convergence of the measures on all subsets. Such families are, in general, homeomorphic to dominated families of distributions indexed by a real parameter.

P. J. Bickel (Berkeley, Calif.)

Denny, J. L. 4084

On continuous sufficient statistics.

Ann. Math. Statist. **35** (1964), 1229–1233.

Questions related to the following are treated. Suppose that $\{P_\theta\}$ is a family of probability distributions dominated by Lebesgue measure and defined on a Borel set in n-space. Let $f = (f_i)$ and $g = (g_{ij})$ $(1 \le i \le k,\ 1 \le j \le n_i)$ be continuous sufficient statistics for $\{P_\theta\}$; the f_i and g_{ij} are real-valued. If each block $g_i = (g_{ij})$ is a.e. equal to a composition $h_i \circ f_i$, is g everywhere a continuous function of f?

J. G. Wendel (Ann Arbor, Mich.)

Barndorff-Nielsen, O. 4085

Characteristic subsequences and limit laws for weighted means.

Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (*Liblice*, 1962), pp. 17–27. *Publ. House Czech. Acad. Sci., Prague*, 1964.

From the author's summary: "In proving limit theorems for sequences $\{Z_n\}$ of random variables, it is a standard method to choose a suitable subsequence, $\{Z_{n_k}\}$ of $\{Z_n\}$, verify that if (and only if) the theorem holds for the subsequence then it holds for $\{Z_n\}$ (so that the subsequence is in a sense 'characteristic' for $\{Z_n\}$ with respect to the convergence property in question) and finally prove the theorem for $\{Z_{n_k}\}$. By analyzing applications of this method we have been able to establish some elementary lemmas of analytic nature, which may be used to sharpen well-known limit theorems in the theory of probability. Section 2 contains the proofs of some of the lemmas mentioned above. In Section 3 we derive strong laws of large numbers for weighted means of random variables, which are uniformly bounded or have uniformly bounded pth moments $(p > 1)$. Theorem 4.1 is an assertion about convergence in distribution for weighted means of indicators (i.e., random variables assuming only the values 0 and 1). By means of this theorem it is possible to obtain certain generalizations of the arcsine law."

F. L. Spitzer (Ithaca, N.Y.)

Georgobiani, D. A. 4086

Proof of existence and uniqueness of a stationary ergodic distribution in a problem on flow control. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **34** (1964), 535–540.

Independent random variables X_t, $t = 0, 1, 2, \cdots$, have common density proportional to $x^{a-1}e^{-bx}$, $x > 0$, and zero elsewhere. The process $\{Z_t\}$ is defined inductively by $Z_{t+1} = \max\{0, \min(X_t + Z_t - \alpha, \beta)\}$, α, β positive constants. The limiting distribution of Z_t is found in the case $\alpha > \beta$.

J. G. Wendel (Ann Arbor, Mich.)

Dwass, Meyer; Karlin, Samuel 4087

Conditioned limit theorems.

Ann. Math. Statist. **34** (1963), 1147–1167.

Suppose that $\{X_n\}$ is a Markov process on the line (with stationary transition probabilities) such that for some sequence of constants $a_n > 0$, $\{X_{[nt]}/a_n\}$ converges as $n \to \infty$ to a limiting process. Suppose also that the recurrent event $E_n = \{X_n = 0\}$ is persistent and not periodic. Then it is of considerable interest to study the limiting behavior of various functionals of $\{X_n\}$, given that E_n occurs, and that is the object of this paper.

Two main types of processes $\{X_n\}$ are considered. The first is sums of independent, integer-valued random variables; they are assumed to belong to the domain of a symmetric stable law of index $1 < \alpha \le 2$, so that the limiting process is a stable one. The other type is a Markov chain which has "local" character, that is, transitions are to neighboring integers only (random walk or birth-and-death). In this case the limiting process must be a diffusion of the "Bessel" type, of which (aside from the Wiener process itself) the best-known example is the radial component of two-dimensional Brownian motion. These things have been discussed by the reviewer [J. Math. Mech. **11** (1962), 749–772; MR **26** #5629; Trans. Amer. Math. Soc. **104** (1962), 62–78; MR **25** #1575] and by C. Stone [Illinois J. Math. **7** (1963), 638–660; MR **28** #1663]. {Incidentally, the reference to the reviewer's paper in Ann. Math. Statist. **33** (1962), 685–696 [MR **25** #632] in the first paragraph of the paper under review ought to have been to the first of the articles cited above.}

The method of attack is to seek relations between the distribution of a functional for the conditioned process and of the same functional for the unconditioned process. Consider, for instance, the number of times $X_m = 0$, $m \le n$, say N_n. The value of N_n for the unconditioned process is equal to N_T, where T is the time of the last zero prior to n.

Thus it is possible to write an equation relating the conditional distributions of N_m, given E_m, to the laws of T and N_n. In the cases studied it is possible to pass to the limit $n\to\infty$ and obtain an equation involving the (known) limit laws of N_n and T, which can be solved by taking Laplace transforms. The result is a statement that if the conditioned limiting law exists, it must have a certain form; the existence generally needs independent verification.

Many results, both for stable and "Bessel" processes, are obtained in more or less this way. The main functionals considered are one- and two-sided maxima, occupation time of state 0, and occupation time of half-lines. In addition to determination of the necessary form of the conditional limiting laws, proofs of existence are given in some cases, but the results here are far from complete. Applications and examples are pointed out, and some open questions are mentioned in a final section.

In the reviewer's opinion, this is a very interesting paper which treats important new questions and should stimulate much further work.

J. W. Lamperti (Hanover, N.H.)

Guiaşu, Silviu 4088

Sur la répartition asymptotique pour les suites aléatoires de variables aléatoires.

C. R. Acad. Sci. Paris **259** (1964), 973–976.

This paper is a research announcement of results stated without proof. Proofs are to appear [Rev. Roumaine Math. Pures Appl. Bucarest **10** (1965)]. The author is interested in limit laws (not necessarily central limit) for sequences of random variables when random normalization is used, and his stated theorems generalize results of Anscombe, Rényi, Blum, Hanson, Rosenblatt, Mogyoródi, and himself.

Let $\{Y_n, n\geq 1\}$ be a random variable sequence, $\{N_r, r\geq 1\}$ a positive integer-valued random variable sequence, $\{\omega_n, n\geq 1\}$ a positive real number sequence, $\{n_r, r\geq 1\}$ an increasing integer sequence with limit infinity, θ a real number and $F(\cdot)$ a distribution function. The author states conditions under which

$$\lim_{r\to\infty} \text{in prob } N_r/n_r = \lambda$$

and

$$\lim_{n\to\infty} P(\{Y_n-\theta < a\omega_n\}) = F(a)$$

imply

$$\lim_{r\to\infty} P(\{Y_{N_r}-\theta < a\omega_{N_r}\}) = F(a).$$

Some of the stated conditions are in terms of the conditional distributions of $\{Y_n, n\geq 1\}$, given the values of additional random variables $\{\nu_n, n\geq 1\}$, thus giving an added degree of generality. *R. H. Farrell* (Ithaca, N.Y.)

Bretagnolle, Jean; Dacunha-Castelle, Didier 4089

Convergence de la n-ième convoluée d'une loi de probabilité.

C. R. Acad. Sci. Paris **258** (1964), 4910–4913.

The main theorem of the paper is the following. Let $X_1, X_2, \cdots$ be a sequence of independent non-lattice distributed random variables with the characteristic function $\varphi(t)$. (D_α) is the hypothesis that there exist constants B_n such that

$$(\varphi(B_n^{-1}t))^n \to \exp(-|t|^\alpha), \qquad n\to\infty,\ 0<\alpha\leq 2,$$

i.e., the random variables belong to the domain of attraction of a symmetric stable distribution with exponent α. By (B) is meant the weak convergence of measures for all continuous functions with compact support. If F^{*n} is the distribution function of $\sum_1^n X_i$, then under (D_α),

$$B_nF^{*n}(dx) \xrightarrow{(B)} \Gamma(\alpha^{-1})(\pi\alpha)^{-1}dx, \qquad n\to\infty,$$

where dx is the Lebesgue measure. The method of proof is indicated. This theorem supplements a result due to Gnedenko [Gnedenko and Kolmogorov, *Limit distributions for sums of independent random variables*, Addison-Wesley, Cambridge, Mass., 1954; MR **16**, 52] and generalizes a theorem by Shepp [#4090 below] where the case $\alpha=2$ is treated. *C. G. Esseen* (Stockholm)

Shepp, L. A. 4090

A local limit theorem.

Ann. Math. Statist. **35** (1964), 419–423.

The results of this paper are related to those of Gnedenko concerning a local limit theorem for a sum of independent random variables [Gnedenko and Kolmogorov, *Limit distributions for sums of independent random variables*, Addison-Wesley, Cambridge, Mass., 1954; MR **16**, 52]. Let $X_1, X_2, \cdots$ be a sequence of identically distributed independent random variables with $E(X_1)=0$, $E(X_1^2)=\sigma^2<\infty$, and let $g(y)$ be a continuous function with compact support. If $H_n(y)=(2\pi n\sigma^2)^{1/2}F_n(y)$, where $F_n(y)$ is the distribution function of $\sum_1^n X_i$, and if X_1 does not have a lattice distribution, then

$$\int g(y)H_n(dy)\to\int g(y)\,dy, \qquad n\to\infty.$$

In the lattice case, let the probability mass be concentrated to $\alpha+k\beta$, $k=0, \pm 1, \pm 2, \cdots$, where β is maximal and chosen equal to one. If $\alpha=0$, then

$$(1)\qquad \int g(y)H_n(dy)\to\int g(y)\nu(dy), \qquad n\to\infty,$$

where the measure $\nu[a,b]$ is the number of integers in $[a,b]$. If α is irrational, then

$$\int g(y)H_n(dy)\to\int g(y)\,dy, \qquad n\to\infty,$$

in $(C,1)$-mean. There is a slight misstatement of the conditions under which (1) is valid.

C. G. Esseen (Stockholm)

Shepp, L. A. 4091

A limit law concerning moving averages.

Ann. Math. Statist. **35** (1964), 424–428.

Let $X_1, X_2, \cdots$ be mutually independent, identically distributed random variables with $E(e^{tX})<\infty$ for some $t>0$. The paper deals with the determination of $T=\limsup_{n\to\infty} T_n$, where $T_n=(S_{n+f(n)}-S_n)/f(n)$ and $f(n)$ is a positive integer-valued nondecreasing function, $n=1, 2, \cdots$. Then T is constant a.e. It is shown that the value of T is closely related to the radius of convergence of the

series $\sum x^{f(n)}$. A theorem of Chernoff [same Ann. 23 (1952), 493–507; MR 15, 241] on "tail" probabilities is essential for the proofs. Application is made to success runs.
C. G. Esseen (Stockholm)

Baklan, V. V. 4092

On the existence of solutions of stochastic equations in Hilbert space. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 1299–1303.

Let A be a set and $\nu(d\alpha)$, $\alpha \in A$, a measure on Borel subsets of A. Further, let $W(d\alpha \times dt)$ be a Wiener measure on $A \times [t_0, T]$, i.e., a Gaussian random measure with zero mean, values of which on disjoint sets are independent. It is supposed that $EW^2(d\alpha \times ds) = \nu(d\alpha)\, ds$. The author considers the equation

$$(1)\quad \xi(t) = \Gamma_\tau^t \xi(\tau) + \int_\tau^t \Gamma_s^t a[s, \xi(s)]\, ds + \int_\tau^t \int_A \Gamma_s^t b[s, \alpha, \xi(s)] W(d\alpha \times ds),$$

where Γ_s^t, $a[s, \cdot]$, $b[s, \alpha, \cdot]$ are families of operators in a Hilbert space H and the following holds for $\varphi \in H$, $\tau \leq s \leq t$: $\Gamma_\tau^t\varphi = \Gamma_s^t\Gamma_\tau^s\varphi$, $\Gamma_t^t\varphi = \varphi$. When $\Gamma_s^t a[s, \varphi]$ and $\Gamma_s^t b[s, \alpha, \varphi]$ are uniformly Hölder-continuous in φ and, for a positive integer j, $\int_\tau^T E\|\Gamma_\tau^t \xi(\tau)\|^{2j}\, dt < \infty$, then equation (1) has a unique solution $\xi(t)$, independent of $W(d\alpha \times ds)$ for $s > t$, satisfying $\int_\tau^T E\|\xi(t)\|^{2j}\, dt < \infty$. The proof of this theorem is based on an inequality for moments of Wiener integrals given in the paper.
Petr Mandl (Prague)

Ikeda, Nobuyuki; Ueno, Tadasi [Ueno, Tadashi]; Tanaka, Hiroshi; Satô, Kenkichi 4093

A boundary-value problem for multi-dimensional diffusion processes. (Japanese)

Sûgaku **12** (1961/62), 37–53.

Consider a bounded domain D in R^n having sufficiently smooth boundary ∂D. A Feller semi-group on $C(\bar{D})$ is a semi-group $\{T_t; t \geq 0\}$ of non-negative, linear contraction operators T_t on the Banach space $C(\bar{D})$ of all continuous functions on $\bar{D}$. In this expository note, the authors first consider the following problem. Find all possible Feller semi-groups on $C(\bar{D})$ whose infinitesimal generator $\mathfrak{G}$ satisfies

$$(*)\qquad \mathfrak{G}u(x) = \tfrac{1}{2}\bar{\Delta}u(x) \quad \text{for } x \in D, \qquad u \in \mathfrak{D}(\mathfrak{G}).$$

Here $\mathfrak{D}(\mathfrak{G})$ denotes the domain of definition of $\mathfrak{G}$ and $\bar{\Delta}$ denotes the closure of the Laplacian Δ in $C(\bar{D})$.

The authors first introduce the boundary conditions of A. D. Ventcel' which must be satisfied by functions in $C^2(\bar{D}) \cap \mathfrak{D}(\mathfrak{G})$ for such semi-groups [Teor. Verojatnost. i Primenen. 4 (1959), 172–185; MR 21 #5246]. The authors then describe the method of constructing semi-groups which satisfy (*) as well as Ventcel''s boundary conditions. This method was first obtained by T. Ueno for more general second-order elliptic operators on D and the results were announced by him in Proc. Japan Acad. 36 (1960), 533–538 [MR 26 #1926]. The authors then discuss "the Markov process on the boundary concerning the diffusion" which was introduced by T. Ueno [ibid. 36 (1960), 625–629; MR 26 #1926]. They present the result of K. Sato [ibid. 39 (1963), 69–73; MR 27 #1983] showing that, in the case of a reflecting barrier diffusion, the process on the boundary can be obtained by means of a random time substitution from the diffusion, and that "the local time on the boundary", whose existence was established by K. Sato and H. Tanaka [ibid. 38 (1962), 699–702; MR 26 #5618], can be used as the time change function in this substitution. Next, the authors show that one can construct the sample paths of the diffusion process corresponding to Ventcel''s boundary conditions using the method of stochastic integral equations. This construction is due to N. Ikeda and the result, together with detailed proofs and applications, has been published in Mem. College Sci. Univ. Kyoto Ser. A Math. 33 (1960/61), 367–427 [MR 23 #A4177].

Finally, in the last section of this note, the authors make a few remarks concerning the probabilistic solutions of Dirichlet and Neumann problems for Δ in D, excessive functions and random additive functionals, and general methods of obtaining a new Markov process out of old ones such as killing, random time change, and so on.
Yuji Itô (Providence, R.I.)

Čantladze, T. L. 4094

A stochastic differential equation in Hilbert space. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **33** (1964), 529–534.

Let H be a separable Hilbert space, $a(t, x)$, $b_k(t, x)$ $(k = 1, 2, \cdots)$ functions from $[s, t] \times H$ into H, and let $b_k(t, x)$ be mutually orthogonal. The author examines the stochastic integral equation

$$(1)\quad \xi(t; s, x) = x + \int_s^t a(z, \xi(z))\, dz + \sum_{k=1}^{\infty} \int_s^t b_k(z, \xi(z))\, dw_k(z),$$

where $w_k(t)$ are mutually independent Brownian motion processes. He gives a sufficient condition for the solvability and uniqueness of (1). We denote by $\bar{H}$ the Hilbert space of square integrable random variables with values in H. The paper contains conditions for strong differentiability of $\xi(t; s, x)$ as a function taking $x \in H$ into $\bar{H}$, conditions for the differentiability of $U(s, x) = Eg(\xi(t; s, x))$ and the backward equation

$$-\frac{\partial U}{\partial t} = \delta U(s, x; a(t, x)) + \frac{1}{2}\sum_{k=1}^{\infty} \delta^2 U(s, x; b_k(t, x), b_k(t, x)).$$

Here $\delta U(s, x; y)$ and $\delta^2 U(s, x; y, z)$ are the first and the second differentials of $U(s, x)$. One theorem concerns the absolute continuity of distributions induced by the solutions of two equations of the form (1).
Petr Mandl (Prague)

Castoldi, Luigi 4095

Continuous stochastic processes and diffusion equations (a short survey). (Italian summary)

Rend. Sem. Fac. Sci. Univ. Cagliari **32** (1962), 263–274.

Expository article.

Dinges, Hermann 4096

Einige Verteilungen, die mit dem Wienerprozess zusammenhängen.

Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (*Liblice*, 1962), *pp.* 75–83. *Publ. House Czech. Acad. Sci., Prague*, 1964.

From the author's summary: "In I wird ein Satz der Fluktuationstheorie erläutert, der zum Problemkreis des Spitzer'schen kombinatorischen Lemmas (siehe F. Spitzer [Trans. Amer. Math. Soc. **82** (1956), 323–339; MR **18**, 156]) gehört. In II verfolgen wir eine ähnliche Situation im Wienerraum. Die Formel, zu der eine Analogiebetrachtung führt, gehört in die Analysis und wird daher hier nicht vollständig beweisen. Wir gehen vielmehr, in III, dem dabei entstandenen Bedürfnis nach, Ausdrücke wie z.B. $\sqrt{(-\partial/\partial t)}$ anzuschreiben; wir geben einen Überblick über den allgemeineren Rahmen, innerhalb dessen solchen Bildungen ein Sinn verliehen wird. Einige weitere Anwendungen in IV beschliessen die Arbeit."

F. L. Spitzer (Ithaca, N.Y.)

Ito, Kiyosi 4097

The expected number of zeros of continuous stationary Gaussian processes.

J. Math. Kyoto Univ. **3** (1963/64), 207–216.

The author proves the following theorem. Let x be a stationary Gaussian process with continuous sample paths, and let N_c be the number of crossings of the sample path of the level c in $0<t<T$. Then

$$E(N_c) = \frac{T}{\pi}\sqrt{\left(-\frac{r''(0)}{r(0)}\right)}\exp\left(-\frac{a^2}{2r(0)}\right),$$

where $a=E[x(t)]$, $r(t)=\operatorname{cov}(x(s), x(s+t))$ and $r''(0)=\lim_{h\downarrow 0}([r(h)-2r(0)+r(-h)]/h^2)$. The purpose of the author is to establish rigorously the above under weaker hypotheses than some previous authors have needed, and more rigorously than certain other authors.

Judah Rosenblatt (Albuquerque, N.M.)

Ivković, Zoran 4098

Sur l'erreur de l'approximation du processus stochastique dans les cas de l'observation aléatoire.

C. R. Acad. Sci. Paris **258** (1964), 5339–5341.

From the author's summary: "On définit l'erreur de l'approximation linéaire d'un processus stochastique dans le cas d'observation aléatoire. Dans le cas de l'extrapolation on donne les évaluations de cette erreur. En astronomie, par exemple, on observe l'éclat des astres avec empêchements aléatoires provoqués par les nuages."

Kerstan, Johannes 4099

Teilprozesse Poissonscher Prozesse.

Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (*Liblice*, 1962), *pp.* 377–403. *Publ. House Czech. Acad. Sci., Prague*, 1964.

Let M be the collection of ordered pairs (t, k) with t real and k in some non-empty set K. Let X be the collection of all subsets x of M such that (i) if p_1 and p_2 are distinct points in x, then they have different first (i.e., t or time) coordinates, and (ii) the collection of t-coordinates of points in x contains no limit points. X is the collection of elementary events x. K may be thought of as a set indexing the various types of events which may occur, and (t, k) in x may be thought of as meaning that if the elementary event x occurs, then the event indexed by k occurs at time t. The author defines a point process as a (stationary) probability measure on a suitable σ-field F of subsets of X.

Let F_A (a sub-σ-field of F) be the collection of all events which are defined in terms of what occurs at time t in A. Think of P_s^x as a conditional probability measure on $F_{[0,s]}$ given $F_{(-\infty,0)}$ evaluated at the point x. Under suitable restrictions the author calls $((P_s^x))$ a chain for a point process.

In special cases the author (i) obtains the existence of a unique chain satisfying certain conditions, (ii) obtains a unique point process with a given chain, and (iii) obtains uniform convergence (as s goes to infinity) of the distribution on $F_{[s,\infty)}$, which is induced by an arbitrary distribution on $F_{(-\infty,0)}$ and a given chain, to the unique stationary distribution on $F_{[s,\infty)}$.

The author's model is both interesting and unusual.

D. L. Hanson (Columbia, Mo.)

Nisio, Makiko 4100

On the representation of strictly stationary processes. (Japanese)

Sûgaku **13** (1961/62), 58–64.

In Section 1 of this expository note, the author describes without giving details the results of approximating a strictly stationary process, continuous in probability, by so-called polynomial processes. These results were first obtained by N. Wiener [Amer. J. Math. **60** (1938), 897–936] and then reproved by the author. A detailed account of these results has been published by the author [J. Math. Soc. Japan **12** (1960), 207–226; MR **22** #6019]. In Section 2, the results obtained by the author on the so-called canonical representation of strictly stationary processes are presented. These, together with detailed proofs, have appeared [J. Math. Kyoto Univ. **1** (1961/62), 129–146; MR **26** #7020]. Finally, in Section 3, the author considers the case of strictly stationary time series and describes in a slightly rephrased form the results of M. Rosenblatt [J. Math. Mech. **8** (1959), 665–681; MR **22** #5073].

Yuji Itô (Providence, R.I.)

Priestley, M. B. 4101

The spectrum of a continuous process derived from a discrete process.

Biometrika **50** (1963), 517–520.

Let $\{X_n, n=0, \pm 1, \cdots\}$ be a stationary stochastic process. The author shows that the $\{X_n\}$ process can be embedded in a continuous-parameter, stationary process by defining $X_t = X_{i+1}$ if $i \leq t-\varphi < i+1$, where φ is an observation on a random variable distributed uniformly on $[0, 1]$. He also derives an expression for the spectral density of the new process.

J. R. Blum (Albuquerque, N.M.)

Rozanov, Ju. A. 4102

On the stability of solutions of linear problems for stationary processes. (Russian. English summary)

Teor. Verojatnost. i Primenen. **9** (1964), 528–530.

Author's summary: "Let $\xi(t)$ be a stationary process with spectral function $F(\lambda)$, prediction error

$$\sigma^2 = \inf \int |e^{i\lambda s} - \sum_{t \in T} c(t) e^{i\lambda t}|^2 \, dF(\lambda)$$

and let

$$\delta(G)^2 = \inf \int |e^{i\lambda s} - \sum_{t \in T} c(t) e^{i\lambda t}|^2 \, dF_1(\lambda),$$

where $F_1(\lambda) = F(\lambda) + G(\lambda)$, $dG(\lambda) \geq 0$ and $\int dG(\lambda) \leq h^2$. Then $\lim_{h \to 0} \sup_G \delta(G) = \sigma$. Other linear problems similar to the prediction one have solutions with the same properties."

Rozanov, Ju. A. 4103

Some problems for stationary processes on a finite interval reducing to integral equations of Wiener-Hopf type. (Russian)

Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (*Liblice*, 1962), *pp.* 605–609. *Publ. House Czech. Acad. Sci., Prague,* 1964.

Let $\xi(t)$ be a stationary process with spectral measure $F(d\lambda)$. There are many statistical problems based on $\xi(t)$ given only on a finite interval $[0, T]$. Such problems are reduced to solving the Wiener-Hopf integral equation

$$(1) \qquad \int_{-\infty}^{\infty} e^{-it\lambda} \varphi(\lambda) F(d\lambda) = A(t), \qquad 0 \leq t \leq T,$$

where $A(t)$ is known, and $\varphi(\lambda)$ is unknown but should belong to the space $L^2(F(d\lambda))$, spanned by $e^{it\lambda}$, $0 \leq t \leq T$.

Linear prediction of $\xi(s)$, $s < 0$ or $s > T$, construction of the least mean-square-error estimates of regression coefficients, and a certain kind of smoothing in automatic regulation are such examples. When $\xi(t)$ is Gaussian, the author also states that for absolute continuity of the probability measure of $\xi(t) + A(t)$, $0 \leq t \leq T$, with respect to that of $\xi(t)$, $0 \leq t \leq T$, it is necessary and sufficient that the solution to (1) exist. *G. Maruyama* (Fukuoka)

Winkler, Wolfgang 4104

Stetigkeitseigenschaften der Realisierungen Gauss'scher zufälliger Felder.

Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (*Liblice*, 1962), *pp.* 831–839. *Publ. House Czech. Acad. Sci., Prague,* 1964.

Entsprechend ihrer Bedeutung in der statistischen Turbulenztheorie werden besonders homogene, bzw. homogene und isotrope, zufällige Felder untersucht. Die homogenen, zufälligen Felder stellen eine Verallgemeinerung des Begriffs des stationären zufälligen Prozesses dar. Für sie wird eine Spektraltheorie entwickelt. Sowohl vom theoretischen Standpunkt als auch vom Standpunkt der Anwendungen ist es wichtig, Bedingungen dafür zu kennen, unter welchen die Realisierungen eines zufälligen Feldes mit Wahrscheinlichkeit 1 stetige Funktionen sind, bzw. einer Lipschitzbedingung genügen. In dieser Arbeit werden insbesondere Gauss'sche zufällige Felder in Hinblick auf die Stetigkeitseigenschaften ihrer Realisierungen untersucht. *E. A. Robinson* (Uppsala)

Heyer, Herbert 4105

Über neuere Ergebnisse aus der Theorie der Wahrscheinlichkeitsmasse auf lokalkompakten und Lieschen Gruppen.

Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (*Liblice*, 1962), *pp.* 277–284. *Publ. House Czech. Acad. Sci., Prague,* 1964.

An excellent survey of the current status of the theory, including a bibliography of 32 items, and careful statements of theorems due to Böge, Urbanik, Hunt and Wehn.

J. G. Wendel (Ann Arbor, Mich.)

Csáki, Péter; Fischer, János 4106

On the general notion of maximal correlation. (Russian summary)

Magyar Tud. Akad. Mat. Kutató Int. Közl. **8** (1963), 27–51.

The authors introduce a functional $S(H_1, H_2)$ on pairs of subspaces H_1, H_2 of a Hilbert space, and characterize it by means of $S(H_1, H_2) = \sup\{(x, y) : x$ and y of norm one, $x \in H_1, y \in H_2\}$. In probabilistic contexts subspaces arising naturally consist of square-integrable functions orthogonal to constants and measurable with respect to specified subalgebras of the basic σ-algebra. Then S is called the maximal correlation of the subalgebras. The authors compute it for a number of examples, and prove the interesting Theorem (5.1): If two such subalgebras are generated by random variables ξ, η, respectively, then the maximal correlation equals $\sup(u, v)$, where u and v range over univalent functions of ξ and η, $(u, 1) = (v, 1) = 0$, $\|u\| = \|v\| = 1$. *J. G. Wendel* (Ann Arbor, Mich.)

Lion, Georges 4107

Principe complet du maximum et semi-groupes sous-markoviens.

C. R. Acad. Sci. Paris **258** (1964), 3621–3623.

Let C_K and C_0 be the spaces of real continuous functions with compact support and continuous functions vanishing at infinity, respectively, on a separable locally compact space X. The author considers a positive linear transformation V which maps C_K into C_0. The author first shows that the complete principle of the maximum and the weak principle of the positive maximum are equivalent for such operators. That the first implies the second is due to G. A. Hunt [Illinois J. Math. **1** (1957), 316–369, esp. pp. 351–357; MR **19**, 951]. Assuming that the operator V satisfies these two principles, a family of positive operators $\{V^\lambda; \lambda > 0\}$ on C_0 is constructed satisfying (1) $(I - \lambda V^\lambda) V = V^\lambda$, $\lambda > 0$; (2) $V^\lambda - V^\mu = (\mu - \lambda) V^\lambda V^\mu$, $\lambda, \mu > 0$; (3) $\|\lambda V^\lambda\| \leq 1$, $\lambda > 0$; (4) if $f \in C_K$, then $\|V^\lambda f - Vf\| \to 0$ as $\lambda \to 0$. $V(C_K)$ is not assumed to be dense in C_0. Letting H denote the closure of the range of V^λ, which is independent of λ, it is shown that there is a semi-group of sub-Markovian transition operators $\{P_t : t \geq 0\}$ taking elements of C_0 into Borel measurable functions such that $Vf(x) = \int_0^\infty P_t f(x)\, dt$ for $f \in C_K$, assuming H separates the points of the Alexandroff compactification of X. This result is a generalization of a theorem due to Hunt in which $V(C_K)$ is assumed to be dense in C_0.

L. L. Helms (Urbana, Ill.)

Chung, Kai Lai 4108

The general theory of Markov processes according to Doeblin.

Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **2**, 230–254 (1964).

The paper deals with discrete-time Markov chains on a general state space X and transition probability $P(X, E)$. When X is countable only the classification and periodicity properties of states are well known [W. Feller, *An introduction to probability theory and its applications*, Vol. I, Chapter 15, second edition, Wiley, New York, 1957; MR **19**, 466]. Here the analogous problems are treated for general X, and even though Doeblin [Ann. Sci. École Norm. Sup. (3) 57 (1940), 61–111; MR **2**, 3] already gave definitive results for this general case, they are not widely known, perhaps because Doeblin's article was not very clear in many places. The present paper simplifies, cleans up and extends Doeblin's paper. There are too many propositions to give a detailed description of the contents, so that we only state a sample result.

A set C is called closed if the Markov chain cannot leave C once it enters C, i.e., $P(x, C) = 1$ for all $x \in C$. X is called indecomposable if it does not contain two disjoint closed sets. A set E is called inessential if the probability of visiting E infinitely often is zero for each starting point. Otherwise it is called essential. Finally, E is called absolutely essential if it is essential but not a countable union of inessential sets. With these definitions the following is the analogue of the decomposition of an irreducible state space into a transient set and a class of persistent states each with the same period. Theorem: If X is indecomposable and absolutely essential, then there exists an integer D such that X can be written as the disjoint union $X = F \cup \bigcup_{i=1}^{D} E_i$, where F is not absolutely essential but the E_i are, and, moreover, $P(X, E_{i+1}) = 1$ for all $x \in E_i$ ($i = 1, \cdots, D$, $E_{k+1} = E_1$). D is maximal with this property and the E_i cannot be broken up further in a certain sense (Propositions 33, 44 and Definition 13).

H. Kesten (Ithaca, N.Y.)

Maruyama, Gishirô; Tanaka, Hiroshi 4109

On recurrent stationary Markoff processes. (Japanese)

Sûgaku **13** (1961/62), 30–37.

The main part of the results presented in this expository note has been published by the authors [Mem. Fac. Sci. Kyushu Univ. Ser. A **13** (1959), 157–172; MR **22** #3030]. In the first four sections of the present note, the authors reproduce almost word for word (but in Japanese) the results and proofs of the paper quoted above, except that they now follow the method of R. Z. Has'minskiĭ [Teor. Verojatnost. i Primenen. 5 (1960), 196–214; MR **24** #A3695] in constructing the invariant measure, which is somewhat simpler than the method given in the original paper. In the last section, the authors make the following remarks: (1) Some of the hypotheses made on the processes in consideration are satisfied by strong Feller processes with some additional properties; (2) If the hypotheses of the paper are satisfied by a strong Feller process, then the transition probabilities of the process are necessarily absolutely continuous with respect to the invariant measure which can be constructed by the method of the paper; (3) For a strong Feller process satisfying the hypotheses of the paper, $\lim_{t\to+\infty} T_t f(a) = (1/m(R^N)) \int f\,dm$ holds for every $f \in C(R^N)$ if it is known that the invariant measure m is finite and that the transition probabilities of the process satisfy one additional condition; (4) For diffusion processes on R^N having Brownian hitting probabilities, the speed measure (the canonical measure in the sense of Feller) is the invariant measure for the process.

Yuji Itô (Providence, R.I.)

Port, Sidney C. 4110

Some theorems on functionals of Markov chains.

Ann. Math. Statist. **35** (1964), 1275–1290.

This paper deals with recurrent events in discrete time. It is shown that the expected waiting time, which determines whether the recurrent event is persistent, transient, or positive, may be expressed in terms of the sequence $E(Y_n)$, where Y_n is the time between n and the last occurrence before n. Subsequently, the author extends results of Dynkin and Lamperti concerning joint limit distributions for some of the functionals usually associated with recurrent events.

F. L. Spitzer (Ithaca, N.Y.)

Port, Sidney C. 4111

Escape probability for a half line.

Ann. Math. Statist. **35** (1964), 1351–1355.

The main result, for the Markov process $S_n = S_0 + X_1 + \cdots + X_n$ of sums of the identically distributed independent random variables X_i, concerns the following escape and hitting probabilities. Let $e(x)$ denote the probability that $S_n > 0$ for all $n \geq 1$, given that the process starts at $S_0 = x$, and let Z be the place of the first visit to $(0, \infty)$, given that $S_0 = 0$. Then if $E|X_1| < \infty$ and $E(X_1) > 0$, $e(x)/E(X_1) = P[Z > -x]/E(Z)$, and $\int_{-\infty}^{\infty} e(x)\,dx = E(X_1)$. (Remark: The integral of $e(x)$ is the capacity of the set $(-\infty, 0]$, and it may be shown that $\int_{-\infty}^{\infty} e_A(x) = |E(X_1)|$ when e_A is the escape probability of any set A of infinite Haar measure, with the property that $P_x[S_n \in A$ infinitely often$] = 0$ for all x. Here $E(X_1)$ may be positive or negative, and when $E|X_1| = \infty$, the above capacity is infinite.)

F. L. Spitzer (Ithaca, N.Y.)

Rosenblatt, M. 4112

Stationary Markov chains and independent random variables.

J. Math. Mech. **9** (1960), 945–949.

Es wird gezeigt: Eine stationäre Markowsche Kette x_n ($n = 0, \pm 1, \cdots$) mit abzählbar vielen Zuständen besitzt genau dann eine Darstellung der Form $x_n = g(\alpha_n, \alpha_{n-1}, \cdots)$ mit Borel-meßbarem g und einer unabhängigen Folge α_n ($n = 0, \pm 1, \cdots$) auf $[0, 1]$ gleichverteilter zufälliger Zahlen, wenn sie ergodisch ist und keine periodischen Zustände besitzt.

K. Matthes (Zbl **96**, 340)

Rosenblatt, M. 4113

Addendum to "Stationary Markov chains and independent random variables".

J. Math. Mech. **11** (1962), 317.

The author indicates that the proof of his main theorem on p. 945 in same J. **9** (1960), 945–949 [#4112 above] is incomplete, and offers a correction.

Teghem, J. 4114

Les processus de Markov et les phénomènes d'attente.

Bull. Soc. Math. Belg. **15** (1963), 45–68.

This is an expository paper about the notion of a Markov process in its elementary form. Birth-and-death processes are discussed in detail; applications to queueing theory are given. The modern notion of a Markov process in the sense of E. B. Dynkin is not mentioned.

H. Bauer (Hamburg)

Neveu, Jacques 4115

Deux remarques sur la théorie des martingales.

Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **3**, 122–127 (1964).

This paper treats two distinct aspects of martingale theory. In the first part, the ergodic theorem of E. Hopf [J. Rational Mech. Anal. **3** (1954), 13–45; MR **15**, 636] is used to prove that $E[f|\mathfrak{A}_n]$ converges in the mean and with probability 1 to $E[f|\mathfrak{A}_\infty]$, where $\mathfrak{A}_\infty$ is the intersection of the decreasing sequence of Borel fields $\{\mathfrak{A}_n\}$ and f belongs to L^1 over a probability measure space (∞, Ω, P). The second part is concerned with a method of constructing a supermartingale by piecing together two supermartingales as follows. Let $\{X_t^{(i)}, \mathfrak{A}_t : t \in T\}$, $i = 1, 2$, be two supermartingales relative to the same family of Borel fields $\mathfrak{A}_t$ and let τ be a stopping time defined on $\Omega_\tau \subset \Omega$ relative to the family $\mathfrak{A}_t$. If $X_\tau^{(2)} \leq X_\tau^{(1)}$ on Ω_τ, then the process X_t equal to $X_t^{(1)}$ on the complement of $\{\tau \leq t\}$ and equal to $X_t^{(2)}$ on $\{\tau \leq t\}$ is again a supermartingale relative to the family $\mathfrak{A}_t$. This method is used to obtain an elementary derivation of an inequality due to L. E. Dubins [Illinois J. Math. **6** (1962), 226–241; MR **25** #5538] for the probability distribution of the number of upcrossings of an interval by a supermartingale.

L. L. Helms (Urbana, Ill.)

Harlamov, B. P. 4116

Some theorems on probabilistic search in a deterministic field. (Russian. English summary)

Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. **19** (1964), no. 3, 69–74.

The author considers asymptotic properties of sequences of points in an affine metric Euclidean space. The sequences are constructed according to a "search" rule that determines the statistical sampling of each next point depending on $m+1$ $(m \geqslant 0)$ immediately preceding points and on $m+1$ real numbers corresponding to these points. The correspondence between the numbers and the points is defined by a nonrandom real-valued function which is constant with respect to time. Boundedness conditions are formulated for the region of the most probable "hit" of the points of the sequence having sufficiently large indexes. Some sufficient conditions are given under which the above are fulfilled.

H. P. Edmundson (Pacific Palisades, Calif.)

Borovkov, A. A. 4117

Analysis of large deviations in boundary-value problems with arbitrary boundaries. II. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 750–767.

Part I appeared in same Ž. **5** (1964), 253–289 [MR **29** #645]. In this second part the author continues the asymptotic analysis of the first exit probabilities from a given region of a random path with vertices $(k/n, S_k/x)$, where S_k is a sum of identically distributed independent random variables. The first passage distribution is considered simultaneously with the distribution of the magnitude of the first jump from the region or with the distribution of S_n/x. In the last paragraph the hypothesis $x \sim n$ is replaced by $x = o(n)$.

Petr Mandl (Prague)

Kesten, Harry 4118

On the number of self-avoiding walks. II.

J. Mathematical Phys. **5** (1964), 1128–1137.

This is a sequel to a recent study [same J. **4** (1963), 960–969; MR **27** #2006] of the number $\chi_n(d)$ of distinct self-avoiding (s.a.) paths of length n, starting at the origin, on the lattice points of d-dimensional Euclidean space. Similarly, $\gamma_n(d)$ is the number of self-avoiding polygons (i.e., paths starting at 0 and ending at a point a distance one from 0 after n steps). The emphasis here is on estimates for the limits $\beta(d) = \lim_{n\to\infty} [\chi_n(d)]^{1/n} = \lim_{n\to\infty} [\gamma_n(d)]^{1/n}$ which are sharp when the dimension d is large. Thus, let $\chi_{n,2r}(d)$ be the number of s.a. paths of length n which do not contain any loops of lengths $2r$ or less (i.e., such paths may visit the same point more than once, but only if there are more than $2r$ steps between consecutive visits). Let $\beta_{2r}(d)$ denote the limit of the nth root of $\chi_{n,2r}(d)$. Then it is shown that $\beta_{2r}(d) - \beta(d) = O(d^{-r})$ as $d \to \infty$, and, in particular, it follows that $\beta(d) = 2d - 1 - 1/(2d) + O(1/d^2)$ as $d \to \infty$. The existence of such an asymptotic expansion was surmised by M. E. Fisher and D. S. Gaunt [Phys. Rev. (2) **133** (1964), A224–A239]. It is also shown that, for suitable constants c_1, c_2,

$$\chi_n(d) \leq [\beta(d)]^n \exp[c_1 n^{2/(d+2)} \log n]$$

and

$$[\beta(d)]^{2n-1} \exp[-c_2 n^{2/(d+1)} \log n] \leq \gamma_{2n-1}(d).$$

F. L. Spitzer (Ithaca, N.Y.)

Dolci, Alba; Bellu, Giuseppina 4119

Il metodo della funzione generatrice nella risoluzione di problemi stocastici. (English summary)

Rend. Sem. Fac. Sci. Univ. Cagliari **33** (1963), 424–432.

The two stochastic problems considered are (i) the determination of transition probabilities and the mean and variance of their distribution, in the "simplest traffic system": Poisson input to an infinite number of exponential servers, and (ii) the determination of the same variables for the same system with Poisson input replaced by limited traffic sources. In the authors' terminology, the first problem is regarded as a birth-and-death process, the second as a machine repair problem. All results given are well known and have been given in the reviewer's book [*Stochastic service systems*, Wiley, New York, 1962; MR **24** #A3703].

J. Riordan (Murray Hill, N.J.)

Dreze, Jean-Pierre 4120

Files d'attente à plusieurs priorités relatives.

Cahiers Centre Études Recherche Opér. **4** (1962), 30–51.

Author's summary: "Cet article est consacré à l'étude des files d'attente à un guichet avec priorités relatives dont la définition et certains principes généraux sont

donnés dans l'introduction. Les arrivées de clients suivent une loi de Poisson et la durée du service, une loi exponentielle. Pour des raisons méthodologiques, nous avons cru préférable d'aborder le problème transitoire par le cas de files à trois priorités relatives pour lequel nous établissons en détail le bilan des probabilités d'états et dont nous esquissons la technique de résolution. Ensuite, nous envisageons le phénomène général à un nombre quelconque de priorités relatives en régime transitoire; la solution théorique de ce problème est présentée sous forme de fonctions génératrices des transformées de Laplace des probabilités d'état. Ces derniers résultats permettent l'étude aisée de la file en régime stationnaire: ici, nous obtenons les fonctions génératrices des probabilités d'états stationnaires, les nombres moyens de clients et, pour terminer, la moyenne et la variance du temps d'attente de chaque type de clients."

Finch, P. D. 4121

On partial sums of Lagrange's series with application to the theory of queues.

J. Austral. Math. Soc. **3** (1963), 488–490.

The author considers the partial sum of Lagrange's expansion series [Whittaker and Watson, *A course of modern analysis*, fourth edition, p. 133, Cambridge Univ. Press, Cambridge, England, 1950].

$$\phi^n(x) = \phi(x) + \sum_{m=1}^{n} (y^m/m!) D^{m-1}[(k(x))^m D\phi(x)],$$

$$D = d/dx,$$

where $k(z)$ is analytic in and on a region C surrounding x, y is such that $|yk(z)| < |z-x|$ for any z on C and $\phi(z)$ is any analytic function in and on C. He gives a recurrence relation for $\phi_j^n(x)$ which is $\phi^n(x)$ with $\phi(x) = x^j$ (the series on the right-hand side of (2.4) must be read as $\sum_{m=1}^{n+1}$). This relation with $x=0$, $y=1$ is seen to be exactly the same, including initial conditions, as one which the probability Q_j^n that the $(n+1)$st arrival finds j or more customers in the queueing system $GI/M/1$ should satisfy. The author then gives an elegant proof of

$$Q_j^n = j \sum_{j}^{n} m^{-1}[(m-j)!]^{-1}[D^{m-j}(k(x))^m]_{x=0}$$

$$(n \geq j \geq 1).$$

The right-hand sides of (2.7) and (3.3) must be replaced by this expression, where

$$k(z) = \sum_{0}^{\infty} k_m z^m \quad \text{and} \quad k_m = (m!)^{-1} \int_0^{\infty} (\mu x)^m e^{-\mu x} dA(x),$$

$A(x)$ being the distribution function of the interarrival time and μ being the parameter of exponential distribution of service time. This result was conjectured by the author [J. Austral. Math. Soc. **3** (1963), 220–236; MR **27** #4298] and was proved by Brockwell [ibid. **3** (1963), 249–256; MR **28** #5490] in a different way.

T. Kawata (Washington, D.C.)

Foster, F. G.; Perera, A. G. A. D. 4122

Queues with batch departures. II.

Ann. Math. Statist. **35** (1964), 1147–1156.

This is a sequel to a paper by the first author and K. M. Nyunt [same Ann. **32** (1961), 1324–1332; MR **25** #629]. The system $E_k/G/1$ is analysed in these papers in a novel way by dissecting each inter-arrival interval into k independent identically negative-exponentially distributed phases, by supposing that a sub-unit arrives at the end of each phase, and by postulating that the sub-units are aggregated as they arrive into batches of size k and served as such, with an arbitrarily specified batch-service-time. This new system is called $E_1/G^k/1$, and it differs from $E_k/G/1$ only in the way in which "the number of customers present in the system" is measured. For $E_1/G^k/1$, this random variable (called $\xi(t)$) is the number of sub-units present, while for $E_k/G/1$ the corresponding random variable (called $\zeta(t)$) is the number of complete batches present, so that $\zeta(t)$ is the integer part of $\xi(t)/k$. Thus $\zeta(t)$ can be studied if enough is known about $\xi(t)$, and the authors carry out this programme in some detail. In particular (for both systems) they compare the limiting distributions (after the lapse of a long time) of "the number present" (i) just before an arrival, (ii) just after a departure, and (iii) at an arbitrary time. For further information on the topics treated in this paper see L. Takács [Trans. Amer. Math. Soc. **100** (1961), 1–28] and R. R. P. Jackson and D. G. Nichols [J. Roy. Statist. Soc. Ser. B **18** (1956), 275–279; MR **18**, 681].

D. G. Kendall (Cambridge, England)

Georgobiani, D. A. 4123

An application of the methods of statistical decision functions to a question of optimal parameters in a control problem. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **35** (1964), 23–28.

Decision-theoretic treatment of a problem arising in queueing theory. *J. G. Wendel* (Ann Arbor, Mich.)

Greenberg, Irwin 4124

The distribution of busy time for a simple queue.

Operations Res. **12** (1964), 503–504.

In a queue with a single channel, Poisson arrival times and exponential service times, let t be the busy time in an interval of given duration. The author proposes to infer the distribution of t by reducing the queueing process to a two-state process, in which the states are "busy" and "idle", and then applying some known results. Unfortunately, this two-state process is not Markovian, and this seems to invalidate the author's method.

The formula for the distribution of the first busy period starting with a queue of length q, cited from Takács [*Introduction to the theory of queues*, p. 38, Oxford Univ. Press, New York, 1962; MR **24** #A3704], can be expressed more simply: see the reviewer's derivation [discussion of Kendall, J. Roy. Statist. Soc. Ser. B **13** (1951), 151–185, pp. 182–183; MR **13**, 957]. *I. J. Good* (Oxford)

Hasofer, A. M. 4125

A dam with inverse Gaussian input.

Proc. Cambridge Philos. Soc. **60** (1964), 931–933.

This paper studies the dam of infinite capacity, with input $X(t) \geq 0$ in time t following an infinitely divisible distribution, and subject to release at constant unit rate ceasing when the dam is empty.

The author points out that for the previously studied

Gamma type input of this kind, functions describing the transient behavior of the content $Z(t)$ of the dam are known, but cannot be calculated in closed form [J. Gani and N. U. Prabhu, same Proc. **50** (1963), 417–430; MR **26** #4425]. He shows that for the inverse Gaussian input with the density function

$$\frac{t}{\sigma\sqrt{(2\pi)}}\left(\frac{\rho}{x}\right)^{3/2}\exp\left\{-\frac{\rho}{2\sigma^2 x}(x-\rho t)^2\right\}$$

it is possible to obtain explicit results for (1) the distribution of time to first emptiness, (2) the probability of emptiness, (3) the expectation of the content, and (4) the Laplace transform of the content $Z(t)$ of the dam.

J. Gani (E. Lansing, Mich.)

STATISTICS

See also 4077, 4084, 4106, 4174, 4177, 4178, 4605.

Freund, John E. 4126

★Mathematical statistics.

Prentice-Hall, Inc., Englewood Cliffs, N.J., 1962. xiii + 390 *pp.* $10.35.

This admittedly non-rigorous textbook is written for an introductory two-semester course with the prerequisite of undergraduate knowledge in calculus. Thus it is destined for a specific age public. The fourteen chapters starts with statements of the problems, followed by a small number of logically well-separated paragraphs and clearly formulated theorems and end with stimulating exercises, some taken from quality control and operations research (without mentioning this name). Solutions for one half of the exercises are shown at the end (why only one half?). References, but in insufficient number, are given after most of the chapters. Articles from journals are only rarely cited. An appendix contains the usual statistical tables and the binomial probabilities. Thus the book tries to be self-contained. Lengthy discussions which would deter from the main point are omitted. On the whole, it meets its aims.

After a short introduction on set theory come three chapters on discrete variables, binomial, multinomial, hypergeometric and Poisson's distributions and their moments. As usual, the binomial distribution is used to obtain the law of large numbers. However, its different forms and philosophical implications are not stated. The discrete uniform, the geometric and the negative binomial distributions are mentioned only in the exercises. Quite late, in the fifth chapter, continuous variables, the uniform, the exponential, the Gamma and the normal distributions are stated. The derivation of the normal distribution from the binomial one is given only in the next chapter. The Pearson system should merit a systematic treatment, not only a notice in the exercises. The logarithmic normal distribution and the use of probability papers are not mentioned. The sums of random variables in Chapter 7 lead to the central limit theorem obtained by the moment-generating function. The next chapter, dealing with sampling distributions, studies the mean, variance, the chi-square, and the F- and t-distributions. The important proof for the independence of mean and variance is left to the reader.

Maximum likelihood method and confidence intervals form Chapters 9 and 10. Two chapters follow on the test of hypothesis against a single alternative. Only one chapter is devoted to regression and correlation and the final one to the analysis of variance. Many bivariate distributions, not only the usual normal one, are studied. This has definite merit, but the conditions for a bivariate function to be a bivariate distribution are not sufficiently clearly stated. In some graphs only one of the regression curves is shown.

The distinction between the probability function $F(x)$ (distribution) as opposed to the density function $f(x)$ is not sufficiently clear because sometimes a density function is also called a distribution. To make things worse, the probability of a certain value for a discontinuous variable is designated by the same symbol as the density of probability. It is sad that such time-dishonored impurities persist in a new book. On page 226, the reader is asked to find the (non-existent) maximum likelihood estimates for the two parameters of a curtailed exponential distribution.

E. J. Gumbel (New York)

Brittain, John A. 4127

Interpolation of frequency distributions of aggregated variables and estimation of the Gini concentration measure.

Metron **22** (1962), 98–109.

Economic research sometimes deals with empirical frequency distributions with gaping unequal class intervals but also with information about the aggregate of the variable within the interval. Consider a basic variable, X, representing income level. Let $F=F(X)$ be the number of incomes greater than X, and let A be the aggregate income accruing to the F incomes greater than X. First, with unequal class intervals of X, the Newton expansion using divided differences for observed F_i can be used in obtaining an approximating polynomial for estimating A for any F, $A(F)$, and this can be differentiated to estimate the X associated with that F, $X(F)$. Second, an adaptation of Simpson's rule can be used in obtaining an approximation to the Gini coefficient. Numerical illustrations are given.

P. S. Dwyer (Ann Arbor, Mich.)

Sevast'janov, B. A.; Čistjakov, V. P. 4128

Asymptotic normality in the classical problem of balls. (Russian. English summary)

Teor. Verojatnost. i Primenen. **9** (1964), 223–237.

Consider a scheme of independent sampling, with replacement, of n objects from a population of size N; μ_r is the number of objects in the population selected exactly r times. (For example, if n balls are dropped into N boxes, μ_r is the number of boxes containing exactly r balls.) The principal result of this paper is a multivariate local limit theorem for $\mathbf{P}(\mu_{r_1}=k_1,\cdots,\mu_{r_s}=k_s)$ as $n, N\to\infty$. With the provision that $0<\alpha_0\leq n/N\leq\alpha_1<+\infty$ for all n, N, the authors show that $\mathbf{P}(\mu_{r_1}=k_1,\cdots,\mu_{r_s}=k_s)=\Phi(k_1,\cdots,k_s)\cdot(1+O(N^{-1/2}))$, where $\Phi(k_1,\cdots,k_s)$ is a multivariate normal density whose covariance structure is determined by N and n/N.

R. F. Gundy (Ann Arbor, Mich.)

Abrahamson, I. G. 4129

Orthant probabilities for the quadrivariate normal distribution.

Ann. Math. Statist. **35** (1964), 1685–1703.

Author's summary: "Let x_1, x_2, x_3, x_4 be jointly distributed with a quadrivariate normal distribution with mean 0, and correlation matrix $\{\rho_{ij}\}$. The orthant probability, i.e., the probability that all the x_i's will be simultaneously positive, is not, in general, given by a closed expression, but is easily computed in a special class of cases, here called orthoscheme probabilities. It is explicitly shown how the general orthant probability can be expressed as a linear combination of six orthoscheme probabilities. Orthoscheme probabilities have been tabulated by the author and instructions for the use of this table are given. In addition, an abridged table is appended."

Beckmann, Petr 4130

Rayleigh distribution and its generalizations.

J. Res. Nat. Bur. Standards Sect. D **68D** (1964), 927–932.

In many problems in radio wave propagation the resultant field is formed by the superposition of a number of elementary vectors:

$$Ee^{i\theta} = \sum_{j=1}^{n} E_j e^{i\phi_j}.$$

The distribution of (E, θ) is considered under various assumptions on the vectors (E_j, ϕ_j) and n, including the following cases: (a) the distributions of phases, θ_j, are not uniform, (b) one or more vector terms predominate, (c) n is a random variable. The arguments are informal and non-rigorous (sometimes erroneous).

M. M. Siddiqui (Fort Collins, Colo.)

Feigel'son, T. S. 4131

On a simple method of establishing the independence of statistics. (Russian. English summary)

Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. **19** (1964), no. 3, 157–158.

Author's summary: "Let **P** be a family of distributions of X. The statistic $f(X)$ is named zonal if its distribution is the same for all $P \in \mathbf{P}$. It is proved that if a family of distributions of a sufficient statistic T for **P** is complete, no zonal statistic depends on T. n-dimensional random variables

$$X_i = \begin{vmatrix} X_{i1} \\ \cdot \\ \cdot \\ \cdot \\ X_{in} \end{vmatrix}, \qquad i = 1, \cdots, N;\ N \geq 4n;\ X_i \in N(A, \Lambda),$$

are considered, and a zonal statistic independent of the system of sufficient statistics is constructed."

J. Wolfowitz (Ithaca, N.Y.)

Gebhardt, Friedrich 4132

Generating normally distributed random numbers by inverting the normal distribution function.

Math. Comp. **18** (1964), 302–306.

Polynomial approximations of the inverse function, $\psi(t)$, of the cumulative normal distribution function are provided for use in transforming uniformly distributed random numbers into normally distributed ones. Table 1 gives values, a_n and b_n, of the linear approximation $a_n + b_n t$ of $\psi(t)$ having maximum error $\varepsilon_M = 0.0004$ for $8 \leq n \leq 15$ and $\varepsilon_M = 0.0001$ for $16 \leq n \leq 49$, with $n/100 \leq t \leq (n+1)/100$. Table 2 presents approximations of $\psi(t)$ by polynomials of second, third, and fourth degree. Explanatory comments lead to a statement about the computer program.

P. S. Dwyer (Ann Arbor, Mich.)

Guenther, William C. 4133

Another derivation of the non-central chi-square distribution.

J. Amer. Statist. Assoc. **59** (1964), 957–960.

The author uses spherical coordinates to give a new and relatively short derivation of the non-central chi-square distribution. *H. B. Mann* (Madison, Wis.)

Gumbel, E. J.; Goldstein, Neil 4134

Analysis of empirical bivariate extremal distributions.

J. Amer. Statist. Assoc. **59** (1964), 794–816.

Two types of bivariate distributions whose marginal distributions are extremal are discussed: independent and homotopically dependent. A graphical test of each of these special cases is proposed; $mm'/(n+1)^2$ is plotted against the values $k/(n+1)$ of the cdf of p, where m and m' are the marginal ranks, k is the rank of mm', n is the sample size and the cdf of p is $p(1-\ln p)$ for independence, while it is $1/(2\sqrt{p})$ for the case of homotopic dependence $(m = m')$. Two chi-square tests of independence, one based on the octants about the medians and the other based on equiprobable regions, are presented. These ideas are illustrated by (a) the distribution of oldest ages at death of the two sexes and (b) floods of the same river recorded at two stations. The marginal distributions are extremal of the first type; (a) illustrates independence, while (b) illustrates homotopic dependence. An appendix contains a proof (due to Pickands) that asymptotic independence of the relative ranks implies independence of the variables themselves. *D. R. Barr* (Dayton, Ohio)

Nadaraja, È. A. 4135

Some new estimates for distribution functions. (Russian. English summary)

Teor. Verojatnost. i Primenen. **9** (1964), 550–554.

Given a sequence of independent, identically distributed random variables $x_1, x_2, \cdots, x_n$ with the common distribution function F and continuous density f, the author examines estimates of the form $F_n(x) = \int_{-\infty}^{x} f_n(x')\,dx'$ for $F(x)$, where

$$(*) \qquad f_n(x) = \frac{1}{nh(n)} \sum_{i=1}^{n} K\left(\frac{x - x_i}{h(n)}\right)$$

and K is a density function.

Under suitable conditions on h and K the asymptotic unbiasedness and consistency of the $F_n(x)$ are proved and sufficient conditions on h, K, and f insuring the consistency and asymptotic normality of sample quantiles ζ_p of the form $F_n(\zeta_p) = p$ are found.

Although it is not mentioned explicitly in the Russian text, this paper is based on and follows closely E. Parzen's paper [Ann. Math. Statist. **33** (1962), 1065–1076; MR **26** #841], where a corresponding but more detailed examination of estimates f_n of the form (*) for the density function and sample mode is presented.

{The Lemma of the paper is a corollary of "Scheffé's convergence theorem" [ibid. **18** (1947), 434–438; MR **9**,

83] but only if the asymptotic unbiasedness of f_n, proved in Parzen's paper [loc. cit.], is assumed.}
S. Kotz (Toronto, Ont.)

Rider, Paul R. 4136
Distribution of product and of quotient of maximum values in samples from a power-function population.
J. Amer. Statist. Assoc. **59** (1964), 877–880.
The function

$$\begin{aligned} F(x) &= 0 && \text{for } x<0, \\ &= (x/\xi)^{k+1} && \text{for } 0 \le x \le \xi, \\ &= 1 && \text{for } x \ge \xi \end{aligned}$$

is called by the author the power function distribution. He considers the maxima L_1 and L_2 of two independent samples of size m and n, respectively. The author determines the distribution of the product $u = L_1 L_2$ and of the quotient $v = L_2/L_1$ and discusses also the estimation of the parameters. *E. Lukacs* (Washington, D.C.)

Siddiqui, M. M. 4137
Statistical inference for Rayleigh distributions.
J. Res. Nat. Bur. Standards Sect. D **68D** (1964), 1005–1010.
The content of this paper can be summed up as follows. The Rayleigh distribution, $F(r) = 1-e^{-r^2/\alpha^2}$, $r \ge 0$, is transformed into the exponential distribution $G(z) = 1-e^{-z/\alpha^2}$, $z \ge 0$, by the change of variable $z = r^2$. Hence methods of statistical inference for the exponential distribution are directly translatable into methods for the Rayleigh distribution.
Benjamin Epstein (Palo Alto, Calif.)

Jones, Richard Hunn 4138
Spectral estimates and their distributions. I, II.
Skand. Aktuarietidskr. **1962**, 39–69 (1963); *ibid.* **1962**, 135–153 (1963).
Much of the material in both of these papers is expository and historical. That which is novel is outlined in the author's summary: "Methods of choosing spectral windows for estimating the spectral density of a stationary time series are discussed, and a generalization of the 'hanning' and 'hamming' spectral windows presented. By using the concept of spectral window bandwidth, 'optimal' discrete spectral windows are derived. Spectral estimates are given for stochastic processes, the time points of which are chosen using an additive random scheme, and a method suggested for handling any unequally spaced data. It is shown that when estimating the two-dimensional spectral density of a harmonizable stochastic process (non-stationary), nothing is gained by smoothing the periodograms. Also, a test for stationarity of a periodic harmonizable stochastic process is given. Two methods are presented for the numerical calculation of the frequency function of a quadratic form in normal variates. A modified Laguerre series using a Type III approximation as its weight is used when the eigenvalues of the matrix are non-negative and closely grouped. The second method uses the saddle-point approximation and method of steepest descent to invert the characteristic function, and is useful for any real eigenvalues. An asymptotic expression for the frequency function for large values of the argument is given, which can be used to obtain an upper confidence limit of an estimate."
L. H. Koopmans (Albuquerque, N.M.)

Weiss, Lionel 4139
On the asymptotic joint normality of quantiles from a multivariate distribution.
J. Res. Nat. Bur. Standards Sect. B **68B** (1964), 65–66.
It is shown that the distribution function of a set of quantiles in samples from a k-variate distribution tends to a multivariate normal distribution function as the sample size goes to infinity under the only condition that the parent distribution function is absolutely continuous. Previously, the reviewer [same J. **64B** (1960), 145–150; MR **25** #4591] showed (for $k=2$), with more stringent conditions, that the probability density function of a set of quantiles tends to the probability density function of a normal. *M. M. Siddiqui* (Fort Collins, Colo.)

Bahadur, R. R. 4140
On Fisher's bound for asymptotic variances.
Ann. Math. Statist. **35** (1964), 1545–1552.
The author provides a simplified proof of a result of LeCam [Univ. California Publ. Statist. **1** (1953), 277–329; MR **14**, 998] to the effect that, in the absence of regularity conditions, the Fisher information bound for the variance of an estimate of a parameter θ may be violated on a set of θ values of Lebesgue measure zero.
G. E. Noether (Boston, Mass.)

Gleser, Leon J. 4141
On a measure of test efficiency proposed by R. R. Bahadur.
Ann. Math. Statist. **35** (1964), 1537–1544.
Discussion and generalization of Bahadur's method of stochastic comparisons of tests [same Ann. **31** (1960), 276–295; MR **22** #7201]. *G. E. Noether* (Boston, Mass.)

Hannan, E. J. 4142
The general theory of canonical correlation and its relation to functional analysis.
J. Austral. Math. Soc. **2** (1961/62), 229–242.
In der klassischen Theorie der kanonischen Korrelation wird die lineare Abbildung A eines Raumes von p zufälligen Variablen $x_1, \cdots, x_p$ in einen Raum von q zufälligen Variablen $y_1, \cdots, y_q$ untersucht. Durch Hauptachsentransformation von A gewinnt man sogenannte kanonische Variable. Für A, lineare Funktionen der Variablen x bzw. y und deren Skalarprodukte gelten bestimmte Zerlegungssätze. Diese Ergebnisse werden mit Hilfe von Spektralzerlegungen linearer Operatoren im Hilbertraum verallgemeinert: Es werden zwei Familien $\{x_s, s \in S\}$, $\{y_t, t \in T\}$ von Zufallsgrößen und die kanonische Korrelation zwischen Funktionen dieser Zufallsgrößen betrachtet. Sind die Indexmengen S und T endlich und ist die gemeinsame Verteilung $H(x, y)$ bezüglich der marginalen Verteilungen $M(x)$ und $N(y)$ absolut stetig, so läßt sich der lineare Operator A durch die Radon-Nikodymsche Ableitung $A(x, y) = \partial H(x, y)/\partial M(x)\partial N(x)$ ausdrücken. Die

im Falle Gaußscher Prozesse bekannte Formel für die in den x-Variablen enthaltene Information über die y-Variablen wird verallgemeinert. Für den Fall unendlicher Indexmengen wird die Theorie für stationäre Gaußsche Prozesse und aufeinanderfolgende Beobachtungen eines vektoriellen stationären Gaußschen Prozesses entwickelt.
M. Peschel (Zbl 107, 351)

Khatri, C. G. 4143
Some more estimates of circular probable error.
J. Indian Statist. Assoc. 1 (1963), 40–47.

Let $\{X_i\}$ ($i=1, \cdots, n$) be NID with mean zero, variance σ^2, and let $F(r)$ denote the distribution of $r=(\sum_{i=1}^n X_i^2)^{1/2}$. The circular probable error (CEP) is given by the root of the equation $F(r)=0.5$ and is therefore a linear function of σ. The author lists eight estimates of CEP that have previously been given; in this paper, two additional estimates are given. These depend on (1) Gini's mean difference, and (2) mean square successive differences of the X_i.

For the second of these new estimates, the author derives the distribution of the estimate, obtaining it in infinite series form involving Laguerre polynomials, thus enabling him to determine the appropriate multiplier so that the estimator is unbiased. He then tabulates the variance and efficiencies (relative to the minimum variance unbiased estimate) of these two estimates.
D. G. Chapman (Seattle, Wash.)

Pratt, John W. 4144
Robustness of some procedures for the two-sample location problem.
J. Amer. Statist. Assoc. 59 (1964), 665–680.

Author's summary: "The level of ordinary two-sample procedures is not preserved if the two populations differ in dispersion or shape. The effect of such differences, especially differences in dispersion, on the t, median, Mann-Whitney, and normal scores procedures is investigated asymptotically, and tables are given comparing the four procedures." *W. Hoeffding* (Calcutta)

Rothenberg, Thomas J.; Fisher, Franklin M.; Tilanus, C. B. 4145
A note on estimation from a Cauchy sample.
J. Amer. Statist. Assoc. 59 (1964), 460–463.

In the estimation of the centre of the Cauchy distribution, it is well known that the sample-mean is an inconsistent estimator, the sample-median is consistent but inefficient, and the maximum likelihood estimator is asymptotically efficient but difficult to calculate.

The authors propose a class of estimators based on the sample order statistics. Each estimator in the class is the arithmetic average of a central subset of the sample of order statistics. Although the sample-median is a member of the proposed class, it is not the most efficient. It is shown that the average of roughly the middle quarter of the ordered sample has the lowest asymptotic variance, and that the use of this estimator in place of the median eliminates nearly half of the efficiency loss.
V. S. Huzurbazar (Poona)

Meredith, William 4146
Canonical correlations with fallible data.
Psychometrika 29 (1964), 55–65.

The canonical correlation coefficient measures the extent to which a linear form of the p variables, x, can be correlated with a linear form of the q variables, y. Even though this correlation for the linear forms of x and y were unity, appreciable errors in the observed x and y would presumably result in a correlation less than unity. In an attempt to compensate for this shrinkage, a correlation coefficient is said to be corrected for attenuation when it is divided by the product of the square roots of the two reliabilities. The author determines sets of weights so as to maximize the correlation between the linear composites after correction for attenuation in the composites and finds that the results may be obtained by determining the canonical correlations and canonical regression weights between the true score components of the measures making up the two sets. In addition, formulae are developed for calculating the correlations between the canonical variates and original measures, both corrected and uncorrected for attenuation. A numerical illustration is provided.
P. S. Dwyer (Ann Arbor, Mich.)

Nadaraja, È. A. 4147
On a regression estimate. (Russian. English summary)
Teor. Verojatnost. i Primenen. 9 (1964), 157–159.

For the bivariate random variable (X, Y), the regression of Y on X is $\bar{y}(x)=E(Y|x)$. Starting with the family of estimators for a probability density studied by Parzen [Ann. Math. Statist. 33 (1962), 1065–1076; MR 26 #841], the author proposes the estimator $\bar{y}_n(x)=\sum_1^n c_i(x)y_i/\sum_1^n c_i(x)$, where $c_i(x)=K((x-x_i)/h(n))$, $K(x)$ is a density function satisfying conditions similar to Parzen's, and $h(n)\to 0$ as $n\to\infty$. Three theorems on asymptotic properties of $\bar{y}_n(x)$ are stated.
J. R. Rosenblatt (Arlington, Va.)

Telser, Lester G. 4148
Iterative estimation of a set of linear regression equations.
J. Amer. Statist. Assoc. 59 (1964), 845–862.

This paper deals with the problem of estimating a set of interdependent linear regression equations. It has been assumed that the explanatory variables are nonstochastic and the disturbances of at least one pair of equations are correlated. Koopmans-Rubin-Leipnik's one-at-a-time least squares estimates are efficient if exactly the same explanatory variables (with the same observations) enter into each of the equations. In case the explanatory variables are different in at least one pair of equations and for this pair the disturbances are correlated, Zellner proposed an estimator based on Aitken's generalized least squares method. He has shown the gain in efficiency by use of this method.

The author considers the case when either (a) different explanatory variables enter various equations or (b) if the same explanatory variables enter, then the observations are different. An iterative method of deriving the coefficient estimates is proposed. The iterative estimates obtained in this way possess the same asymptotic properties as the estimates given by Zellner. Although Zellner's estimates are computationally simpler, the analysis of the iterative

procedure shows how the relations among the residuals of various equations can be exploited to obtain asymptotically efficient estimates. *A. L. Nagar* (Delhi)

Rossberg, Hans-Joachim 4149

Über das asymptotische Verhalten der Rand- und Zentralglieder einer Variationsreihe. I. (Russian summary)

Magyar Tud. Akad. Mat. Kutató Int. Közl. 8 (1963), 463–468 (1964).

The author proves the following result. Two order statistics x_h and x_k of a continuous variable are asymptotically independent if $\lim_{n\to\infty} (h/k) = 0$. The same holds for x_{n+1-h} and x_{n+1-k}. The proof follows a method of Rényi [Acta Math. Acad. Sci. Hungar. 4 (1953), 191–231; MR 15, 885] and is based on the exponential distribution of $y = -\lg[1-F(x)]$. It eliminates the condition on the initial variable made previously by this reviewer [Ann. Math. Statist. 17 (1946), 78–81; MR 7, 464]. *E. J. Gumbel* (New York)

Krem, Alajos 4150

On the independence in the limit of extreme and central order statistics. (Russian summary)

Magyar Tud. Akad. Mat. Kutató Int. Közl. 8 (1963), 469–474 (1964).

This paper gives another proof for the statement in the previous review [#4149]. The asymptotic independence of the order statistics does not hold for the central members of a normal distribution. *E. J. Gumbel* (New York)

Snow, Barbara A. S. 4151

The distribution of Kendall's tau for samples of four from a normal bivariate population with correlation ρ.

Biometrika 50 (1963), 538–540.

The only possible values of tau, in samples of four, are the seven values -1 (1/3) 1. The permutations of 1234 having values are shown in Table 1. The frequencies of positive tau as a power series in $\theta_1 = \sin^{-1}\rho$ and $\theta_2 = \sin^{-1}\frac{1}{2}\rho$ are given in Table 2, and the power series in ρ in Table 3, where the population is normal bivariate. A sampling experiment with $n=4$, $\rho = 1/\sqrt{2}$ is used as an illustration. *P. S. Dwyer* (Ann Arbor, Mich.)

Pearson, E. S.; Stephens, M. A. 4152

The goodness-of-fit tests based on W_N^2 and U_N^2.

Biometrika 49 (1962), 397–402.

The authors give some results on the performance in small samples of the Cramér-von Mises-Smirnov type of statistics

$$W_N^2 = N\int_{-\infty}^{\infty} [F_N(x)-F(x)]^2\,dF(x),$$

$$U_N^2 = N\int_{-\infty}^{\infty}\left\{F_N(x)-F(x)-\int_{-\infty}^{\infty}[F_N(y)-F(y)]\,dF(y)\right\}^2 dF(x),$$

the latter being a statistic introduced by G. S. Watson [Biometrika 48 (1961), 109–114; MR 24 #A1777; ibid. 49 (1962), 57–63; MR 25 #1626], in discussing goodness-of-fit tests on a circle. In the expressions above, $F(x)$ is the theoretical cumulative distribution function (c.d.f.) of the random variable x, and $F_N(x)$ the sample c.d.f. for a sample of size N.

Percentage points of W_5^2 and W_{10}^2 have been obtained from Johnson's four-parameter S_B curves having the correct first four moments. Large-scale Monte Carlo sampling has been used to derive percentage points of W_{10}^2 and U_{10}^2. These are compared, respectively, with the percentage points of W_∞^2 and U_∞^2. A relationship between the cumulants of W_∞^2 and U_∞^2 has been established. *I. M. Chakravarti* (Chapel Hill, N.C.)

Stephens, M. A. 4153

The distribution of the goodness-of-fit statistic U_N^2. I.

Biometrika 50 (1963), 303–313.

The goodness-of-fit statistic

$$U_N^2 = N\int_{-\infty}^{+\infty}\left\{F_N(x)-F(x)-\int_{-\infty}^{\infty}[F_N(y)-F(y)]\,dF(y)\right\}^2 dF(x)$$

introduced by Watson [Biometrika 48 (1961), 109–114; MR 24 #A1777] is independent of the origin and hence can also be used to test the randomness on a circle. On the null hypothesis that $F(x)$ is the true probability function, U_N^2 is distribution-free. Therefore, $F(x)$ may be assumed to be the uniform distribution (0, 1). For the computation the sample value $F_N(x_i)$ is replaced by $(i-0.5)/N$, where i is the rank. {This reviewer objects, and prefers the expected value $i/(N+1)$ of $F_N(x_i)$.} The author gives the first four moments of U_N^2 for any N, the exact probability function for $N = 1, 2, 3$, its lower tail for $N > 3$ and associated tables of moments and significance points. They indicate rapid convergence to the asymptotic distribution. The derivations, based on geometric interpretations, are very complicated. *E. J. Gumbel* (New York)

Pearson, E. S. 4154

Comparison of tests for randomness of points on a line.

Biometrika 50 (1963), 315–325.

This article is a sequel to the preceding one [#4153]. In addition to U_N^2, it lists three other similar goodness-of-fit tests, namely,

$$W_N^2 = N\int_{-\infty}^{+\infty}[F_N(x)-F(x)]^2\,dF(x),$$

$$D_N = \sup_{-\infty\le x\le\infty} |F_N(x)-F(x)|,$$

$$V_N = \sup_{-\infty\le x\le\infty}\{F_N(x)-F(x)\} - \inf_{-\infty\le x\le\infty}\{F_N(x)-F(x)\}.$$

Their distribution on the null hypothesis is distribution-free. Therefore, it is sufficient to consider the distribution of points on a line. The assumption for $F_N(x)$ is the same as in the previous article. The results of the four tests are compared for different sets of data which are not independent, warp breaks on a loom, times of road accidents, fatal explosions in a coal mine, and the lengths of reigns of English kings and queens; finally, artificial series with changing expectations. The U_N^2 and V_N tests turn out to

be more sensitive for departures from randomness than $W_N{}^2$ and D_N. However, as the author states, such numerical work cannot establish anything final.

E. J. Gumbel (New York)

Čibisov, D. M. 4155

On the asymptotic power of goodness-of-fit tests for near alternatives. (Russian. English summary)

Teor. Verojatnost. i Primenen. **9** (1964), 561–562.

Let $G_n{}^*(u)$ be the empirical distribution function (df) of a sample of size n from a df $G_n(u)$ on $[0, 1]$, and let $\beta_n(u) = n^{1/2}(G_n{}^*(u) - u)$. Let $\beta(u)$ be the Gaussian process with $M\beta(u) = 0$, $M\beta(u)\beta(v) = \min(u, v) - uv$. Suppose that $n^{1/2}(G_n(u) - u) \to \delta(u)$ as $n \to \infty$. Sufficient conditions are given for $\beta_n(u)$ to converge to $\beta(u) + \delta(u)$, in a sense defined by Prohorov [Teor. Verojatnost. i Primenen. 1 (1956), 177–238; MR 18, 943].

W. Hoeffding (Calcutta)

Fellingham, S. A.; Stoker, D. J. 4156

An approximation for the exact distribution of the Wilcoxon test for symmetry.

J. Amer. Statist. Assoc. **59** (1964), 899–905.

The authors approximate the distribution of the Wilcoxon test statistic by means of an Edgeworth series. They compare their approximation and the normal approximation and conclude that the approximation by an Edgeworth series is more accurate.

E. Lukacs (Washington, D.C.)

Milton, Roy C. 4157

An extended table of critical values for the Mann-Whitney (Wilcoxon) two-sample statistic.

J. Amer. Statist. Assoc. **59** (1964), 925–934.

A table of critical values for the Mann-Whitney two-sample statistic is given for sample sizes $n = 1(1)20$, $m = 1(1)40$ and significance levels .0005, .0025, .005, .001, .01, .025, .05, .1.

E. Lukacs (Washington, D.C.)

Lindley, Dennis V. 4158

The Bayesian analysis of contingency tables.

Ann. Math. Statist. **35** (1964), 1622–1643.

The author describes how data from a multinomial distribution, and, in particular, data in the form of a contingency table, may be studied by using a prior distribution of the parameters and expressing the results in the form of a posterior distribution, or some aspects thereof, of the parameters. The analysis used depends on the prior distribution, and the form given in Theorem 1 only applies to a certain type of prior knowledge. Theorem 1: If the random variables $n_1, n_2, \cdots, n_k$ have a multinomial distribution with parameters $\Theta_1, \Theta_2, \cdots, \Theta_k$; and if the prior distribution of the Θ_i has density proportional to $(\prod \Theta_i)^{-1}$ over the region $\Theta_i \geq 0$, $\sum \Theta_i = 1$; then if the constants a_{pi} ($p = 1, 2, \cdots, m$; $i = 1, 2, \cdots, k$; $m < k$) satisfy $\sum_i a_{pi} = 0$, the joint posterior distribution of the contrasts $\sum_i a_{pi} \ln \Theta_i$ ($p = 1, 2, \cdots, m$) is approximately normal with means $\sum_i a_{pi} \ln n_i$ and covariances $\sum_i a_{pi} a_{qi} n_i{}^{-1}$. The author gives reasons for believing the type of prior distribution used to be of frequent occurrence. For application to contingency tables the author proves Theorem 2: If the prior distribution of Θ_{ij} ($i = 1, 2, \cdots, r$; $j = 1, 2, \cdots, s$) is proportional to $\prod_{ij} \Theta_{ij}{}^{-1}$, then the prior distribution of Θ_i and $\Phi_{ij} = \Theta_{ij}/\Theta_i$ is proportional to $\prod_i \Theta_i{}^{-1} \prod_{i,j} \Phi_{ij}{}^{-1}$. The author examines the connection between the methods he proposes and the analysis of variance, particularly with a view to simplifying the analysis of contingency tables involving three or more factors.

S. Kullback (Washington, D.C.)

Zellner, Arnold; Tiao, George C. 4159

Bayesian analysis of the regression model with autocorrelated errors.

J. Amer. Statist. Assoc. **59** (1964), 763–778.

The title subject is developed in the spirit of Jeffreys with locally uniform and independent distributions for the parameters.

I. R. Savage (Tallahassee, Fla.)

Nef, Walter 4160

Über die Differenz zwischen theoretischer und empirischer Verteilungsfunktion.

Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **3**, 154–162 (1964).

Let $X_1, \cdots, X_n$ be a sample of size n taken from a population with distribution function $F(x)$, and denote the corresponding empirical distribution by $S_n(x)$. Let r be an integer and write $Z_n(r)$ for the number of values X for which $S_n(x) - F(x) = r/n$. The author determines the distribution of the random variable $Z_n(r)$. His result can be used to obtain Smirnov's limit distribution.

E. Lukacs (Washington, D.C.)

Hanani, Haim 4161

The existence and construction of balanced incomplete block designs.

Ann. Math. Statist. **32** (1961), 361–386.

It is well known that the following conditions are necessary and sufficient for the existence of BIB designs: $\lambda(v-1) \equiv 0 \bmod (k-1)$ and $\lambda v(v-1) \equiv 0 \bmod k(k-1)$. The author shows that, for $k = 3$ and $k = 4$, and also in several other cases, these conditions are also sufficient.

L. Mešalkin (RŽMat 1962 #5 A128)

Shrikhande, S. S. 4162

Generalized Hadamard matrices and orthogonal arrays of strength two.

Canad. J. Math. **16** (1964), 736–740.

The author indicates some relationships among generalized Hadamard matrices, group divisible designs, affine resolvable balanced incomplete block designs and orthogonal arrays of strength two. Several theorems which pertain to the construction of orthogonal arrays are also presented.

S. Addelman (Durham, N.C.)

Paulson, Edward 4163

Sequential estimation and closed sequential decision procedures.

Ann. Math. Statist. **35** (1964), 1048–1058.

A sequential procedure for estimating the mean of a normal distribution is given and the results applied to the problem of deciding which of k disjoint intervals contains the mean. Sequential confidence limits are derived for the variance and the ratio of variances of normal distributions.

The results are applied to get closed decision procedures in (a) testing a hypothesis on the mean or variance of a normal distribution, (b) comparing the means or variances of k experimental categories with a standard, (c) testing hypotheses about the ratio of variances. A discussion about the construction of confidence intervals for a parameter after a decision has been reached is provided.
M. Katz (Detroit, Mich.)

Phatarfod, R. M. 4164

Large sample sequential analysis of Markovian observations.

J. Indian Statist. Assoc. 1 (1963), 152–160.

Asymptotic sampling properties of Wald's sequential probability ratio test are obtained where (a) the two hypotheses are close, requiring a large sample for discrimination, and (b) the successive observations form an ergodic Markov chain. Hypotheses depending on one and two parameters are discussed. The results extend those of Bartlett [Proc. Cambridge Philos. Soc. **42** (1946), 239–244; MR 8, 471], who derives properties assuming that the observations are independent. *M. Katz* (Detroit, Mich.)

Grettenberg, Thomas L. 4165

The ordering of finite experiments.

Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (*Liblice*, 1962), *pp.* 193–206. *Publ. House Czech. Acad. Sci., Prague*, 1964.

The author considers the expected-loss criteria in decision problems for comparing experiments with a common input space and with different input spaces. Of the two experiments P and Q, suitably defined in terms of a Markov matrix with a common input space, P is preferred to Q if each loss vector attainable with Q is also attainable with P. If P and Q are comparable, then the expected values of their information measures I_P and I_Q are also comparable. Several equivalent conditions are reviewed relating to a common input space and also to input spaces of unequal dimensions. Then various implications of the expected-loss comparison are discussed. In particular, it is shown that a discriminatory-information inequality may also be obtained from the expected loss comparison.
M. Atiqullah (Dacca)

Pathak, P. K. 4166

On sampling schemes providing unbiased ratio estimators.

Ann. Math. Statist. **35** (1964), 222–231.

Ratio estimators are valuable in sampling problems where the value of an auxiliary variable is known for the entire population. Thus, if one samples Y and W, where W is the auxiliary random variable, an estimate of the ratio $R=\sum Y/\sum W$, where the sums are taken over the entire population, together with the known value $\sum W$, can be used to estimate $\sum Y$. The author extends a result of Nanjamma, Murthy, and Sethi [Sankhyā **21** (1959), 299–314; MR **22** #3086], who discussed a ratio estimator $\hat{R}=F/G$ which is a ratio of estimates of $\sum Y$ and $\sum W$ and would be biased under the original sampling scheme, but is unbiased if the sampling scheme is modified slightly. A uniformly better unbiased estimator, the conditional expectation of $\hat{R}$ given a sufficient statistic, is shown to be equal to F_T/G_T, where F_T and G_T would be the conditional expectations of F and G under the original sampling scheme. This result is applied with sophistication to a number of sampling schemes, including sampling with equal and unequal probabilities (with replacement), stratified sampling and two-stage sampling. The key to the easily derived result and its main contribution lies in the set-up which, partly because of its generality, is vaguely stated and was difficult for the reviewer to comprehend. For example, it is not made clear in the original description of the set-up that the sample ω may be described by the frequencies with which each element of the population is observed, ignoring the order in which they are observed. But this is precisely the case for the simplest applications. The main idea seems to have considerable potential value. *H. Chernoff* (Stanford, Calif.)

Roberts, Charles DeWitt 4167

An asymptotically optimal fixed sample size procedure for comparing several experimental categories with a control.

Ann. Math. Statist. **35** (1964), 1571–1575.

Author's summary: "The basic problem considered here involves k experimental categories. The experimenter must decide none of the k categories is better than the control or decide a certain category is better. For this problem a fixed sample size procedure δ_n^* is given. With a definite loss function and a cost $c>0$ per observation, δ_n^* and other fixed sample size procedures are compared in a certain asymptotic sense as $c\to 0$. In particular, δ_n^* is shown to be an optimal fixed sample size procedure in this asymptotic sense. By appealing to asymptotic results the procedure δ_n^* is compared with sequentially designed procedures."

Seber, G. A. F. 4168

The linear hypothesis and large sample theory.

Ann. Math. Statist. **35** (1964), 773–779.

Let us consider n independent observations $x_1, \cdots, x_n$ from a population with a density $f(x, \theta)$, where $\theta=(\theta_1, \theta_2, \cdots, \theta_s)$ is an unknown vector with real parameters belonging to a subset Ω of s-dimensional Euclidean space. Further, let ω_1, ω_2 be $(s-r)$-dimensional subsets of Ω and suppose that the hypotheses $\theta\in\omega_i$ have to be tested. Following Aitchison [J. Roy. Statist. Soc. Ser. B **24** (1962), 234–250; MR **25** #3586] the author calls the hypotheses ω_1, ω_2 separable with respect to Ω if and only if the critical region for testing $\omega_i\cap\omega_j$ against $\omega_j-\omega_i\cap\omega_j$ is the same as that for testing ω_i against $\Omega-\omega_i$. Assuming that the sets ω_i are defined in terms of freedom equations, i.e., $\omega_i=\{\theta:\theta\in\Omega, \theta=\theta_i(\beta_i)\}$, where $\theta_i(\beta_i)$ is a function of an $(s-r_i)$-dimensional vector β_i, the author demonstrates asymptotic equivalence of the hypotheses considered as certain linear hypotheses and derives sufficient conditions for the separability of ω_1 and ω_2.
J. Machek (Prague)

Gardner, L. A., Jr. 4169

Adaptive predictors.

Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (*Liblice*, 1962), *pp.* 123–192. *Publ. House Czech. Acad. Sci., Prague*, 1964.

When the covariance function of a second-order stationary time series is known, then the optimum linear predictor based on the "last" p observations is known and the p coefficients can be obtained by inverting an appropriate matrix. The author is concerned with estimating the optimum coefficients when the covariance function is unknown and observations are taken sequentially. He discusses the asymptotic behavior of three methods, the first of which is to estimate at time n the covariance function and then use it as if it were the "true" covariance function, while the other two are recursive schemes which avoid the complexity of matrix inversion. These recursive schemes are stochastic approximation procedures which are closely related to the generalizations of the Robbins-Monro procedure in p-dimensions (see, for example, this reviewer [Ann. Math. Statist. **29** (1958), 373–405; MR **20** #4886]) but differ, primarily, in that the errors are not conditionally independent as they usually are in the literature of stochastic approximation. However, the author is able to successfully adapt the methods of previous authors, such as in the above reference, to prove convergence and asymptotic normality in the present context.

J. Sacks (Evanston, Ill.)

Jorgenson, Dale W. 4170

Minimum variance, linear, unbiased seasonal adjustment of economic time series.

J. Amer. Statist. Assoc. **59** (1964), 681–724.

The time series $\{y_t\}$ is assumed to satisfy the model $y_t = d_t + s_t + \varepsilon_t$, $t = 1, \cdots, N$, where $\{d_t\}$ is a deterministic trend, $\{s_t\}$ is a seasonal component and $\{\varepsilon_t\}$ is a series of uncorrelated errors with zero mean and constant variance. One method of estimating the seasonal component is to filter the series by means of a filter L designed to eliminate as much of the trend as possible. The filtered seasonal component $\{Ls_t\}$ is efficiently estimated by Aitken's generalised least squares, and from this an estimate of the original component $\{s_t\}$ is obtained. An alternative method is to fit the model $y_t = d_t' + s_t + \varepsilon_t$, where d_t' is the general solution of the difference equation $Ld_t' = 0$, by ordinary least squares. The paper demonstrates that the two estimates are identical and discusses some implications.

J. Durbin (London)

Šeff', O. [Šefl, O.] 4171

Some questions of automatic control of processes with unknown characteristics. (Russian)

Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (*Liblice*, 1962), pp. 611–620. *Publ. House Czech. Acad. Sci., Prague*, 1964.

The author discusses some well-known examples of efficient estimates and obtains such estimates for the unknown parameters a_1 and a_2 defined by the model $E(Y_i) = x_{i-1} + a_1 + ia_2$ $(i = 1, \cdots, n)$, where the Y_i's are independent normal random variables with a common variance and the x_i's are the observed (input) values. These estimates are then used for estimating the optimal control values in a special type of discrete control process.

S. Kotz (Toronto, Ont.)

NUMERICAL METHODS

See also 3445, 3674, 3675, 3676a–b, 3808, 3809a–b, 3846, 3883, 3892, 4236, 4282, 4637.

★Numerical methods and programming. I 4172

[Вычислительные методы и программирование. I].

A collection of papers from the Computing Center of Moscow University. Edited by N. P. Trifonov, G. S. Rosljakov, and E. A. Žogolev.

Izdat. Moskov. Univ., Moscow, 1962. 350 pp. 1.30 r.

This collection is in three parts: (I) 13 papers on some questions of a mathematico-computational sort, relating mostly to problems cast as differential equations with finite differences figuring prominently in many of the numerical solution methods; (II) 8 papers describing standard numerical algorithms and programs commonly used for solving typical problems on the STRELA at the Moscow State University Computing Center; and (III) 5 notes on standard routines for the computation of five elementary functions. The book's style is primarily expository with the emphasis on "how to do it", with the more theoretical papers, with a few exceptions, having more of a physics or engineering physics computational, rather than a mathematical, flavor with definitions, theorems, and the like. Much of the work is reasonably standard in computing circles, although the material in Part I seems to be more up to date than in the other parts and is definitely much more sophisticated and mathematical. Most of the papers cite results of computational work done on the STRELA.

The titles of the papers of Part I (and the other parts) describe quite well their contents therefore and, because of the number of papers as well, comments will be brief. B. M. Budak and A. D. Gorbunov, "On Multipoint Difference Methods for the Solution of the Cauchy Problem" for the equation $y' = f(x, y)$, discuss and present in the form of theorems questions of convergence and bounds on errors in the use of these methods (cf. P. Henrici [*Discrete variable methods in ordinary differential equations*, Wiley, New York, 1962; MR **24** #B1772], with this and part of the following two papers). N. S. Bahvalov, "On the Accumulation of Computational Errors in the Numerical Solution of Differential Equations", sketches error estimation procedures for the numerical solution of the Cauchy problem for ordinary differential equations, a two-point boundary-value problem, and the one-dimensional heat diffusion problem. In his second paper, "On the Determination of the Initial Step and an Estimate of the Principal Error Term in Numerical Integration with Automatic Step Selection", Bahvalov applies this type of analysis to the Cauchy problem for ordinary differential equations with particular interest in the use of one-step methods such as those of Runge-Kutta type. N. M. Korobov, "On an Application of Number-Theoretic Grids", is concerned with the evaluation of multiple integrals of periodic and non-periodic functions by finite sums of the integrand evaluated at non-uniformly spaced points determined by number-theoretic considerations which enhance the reduction of the order of the truncation error. Numerical results with several examples in 4 dimensions with a method of non-uniform nets and a method of "optimal coefficients" are compared with Monte Carlo computations.

A. B. Bakušinskiĭ and V. K. Vlasov, "Computation of

Exciton Energy Levels with the Aid of a Continuum-dimensional Integral", are concerned with the application of concepts of M. Kac [Proc. Second Berkeley Sympos. Math. Statist. and Prob. (1950), pp. 189–215, Univ. California Press, Berkeley, Calif., 1951; MR 13, 568] and R. P. Feynman [Rev. Modern Phys. 20 (1948), 367–387; MR 10, 224] to the estimation of the lower quantum-mechanical energy levels of exciton (of interest in the study of semiconducting crystals). Monte Carlo methods to obtain random paths for Feynman integrals are used. Details of the method and program as well as numerical results are given. In a second paper, "On a Method of Numerical Solution of the Dirichlet Problem for Laplace's Equation", they apply an approximation to some old formulas of Krylov and Bogoljubov [Dokl. Akad. Nauk SSSR A 1929, 283–288] which solve the interior and exterior Dirichlet problems where the boundary is a smooth convex curve. Some numerical results on examples with circular and elliptical boundaries with relatively simple boundary functions were obtained on a STRELA computer and are reported in the paper. (See also L. V. Kantorovič and V. I. Krylov [*Approximate methods of higher analysis* (Russian), third edition, GITTL, Moscow, 1950; MR 13, 77; English transl., Interscience, New York, 1955; MR 21 #5268] for a very similar but more detailed description of the same method.)

R. M. Džabar-zade, "The Schwarz Alternating Method for the Solution of the Dirichlet Problem for Nonlinear Equations", establishes the validity of the method for the sum of two regions, and then applies some results of Motzkin and Wasow [J. Math. and Phys. 31 (1953), 253–259; MR 14, 693] to the question of difference equation analogues of the problem, closing with an application to a system of Monge-Ampère type arising in weather prediction. The paper is thoroughly mathematical with no numerical results. N. P. Židkov, A. A. Kornelčuk, A. L. Krylov, and S. B. Mostinskaja, in "Plane-Parallel Flow of a Viscous Fluid Between Rotating Cylinders" considering the cylinders to be infinite in the axial direction, introduce finite-difference approximations to the Navier-Stokes equations which are to be solved by the "metod progonki" (sweep method, or driving-through method) [see Berezin and Židkov, *Computational methods* (Russian), Vol. 2, Fizmatgiz, Moscow, 1959; MR 22 #12685]. Some STRELA-obtained work is discussed. I. Ju. Brallovskaja and L. A. Čudov, in "The Solution of Boundary Layer Equations by Difference Methods", apply standard techniques to a computationally relatively difficult (because of a boundary condition at infinity) problem and present the most extensive numerical results in the entire book.

S. M. Belonosov, A. P. Pavlenko, B. M. Pavlov, and G. S. Rosljakov, in "A Transverse Blow on a Membrane with a Circular Hole", present primarily an engineering type of paper featuring the basic partial differential equations describing the transverse wave motions, the equations of the characteristics, numerical results, diagrams and graphs. E. A. Žogolev, N. P. Trifonov, and D. N. Šahsuvarov, "Computation of Electromagnetic Fields in Laminar Media", investigate the steady field of a dipole in a sheaf of a homogeneous, isotropic, flat plates of varying thickness. The solution involves extensive calculation of infinite integrals whose integrands contain Bessel functions. Derivations, some description of the program, numerical results, graphs, and discussion are given. A. G. Svešnikov, I. P. Kotik, and Ju. S. Černyšev, "On a Method for Calculating the Matchings of Plane Waveguides", are concerned with numerical methods for solving Maxwell's equations with boundary conditions for radio waves traveling in a region determined by two semi-infinite waveguides in the shape of rectangular parallelepipeds joined by a piecewise smooth non-regular coupler. The technique is to use essentially a variation-of-parameters method to determine ordinary differential equations for the longitude varying amplitudes of the component waves. Again, some numerical results are cited. V. V. Voevodin in the final paper of this section, "Computation of the First Eigenfunction and Eigenvalue of the Hill Operator", is concerned with the equation

$$y'' + (\lambda - 2\phi(Z))y = 0,$$

where ϕ is given as a convergent Fourier cosine series simplified for computation purposes to only three terms, leading to an infinite positive definite symmetric matrix with only seven centrally clustered non-vanishing diagonals, which is truncated to one of 19th order for computation.

Part II concerns descriptions of standard algorithms for determining roots of polynomials, determining the characteristic polynomial of a matrix, determining the eigenvalues and eigenvectors of a symmetric matrix by the Jacobi rotation method, integration of ordinary differential equations by the Runge-Kutta method, and, for those of second order, by Störmer's method, interpolation and numerical differentiation, Aitken interpolation, and the computation of Bessel functions.

Part III deals with standard algorithms for computing e^x, $\ln x$, $\sin x$, $\tan x$, and $\arcsin x$, essentially by approximations using Chebyshev polynomials rather than Taylor series to economize on the number of terms needed.

M. L. Juncosa (Santa Monica, Calif.)

Henrici, Peter 4173

★**Elements of numerical analysis.**

John Wiley & Sons, Inc., New York-London-Sydney, 1964. xv + 328 *pp.* $8.00.

This book marks a new level of excellence for introductory texts in numerical analysis. One reason for this is that the author does not try to squeeze in a large number of techniques, but rather takes a few selected topics and studies them with care and in some depth and detail. One may perhaps quibble with the particular selection, but this approach is especially sound in numerical analysis. More fundamental, however, is that the development is placed on scientific and mathematical foundations. Thus a clear distinction is made between algorithms and things we know about them and, more generally, between things we can prove and things we suspect are true.

It is significant that the first section is entitled "What is numerical analysis?". The author obviously believes that it is a science and not an art. He notes that this is not generally the impression given by many otherwise excellent texts and by many instructors who are not primarily numerical analysts. Nor is this belief widespread among mathematicians. Thus a major contribution of this book is to go down toward the foundations that this science must have. Few mathematicians or even numerical analysts seem to be aware of how deep these foundations are or how far we are from them. The author points out

the gross inadequacy of classical and modern mathematics to describe many phenomena in computation. Such fundamental things as stability, conditioning, error propagation and numerical convergence are still not well-defined and hence not understood.

In spite of the fact that this book is entitled "elements", it reaches the frontiers of knowledge with supriaing frequency. This is not done artificially and it is symptomatic of the youth of this science that so many basic, but unanswered, questions present themselves in an elementary text. The author has exploited this fact to present "research problems" at the end of some chapters. A reasonably bright student can actually explore the frontiers of knowledge and gain a taste of the excitement of research. One can only regret that there are not more of these problems. There is also some recent material in the text. Such things as Romberg integration, Müller's method, round-off propagation, Steffensen iteration and the QD algorithm are fitted naturally into the presentation.

In view of the above discussion, one wonders if the prerequisites suggested in the preface are adequate. The development does not depend on any body of material beyond the suggested analytical geometry, calculus and differential equations (though Dedekind cuts are mentioned near the end). However, the student must have enough maturity to handle the concepts of limit and convergence from the beginning.

Among the special features of this book is the widespread and natural use of extrapolation to the limit. This is well done and is placed in its proper perspective. This reviewer heartily applauds the deëmphasis of the difference calculus. It is used only where it is useful and natural, which is to say, not too often. Some authors tend to develop this calculus for its own sake, far beyond the point of usefulness or interest span of the students. The treatment of differential equations is made entirely with Taylor's series. Finally, the author discusses and encourages the experimental study of numerical analysis. This approach is in disrepute among mathematicians, but so little is known about numerical analysis that it is essential to the rapid development of this science. He notes that this is a standard tool in many sciences. Indeed, when a patient takes a drug, no one can prove it will not kill him, one can merely point to the fact that it did not kill the first 1000 people to take it.

For use as a text, the exposition is generally clear and well motivated. There are frequent pertinent examples. There is a large number of problems in addition to the "research" problems. There is an adequate index and bibliography. The book has a pleasant appearance, good figures and the usual number of misprints. The quality of printing is less than one would expect from this publisher. There are several instances in the reviewer's copy of print showing through the back of a page to an objectionable extent and there is an instance of badly broken type on page 188.

A brief description of the contents follows. The introduction contains three chapters. The second presents an elementary account of the properties of complex numbers, polynomials in the complex field and the complex exponential function. The third considers first-order difference equations.

Part One—Solution of Equations—contains five chapters. Chapter 4 considers one equation in one unknown and presents the method of successive substitutions, Newton's method, Steffensen's method and related topics. The next chapter considers systems of equations and the development of the material in parallel with Chapter 4 is well done. Contraction mappings are introduced to discuss successive substitutions. Chapter 6 considers linear difference equations of second order in detail and then neatly extends the results to nth-order systems. Some instructors will find more material on difference equations in this book than they would want to present in a course. The next two chapters present two methods for solving polynomial equations. The first, Bernoulli's method, is analysed in detail. The second, the QD algorithm, seems out of place. Not that it is unimportant, but the author simply cannot analyse it at the level of this book. Although proofs of stated facts are carefully referenced, it does go counter to the general philosophy of the book. The author feels that the algorithm is important enough to warrant its presentation even though its theoretical background cannot be fully exposed. This material illustrates vividly the closeness of the frontiers of numerical analysis. This chapter also discusses briefly the fusing of two (or more) algorithms to obtain an algorithm which is efficient for a large class of problems. Such fusing deserves more attention in numerical analysis.

Part Two—Interpolation and Approximation—considers interpolation by polynomials and approximation of derivations, integrals and differential equations. Chapter 9 considers classical interpolation theory, and though a deep analysis is not possible at this level, the possible pitfalls of interpolation by polynomials of high degree are pointed out. One wonders why a better reference was not suggested for further reading on this topic. It is refreshing to see an author differentiate between a polynomial and its representations. The next two chapters develop various methods of computing interpolating polynomials, using either ordinates or differences. There is a nice discussion of throwback. Chapter 11 presents approximations for derivatives and contains a nice presentation of the general method of extrapolation to the limit. The research problem is especially pertinent and is the type of problem which practicing numerical analysts should consider more frequently. Several of the classical integration formulas are derived in Chapter 12 and then two methods, the trapezoidal rule with end correction and Romberg integration, are discussed in detail. The exposition is good. The next chapter develops difference methods for ordinary differential equations using developments in Taylor series. The discussion of the Runge-Kutta method is brief and leaves the wrong impression of the method. The predictor-corrector methods are nicely done. Some of the pitfalls of instability are neatly illustrated by an example.

Part Three—Computation—considers two topics, number representations and their arithmetics and the propagation of round-off error. The characteristics of fixed-point and floating-point computations are discussed in some detail, as well as the effect of rounding errors in each case. The final chapter discusses the two approaches to round-off error analysis, that of finding strict bounds and statistical estimates. Both have conceptual and practical difficulties. The presentation ends with five stimulating examples of the various possibilities in the nature of round-off error propagation.

In spite of the many good qualities of this book, there is a class of potential users (perhaps the majority) who

will not find this book to their taste. Those who want an exposition of "how-to-do-it" rather than "why-it's-done" will perhaps prefer one of the other books available. Those students who intend to study numerical analysis more than superficially will find this book rewarding.

J. R. Rice (W. Lafayette, Ind.)

Kronmal, Richard 4174

Evaluation of a pseudorandom normal number generator.

J. Assoc. Comput. Mach. **11** (1964), 357–363.

This is a report on some tests for randomness of pseudorandom normally distributed numbers. Two independent random variables $X_1 = A\cos 2\pi U_2$, $X_2 = A\sin 2\pi U_2$, where $A = |2\log U_1|^{1/2}$, were studied. They are normally distributed with mean zero and variance unity in case U_1 and U_2 are uniformly distributed in (0, 1). The U's were generated by a method suggested by Rotenberg [same J. 7 (1960), 75–77; MR 22 #8642], namely, $U = r_i 2^{-35}$, where

$$(1)\qquad r_{i+1} \equiv (2^n+1)r_i + c \pmod{2^{35}}.$$

After some experimenting, n was taken as 12 for U_1 and 7 for U_2, and every other term in U_2 was discarded. Eight samples of size 10^6 were tested for serial correlation, runs up and down, runs above and below zero, tests for normality, and order statistics with all data duly tabulated. All tests seem adequately satisfied. The generation of one of the X's requires 200 machine cycles.

{The reviewer would like to question the use of the coefficient 2^n+1 in (1). He realizes that the purpose of this coefficient is to cheapen the cost of producing the r's by eliminating a multiplication; but this also may fail to "stir up" adequately the digits of r_i in the production of r_{i+1} in a binary arithmetic unit. Almost any other large coefficient would be better. If one is going on to calculate X_1 and X_2, one extra multiplication of 14 cycles is almost negligible.}

D. H. Lehmer (Berkeley, Calif.)

Wynn, P. 4175

Note on a converging factor for a certain continued fraction.

Numer. Math. **5** (1963), 332–352.

A convergence factor (that is, a device for acceleration of convergence) is developed for a certain class of continued fractions. A complete ALGOL program is given for the procedure, and it is illustrated with numerical results.

E. Frank (Chicago, Ill.)

Elliott, David 4176

The evaluation and estimation of the coefficients in the Chebyshev series expansion of a function.

Math. Comp. **18** (1964), 274–284.

The efficient computation of the coefficients a_n in the Chebyshev series expansion $f(x) = \sum' a_n T_n(x)$ is often easier if there is prior knowledge of their behaviour with increasing n. The author shows how such knowledge may be obtained for certain classes of functions by studying a contour integral definition of a_n. His estimates are supported by examples of (i) a function with poles in the complex plane, (ii) an entire function, (iii) a function regular except at $x = \pm 1$, and (iv) a function with a branch point on the real axis.

C. W. Clenshaw (Teddington)

Jaech, J. L. 4177

A note on the equivalence of two methods of fitting a straight line through cumulative data.

J. Amer. Statist. Assoc. **59** (1964), 863–866.

Author's summary: "When measurements are made on the same experimental item at successive stages of an experiment, the experimental errors include components which are cumulative in nature, and are hence not independent. This dependence may be removed by using incremental changes in the experimental item as the yield variable, and then using a weighted least squares approach to estimate the parameters of the model, assumed to be linear. Alternately, the original cumulative data may be used directly, and Aitken's method of weighted least squares applied to obtain the parameter estimates. These estimates are shown to be identical."

G. Elfving (Helsinki)

Fisher, Donald D. 4178

Minimum ellipsoids.

Math. Comp. **18** (1964), 669–673.

Author's introduction: "A well-known statistical problem is to determine an ellipsoid R_S^e in E_n which contains a certain fraction of the points from a set S. Here we use the word 'contains' to mean that a point $p_i \in S$ is either in the interior or on the surface of the ellipsoid R_S^e. Although the determination of R_S^e is easy computationally, the determination of the ellipsoid of minimum volume, say, which contains all the points of S is quite difficult. In this note we give a method for determining ellipsoids satisfying a certain minimum property and compare these with ones obtained by statistical methods."

Hofsommer, D. J.; van de Riet, R. P. 4179

On the numerical calculation of elliptic integrals of the first and second kind and the elliptic functions of Jacobi.

Numer. Math. **5** (1963), 291–302.

Formulas are given for the computation of elliptic integrals and the elliptic functions of Jacobi. An ALGOL 60 program for the calculation is shown.

E. Frank (Chicago, Ill.)

Wetterling, W. 4180

Anwendung des Newtonschen Iterationsverfahrens bei der Tschebyscheff-Approximation, insbesondere mit nichtlinear auftretenden Parametern. II.

Math.-Tech.-Wirtschaft **10** (1963), 112–115.

The author gives further applications of his adaptation of Newton's method to the Čebyšev approximation of a function $f(x)$ by a family $F(x, \mathbf{a})$ (where $\mathbf{a}$ is a real n-vector) [see I. Teil, Math.-Tech.-Wirtschaft 10 (1963), 61–63; MR 28 #704]. The case when $F(x, \mathbf{a})$ is a family of rational functions is discussed in some detail. One instance of an exponential polynomial family, $F(x, \mathbf{a}) = a_1 e^{a_2 x^2} + a_3$, and two instances of a function $f(x, y)$ approximated by families $F(x, y, \mathbf{a})$ of quadratic polynomials, are briefly worked out. Each method is accompanied by a numerical example.

I. Marx (Lafayette, Ind.)

Hsieh, H. C. 4181

An on-line identification scheme for multivariable nonlinear systems.

Computing Methods in Optimization Problems (*Proc. Conf., Univ. California, Los Angeles, Calif.*, 1964), *pp.* 193–210. *Academic Press, New York*, 1964.

Author's summary: "The multivariable nonlinear system considered here is assumed to be characterized by a truncated functional power series. The system is to be identified under normal operating conditions. Least square error criterion is used to estimate the system weighting function matrix and a steepest descent method is employed for solving this problem. A recursive estimation scheme is devised for up-dating its estimation. An 'adjoint space approach', which has the advantage of reducing the dimensionality of the identification problem is explored."

Faddejew, D. K. [Faddeev, D. K.]; 4182
Faddejewa, W. N. [Faddeeva, V. N.]

★Numerische Methoden der linearen Algebra.

Mathematik für Naturwissenschaft und Technik, Band 10.

VEB Deutscher Verlag der Wissenschaften, Berlin, 1964. 771 *pp.*

This is a translation of the second edition. The first edition appeared in 1960 [*Computational methods of linear algebra* (Russian), Fizmatgiz, Moscow, 1960]; the second in 1963 [second, augmented edition, Fizmatgiz, Moscow, 1963; MR 28 #4659]; and an English translation of the first appeared also in 1963 [Freeman, San Francisco, Calif., 1963; MR 28 #1742]. By contrast with the English translation, this translation into German is excellent. Clearly the translator understands the subject as well as the language.

A six-page index has been added. The bibliography has been rearranged, names given in Cyrillic in the original and listed separately are here transliterated and placed accordingly. Some errors in the bibliography that remained in the second edition are corrected, and throughout the proofreading has been most painstaking. It is regrettable that not more English translations are of this quality.

A. S. Householder (Oak Ridge, Tenn.)

Bittner, L. 4183

Über ein mehrstufiges Iterationsverfahren zur Lösung von linearen Gleichungen.

Numer. Math. **6** (1964), 161–180.

For solving the system $Ax = y$, the iteration

$$x_n = \sum_1^q C_k x_{n-k} + \sum_1^q D_k(Ax_{n-k} - y)$$

is considered, with special attention to the case where C_k and D_k are polynomials in A. The fundamental lemma states that convergence for arbitrary initial x_i occurs when the determinant of $-\lambda^q I + \sum \lambda^k (C_{q-k} + D_{q-k}A)$ vanishes for no λ with $|\lambda| \geq 1$. The author obtains an estimate of error of a given iterate, and applies the method to the solution of nonlinear systems.

A. S. Householder (Oak Ridge, Tenn.)

Fiedler, M.; Pták, V. 4184

On aggregation in matrix theory and its application to numerical inverting of large matrices.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 11 (1963), 757–759.

The authors describe some methods of reducing the problem of inverting a large matrix to the inversion of a smaller one. These methods can be applied to the inversion of badly conditioned Leontiev matrices.

Let $N = N_1 \cup \cdots \cup N_r$ be a decomposition of the numbers $1, \cdots, n$, where N_i has n_i members. If A is an (n, n) matrix, then $c(A)$ is the (r, r) matrix whose i, k element is $(1/n_i n_k) \sum_{p \in N_i, q \in N_k} a_{pq}$. The summation operator $s(A)$ is defined like the contraction operator $c(A)$, omitting the factor $1/n_i n_k$. $\mathscr{L}$ denotes the set of all (n, n) matrices whose elements are constant on each block $N_r \times N_s$.

The main result proved is that if H and S are two (n, n) matrices, H being non-singular and $S \in \mathscr{L}$, then $H - S$ is non-singular if and only if the (r, r) matrix $E - s(H^{-1})c(S)$ is non-singular, and in this case,

$$(H-S)^{-1} = H^{-1} + H^{-1}BH^{-1},$$

where $c(B) = c(S)[E - s(H^{-1})c(S)]^{-1}$, $B \in \mathscr{L}$, and E is the identity matrix.

This and some related results are proved by applying the following lemma. Let H be a non-singular (r, n) matrix and U', V two (r, n) matrices. Then

$$\det(H - U'V) = \det H \det(E - VH^{-1}U'),$$

and

$$(H - U'V)^{-1} = H^{-1} + H^{-1}U'(E - VH^{-1}U')^{-1}VH^{-1},$$

provided $(E - VH^{-1}U')$ is non-singular.

A Leontiev matrix x has the form $E - A$, where A has non-negative elements and row sums smaller than 1. To apply the theorem given above, choose $S \in \mathscr{L}$ (with suitable N_i) such that $Y = A - S$ is small enough (according to the norm $\|Y\| = \max_i \sum_k |y_{ik}|$), and take $H = E - Y$. If $E - A$ is badly conditioned, i.e., $\|A\|$ is not much less than 1, then H will be better conditioned, and the difficulties of inverting an ill-conditioned matrix are transferred to a smaller matrix. *A. M. Duguid* (London)

Hageman, Louis A.; Varga, Richard S. 4185

Block iterative methods for cyclically reduced matrix equations.

Numer. Math. **6** (1964), 106–119.

Bei Anwendung des Differenzenverfahrens auf Randwertaufgaben für elliptische Differentialgleichungen zweiter Ordnung erhält man gewöhnlich lineare Gleichungssysteme der Form

$$(*) \qquad \begin{bmatrix} I_{1,1} & -B_{1,2} \\ -B_{2,1} & I_{2,2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix},$$

wobei $I_{1,1}$ und $I_{2,2}$ Einheitsmatrizen bedeuten. Bei einem solchen Gleichungssystem kann man der Vektor x_1 auch als Lösung des reduzierten Systems

$$(I_{1,1} - B_{1,2}B_{2,1})x_1 = k_1 + B_{1,2}k_2$$

und anschliessend x_2 aus (*) berechnen. Im Falle der Differenzengleichungen enthält dieses reduzierte System

etwa halb so viel Unbekannte, es kann ohne Matrizenmultiplikation aufgestellt werden, und das Gauss-Seidel-Verfahren konvergiert für das reduzierte System schneller [siehe Referent, Z. Angew. Math. Mech. **34** (1954), 241–253; MR **16**, 406; R. S. Varga, *Matrix iterative analysis*, Prentice-Hall, Englewood Cliffs, N.J., 1962; MR **28** #1725]. Für Differenzenverfahren der erwähnten Art (*) sind verschiedene verbesserte Iterationsverfahren entwickelt worden, z.B. die "successive over-relaxation iterative method" [S. P. Frankel, MTAC **4** (1950), 65–75; MR **13**, 692; D. M. Young, Trans. Amer. Math. Soc. **76** (1954), 92–111; MR **15**, 562] und die "cyclic Chebyshev semi-iterative method" [G. H. Golub und R. S. Varga, Numer. Math. **3** (1961), 147–156; MR **26** #3207; ibid. **3** (1961), 157–168; MR **26** #3208], gekoppelt mit "block or multi-line techniques". Die Verfasser definieren nun diesen Verfahren zugehörige "induzierte" Iterationsverfahren für das reduzierte System und untersuchen deren Konvergenz. Die zugehörigen Iterationsmatrizen besitzen einen kleineren Spektralradius. Im Falle eines Modellproblems wird der erforderliche Rechenaufwand genauer verglichen. Numerische Beispiele bestätigen, dass die induzierten Verfahren weniger Zeit und Iterationsschritte benötigen.

J. Schröder (Seattle, Wash.)

Kublanovskaja, V. N. 4186

A numerical scheme for the Jacobi process. **(Russian)**

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 732–733.

The author gives descriptions of algorithms, based on sequences of right multiplications of A by two-dimensional rotation matrices, $T^{(k)}$, which generate sequences of matrices $A^{(k)}$ such that $A^{(k)'}A^{(k)}$ tends to a diagonal matrix and such that $T_k = \prod_0^k T^{(i)}$ tends to the modal matrix of $A'A$. Thus a solution of the complete eigenvalue problem for $A'A$ is obtained without actually forming $A'A$. Acceleration of the process is discussed, and an application to the orthogonalization, with respect to a given metric, of a system of vectors, is mentioned.

John Todd (Pasadena, Calif.)

Lietzke, M. H.; Stoughton, R. W.; 4187
Lietzke, Marjorie P.

A comparison of several methods for inverting large symmetric positive definite matrices.

Math. Comp. **18** (1964), 449–456.

The $n \times n$ matrices of diagonal 2's and super- and sub-diagonal -1's and a few of its low powers are inverted numerically for various n in order to compare four methods of inversion: Gauss-Jordan, Choleski, congruent transformation, and rank annihilation. The Gauss-Jordan reduction seems to fare somewhat the best.

H. S. Wilf (Philadelphia, Pa.)

Yamashita, Shin-ichiro 4188

Numerical solution of algebraic equations.

Information Processing in Japan **2** (1962), 13–16.

This paper first appeared in Japanese in the Journal of the Information Processing Society of Japan **2** (1961), no. 5, 249–252. The version reviewed in the following has obscurities but is still fairly easy to read.

A part of the paper considers a method for computing roots of polynomials. The author's description of the method is as follows: $f(x) = (x+p_n)Q(x) + R$, $Q(x) = (x+p_n)Q'(x) + S$, $p_{n+1} = p_n + R/S$, convergence test: $|R/p_{n+1}S|$. Here $f(x)$ is the given polynomial but the author does not define or discuss any other symbols. He omits any discussion of procedure or of what he means by convergence test. No derivation is given nor is an example worked out. The method appears to be a variant of Hitchcock's method designed to extract linear factors from $f(x)$.

The remainder of the paper is concerned with polynomials having ill-conditioned roots. The paper calls such polynomials "bad-charactered". Two examples are given to illustrate the difficulties in computing such roots, using a fixed number of digits.

Nothing new is presented in the discussion of ill-conditioned roots. A more thorough discussion of the problem is given, for example, by Wilkinson [Numer. Math. **1** (1959), 150–180; MR **22** #321].

{The paper suffers from a lack of preciseness. Statements occur which vary from the ambiguous to the incorrect.}

E. R. Hansen (Palo Alto, Calif.)

Kowel, Stephen T. 4189

Absolute iteration and the solution of transcendental equations.

J. Franklin Inst. **278** (1964), 210–218.

To solve the equation $\theta(x) = 0$, it is proposed that the equation be rewritten in the form $f(x) = g(x)$ and the iterative scheme $f(x_{n+1}) = g(x_n)$ be used. Under appropriate conditions it is shown that a sufficient condition for convergence is $f'(x) > g'(x)$ for x in a neighborhood of a solution. The ratio $f'(x)/g'(x)$ is a measure of the rapidity of convergence. Some examples are given.

L. W. Ehrlich (Silver Spring, Md.)

Našul', A. B. 4190

Improving the convergence of methods of successive approximation for linear equations. **(Russian)**

Dokl. Akad. Nauk SSSR **158** (1964), 279–280.

The author describes a method for improving the convergence of the process of successive approximation $x_n = Bx_{n-1} + b$ for the solution of linear algebraic equations $x = Bx + b$. Noting that, for $n = 2^{s-1}$,

$$U_n \equiv (I + B + B^2 + \cdots + B^{n-1}) =$$

$$(I + B^{2^{s-2}})(I + B^{2^{s-3}}) \cdots (I + B^2)(I + B),$$

the author suggests the use of the iteration process $x_{2^{s-1}} = B^{2^{s-1}}x_0 + U_{2^{s-1}}b$ in which $U_{2^{s-1}}$ and $B^{2^{s-1}}$ are successively computed by the formulas

$$U_{2^{s-1}} = (I + B^{2^{s-2}})U_{2^{s-2}}, \quad U_1 = 1, \quad B^{2^{s-1}} = (B^{2^{s-2}})^2.$$

In this way, s steps with the improved process are equivalent to $n = 2^{s-1}$ steps with the ordinary process.

W. V. Petryshyn (Chicago, Ill.)

Obreškov, Nikola 4191

On the numerical solution of equations. **(Bulgarian. French summary)**

Annuaire Univ. Sofia Fac. Sci. Phys. Math. Livre 1 Math. **56** (1961/62), 73–83 (1963).

In the first part of this paper the author discusses the

Laguerre method for determining a root of an algebraic equation of nth degree, $f(x)=0$, with n real roots. Consider a sequence of successive approximate values $a_1, a_2, \cdots$ of a simple root α, obtained by the Laguerre method. It is known that in this case the process of approximation is of third order. The author shows that if α is a root of multiplicity $p>1$, then the Laguerre method gives only a first-order process. For improving the order of approximation the author modifies the Laguerre method:

$$a_{n+1} = a_n - \frac{nf(a_n)}{f(a_n) \pm \sqrt{((n/p-1)[(n-1)f'(a_n)^2 - nf(a_n)f''(a_n)])}}$$

and shows that this gives a third-order process.

In the second part of the paper the author makes similar considerations in the case of Newton's method. He shows that if α is a root of multiplicity $p>1$, then the Newton method represents a first-order procedure. In this case he presents a modification of Newton's method:

$$a_{n+1} = a_n - \frac{2pf(a_n)f'(a_n)}{(p+1)f'(a_n)^2 - pf(a_n)f''(a_n)},$$

which is a third-order process of approximation. Finally, he gives a generalization of the Chebyshev method.

D. D. Stancu (Cluj)

Emanuel, George 4192

The Wilf stability criterion for numerical integration.

J. Assoc. Comput. Mach. **10** (1963), 557–561.

The author shows how the calculations involved in applying a stability criterion of this reviewer can be simplified by taking advantage of certain symmetries which are always present in the matrices considered.

H. S. Wilf (Philadelphia, Pa.)

Gatteschi, Luigi 4193

Su una formula di quadratura "quasi gaussiana". Tabulazione delle ascisse d'integrazione e delle relative costanti di Christoffel.

Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **98** (1963/64), 641–661.

The author derives the quadrature formula

$$\int_{-1}^{1} f(x)\,dx \doteq \sum_{i=1}^{n} H_{n,i} f(\nu_{n,i}) + K_n[f(-1)+f(1)] + K_n^*[f'(-1)-f'(1)]$$

having maximum degree of exactness $2n+3$. The abscissas $\nu_{n,i}$ are the zeros of the ultraspherical polynomial $P_n^{(5/2)}$. The weights $H_{n,i}$ are expressed in terms of $P_{n+1}^{(5/2)}$, whereas the weights K_n and K_n^* turn out to be rational functions of n. An estimate of the remainder is also included. Numerical values of $H_{n,i}$, $\nu_{n,i}$, K_n, K_n^* are tabulated to 12 decimal places for $n=0(1)16$. {The author's table overlaps with Table 1 in D. D. Stancu and A. H. Stroud [Math. Comp. **17** (1963), 384–394; MR **28** #718] which gives 20D values for $n=2(1)5$.}

Walter Gautschi (Lafayette, Ind.)

Goodisman, Jerry 4194

Hermiticity and Gaussian quadrature.

J. Chem. Phys. **41** (1964), 2365–2368.

Author's summary: "It is often convenient to use Gaussian quadrature techniques to calculate integrals for variational calculations. A basis set may be constructed which is orthonormal under the summations used to approximate the integrals. For this set, the Hamiltonian matrix is Hermitian, in spite of the matrix elements not being calculated exactly."

Lubomirsky, Wadim 4195

Extension of a method for the approximate evaluation of Fermi-Dirac integrals. (Spanish)

Rev. Un. Mat. Argentina **21**, 165–172 (1963).

The Fermi-Dirac integrals $I=\int_{-W/kT}^{\infty} F'(v)C(v)\,dv$, $F(v)=(1+e^v)^{-1}$, are approximated by replacing $F'(v)$ by the partial sums of its expansion in Hermite polynomials of even degree.

A. E. Danese (Buffalo, N.Y.)

Spiridonov, V. 4196

On a quadrature formula of S. N. Bernstein. (Russian)

C. R. Acad. Bulgare Sci. **17** (1964), 339–342.

The concern here is with quadrature formulas

$$\int_{-1}^{1} f(x)\,dx \doteq \sum_{i=1}^{l-1} p_i[f(x_i)+f(-x_i)] + p_0 f(0)$$

having the following properties: (i) they are exact for all polynomials of degree $2l-1$; (ii) all coefficients p_i are positive; (iii) the abscissas x_i are rational numbers in $[-1,1]$ which can be represented by fractions with denominators not exceeding 10. S. N. Bernšteĭn [*Collected Works* (Russian), Vol. II, pp. 228–230, Izdat. Akad. Nauk SSSR, Moscow, 1954; MR **16**, 433] showed that such formulas exist if $l \le 7$, but do not exist if $l \ge 9$. The case $l=8$ remained undecided. The author now settles this case in the negative.

Walter Gautschi (Lafayette, Ind.)

Adrianova, L. Ja. 4197

A strict error bound for the integration of differential equations by Störmer's method. (Russian. English summary)

Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. **19** (1964), no. 3, 5–17.

The author deals with systems of the form $d^2x/dt^2 = f(t,x)$, $x(t_0)=x_0$, $x'(t_0)=x_0'$, where x and $f(t,x)$ are real vectors from E_n and t is a scalar variable. The object is to get an a priori estimate for the truncation error of Störmer's method. To this end she uses the method of Lozinskiĭ [Izv. Vysš. Učebn. Zaved. Matematika **1958**, no. 5 (6), 52–90; errata, ibid. **1959**, no. 5 (12), 222; MR **26** #3191] which was used by Lozinskiĭ for deriving a priori estimates for the truncation error when solving systems of the first order by the Adams method. The theorem proved by the author is too lengthy to be stated here.

M. Zlámal (College Park, Md.)

Axelsson, Owe 4198

Global integration of differential equations through Lobatto quadrature.

Nordisk Tidskr. Informations-Behandling **4** (1964), 69–86.

Zur numerischen Integration von $y'=f(x,y)$, $y(-1)=y_0$ im Intervall $-1\leq x\leq 1$ wird ein System von nichtlinearen Gleichungen

$$y_{i,n}=y_0+\sum_{k=1}^{n}A_{i,k}^{(n)}f(x_{k,n},y_{k,n})\qquad (i=1,2,\cdots,n)$$

gelöst. Die $A_{ik}^{(n)}$ werden durch Integration von Interpolationsformeln gewonnen; ihre Matrix ist wertemäßig nahezu eine Dreiecksmatrix; für große n streben die Elemente oberhalb der Hauptdiagonalen gegen 0; das nichtlineare Gleichungssystem ist daher bequem mit Iterationen zu behandeln. Nimmt man als Teilpunkte $x_{k,n}$ die Nullstellen von $P_n(x)-P_{n-2}(x)$ (mit $P_n(x)$ als Legendresche Polynome), so wird

$$\operatorname*{Max}_i |y_{i,n}-y(x_{i,n})|\to 0\quad \text{für } n\to\infty$$

gezeigt. Aus den Werten $f(x_{i,n},y_{i,n})$ für $i=1,\cdots,n$ werden Approximationspolynome $y_n(x)$ konstruiert, die in $\langle -1,1\rangle$ gleichmäßig gegen $y(x)$ konvergieren. Speziell am Intervallende ist der Fehler verhältnismäßig klein.

Es wäre schön gewesen, wenn der Autor bei seinen Sätzen die Voraussetzungen angegeben hätte, die er beim Beweis gelegentlich braucht, wie z.B. Beschränktheit vom f_{yy}. In der Einleitung wird gesagt: Die Stellen des maximalen Fehlers sind bekannt; auch hier wäre die Angabe der speziellen Voraussetzungen, unter denen dies gilt, sehr erwünscht. *L. Collatz* (Hamburg)

Krylov, V. I.; Monastyrnyi, P. I. 4199

The sweep method for solving a fourth-order differential equation. (Russian)

Vesci Akad. Navuk BSSR Ser. Fiz.-Tèhn. Navuk **1964**, no. 2, 5–11.

The authors consider the fourth-order differential equation

$$(1)\qquad y^{(4)}+\sum_{k=1}^{4}P_k(x)y^{(4-k)}=F(x)$$

subject to the boundary conditions

$$(2)\qquad \sum_{k=1}^{4}a_{ik}y^{(4-k)}(a)=A_i,\qquad \sum_{k=1}^{4}b_{ik}y^{(4-k)}(b)=B_i\qquad (i=1,2).$$

The method of solution discussed here derives from the observation that the general solution of (1), subject to the boundary conditions at $x=a$, coincides with the general solution of

$$(3)\qquad y''-p(x)y'-q(x)=r(x),$$

where p, q, r are obtainable from a nonlinear Cauchy problem with initial data at $x=a$. The desired solution of (1), (2) may then be found by solving a Cauchy problem for (3), with initial data given at $x=b$. This data depends on the values at $x=b$ of p, q, r and their derivatives. Possible singularities in the first Cauchy problem can be handled by working with new dependent variables defined by $p(x)=\tan\alpha(x)$, $q(x)=\tan\beta(x)\sec\alpha(x)$, $r(x)=r_1(x)\sec\alpha(x)\sec\beta(x)$. *Walter Gautschi* (Lafayette, Ind.)

Meshaka, P. 4200

Deux méthodes d'intégration numérique pour systèmes différentiels.

Rev. Française Traitement Information (Chiffres) **7** (1964), 135–148.

First, a stable, implicit, linear 5-step method, $p=k=5$, is suggested, the characteristic feature of which is that no values of y' need to be stored, $\sigma(\zeta)=\zeta^5$. (The 4-step Adams-Moulton method, $\rho(\zeta)=\zeta^4-\zeta^3$, has a similar merit, and the reviewer cannot find in what respect it is inferior to the author's method.) Next, an implicit, stable method for systems of the first order is suggested, which requires the computation of y''. The corrector is based on the relation

$$7y_{n+1}=(8y_n-y_{n-1})+2h(y'_{n+1}+2y_n')+2h^2y_n''+O(h^5).$$

It is not mentioned why this particular formula is preferable to other, more accurate, formulas based on essentially the same information. *G. Dahlquist* (Stockholm)

Pelczar, A. 4201

On a modification of the method of Euler polygons for the ordinary differential equation.

Ann. Polon. Math. **15** (1964), 195–202.

To solve the ordinary differential equation

$$(1)\qquad y'=f(x,y),\qquad y(a)=c,\qquad a\leq x\leq\xi,$$

the author constructs a sequence of functions which converge to its solution. Let $Z_0(x,\xi)=c$. Let $Z_n(x,\xi)$, $n=1,2,\cdots$, be the Euler polygon constructed for the interval (a,ξ) and the division of this interval into points

$$a_j=a+\frac{j}{n}(\xi-a),\qquad j=0,1,\cdots,n.$$

Define $\phi_n(x)=Z_n(x,x)$, $n=0,1,2,\cdots$. Provided that $f(x,y)\in C$ and is bounded in $a\leq x\leq b$, $|y-c|\leq M$, where $b-a<1$, and a unique solution $y(x)$ to (1) exists in $[a,b]$, then the sequence $\phi_n(x)$ converges uniformly to $y(x)$. Further theorems are given which give conditions under which the derivatives of $\phi_n(x)$ converge uniformly to $y^{(n)}(x)$ in $[a,b]$. *A. O. Garder, Jr.* (St. Louis, Mo.)

v. Sanden, H. 4202

Eine Bemerkung zur numerisch-tabellarischen Integration gewöhnlicher Differentialgleichungen nach Runge-Kutta.

Z. Angew. Math. Mech. **43** (1963), 561.

The presentation of the classical Runge-Kutta method in a course is discussed. The author first gives a short proof that the error in one step is $O(h^5)$ for $y'=f(x,y)$, $y(0)=0$, if $f(0,0)=0$. In the extension to the general case it is tacitly assumed that the relation $y(h)=z(h)+rh$ holds exactly for the approximate solutions of the problems $y'=r+f(x,y)$, $y(0)=0$ and $z'=f(x,z+rx)$, $z(0)=0$. This is correct, though not self-evident. *G. Dahlquist* (Stockholm)

Veliev, M. A. 4203

A study of the stability of the Bubnov-Galerkin method for non-stationary problems. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 16–18.

For the equation $Au=f$, where A is a positive definite operator defined on a Hilbert space, Mihlin proved stability of the Ritz method provided the system of coordinate functions is "strongly minimal" [same Dokl. **135** (1960), 16–19; MR **27** #4362; Vestnik Leningrad. Univ. **16**

(1961), no. 13, 40–51; MR 25 #2698]. In the present paper the author announces analogous results for the Bubnov-Galerkin method when applied to the Cauchy problem for the second-order differential equation $Au''(t)+Bu'(t)+Cu(t)=f(t)$ with constant operator coefficients.

Walter Gautschi (Lafayette, Ind.)

Zee, Chong-Hung 4204

On solving second order nonlinear differential equations.

Quart. Appl. Math. **22** (1964), 71–73.

In solving the equation $y''=f(x, y, y')$ by power series using n terms, the author proposes to treat the points at which derivatives are evaluated in the expressions for the remainders of y, y', and y'' as if at all three points the $(n+1)$st derivative of y has the same value. Then if R_{n0}, R_{n1}, R_{n2} are the remainders in the series for y, y', and y'', respectively, we can write

$$R_{n1}=\frac{n+1}{\Delta X}R_{n0}, \qquad R_{n2}=\frac{n(n+1)}{\Delta X^2}R_{n0}.$$

If the three truncated series with remainders are then substituted into the differential equation, it may be solved for the one unknown, R_{n0}. Unfortunately, the author makes no effort to investigate how good this approximation is. It would seem natural to test it on simple equations where remainders are known exactly.

A. O. Garder, Jr. (St. Louis, Mo.)

Bellman, R.; Kalaba, R. 4205

Dynamic programming, invariant imbedding and quasilinearization: Comparisons and interconnections.

Computing Methods in Optimization Problems (Proc. Conf., Univ. California, Los Angeles, Calif., 1964), pp. 135–145. *Academic Press, New York*, 1964.

Authors' introduction: "In this paper we shall treat a simple class of variational problems from three different points of view: quasilinearization, dynamic programming, and invariant imbedding. Our aim is to adumbrate the point of view that leads to the various equations associated with each method. Primary emphasis is upon the computational aspects, though the development of effective numerical schemes requires significant analytical advances. The general theme is the computational solution of nonlinear two-point boundary value problems via a direct assault and via conversions into initial-value problems."

Bhargava, R. D.; Radhakrishna, H. C. 4206

Numerical solution of two-dimensional Laplace's equation.

Proc. Nat. Inst. Sci. India Part A **29** (1963), 283–293.

Green's formula gives a singular linear integral equation from which the boundary values of a harmonic function or its normal derivative can be solved when the boundary values of the other (or of a linear combination of them) are given. The value at an interior point can then be obtained by quadrature. A technique for numerical integration of this equation is applied to a symmetric Dirichlet problem with four arcs in a quadrant.

The numerical aspects of the technique are not extensively discussed. {It is worth mentioning that the integral equation method requires little storage compared to finite-difference methods, and that it can be applied when the boundary conditions are complicated.}

G. Dahlquist (Stockholm)

Zenkin, O. V. 4207

On the relation between the algorithms for constructing solutions of certain equations of parabolic and elliptic types. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 23–26.

The author considers the following boundary-value problem

$$au_t = u_{xx}+u_{yy}+f(x, y), \qquad a>0,$$

when initially $u(x, y, 0)=0$, and when on the boundary $u=0$. The author assumes that the solution is in the form

$$u(x, y, t) = U(x, y)+v(x, y, t),$$

which, when substituted in the original problem, leads to two new boundary-value problems, one determining v, another determining U. The problem for v can be solved in series, but the one for U requires a numerical solution.

The author solves numerically the original problem for $u(x, y, t)$ and the modified problem for $U(x, y)$ using finite differences with the subsequent application of the Gauss-Seidel process. Comparing the two solutions, the author shows that the modified problem needs only half of the number of iterations as compared with the original one.

T. Leser (Aberdeen, Md.)

Bramble, J. H.; Hubbard, B. E. 4208

Approximation of derivatives by finite difference methods in elliptic boundary value problems.

Contributions to Differential Equations **3** (1964), 399–410.

The authors consider the problem of approximating the solution of the Dirichlet problem for

$$Lu \equiv au_{xx}+2bu_{xy}+cu_{yy}+du_x+eu_y+fu = F$$

in a connected region R with smooth boundary C in the (x, y)-plane, where $|b|<a, c$; $f<0$ and the coefficients and data are smooth. Results are obtained for problems in this class which have solutions with bounded fifth derivatives.

A finite-difference analogue $L_hU(p)=F(p)$, where h is the grid size and where p is a point of the difference mesh, is constructed in such a way that L is approximated no worse than $O(h)$ at points near and on the boundary C and $O(h^2)$ in the interior. Then if $e(p)=U(p)-u(p)$ and p and p' are neighboring points with $\overline{pp'}$ the distance from p to p', it is shown that

$$|e|_{R_h} = \max_{p\in R_h}|e(p)| = O(h^2),$$

where R_h is the set of mesh points each having its eight nearest neighbors in R. Also

$$\left|\frac{e(p)-e(p')}{\overline{pp'}}\right| = O(h^2).$$

Using the Laplace operator, an example is constructed to show that a uniform estimate of the latter type cannot hold if the approximation is $O(1)$ on the set of points which have at least one horizontal or vertical neighbor not in R.

At interior points, however, it is further shown that, in

spite of poorer convergence near the boundary, one may still have an approximation of order $O(h^2)$. In fact, if D_h^n is an nth-order difference quotient, it is shown that, at interior points,

$$|D_h^n e(p)| \leq K_n |e|_{B_h} + O(h^2),$$

provided b does not change sign in the region and the coefficients of L and the solution u are sufficiently smooth.

A. O. Garder, Jr. (St. Louis, Mo.)

Johansson, Olov; Kreiss, Heinz-Otto 4209

Über das Verfahren der zentralen Differenzen zur Lösung des Cauchyproblems für partielle Differentialgleichungen. (English summary)

Nordisk Tidskr. Informations-Behandling **3** (1963), 97–107.

Difference methods are investigated for the Cauchy problem for a system of n partial differential equations in s space variables,

$$(*) \qquad \partial u/\partial t = P(u), \qquad u(x, 0) = f(x),$$

$$x = (x_1, x_2, \cdots, x_s) \in R_s, \qquad u = (u_1, u_2, \cdots, u_n) \in S_n,$$

$$Pu = \sum_{i=1}^{s} [\partial_i(A_i(x)\partial_i u) + \tfrac{1}{2}(B_i(x)\partial_i u + \partial_i(B_i(x)u))] + C(x)u,$$

where the $u_i = u_i(x, t)$ are complex-valued. The A_i, B_i, C are complex matrix-valued, twice differentiable, bounded functions of x such that the $A_i^* + A_i$ are positive semi-definite and the B_i are hermitian for all $x \in R_s$. If for all $w \in L_2(R_s)$,

$$\operatorname{Re}(w, Pw) \leq \operatorname{Re}(w, Cw) \leq \delta\|w\|^2,$$

then the solutions of (*) which belong to $L_2(R_s)$, together with their derivatives of second order, satisfy the inequality

$$(**) \qquad \|u(\cdot, t)\| \leq e^{\delta t}\|u(\cdot, 0)\| \qquad (t > 0).$$

Let $P = P_0 + P_1$, where P_0 and P_1 are selfadjoint and anti-selfadjoint operators in $L_2(R_s)$, respectively.

The authors suggest difference methods of the form

$$(I - kQ_0)v(\cdot, t+k) = (I + kQ_0)v(\cdot, t-k) + 2kQ_1 v(\cdot, t),$$

where Q_0, Q_1 are difference operators defined on the grid $\{(\nu_1 h^{(1)}, \nu_2 h^{(2)}, \cdots, \nu_s h^{(s)})\}$ in R_s (ν_i integers) which converge formally to P_0, P_1, when $h = (h^{(1)}, h^{(2)}, \cdots, h^{(s)}) \to 0$. The characteristic requirement of this paper is that there should exist a sequence $S = \{(h_i, k_i)\}_{i=0}^{\infty}$ converging to $(0, 0)$, such that for all $(h, k) \in S$ and for all $t = \nu k$, $\nu = 0, 1, 2, \cdots$,

$$(***) \quad \|v(\cdot, t)\|_h^2 \leq$$
$$K \exp(2\delta t(1 + O(k^2\delta^2)))(\|v(\cdot, 0)\|_h^2 + \|v(\cdot, k)\|_h^2).$$

(The norm $\|\cdot\|_h$ converges to the L_2-norm as $h \to 0$.)

The authors prove a convenient sufficient condition for (***) and apply it to the case where Q_0, Q_1 are the simplest central difference operators consistent with P_0, P_1. If the $A_i(x)$ are hermitian for all x, then (***) holds provided that $k/|h|$ does not exceed a certain bound, otherwise a bound for $k/|h|^2$ is also required. (All $h^{(i)}$ are assumed to be equal.)

Finally, it is remarked that if a first-order term is given in the form $B_i(x)\partial_i u$, then its anti-selfadjoint part, $-\tfrac{1}{2}(B_i\partial_i u) + \tfrac{1}{2}\partial_i(B_i u) = \tfrac{1}{2}(\partial B_i)u$, is a bounded operator which has to be added to C, and hence included in P_0, in order to warrant the validity of (***).

G. Dahlquist (Stockholm)

Kreiss, Heinz-Otto 4210

On difference approximations of the dissipative type for hyperbolic differential equations.

Comm. Pure Appl. Math. 17 (1964), 335–353.

Let $(M_h u)(x) = \sum c_j(x)u(x + jh)$ define an explicit difference operator M_h with bounded, Lipschitz continuous, Hermitian matrix coefficients c_j. Suppose the eigenvalues $\lambda_k(x, \xi)$ of $\sum c_j(x)e^{ij\xi}$ satisfy $|\lambda| \leq 1 - \delta|\xi|^{2r}$, with fixed positive δ and r, and suppose $u(t+h) = M_h u(t)$ approximates some symmetric hyperbolic system with local accuracy $2r-1$ or $2r-2$. Then the author establishes $\|M_h\| \leq 1 + O(h)$ in a norm equivalent to the L_2 norm, so that M_h is stable in L_2. Although simplifications have since been found for the proof, the result itself is the climax of the theory of symmetric hyperbolic difference methods.

W. G. Strang (Cambridge, Mass.)

Konoval'cev, I. V. 4211

A difference boundary-value problem for a self-adjoint parabolic system. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. 4 (1964), 765–771.

Bounds are derived for the difference solution of the first initial boundary value problem for the parabolic system

$$\frac{\partial u}{\partial t} = \sum_{i=0}^{p} \frac{\partial^{p-i}}{\partial x^{p-i}}\left(a^{(i)}(x, t)\frac{\partial^{p-i}u}{\partial x^{p-i}}\right) + f$$

in $\{x, t : 0 \leq x \leq \varepsilon,\ 0 \leq t < \infty\}$ with

$$u\bigg|_{t=0} = \phi(x), \quad \frac{\partial^k u}{\partial x^k}\bigg|_{x=0} = \psi^{(k)}(t), \quad \frac{\partial^k u}{\partial x^k}\bigg|_{x=\varepsilon} = \chi^{(k)}(t),$$
$$k = 0, 1, \cdots, p-1,$$

which prove that the principle of non-growth mesh ratio is also applicable in this case. The asymptotic behaviour of the solutions for $\varepsilon \to 0$ as well as for $t = \infty$ is considered.

J. R. M. Radok (Adelaide)

Konovalov, A. N. 4212

Application of the splitting method to the numerical solution of dynamics problems in the theory of elasticity. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. 4 (1964), 760–764.

A finite-difference technique is employed to solve the following dynamical problem in the theory of isotropic elasticity. In the cylinder

$$D = Q \times [0 \leq t \leq T],$$
$$Q = \{[\alpha_1 \leq x_1 \leq \beta_1] \times [\alpha_2 \leq x_2 \leq \beta_2]\},$$

it is required to find the displacement vector $\mathbf{u}(x_1, x_2)$ which (a) is a solution of the equation

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = \mu\Delta\mathbf{u} + (\lambda + \mu)\operatorname{grad}\operatorname{div}\mathbf{u} + \mathbf{f};$$

(b) at $t = 0$ satisfies the conditions

$$\mathbf{u}(x_1, x_2, 0) = \boldsymbol{\varphi}(x_1, x_2), \qquad \frac{\partial \mathbf{u}}{\partial t}(x_1, x_2, 0) = \boldsymbol{\psi}(x_1, x_2);$$

and (c) attains on the lateral surface S of the cylinder the given value

$$u(x_1, x_2, t)|_S = g(x_1, x_2, t) \qquad (x_1, x_2, t) \in S.$$

J. Hubert Wilkinson (Battersea)

Nemčinov, S. V.; Libov, S. L. 4213

A direct method for increased accuracy in solving boundary-value problems for the Helmholtz equation on a grid of points in a rectangle. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 771–773.

This note improves the first author's solution of the Helmholtz equation by the grid method [same Ž. **2** (1962), 418–436; MR **27** #4372] in that, instead of central differences, the formula

$$\varphi(x_{j+1}) - 2\varphi(x_j) + \varphi(x_{j-1}) = \tfrac{1}{12} h^2[\varphi''(x_{j+1}) + 10\varphi''(x_j) + \varphi''(x_{j-1})] + O(h^6)$$

is used to approximate the second derivative in the differential equation and the formula

$$2h\varphi'(x_i) = \varphi(x_{i+1}) - \varphi(x_{i-1}) - \tfrac{1}{3}h^3\varphi'''(x_i) + O(h^5),$$

the first derivative in the boundary conditions. Use of the differential equation eliminates $\varphi'''(x)$. This method yields a higher order of accuracy without complicating the algorithm or increasing the number of arithmetical steps.

R. N. Goss (San Diego, Calif.)

Pavljuk, I. A. 4214

Asymptotic representation of the solution (by the method of lines) of a mixed problem for an hyperbolic equation. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 726–729.

The following mixed problem is considered in the region $0 \le x \le l$, $0 \le t \le L$:

$$(1) \qquad \begin{gathered} \frac{\partial^2 u}{\partial t^2} - b(x, t)\frac{\partial^2 u}{\partial x^2} = f(x, t), \qquad b(x, t) > 0, \\ u(x, 0) = p_1(x), \qquad u_t(x, 0) = p_2(x), \\ u(0, t) = r_1(t), \qquad u(l, t) = r_2(t). \end{gathered}$$

Using the method of lines the author transforms the equation (1) into the system of linear differential equations

$$u_k'' - h^{-2}b_k(t)[u_{k+1} - 2u_k + u_{k-1}] = f_k(t) + O(h^2),$$

where $u_k(t) = u(kh, t)$, $f_k(t) = f(kh, t)$, $b_k(t) = b(kh, t)$ and $k = 0, 1, \cdots, n+1$, $h = l/(n+1)$. By examining the asymptotic form of the solution of this system he derives the asymptotic formula of the form $u(t, h) = A(t)z_1(t, h) + hA(t)z_2(t, h) + O(h^2)$, where $u(t, h)$ is the solution-vector, i.e., the vector whose components are the functions $u_1(t)$, $u_2(t)$, $\cdots$, $u_n(t)$, and $A(t)$ is a certain square matrix and $z_1(t, h)$ and $z_2(t, h)$ certain column-vectors.

M. Zlámal (College Park, Md.)

Strang, Gilbert 4215

Accurate partial difference methods. II. Non-linear problems.

Numer. Math. **6** (1964), 37–46.

Part I appeared in Arch. Rational Mech. Anal. **12** (1963), 392–402 [MR **26** #4489]. Consider the initial-value problem for a quasilinear hyperbolic system

$$(1) \qquad \frac{\partial u}{\partial t} = \sum_{i=1}^{d} A_i(u, x, t)\frac{\partial u}{\partial x_i}, \qquad u(0, t) = u_0(x), \qquad -\infty < x_i < +\infty,$$

and approximate it by a difference approximation which, in its linearized form, is stable in the sense of Lax and Richtmyer. Assuming that (1) has a sufficiently smooth solution $u(x, t)$, the author shows that the solutions of the difference approximation converge to $u(x, t)$. Then he applies this theorem to approximations of the Runge-Kutta and Lax-Wendroff type. *H.-O. Kreiss* (Göteborg)

Tee, G. J. 4216

A new technique for solving elliptic partial differential equations.

J. Soc. Indust. Appl. Math. **12** (1964), 311–347.

By replacing a nine-point difference equation with a pair of equations of the form

$$(1) \qquad \phi_{i,k} = \alpha_{i,k}\phi_{i-1,k} + \beta_{i,k}\phi_{i-1,k-1} + \gamma_{i,k}\phi_{i,k-1} + h^2\delta_{i,k},$$

$$(2) \qquad v_{i,k} = a_{i,k}v_{i+1,k} + b_{i,k}v_{i+1,k+1} + c_{i,k}v_{i,k+1} + \phi_{i,k},$$

the author explores the possibility of approximating the solution of a linear elliptic boundary-value problem by a method of sweeping up and to the right by means of (1) and then down and to the left by means of (2). A large portion of the paper is devoted to the determination of $\alpha, \beta, \gamma, a, b$, and c for various boundary conditions and to various computational considerations. No convergence criteria are established, and the boundary of the region is severely restricted in shape. The method is applied in detail to a specific Dirichlet problem for the Laplace equation on a rectangular region, a problem for which, of course, there is no need for a numerical method.

D. Greenspan (Madison, Wis.)

Wynn, Peter 4217

Partial differential equations associated with certain non-linear algorithms. (German summary)

Z. Angew. Math. Phys. **15** (1964), 273–289.

In this paper are discussed certain algorithms (for example, the qd-algorithm) that relate four quantities connected by rational non-linear relationships and lying at the vertices of a rhombus. When the dimension of this rhombus is made infinitesimally small, these algorithms lead to systems of two simultaneous partial differential equations of the first order. These partial differential equations are derived for the algorithms given here.

E. Frank (Chicago, Ill.)

Wynn, P. [Wynn, Peter] 4218

An arsenal of ALGOL procedures for complex arithmetic.

Nordisk Tidskr. Informations-Behandling **2** (1962), 232–255.

Here are given a number of ALGOL procedures for arithmetic operations with complex numbers, and for the evaluation of certain elementary functions of a complex variable. As examples of these procedures, the author gives programs for the computation of the confluent hypergeometric function, the Weber parabolic cylinder

function, the application of the ε-algorithm to complex series, S-continued fractions, C-continued fractions, and a general continued fraction. *E. Frank* (Chicago, Ill.)

Wynn, P. [Wynn, Peter] 4219

Singular rules for certain non-linear algorithms.

Nordisk Tidskr. Informations-Behandling **3** (1963), 175–195.

The author discusses methods one can apply in certain singular cases that occur in the ε-, ρ-, and qd-algorithms. ALGOL programs are developed to show how the derived formulae can be used. *E. Frank* (Chicago, Ill.)

Wynn, P. [Wynn, Peter] 4220

General purpose vector epsilon algorithm ALGOL procedures.

Numer. Math. **6** (1964), 22–36.

The epsilon algorithm is a computational device for the acceleration of the convergence of a slowly convergent sequence. Here it is applied to slowly convergent vector sequences, and ALGOL programs are given for the procedures. *E. Frank* (Chicago, Ill.)

Volkov, E. A. 4221

The method of nets for boundary-value problems with skew and normal derivatives. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **1** (1961), 607–621.

The author considers the numerical solution of

$$\Delta\varphi = f(x_1, x_2), \qquad (x, y) \in G,$$

$$\frac{\partial\varphi}{\partial\rho} = \alpha, \qquad (x, y) \in \partial G.$$

He assumes that G is simply connected and contains the origin. Also, he assumes that ∂G, f, $\alpha(s)$, and $\rho(s)$, where s is the arc length on ∂G, satisfy regularity conditions and that $\varphi \in C^4(G^-)$. He imposes a square lattice on the region and divides the grid points in G into interior grid points D and points D_0 near the boundary. His difference equation is of the form

$$w_{0,0} = 0,$$

$$\Delta_h w_{ij} = f_{ij}, \qquad (x_i, y_j) \in D - \{(0, 0)\},$$

$$l_h w_{ij} = \sum_r \mu_{ijr}\alpha(s_r) + h\left[\sum_r \zeta_{ijr} \frac{\partial\alpha}{\partial s}(s_r) + \nu_{ij} f_{ij}\right],$$

$$(x_i, y_j) \in D_0,$$

where Δ_h is the usual five-point difference analogue and l_h has the form

$$l_h w_{ij} = h^{-1}\left[\sum_{(i-\xi)^2+(j-\eta)^2 \neq 0} \lambda_{ij\xi\eta} w_{\xi\eta} - w_{ij}\right].$$

The parameters satisfy the relations

$$\lambda_{ij\xi\eta} \geq 0, \quad \lambda_{ij\xi\eta} = 0 \quad \text{if } (i-\xi)^2 + (j-\eta)^2 \geq C_0^2,$$

$$\sum_{(\xi,\eta)\in D\cup D_0} \lambda_{ij\xi\eta} = 1, \qquad \sum_{(\xi,\eta)\in D} \lambda_{ij\xi\eta} \geq C_1, \qquad C_1 > 0,$$

and can be chosen so that

$$l_h\varphi_{ij} = \sum_r \mu_{ijr} \frac{\partial\varphi}{\partial\rho}(s_r) + h\left[\sum_r \zeta_{ijr} \frac{\partial^2\varphi}{\partial\rho\,\partial s}(s_r) + \nu_{ij}\Delta\varphi(x_i, y_j)\right] + h^2\vartheta_{ij}\varphi_{ij}^{(3)},$$

where

$$\mu_{ijr} \geq 0, \quad \sum_r \mu_{ijr} \geq C_2, \quad \sum_r |\zeta_{ijr}| + |\nu_{ij}| < C_3, \quad |\vartheta_{ij}| < C_4,$$

and $\varphi_{ij}^{(3)}$ bounds all third derivatives of φ in the neighborhood of (x_i, y_j). The boundary condition is called starlike with respect to a point $p \in G$ if $\cos(\rho, R) \leq -C_5$, where R is the radius vector from p, on all of ∂G. He proves that $|\varphi - w| = O(h^{2-2/|\log h|}(\log h)^2)$ for starlike regions under the smoothness conditions mentioned above. He then shows that the same estimate holds if the requirement that G be starlike is replaced by the requirement that $\cos(n, \rho) \leq C_6 < 1$, where n is the interior normal, be valid on ∂G. He follows these results with a number of remarks on weaker hypotheses and higher dimensions.

J. Douglas, Jr. (Houston, Tex.)

Nikolaev, P. V. 4222

On the representation of equations by nomograms of the second kind. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 1293–1296.

The author indicates that the conditions for a function $t_3 = f(t_1, t_2)$ to be represented by a nomogram in which the scale t_3 is on a straight line, notwithstanding the great number of papers published on this subject, have never been stated. {T.-H. Gronwall [J. Math. Pures Appl. (6) 8 (1912), 59–102] considered the general conditions and analysed the nomograms with three and two scales along a straight line, as well as those with two scales on the same conic. He showed that for nomograms in which two scales are on straight lines the representation problem can be solved without quadratures.} The author proves, starting with the equation of Massau, that the conditions given by him in his paper are necessary and sufficient for a function to be represented by the required nomogram. He solves the representation problem by quadratures, the general formulas being explicitly given, and shows that, but for a collineation, the representation is unique.

E. M. Bruins (Amsterdam)

COMPUTING MACHINES

See also 3429, 4076, 4181, 4217–4220, 4308, 4418, 4604, 4624, 4625, 4629.

Brudno, A. L. 4223

Stability and efficient machine time. (Russian)

Problemy Kibernet. No. 8 (1962), 243–252.

Harčenko, V. L. 4224

A machine method of designing junctions. (Russian)

Vyčisl. Sistemy No. 6 (1963), 32–40.

The author prescribes machine methods for joining pairs of distinct cells of a lattice diagram by a minimum path if (a) certain preassigned cells of the diagram are not to

contain segments of the path, and (b) the number of intersections with a preassigned path is the least possible.
E. J. Cogan (Bronxville, N.Y.)

Hillsley, R. H.; Robbins, H. M. 4225

A steepest-ascent trajectory optimization method which reduces memory requirements.
Computing Methods in Optimization Problems (*Proc. Conf., Univ. California, Los Angeles, Calif.*, 1964), *pp.* 107–133. *Academic Press, New York*, 1964.

From the authors' introduction: "This paper describes an iterative process for steepest-ascent optimization of orbit transfer trajectories. The method is designed to minimize storage requirements, at the expense of a small amount of additional computation. The reduced memory requirements eliminate the need for tape operations on the IBM 7090 when the method is used to compute nominal trajectories. In-flight use, by a guidance computer of limited memory, appears feasible."

Hrapčenko, V. M. 4226

An error bound for binary multiplication. (Russian)
Problemy Kibernet. No. 10 (1963), 165–177.

In the multiplication of two n-place numbers in a machine with a register of length $2n$, roundoff can be accomplished after the multiplication, and the error is easily calculated. Under some conditions, however, it is preferable to round at the end of each product of the multiplicand by a digit of the multiplier. This paper is a study of the error resulting from the second process.

Let $A=\sum_{i=1}^n a_i 2^{-i}$ be the multiplier and $B=\sum_{j=1}^n b_j 2^{-j}$ the multiplicand, with each a_i, b_j equal to 0 or 1. Consider roundoff, by the usual rule, of the product $Ba_i 2^{-i}$ for each i, $1\leqq i\leqq n$, to $n+p$ bits, where $p<n$. The total error in the product is $2^{-(n+p)}\delta$, where

$$\delta = 2^{-1}\sum_{i=p+1}^{n} a_i b_{n+p-i+1} - \sum_{i=p+2}^{n}\sum_{k=2}^{i-p} a_i b_{n+p-i+k} 2^{-k}$$

and $k=i+j-n-p$. It is proved that δ attains its greatest value when $a_{p+1}=a_n=1$, the sequence $a_{p+2}\cdots a_{n-1}$ is either $101010\cdots$ or $010101\cdots$, and the bits $b_{p+1},\cdots,b_n$ are related to the a's by the equation $a_i=b_{n+p-i+1}$, $p+1\leqq i\leqq n$. This largest value is $[3(n-p)+7-2^{p-n+1}]/18$. Analogous results are obtained for min δ. The inequality $n-p+2<3\cdot 2^p$ is derived for the determination of p when the error is given in advance. A comparison is made between the smallest value of p obtained for a given error under true roundoff of each product $Ba_i 2^{-i}$ (as above) and under truncation of this product at the $(n+p)$th place.
R. N. Goss (San Diego, Calif.)

Gilbert, Elmer G. 4227

The application of hybrid computers to the iterative solution of optimal control problems.
Computing Methods in Optimization Problems (*Proc. Conf., Univ. California, Los Angeles, Calif.*, 1964), *pp.* 261–284. *Academic Press, New York*, 1964.

From the author's introduction: "This paper explores certain programming aspects of hybrid computation as they pertain to the iterative solution of time optimal control problems. After reviewing the iterative procedures due to Eaton [J. Math. Anal. Appl. 5 (1962), 329–344; MR 25 #4214] and Neustadt [ibid. 1 (1960), 484–493; MR 23 #B1612; 'On synthesizing time optimal control systems', Second Congr. Internat. Federation of Automatic Control (Basel, 1963)], related procedures are derived which are better suited to hybrid computation. Then the programming of these procedures is described, along with a brief discussion of the characteristics of hybrid computers. Finally, some computer results are presented."

GENERAL APPLIED MATHEMATICS

See also 4230.

Tola, José [Tola Pasquel, José] 4228

On the use of finite reasoning in establishing the differential equations of engineering problems. (Spanish)
Univ. Nac. Ingen. Inst. Mat. Puras Apl. Notas Mat. 1 (1962/63), 97–106.

MECHANICS OF PARTICLES AND SYSTEMS

See also 3340, 3341, 3731, 3919, 3964–3966, 4347, 4372, 4433, 4588, 4596.

Ziegler, Hans (Editor) 4229

★Kreiselprobleme [Gyrodynamics].
Symposion Celerina, 20. bis 23. August, 1962. International Union of Theoretical and Applied Mechanics.
Springer-Verlag, Berlin, 1963. xi + 303 *pp.* *DM* 56.00.

This volume contains papers presented at the above symposium. Those papers of mathematical interest will be reviewed individually.

Lanczos, Cornelius 4230

★The variational principles of mechanics.
Second edition. Mathematical Expositions, No. 4.
University of Toronto Press, Toronto, Ont., 1962 (*reprinted* 1964). xxv + 367 *pp.* $7.50.

This is an excellent book which covers a great deal of information on variational methods. The brilliant accomplishments as explained in the text would definitely be welcomed by students and teachers of analytical mechanics.

This book is a corrected edition of the previous (1949) edition [MR 11, 549] with the excellent addition of a chapter on relativity. The author traces the historical development, beginning from the basic ideas of mechanics to the more powerful and advanced methods of D'Alembert, Lagrange, Hamilton, canonical transformations and the Hamilton-Jacobi equation. A number of clear specific examples are presented and terse, clear summaries are given in each section. The fundamental results explained in each section enable the readers to follow the trend and general line of thought in a most clear and lucid fashion. A good illustration is Chapter V (50 pages) entitled, "The Lagrangian Equations of Motion". Using the principles in the previous chapter entitled "D'Alembert Principle", the reader is introduced to the "Hamilton's Principle". This method is the most

direct and most natural transformation of D'Alembert into a minimum principle or, more generally, the stationary value of a definite integral. The step-by-step procedure leads from the Hamilton principle to the Lagrange equation, Jacobi's principle and Lagrangian multiplier. The holonomic and scleronomic systems, i.e., the Lagrangian L which does not contain the time explicitly, are explained very clearly. The final portion of this chapter concludes with the application of the principles to the theory of small vibrations. The last chapter (IX) is an excellent introduction to relativistic mechanics. Based upon the previous Hamiltonian formulation, additional information on Einstein's observations plus Minkowski's classic paper on the four-dimensional world and Lorentz transformations, the basic elementary concept of "General Relativity" is explained in a very simple fashion.

The reviewer regrets that the author has not expanded his work to encompass the variational principles of solid mechanics. The application of Rayleigh-Ritz, Galerkin and Reissner's method would be a most welcome addition to this book. With the advent of digital computers, these variational applications to solid mechanics play a most important part. *H. Saunders* (Philadelphia, Pa.)

Dupont, Pascal 4231

Sulla determinazione della curvatura delle polari nei moti rigidi piani.

Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **98** (1963/64), 397–417.

The author indicates that geometrical and analytical methods are equally valuable in the study of the kinematics of rigid moving planes, his interest here being in the points of contact of the polars and their radii of curvature, two pairs of conjugate profiles being given. The analytical treatment by Grübler in his articles on the motion of the poles [Z. Math. Phys. **29** (1884), 212–221; ibid. **34** (1889), 305–310] is criticised for lack of generality. Certain particular cases of moving profiles are considered, notably the beam of an articulated quadrilateral.

G. Huxley (Belfast)

Dupont, Pascal 4232

Il teorema di Bobillier, il problema della retta tangente alle polari e sua applicazione per la determinazione del centro delle accelerazioni.

Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **98** (1963/64), 442–471.

The contents are adequately indicated by the title. Two theorems of Bobillier (1797–1832) are considered, together with their kinematical interpretation by Aronhold. Burmester's treatment of Bobillier's construction [Burmester, *Lehrbuch der Kinematik*, Leipzig, 1888] is touched upon, and a trigonometrical demonstration of his theorems provided. The critical survey ends with an application of Bobillier's ideas to the study of centres of acceleration.

G. Huxley (Belfast)

Dupont, Pascal 4233

Determinazione grafica del centro di curvatura della polare fissa nei moti rigidi piani.

Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **98** (1963/64), 489–513.

The author considers the problem of determining the centre of curvature of the fixed polar, given the position of a moving plane by means of two pairs of conjugate profiles. L. Burmester's graphical solution [*Lehrbuch der Kinematik*, p. 106, Leipzig, 1888] is examined and corrected, and then another graphical solution is offered, involving quite complicated constructions. The tendency is less historical than in the other papers in this series.

G. Huxley (Belfast)

Arens, Richard 4234

Differential-geometric elements of analytic dynamics.

J. Math. Anal. Appl. **9** (1964), 165–202.

The author describes at some length an approach to classical mechanics via the theory of Pfaffian forms. This approach is based to a large extent on his point of view according to which "the Hamilton-Jacobi technique ... is just a procedure for reducing Pfaffians (that is, covariant vector fields) to normal form". The first section of the paper is devoted to this topic (largely in the form of examples). This is followed by some remarks concerning the connection between the calculus of variations and the theory of Pfaffians. The usual phase space is described at some length in terms of the tangent bundle of the underlying configuration space. This leads to a discussion of extremals of variational problems, including geodesics of Riemannian geometry. The so-called absolute mechanical systems are then defined in such a way as to facilitate a direct application of the method of Pfaffians, which in turn gives rise to a comparison of classical with relativistic mechanics. *H. Rund* (Pretoria)

Pollard, Harry 4235

A sharp form of the Virial Theorem.

Bull. Amer. Math. Soc. **70** (1964), 703–705.

Soit un système S de n points matériels, soumis à l'attraction newtonienne, le centre de masses étant fixe, T son énergie cinétique totale, h la constante d'énergie ($T+V=h$). l'auteur prouve que l'égalité

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t T(\tau)\,d\tau = -h$$

est vraie si et seulement si, $R(t)$ désignant le maximum des distances mutuelles des points, on a l'égalité $R(t)=o(t)$ pour $t\to\infty$. La partie délicate de la démonstration utilise un certain théorème de Landau, pour lequel on pourra se reporter à un article de Boas [Duke Math. J. **3** (1937), 637–646].

Sans aucune hypothèse a priori sur la croissance du système S, l'auteur prouve d'autre part le théorème suivant: $\limsup_{t\to\infty} (1/t)\int_0^t T(\tau)\,d\tau \ge |h|$.

M. Janet (Paris)

Zimmerman, J. R. 4236

Five precision point synthesis of the four-bar function generator.

Trans. ASME Ser. E. J. Appl. Mech. **31** (1964), 563–564.

Author's introduction: "Presently used algebraic methods for synthesizing the four-bar function generator usually yield compatibility equations having extraneous roots. Use of a superposition technique for the case of five precision points leads to a cubic compatibility equation

free of extraneous roots. The method presented here lends itself well to programming for a digital computer, available library subroutines sufficing for the more critical and tedious portions of the computation."

Raitzin, Carlos 4237

On the variational principles of dynamics. (Spanish)

An. Soc. Ci. Argentina **176** (1963), 62–70.

In a previous paper [same An. **171** (1961), 50–65; MR **23** #B1711] the author has introduced a new function $R(s_i, r_i) = L - \sum q_i r_i$ with L: Lagrange's function, $s_i = \dot{q}_i$, $r_i = \dot{p}_i$. In this paper, the author develops four variational properties, using Hölder's operator $\Delta = (\partial/\partial\varepsilon + (d/dt) \times (dt/d\varepsilon))\delta\varepsilon$ (which amounts to considering non-synchronous variations). (1) $\Delta \int_{t_1}^{t_2} R\,dt = 0$; (2) $\Delta(A_c + A_p) = 0$, which is a generalization of Maupertuis' principle, A_c being the kinetic action $\int_{t_1}^{t_2} \sum p_i s_i\,dt = \int_{t_1}^{t_2} \sum (\partial L/\partial \dot{q}_i)\dot{q}_i\,dt$ and A_p being the newly defined potential action $\int_{t_1}^{t_2} \sum r_i q_i\,dt = \int_{t_1}^{t_2} \sum (\partial L/\partial q_i) q_i\,dt$ (this result is valid for non-conservative systems having a Lagrangian); (3) $\delta \int_{t_1}^{t_2} (R - H)\,dt = 0$, which is an extension of Hamilton's principle; (4) Proof of Hamilton's principle $\delta \int_{t_1}^{t_2} L\,dt = 0$ for non-synchronous paths, i.e., $\delta t \neq 0$. *J. Kestens* (Brussels)

Udeschini Brinis, Elisa 4238

Sul divario fra due campi cinetici.

Ist. Lombardo Accad. Sci. Lett. Rend. A **97** (1963), 274–281.

Résumé de l'auteur: "Calcolato il divario fra due campi cinetici dati, lo si interpreta come distribuzione delle velocità 'relative' ad uno di essi. Si stabilisce, di conseguenza, una estensione sia del teorema di Coriolis, sia delle formule di Poisson."

J. Charles-Renaudie (Montpellier)

Udeschini Brinis, Elisa 4239

Sui sistemi meccanici dinamicamente equivalenti.

Ist. Lombardo Accad. Sci. Lett. Rend. A **97** (1963), 592–602.

The dynamical systems under consideration are holonomic with constraints which are independent of the time. The configuration space of a system having kinetic energy T is regarded as a Riemannian space with the metric $ds^2 = 2T\,dt^2$. Two systems, m and μ, each with n degrees of freedom, are said to be dynamically equivalent if coordinates $q^1, q^2, \cdots, q^n$ for m, and coordinates $\chi^1, \chi^2, \cdots, \chi^n$ for μ, can be chosen so that q^i and χ^i are identically equal, as functions of t, for all values of the index i. It is shown that a necessary and sufficient condition that two systems, m and μ, with kinetic energies T and $\mathcal{T}$, respectively, be dynamically equivalent is that $D^i{}_{hk}\dot{q}^h\dot{q}^k = \Phi^i - F^i$ ($i = 1, 2, \cdots, n$). Here F^i and Φ^i are components of the (contravariant) generalized forces applied to m and μ, and $D^i{}_{hk} = \left\{\begin{Bmatrix} i \\ h\ k \end{Bmatrix}\right\} - \begin{Bmatrix} i \\ h\ k \end{Bmatrix}$, the terms in the right-hand member being the Christoffel symbols of the second kind calculated with respect to the metrics $d\sigma^2 = 2\mathcal{T}\,dt^2$ and $ds^2 = 2T\,dt^2$, respectively. (It is understood that the calculation of $D^i{}_{hk}$ and Φ^i involves setting $\chi^i = q^i$ for $i = 1, 2, \cdots, n$.) The paper concludes with an extended discussion of the case in which m is subjected to a conservative force derived from a potential U, and in which the metrics of the configuration spaces of m and μ satisfy the relation $d\sigma^2 = 2(E + U)E^{-1}\,ds^2$, the constant E being the total energy of the system m. In this case, if the two systems are dynamically equivalent, the dynamical trajectories of μ in its configuration space are geodesics. The case includes several subcases presenting various features of interest. *L. A. MacColl* (New York)

Datzeff, Assène [Dacev, A.] 4240

Sur les principes d'extremum en mécanique classique.

C. R. Acad. Sci. Paris **259** (1964), 56–59.

In applying Hamilton's principle to a conservative holonomic system, one considers a natural motion from one configuration to another and a family of generally unnatural neighboring motions joining the same two configurations. In the present note the author considers each of the varied motions as a natural motion under appropriately defined forces, and he discusses relations between the values of the action integral calculated for the original natural motion and for the several varied motions. The purpose of these unusual considerations is not clearly explained. *L. A. MacColl* (New York)

Vacca, Maria Teresa 4241

Sui sistemi anolonomi riducibili a forme lagrangiane.

Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **98** (1963/64), 375–382.

A discussion of some situations in which a nonholonomic dynamical system is equivalent to a holonomic system admitting certain integrals, linear or quadratic in the velocities. The details of the argument are not entirely clear to the reviewer. *L. A. MacColl* (New York)

Vujicić, Veljko [Vujičić, V. A.] 4242

Sur certaines questions de la mécanique analytique des systèmes non holonomes.

C. R. Acad. Sci. Paris **259** (1964), 709–711.

From the author's summary: "Nous avons mis en évidence la possibilité de former toutes les équations de la dynamique, ainsi que celles, plus générales, de la physique à partir d'un principe général de la mécanique, connu comme méthode de Pfaff ou de Pfaff-Bilimović. Dans ce travail nous montrerons comment cette méthode peut être utilement appliquée à l'étude des systèmes non holonomes grâce à l'emploi des quasi-coordonnées."

Czarnecki, Adam Z. 4243

Physical systems S_k of curves in a central field of force.

Math. Notae **19** (1964), 137–146.

Ein Massenpunkt (Ortsvektor $\mathbf{r}(x, y, z)$, Masse $m = 1$) sei durch das Kraftfeld $\mathbf{F}(x, y, z)$ gezwungen, eine Kurve C (Ortsvektor $\mathbf{r}(t)$) zu durchlaufen. Es seien s die Bogenlänge und ρ der Krümmungsradius der Bahn C, ferner v die Geschwindigkeit des Teilchens, schließlich $r' = dr/ds$ usw. Ein physikalisches System S_k besteht dann nach E. Kasner und J. DeCicco [Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 106–108; MR **10**, 398] aus allen Kurven C, längs denen eine Zwangsbewegung mit der Eigenschaft möglich ist, daß die Schmiegebene in jedem Punkt der Kurve C den dort wirkenden Kraftvektor $\mathbf{F}$ enthält und der Druck P,

definiert durch $P = mv^2/\rho - F_N$ mit einem Faktor k proportional ist zu der Normalkomponente $F_N = \rho(\mathbf{F}\cdot\mathbf{r}'')$ von $\mathbf{F}$. Längs jeder Kurve C eines physikalischen Systems S_k gilt also $P = kF_N$. Besondere Fälle ergeben sich für $k = 0$ (Kraftlinien S_0), $k = 1$ (allgemeine Kettenlinien S_1), $k = -2$ (allgemeine Brachystochronen S_{-2}), $k \to \infty$ (Geschwindigkeitssystem S_∞; die Bahnkurven der Massenpunkte, die mit fester Geschwindigkeit in Kraftrichtung starten, haben dabei Bahnen, welche die Kraftlinien oskulieren). Der Fall $k = -1$ ist auszuschließen.

Die Arbeit studiert hauptsächlich den Fall eines zentralen Kraftfeldes. Die Kraftvektoren sind dann dem Ortsvektor proportional: $\mathbf{F} = \phi(x, y, z)\mathbf{r}$, wobei im konservativen Fall $\phi = \phi(r)$ ist. Die ∞^6 Kurven C von S_k sondern sich dann in ∞^2 Familien von ebenen Kurven. Auch die Umkehrung gilt ($k = 0$ gibt den Satz von Halphen). Verallgemeinerungen des Keplerschen Flächensatzes ($k = 0$) auf den Fall $k \neq 0$ ergeben sich als Folgerungen aus der Differentialgleichung des physikalischen Systems S_k.

Schließlich wird noch der Fall eines konservativen Zentralkraftfeldes studiert. Das physikalische System S_k ($k \neq -1$) oder S_∞ in Polarkoordinaten zu beschreiben erfordert dann lediglich zwei Integrationen, wobei die erste das Energieintegral liefert. Kennt man die Potentialfunktion des Zentralfeldes, so reicht eine Integration aus.

Das Problem, physikalische Systeme S_k im einfachen Fall eines Newtonschen Zentralfeldes zu finden, ist damit auf zwei Quadraturen zurückgeführt, die aber nur in bestimmten Fällen (z.B. für $k = 0$) elementar ausfallen.

K. Strubecker (Karlsruhe)

Arkhangel'skii, Iu. A. [Arhangel'skii, Ju. A.] 4244

On a motion of an equilibrated gyroscope in the Newtonian force field.

Prikl. Mat. Meh. **27** (1963), 1099–1101 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1684–1688.

The author shows that the equations of motion of a heavy solid about a point fixed in its center of gravity can be reduced to classical equations of motion of a solid about a fixed center as in the case of Euler, and solves the equations when the initial spin velocity is large.

Y. Kozai (Tokyo)

Gončarenko, V. M. 4245

On the time of fluctuational escape of a dynamical system from a given region. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 583–586.

Author's summary: "The author considers a dynamic system, the change of phase variables of which is a multidimensional Markov system. Differential equations are deduced for the dispersion and for higher moments of a random interval of time, which lasts till the first escape from the given region. The dispersion is negligible if the perturbations are weak."

Mayné, Georges 4246

Détermination des cas de séparabilité pour les systèmes dynamiques scléronomes de fonction hamiltonienne non homogène en les momentoïdes.

C. R. Acad. Sci. Paris **259** (1964), 60–63.

By a generalization of a method of P. Burgatti [Atti Accad. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (5) **20** (1911), 1° semestre, 108–111] and Dall'Acqua [Rend. Circ. Mat. Palermo **33** (1912), 341–351] the problem of finding the types of separable dynamic systems with non-homogeneous Hamiltonian is solved. Application to three-dimensional systems is discussed. *K. Forster* (Los Angeles, Calif.)

Goroško, O. O.; Krasil'nikov, K. V. 4247

Transverse vibrations of a string of variable length. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 319–322.

Authors' summary: "With the help of asymptotic methods a solution is obtained for the problem of transverse vibrations of a string of variable length with mobile ends. It was found that in the first approximation the amplitudes of the free tones are constant for a string of variable length, while the total energy of the transverse vibrations vary in inverse proportion to the length of the vibrating string $l_0 - l$. The form of the weak dependence between various tones of the free vibrations of a string of variable length is determined. The solutions obtained may be used for calculating the flexible elements of transporting and hoisting devices."

Quick, William H. 4248

Theory of the vibrating string as an angular motion sensor.

Trans. ASME Ser. E. J. Appl. Mech. **31** (1964), 523–534.

Equations are presented for the vibration of a string driven by longitudinal motion at one end. It is assumed that axial frequencies in the string are much higher than lateral frequencies. It is shown that non-linear effects play a predominant influence in the usefulness of this concept as a "gyro". With large amplitude in one direction and smaller amplitude transversely, frequencies are unequal in the two directions, and precession occurs. Other conditions for precession are also presented.

Samuel Levy (Schenectady, N.Y.)

Harlamov, S. A. 4249

An example of a heteroparametric perturbation of a pendulum by quasi-periodic oscillations of the suspension. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 1311–1313.

Im Anschluß an Arbeiten von Bogoljubov, Mitropolski und Caughey untersucht der Verfasser parametererregte Schwingungen eines mathematischen Pendels mit vertikaler Achse. Die vertikale Achse wird in horizontaler Richtung quasiperiodisch nach dem Gesetz $\xi = \xi_0 \sin \Omega t \times \sin \omega t$ bewegt. Es wird gezeigt, daß unter diesen Bedingungen Rotationen des Pendels um die vertikale Achse mit der Schwebungsfrequenz der Parametererregung möglich sind. Das Phasenporträt für die möglichen Bewegungen wird skizziert. *K. Magnus* (Stuttgart)

Morozov, V. M. 4250

A case of stability of the unsteady motion of a top on a plane. (Russian. English summary)

Vestnik Moskov. Univ. Ser. I Mat. Meh. **1964**, no. 3, 70–74.

The paper is concerned with the stability of the "tippe-top" supported by a smooth horizontal plane and subject to forces, the moment of which is proportional to the projection of the angular velocity on the vertical. An estimate for the domain of admissible perturbations is given [cf. also V. V. Rumjancev, Prikl. Mat. Meh. **25** (1961), 778–784; MR **24** #B2145; S. O'Brien and J. L. Synge, Proc. Roy. Irish Acad. Sect. A **56** (1954), 23–35; MR **15**, 659; L. S. Isaeva, Prikl. Mat. Meh. **23** (1959), 403–406; MR **22** #1121; F. Schuh, Nederl. Akad. Wetensch. Proc. Ser. A **56** (1953), 423–432; MR **15**, 567; A. P. Duvakin, Inž. Ž. **2** (1962), 222–230].

E. Leimanis (Vancouver, B.C.)

Herrmann, G.; Bungay, R. W. 4251

On the stability of elastic systems subjected to nonconservative forces.

Trans. ASME Ser. E. J. Appl. Mech. **31** (1964), 435–440.

From the authors' summary: "Free motions of a linear elastic, nondissipative, two-degree-of-freedom system, subjected to a static nonconservative loading, are analyzed with the aim of studying the connection between the two instability mechanisms (termed divergence and flutter by analogy to aeroelastic phenomena) known to be possible for such systems." *R. A. Struble* (Raleigh, N.C.)

Schmidt, Günter 4252

Zur Stabilität der Längs- und Querschwingungen eines längs pulsierend belasteten Stabes.

Math. Nachr. **27** (1963/64), 341–351.

Unter Verwendung der Floquet'schen Theorie werden die Stabilitätsbedingungen für ein System linearer Differentialgleichungen mit periodischen Koeffizienten in sehr allgemeiner Form angegeben. Diese Ergebnisse werden dann auf das Problem der parametererregten Schwingungen eines gelenkig gelagerten, in der Längsrichtung pulsierend belasteten Stabes angewendet. Für den Fall der Längsschwingungen kommt man auf diese Weise zu expliziten Angaben über die Stabilität. Bei den Querschwingungen müssen die Fälle der einfachen und der Kombinationsresonanz unterschieden werden. Hier werden die Bereiche des Überhängens und die Existenz periodischer Lösungen berechnet. *K. Magnus* (Stuttgart)

Yamamoto, Toshio; Ōta, Hiroshi 4253

On the unstable vibrations of a shaft carrying an unsymmetrical rotor.

Trans. ASME Ser. E. J. Appl. Mech. **31** (1964), 515–522.

Authors' summary: "In a rotating shaft system carrying an unsymmetrical rotor, there is always one unstable region in the neighborhood of the rotating speed at which the sum of two natural frequencies of the system is equal to twice the rotating speed of the shaft. In this unstable region two unstable lateral vibrations with frequencies P_1 and P_2 take place simultaneously and grow up steadily. Generally, frequencies P_1 and P_2 are not equal to the rotating speed ω of the shaft and the sum of these P_1+P_2 is always equal to 2ω. Of course there are other unstable regions which appear at the major critical speeds."

Kolomiec', V. G. 4254

The effect of harmonic and random forces on a self-oscillatory system. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 701–704.

Author's summary: "Using the Krylov-Bogolyubov methods and the Fokker-Planck-Kolmogorov equations, the author finds the stationary density of the joint distribution of probabilities of amplitude and phase fluctuations of a self-oscillatory system. The result is a statistical description of the self-oscillations of the system."

Struble, R. A.; Warmbrod, G. K. 4255

Free resonant oscillations of a conservative two-degree-of-freedom system.

J. Franklin Inst. **278** (1964), 195–209.

From the authors' summary: "The resonance phenomena associated with the free oscillations of a pair of coupled nonlinear differential equations of the Duffing type are analyzed in the general case. A method for applying the analysis to the more general case is illustrated for a particular perturbational technique—the method of harmonic balance." *N. D. SenGupta* (Bombay)

ELASTICITY, PLASTICITY

See also 4172, 4212, 4383.

★**Stress waves in anelastic solids.** 4256

Symposium held at Brown University, Providence, R.I., April 3–5, 1963. Edited by Herbert Kolsky and William Prager. International Union of Theoretical and Applied Mechanics.

Springer-Verlag, Berlin, 1964. xii+342 pp. *DM* 67.50.

This volume contains the published papers given at the symposium described in the heading. Most of the papers will be reviewed individually.

Duvaut, Georges 4257

Application du principe de l'indifférence matérielle à un milieu élastique matériellement polarisé.

C. R. Acad. Sci. Paris **258** (1964), 3631–3634.

En appliquant le principe de l'indifférence matérielle (les lois de comportement, qui lient les tensions et les déformations, ne dépendent pas de l'observateur) pour deux observateurs en rotation l'un par rapport à l'autre, on étudie les lois de comportement de matériaux élastiques sans énergie de déformation.

P. P. Teodorescu (Bucharest)

Galletto, Dionigi 4258

Nuove forme per le equazioni in coordinate generali della statica dei continui con caratteristiche di tensione asimmetriche.

Ann. Scuola Norm. Sup. Pisa (3) **17** (1963), 297–317.

The author studies the tensor formulas for various physical quantities associated with two sets of reference frames. The quantities studied are force, velocity,

acceleration, moments, strain components, etc. Formulas for the behavior of these quantities under coordinate change are given, and a detailed examination of the quantities is furnished. The work is an extension of work by C. Truesdell and R. Toupin [*Handbuch der Physik*, Bd. III/1, pp. 226–793, Springer, Berlin, 1960; MR **22** #8778] and other authors [see, for instance, A. Signorini, Ann. Mat. Pura Appl. (4) **22** (1943), 33–143; MR 8, 240; errata, MR 8, 708; C. Tolotti, Rend. Accad. Sci. Fis. Mat. Napoli (4) **13** (1945), 69–77; MR 8, 240; A. Tonolo, Rend. Sem. Mat. Univ. Padova **14** (1943), 43–117; MR 8, 356].
N. Coburn (Ann Arbor, Mich.)

Knops, R. J. 4259

Uniqueness for the whole space in classical elasticity.

J. London Math. Soc. **39** (1964), 708–712.

The author is concerned with the equations of classical elasticity in which the Lamé constants λ and μ are permitted to vary outside the range of physical interest. He shows that (i) If the displacement vector tends uniformly to zero at infinity and satisfies the field equations of elasticity in the whole space, then it vanishes identically everywhere provided $\lambda+2\mu\neq 0$, $\mu\neq 0$; (ii) If the stress components tend uniformly to zero at infinity and satisfy the field equations of elasticity in the whole space, then they vanish identically everywhere provided $\lambda+2\mu\neq 0$.

From this and appropriate examples it follows that for solutions of the field equations which satisfy certain order conditions at infinity, the requirement $\lambda+2\mu\neq 0$, $\mu\neq 0$ is both necessary and sufficient for uniqueness in the displacement boundary-value problem. On the other hand, if the stresses vanish appropriately at infinity, a necessary and sufficient condition that the stress be unique is that $\lambda+2\mu\neq 0$.
L. E. Payne (College Park, Md.)

Adler, György 4260

Majoration des tensions dans un corps élastique à l'aide des déplacements superficiels.

Arch. Rational Mech. Anal. **16** (1964), 354–372.

The system of equations

$$\Delta u_i + k\frac{\partial}{\partial x_i}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right) = 0 \qquad (i = 1, 2, 3),$$

where $k > -1$ is a constant, is considered in a domain Ω with frontier Σ. Certain upper bounds for

$$\max_{\Omega+\Sigma}\max_{i=1,2,3}|\operatorname{grad} u_i|$$

are found in terms of the geometrical properties of Σ and of the values of grad u_i at points of Σ.
L. A. MacColl (New York)

Resende, Eliseu 4261

Generalized progressing wave expansion for the equations of elasticity.

An. Acad. Brasil. Ci. **36** (1964), 7–11.

Author's introduction: "The purpose of this work may be described as an application of the methods of geometrical optics to initial-boundary value problems in elasticity. A generalized progressing wave expansion is constructed by performing the transition from the partial differential equations of motion in elasticity to ordinary differential equations along bicharacteristics. In the special case of periodic solutions these generalized progressing wave expansions become asymptotic expansions with respect to the frequency [F. C. Karal and J. B. Keller, J. Acoust. Soc. Amer. **31** (1959), 694–705; MR **21** #2416]. By letting the wave form a distribution, these expansions permit the discussion of the propagation of discontinuities [R. Courant and P. Lax, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 872–876; MR 18, 399] in an elastic medium."

Spencer, A. J. M. 4262

Finite deformations of an almost incompressible elastic solid.

Second-order Effects in Elasticity, Plasticity and Fluid Dynamics (Internat. Sympos., Haifa, 1962), pp. 200–216. Jerusalem Academic Press, Jerusalem; Pergamon, Oxford, 1964.

This paper deals with isotropic elastic materials which, under finite deformation, allow only small relative changes in elemental volume of order ε. These materials are classified as almost incompressible elastic solids. A theory is developed which permits treatment of finite elasticity problems for this class of solids once the solution to the geometrically equivalent but strictly incompressible solid is obtained. The approach is based on expansion of the strain-energy function in powers of (I_3-1), where I_3 denotes the third strain invariant. The resulting equations possess certain similarities to those obtained in the theory of small deformations superposed on large deformations of elastic solids.

An illustration is given in which the author treats the problem of simultaneous extension, inflation and torsion of an almost incompressible circular cylinder.
R. L. Fosdick (Chicago, Ill.)

Mahovikov, V. I. 4263

A dynamic problem of elasticity theory for a plate of transversely isotropic material. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, no. 4 (41), 111–117.

Four boundary conditions are considered in this paper. With the usual notation in thin-plate theory they are, on $z=\pm h$:

(1) $\sigma_z = q\sin\gamma ct,\qquad \tau_{xz}-i\tau_{yz} = \pm p\sin\gamma ct;$

(2) $w = \pm q\sin\gamma ct,\qquad u-iv = p\sin\gamma ct;$

(3) $w = \pm q\sin\gamma ct,\qquad \tau_{xz}-i\tau_{yz} = \pm p\sin\gamma ct;$

(4) $\sigma_z = q\sin\gamma ct,\qquad u-iv = p\sin\gamma ct.$

J. Hubert Wilkinson (Battersea)

Pan, S. K. 4264

Thermal stresses in an infinite elastic plate containing two unequal circular holes the boundaries of which are kept at different temperatures.

Bull. Calcutta Math. Soc. **55** (1963), 63–77.

Author's summary: "Stresses in an infinite elastic plate containing two unequal circular holes can be found by means of bipolar coordinates. In the case considered, the

stresses are caused by the difference in temperature of the boundaries of the holes. A method is given here to calculate the stresses. Numerical values of stresses are also given when the radii of the holes are 0.1653*a* and −0.4692*a* respectively."

Westmann, R. A. 4265

Pressurized star crack.

J. Math. and Phys. **43** (1964), 191–198.

A thin elastic plate contains *n* linear cracks, each of length *a*, radiating symmetrically from the origin. The crack faces are free from shear and subject to uniform normal pressure. The elastic field is found by using Mellin transforms, and the stress concentration factor at the tip of each crack is determined. An asymptotic formula is found which is accurate for $n > 6$.

F. R. N. Nabarro (Cleveland, Ohio)

Ufljand, Ja. S. 4266

The behaviour of stresses at an edge point of a wedge. (Russian)

Leningrad. Politehn. Inst. Trudy No. 226 (1963), 109–113.

En utilisant la transformation de Mellin, on étudie l'état de tension autour du sommet d'un coin plan élastique d'angle 2α, actionné par des charges concentrées dans cette région. La discussion d'après divers angles α peut être mise en liaison avec des études antérieures qu'on a fait dans le dernier temps dans cette direction. Remarquons les considérations relativement simples qui ont permis de trouver ces résultats.

P. P. Teodorescu (Bucharest)

Arutjunjan, N. H.; Abramjan, B. L.; Bablojan, A. A. 4267

On the compressibility of an elastic sphere with a rigid annular housing. (Russian. Armenian summary)

Izv. Akad. Nauk Armjan. SSR Ser. Fiz.-Mat. Nauk **17** (1964), *no.* 3, 55–63.

En utilisant des développements en série en termes de polynômes de Legendre, on considère le problème de la sphère comprimée par des charges reparties sur deux calottes opposées et encerclée par un anneau rigide. L'étude du problème est réduite à la solution d'un système infini d'équations algébriques.

P. P. Teodorescu (Bucharest)

Eason, G. 4268

On the torsional impulsive body force within an elastic half space.

Mathematika 11 (1964), 75–82.

The title problem of linear elasticity is solved using Hankel, Fourier and Laplace transforms. Impulsive (delta function) torque acts in a plane parallel to the surface, which is assumed to be free of stress or rigidly clamped. The solution is given in closed form.

H. Parkus (Vienna)

Heymann, Joachim 4269

Halbraum unter elliptisch begrenzter, gleichmässiger Flächenlast.

Z. Angew. Math. Mech. **43** (1963), 568–572.

This paper deals with the stress in a semi-infinite solid due to a uniformly distributed load over an elliptic area on its plane boundary.

R. M. Morris (Cardiff)

Keer, L. M. 4270

The torsion of a rigid punch in contact with an elastic layer where the friction law is arbitrary.

Trans. ASME Ser. E. J. Appl. Mech. **31** (1964), 430–434.

Author's summary: "The problem of torsion of a rigid punch in contact with an elastic layer where the friction law is arbitrary is solved by the method of A. H. England [Proc. Cambridge Philos. Soc. **58** (1962), 539–547]. The solution is seen to depend upon the further solution of two independent Fredholm integral equations. As a first approximation to these equations the problem for torsion on an elastic half-space is solved."

Westmann, R. A. 4271

Layered systems subjected to asymmetric surface shears.

Proc. Roy. Soc. Edinburgh Sect. A **66** (1962/63), 140–149 (1964).

Author's summary: "This paper formulates a general solution, within the scope of classical elastostatic theory, for the problem of layered systems subjected to asymmetric surface shears. As an illustrative example the solution for the problem of an elastic layer supported on an elastic half-space is presented for the particular loading consisting of a surface shearing force uniformly distributed over a circular area. Numerical results are included indicating some displacement and stress components of interest."

R. Muki (Tokyo)

Ghosh, Basudev 4272

Torsion of a composite beam of rectangular cross-section.

J. Tech. Bengal Engrg. College 7 (1962), 121–130.

The torsion problem for a composite beam of rectangular section is solved by means of the classical method of constructing the Green's function for the two regions involved.

B. R. Seth (Kharagpur)

Golubev, O. B. 4273

A generalization of the theory of thin rods. (Russian)

Leningrad. Politehn. Inst. Trudy No. 226 (1963), 83–92.

The classical theory of the small deformation of thin rods is due to Kirchhoff and Clebsch; a modern account can be found in a paper by Lur'e [same Trudy No. 3 (1941), 148–157]. The theory of equilibrium and stability of thin-walled elastic rods with rectilinear axes was developed by Adadurov, Vlasov, Umanskiĭ and others, while rods with curvilinear axes were investigated by Džanelidze. (Full references to Soviet work are given in the paper under review.) The author considers the small deformation of a thin solid (or a thin-walled) elastic isotropic rod whose axis is a general space curve. Two kinematical assumptions are made: (1) deformations in the plane of a cross-section of the rod are absent, that is, the displacement of a point of the section consists of a displacement of the section as a rigid whole together with warping of the section in the perpendicular direction;

and (2) warping of the section is proportional to a certain known function, here taken to be the Saint-Venant warping function. Expressions, in terms of integrals taken over a general cross-section, are obtained for the potential energy of deformation and for the work done by the external forces applied to the rod. The complete set of equations pertaining to the theory is displayed in the final section of the paper. *J. Hubert Wilkinson* (Battersea)

Herrmann, L. R. 4274

A three-dimensional elasticity solution for continuous beams.

J. Franklin Inst. **278** (1964), 75–83.

Author's summary: "A method for the development of solutions, within the framework of three-dimensional elasticity, for continuous beam problems is illustrated. A three-dimensional continuous beam analysis will find utility in analyzing beam-like structures whose cross-sectional dimensions, as compared to their span lengths, are such that neither the usual beam assumption (which yields a one-dimensional problem) or the plane stress assumption (which yields a two-dimensional problem) can be invoked. Thus, since neither the beam nor plane stress assumptions have been applied, the structural element under consideration may be as deep or wide as desired. Additionally, a beam subjected to a load which varies with the transverse direction may be rigorously analyzed. The solution is restricted to a continuous beam in which the loading conditions are identical over each span. The solution is in the form of two double series, the first of which is a double Fourier series, the second a combination Fourier and Fadle eigenfunction series."

W. D. Kroll (Washington, D.C.)

Krul, I. I. 4275

An investigation of forced vibrations of a rod allowing for after-effects of the material. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 589–592.

From the author's summary: "The solution of the integro-differential equation

$$\rho s \frac{\partial^2 y}{\partial t^2} + EI \frac{\partial^4 y}{\partial x^4} = F(x, t)$$

of the forced cross oscillations of a rod is obtained by the integro-operator method. The rod material is visco-elastic."

Sastry, U. A. 4276

Torsion of hollow prismatic cylinders.

Boll. Un. Mat. Ital. (3) **19** (1964), 114–120.

Author's summary: "In this paper (a) the torsion of a hollow cylinder whose outer cross-section is a circle and inner cross-section a square with rounded corners and (b) the torsion of a hollow prismatic beam bounded externally by a quartic curve and internally by a circle have been studied by using Schwarz's alternating method."

W. D. Kroll (Washington, D.C.)

Makai, E. 4277

On the principal frequency of a membrane and the torsional rigidity of a beam.

Studies in mathematical analysis and related topics, pp. 227–231. *Stanford Univ. Press, Stanford, Calif.*, 1962.

For a beam with a cross-section given by any simply connected plane convex domain D, the author finds two upper bounds for the torsional rigidity

$$P \leq \frac{8}{3}\frac{A^3}{L^2}, \qquad P < \frac{4}{3}A\rho^2,$$

where A is the area of D, L is the length of the perimeter of D, and ρ is the radius of the greatest circle contained in D.

For a membrane over any simply connected plane convex domain D, the author finds a lower bound for the principal frequency

$$\Lambda \geq \frac{\pi}{4}\frac{L}{A},$$

where L and A are defined as above.

A. Phillips (New Haven, Conn.)

Tola, José [Tola Pasquel, José] 4278

Application of Kani's method to structures with variable cross-section. (Spanish)

Univ. Nac. Ingen. Inst. Mat. Puras Apl. Notas Mat. **2** (1964), 8–15.

This paper deals with the problem of building frames made out of beams of variable cross-sections. The author refers to equations established by him previously in 1954. As it is classical, three states of the structure are considered. (1) The "initial" state where the structure is deformed by the given vertical loads and such that the joints do not move or rotate; this is possible by introducing horizontal forces and end moments to the beams. (2) The "complementary" state where the structure is subjected to these forces and moments, with reversed signs. (3) The "actual" state which results as the superposition of (1) and (2).

State (1) can be solved using tables established by Kani (1958); the coefficients which are to be computed are those appearing in H. Cross's method. The author shows how to solve state (2). A numerical example would have permitted one to follow the different steps of the computation. *J. Kestens* (Brussels)

Westbrook, D. R. 4279

The torsional rigidity of tubes of constant normal thickness.

Proc. Cambridge Philos. Soc. **60** (1964), 1023–1026.

An inequality given by J. L. Synge is used as the basis of a method of approximating the torsional rigidity of a thin tube. *J. W. Craggs* (Melbourne)

Zargarjan, S. S. 4280

Torsion of a circular cylinder having a non-coaxial polyhedral cavity. (Russian. Armenian summary)

Akad. Nauk Armjan. SSR Dokl. **38** (1964), 217–224.

The author considers a bar with doubly connected constant cross-section. The inner boundary is a square with rounded-off corners and the external boundary is a circle. The center of the circle does not coincide with that of the square. A conformal transformation is used with the aid of the generalized Laurent power series. For the

example given in the paper the transformation $z = D(1/w - w^3/b)$, $w = (\zeta_1 - \alpha)/(1 - \bar{\alpha}\zeta_1)$, $\zeta_1 = re^{i\varphi}$ is employed. The torsion rigidity and the stresses are obtained.

M. Mişicu (Bucharest)

Curpal, I. A. 4281

A physically non-linear elastic plate with a reinforced circular aperture. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 340–344.

Izdebs'ka, G. A.; Kil'čevs'kiĭ, M. O. 4282

On the convergence of the collocation method and the optimal choice of the collocation points in connection with the integro-differential equations of equilibrium in the theory of plates. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 469–472.

Authors' summary: "The authors consider the development of the method of collocation as applied to integro-differential equations of equilibrium of rectangular plates of variable thickness. An integro-differential equation is deduced, defining the error of the initial approximation, and the process of obtaining successive approximations is indicated. The general principles of the optimal choice of collocation points are outlined."

Morley, L. S. D. 4283

Bending of clamped rectilinear plates.

Quart. J. Mech. Appl. Math. **17** (1964), 293–317.

In a recent paper, the author showed [same J. **16** (1963), 451–471; MR **28** #2691] that the small deflection bending of a clamped plate can be simplified by the reduction to the variational solution of two successive membrane boundary-value problems. In the present paper he uses this reduction to derive a solution which is strictly valid when the boundary is finite, simply connected and smooth—and which is also satisfactory in the engineering sense for many problems where the boundary is rectilinear. Particular attention is given to the irregular behaviour which generally occurs in the corners. A numerical example is treated, and data are given for the clamped rhombus plate under a uniformly distributed load and under a central concentrated load.

H. D. Conway (Ithaca, N.Y.)

Landau, H. G. 4284

The elastic-plastic plate under cycles of moving dilatations and small applied load.

J. Mech. Phys. Solids **11** (1963), 97–117.

Author's summary: "An elastic and perfectly plastic plate is subjected to cycles of phase transformations which produce alternately positive and negative dilatations. The dilatations move symmetrically from the faces to the centre of the plate and vary only in the direction perpendicular to the faces. Simultaneously a constant load, which is small relative to the yield stress, is applied in a direction parallel to the faces of the plate.

"Equations based on the Tresca yield and flow condition are given for the stress and strain rates at any time, in terms of the existing stress distribution and the rate of change of the dilatation. Conditions for plastic flow to occur are derived, and it is shown that shakedown does not take place. For the special case of dilatation occurring in a zone of zero width, the stress-rate equations are integrated analytically for the first half-cycle of dilatation and for repeated cycles without the applied load. For the general case numerical integration is used. It is found that a periodic stress distribution is established after a few cycles, with strains which change at a constant rate per cycle and are proportional to the applied load."

A. Phillips (New Haven, Conn.)

Guo, Zhong-heng 4285

Certain problems of initially deformed plates. (Polish and Russian summaries)

Arch. Mech. Stos. **14** (1962), 779–788.

The equations of motion are written for rectangular, square, triangular and elliptical plates, assuming homogeneous finite deformations with additional small deformations. Proceeding from certain boundary conditions, the author gives formal solutions for the material frequencies and buckling conditions.

M. Mişicu (Bucharest)

Abubakar, Iya 4286

Free vibrations of a transversely isotropic plate.

Quart. J. Mech. Appl. Math. **15** (1962), 129–136.

Author's summary: "In this paper the boundary value problem concerning the propagation of plane harmonic waves in a thin, flat, homogeneous, transversely isotropic plate of finite width and infinite length in vacuum is solved. The frequency equations corresponding to the symmetric and antisymmetric modes of vibration of the plate are obtained, and some limiting cases of the frequency equations are then discussed. Finally, numerical solution of the frequency equations for a beryl plate is carried out, and the dispersion curves for the first two modes of the symmetric and antisymmetric vibrations are presented."

Prasad, B. 4287

Fracture of thick flat plates of elastic-plastic material under uniform stress.

J. Math. Anal. Appl. **9** (1964), 119–137.

A thick rectangular plate is loaded in uniform inplane unaxial tension under plane strain conditions. It is assumed that at a critical load τ the plate will be divided into elastic and plastic regions by a plane Σ which is perpendicular to the plane of the plate and whose normal makes an angle θ with the tensile direction. It is further assumed that Hooke's law applies in the elastic region and Hencky's law in the plastic region. A static analysis is made first. If the material is incompressible, it is shown that θ must be 45° to satisfy these assumptions. For a compressible material θ is determined from the further assumption that the plastic energy density function is an extremum to be either 0° or given by $\theta = \cos^{-1}[2(1-\nu+\nu^2)/3]^{1/2}$, which is slightly less than 45°. It is suggested that $\theta = 0°$ corresponds to cleavage fracture and θ near 45° to shear failure.

It is further shown that for the compressible material

there is discontinuity of the stress tensor across Σ according to the static analysis. Therefore dynamic conditions are introduced, and it is shown that for $\theta=45°$ a discontinuity in tangential velocity remains constant, for θ near 45° it increases continuously, and for $\theta=0°$ it increases infinitely, thus bearing out the suggested modes of failure.

{The reviewer notes that the predicted discontinuity of the stress tensor across Σ implies a rapid change in time of the orientation of the principal directions (since for the load infinitesimally less than τ, the elastic solution applies everywhere) and that under these conditions the Hencky theory of plastic flow is not generally a good approximation to the more realistic Prandtl-Reuss theory. Also, he questions the extremization of the plastic energy density rather than the total plastic energy which would be proportional $1/\cos\theta$ times the density and would presumably predict different values of θ.}

P. G. Hodge, Jr. (Chicago, Ill.)

Prokopov, V. K. 4288

The flexure of a heavy circular plate. (Russian)

Leningrad. Politehn. Inst. Trudy No. 226 (1963), 103–108.

The solution of the Navier equations is given for an axially symmetric problem of a thick circular plate such that both the radial displacement and the transverse shear stress vanish along the cylinder boundary. The plate is subjected to its own weight and supported either over a circular area or by a ring of forces. The solution is sought in the form of a product of a zero-order Bessel function defining the variation of the field quantities in the radial direction and an appropriate biharmonic function of the transverse variable. The resulting series is tabulated.

A. Sawczuk (Warsaw)

Wilson, P. E.; Boresi, A. P. 4289

Large deflection of a clamped circular plate including effects of transverse shear.

Trans. ASME Ser. E. J. Appl. Mech. **31** (1964), 540–541.

Authors' introduction: "A survey of literature on the analysis of flat plates reveals that considerable progress has been made in the formulation of plate theories that include both transverse shear and large deflection terms. However, numerical results for specific problems are still quite limited. In this note the influence of transverse shear on the large axisymmetric deflection of a clamped circular plate subjected to lateral loading is investigated. Numerical results indicate that there exist geometries for which effects of both transverse shear and large deflection are of interest."

Armenàkas, Anthony E. 4290

Influence of initial stress on the vibrations of simply supported circular cylindrical shells.

AIAA J. **2** (1964), 1607–1612.

The author investigates the influence of uniform circumferential and axial stress on the vibration frequency of thin cylindrical shells with hinged edges, using a more complete theory than previously applied to this problem. He finds previous results obtained by Reissner using shallow shell theory to be accurate except for modes involving a small number of circumferential waves with rather long axial wavelength. His theory permits him to distinguish between circumferential stress produced by hydrostatic and strictly radial pressure, the former resulting in up to 12 percent higher frequencies than the latter.

L. H. Donnell (Chesterton, Ind.)

Clark, Robert A. 4291

Asymptotic solutions of elastic shell problems.

Asymptotic Solutions of Differential Equations and Their Applications (*Proc. Sympos., Math. Res. Center, U.S. Army, Univ. Wisconsin, Madison, Wis.*, 1964), *pp.* 185–209. *Wiley, New York*, 1964.

This paper offers a succinct and informative synopsis of work on asymptotic solution of problems in elastic shell theory, with special references to toroidal shells.

J. W. Craggs (Melbourne)

Iyengar, K. T. Sundara Raja; Yogananda, C. V. 4292

Long circular cylindrical laminated shells subjected to axisymmetric external loads.

Z. Angew. Math. Mech. **44** (1964), 270–272.

From the authors' introduction: "It is intended in this note to present an elasticity solution for the stresses in a long circular cylindrical sandwich shell of two layers subjected to axially symmetric radial pressure on the outer surface. The solution presented here can easily be extended to the cases when the shell has elastic or rigid inclusions."

Jahanshahi, A. 4293

Some notes on singular solutions and the Green's functions in the theory of plates and shells.

Trans. ASME Ser. E. J. Appl. Mech. **31** (1964), 441–446.

From the author's summary: "Singular solutions are constructed which generate the singularities for many types of concentrated action on shallow spherical and cylindrical shells. Subsequently, a technique is introduced to construct the Green's functions for closed circular cylindrical shells. Also, the deformation of thin plates subjected to moving hot spots is discussed briefly."

H. D. Conway (Ithaca, N.Y.)

Mitkevič, V. M. 4294

Allowance for the local nature of the boundary effect in the asymptotic solution of an axi-symmetric problem for shells of revolution. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 476–479.

Author's summary: "It is shown that the main influence of the boundary effect in the general solution of the axisymmetrical problem of shells of rotation is confined to small values $(\Theta-\Theta_H)$ of the order $(h/R)^{1/2}$ (where Θ_H is the value of the argument at the shell edge). Allowing for this fact permits one to eliminate the contradictions present in the estimate of the error of the asymptotic solution of the homogeneous problem and to write this solution in simplified form."

Naghdi, P. M. 4295

Further results in the derivation of the general equations of elastic shells. (French, German, Italian, and Russian summaries)

Internat. J. Engrg. Sci. **2** (1964), 269–273.

The author supplements and modifies his earlier derivation of shell theory under the Kirchhoff-Love hypothesis [same J. **1** (1963), 509–522; MR **28** #5619] leading to simplification of the field equations. The resulting equations are compared with some recent results in the literature, and agreement in substance is observed in spite of formal differences. *W. T. Koiter* (Delft)

Newman, Malcolm; Reiss, Edward L. 4296

Nonlinear axisymmetric deformations of conical shells.

J. Soc. Indust. Appl. Math. **12** (1964), 386–402.

Buckling stresses for conical shells are obtained by use of a finite-difference method based on the variational method. *J. W. Craggs* (Melbourne)

Reissner, Eric 4297

On the form of variationally derived shell equations.

Trans. ASME Ser. E. J. Appl. Mech. **31** (1964), 233–238.

Equations for the linear theory of thin elastic shells are derived using a new three-dimensional variational theorem which admits a non-symmetry in the stress tensor and yields the symmetry condition as an Euler equation. This generalization permits assumption of convenient forms for the variation of stress components across the shell thickness. The resulting constitutive equations are simplified to those given by Koiter [Proc. Sympos. Thin Elastic Shells (Delft, 1959), pp. 12–33, North-Holland, Amsterdam 1960; MR **25** #5634] and Sanders [NASA Tech. Rep. R-24 (1959)]. An application to torsion of a circumferentially nonhomogeneous circular cylindrical shell illustrates the importance in some problems of certain terms which are often negligible and are not present in more approximate shell equations.

R. P. Nordgren (Houston, Tex.)

Reissner, Eric 4298

On asymptotic expansions for circular cylindrical shells.

Trans. ASME Ser. E. J. Appl. Mech. **31** (1964), 245–252.

Author's summary: "The paper presents an asymptotic expansion procedure (with reference to decreasing values of wall-thickness shell-radius ratio) for the solutions of the linear theory of the three simultaneous equations for displacements in isotropic circular cylindrical shells. The resultant expansions are applied to a number of fundamental boundary-value problems for a semi-infinite shell. The analysis shows the importance of the joint consideration of differential equations and boundary conditions for a clarification of the asymptotic nature of the solution to be determined." *R. P. Nordgren* (Houston, Tex.)

Karp, Samuel N.; Karal, Frank C., Jr. 4299

The elastic-field behavior in the neighborhood of a crack of arbitrary angle.

Comm. Pure Appl. Math. **15** (1962), 413–421.

A homogeneous elastic medium is considered in which a wedge-shaped section of angle 2α has been removed. A stress-free surface of the remaining medium is taken as boundary condition. The authors examine the types of static deformation which are possible if one imposes the condition that the strain energy density of the elastic field be integrable over any finite region, in particular, one which includes the edge of the wedge. It is assumed that the leading terms of an expansion of the displacement components in powers of r (distance from the edge) are proportional to r^ν. Then it turns out that, due to the boundary condition, the only permissible values are the roots of the equation $(1/2\beta)\sin 2\beta = \pm(1/2\nu\beta)\sin 2\nu\beta$, where $\beta=\pi-\alpha$. The energy condition excludes all roots ν with real part equal to or less than zero. Then the displacements are finite, although the stress becomes infinite at the edge for $0<\operatorname{Re}\nu<1$. It is conjectured that the energy condition can be replaced by the condition of finite displacements. An interesting feature is that the number of real roots ν is always finite, but there are, in addition, always infinitely many complex roots. The results are also significant for the dynamic case and for an elastic medium which has only locally a practically wedge-shaped portion.

J. Meixner (Aachen)

Sih, G. C. 4300

Fracture strength of a rectangular beam with surface cracks.

J. Soc. Indust. Appl. Math. **12** (1964), 403–412.

The solution is obtained for the stresses in the neighborhood of the tip of surface cracks located symmetrically at the midpoint of a rectangular cantilever beam under an end transverse load. *B. A. Boley* (New York)

Mura, T. 4301

Periodic distributions of dislocations.

Proc. Roy. Soc. Ser. A **280** (1964), 528–544.

The elastic displacement and stress fields are obtained in an anisotropic crystal through which is passing a prescribed sinusoidal plane wave of plastic deformation. The plastic wave is then represented by the corresponding dislocation motion tensor. The results are specialised to periodic fields of stationary dislocations and applied to the example of a hexagonal dislocation network. It is then shown that the equilibrium of any array of dislocation under constant external stress is stable. {The reviewer notes that this result is not true for finite dislocations, as follows from the example of two edge dislocations of opposite sign held in unstable equilibrium on the same glide plane by an external shear stress.} The possible equilibrium states of a periodic distribution are determined, and Fourier analysis is used to determine non-periodic distributions.

F. R. N. Nabarro (Cleveland, Ohio)

Amba-Rao, Chintakindi L. 4302

On the vibration of a rectangular plate carrying a concentrated mass.

Trans. ASME Ser. E. J. Appl. Mech. **31** (1964), 550–551.

Author's introduction: "A new formulation in a series form is given to the problem of vibration of simply supported rectangular plates carrying a concentrated mass in which the Dirac δ-function and finite Fourier sine transforms are used. The infinite set of eigenvalues is

obtained as the roots of a double infinite series, and the 'natural frequencies' of a simply supported rectangular plate occur as a special case of this analysis."

Bolotin, V. V. 4303

Vibration of layered elastic plates. (Polish and Russian summaries)

Proc. Vibration Problems **4** (1963), 331–346.

The author presents the theory of vibration of elastic plates composed of alternate "rigid" and "soft" layers. The theory can be considered as a generalization of the bending of sandwich plates with a soft filler. The plate is considered to be of n equal layers (rigid) and $n-1$ equal layers "soft", each "rigid" layer obeys the Kirchhoff-Love law and in each "soft" layer the stresses σ_x, σ_y and τ_{xy} are negligibly small. The other shear stresses τ_{xz} and τ_{yz} are constant across each "soft" layer and are proportional to the shear distortions γ_{xz} and γ_{yz}. The tangential components of the inertia force are neglected. The author derives the kinetic and potential energy of the plate and load. By application of the Ostrogradsky-Euler equations, the simultaneous partial differential equations for the plate are derived. After further simplifying assumptions and considering the boundary conditions for the simply supported plate, the frequencies for the composite plate are derived. The author extends the method to that of an equivalent anisotropic one-layer plate.

{The reviewer believes that this is an excellent contribution to multilayer plate theory. Present-day designs incorporate a bond between two hard layers and the frequencies can be calculated using this procedure. However, the theory could have been made more general by considering layers of unequal thickness which would be more useful for design purposes. The application to other support conditions can be easily accomplished by employing deflection equations satisfying the boundary conditions.} *H. Saunders* (Philadelphia, Pa.)

Kaczkowski, Zbigniew 4304

Vibration of a beam under a moving load. (Polish and Russian summaries)

Proc. Vibration Problems **4** (1963), 357–373.

L'autore espone un suo metodo di trattazione del problema delle vibrazioni di una trave sollecitata da un carico mobile, argomento questo fondamentale per la teoria dei ponti.

Scritta l'equazione del problema che, con evidente significato dei simboli, è:

$$EJ\frac{\partial^4 y}{\partial x^4}+\mu\frac{\partial^2 y}{\partial t^2} = P\delta(x-vt)$$

la trasforma (con l'introduzione di coordinate adimensionali e facendo opportune posizioni) in una equazione della forma:

$$\frac{\partial^4 y}{\partial \xi^4}+\sigma^2\frac{\partial^2 y}{\partial \tau^2} = \frac{Pl^3}{EJ}\delta(\xi-\tau).$$

L'introduzione di nuove variabili consente l'ulteriore trasformazione in una equazione differenziale alle derivate parziali del quarto ordine simmetrica nelle nuove variabili e di questa ricerca la soluzione nella forma:

$$y(\xi,\tau) = y_1(\zeta)+y_2(-\eta).$$

Per quanto riguarda le condizioni ai limiti Egli rimane nell'ambito della trave soddisfacente le "simple support conditions" e conclude intravedendo possibili estensioni alle travi ed alle piastre di materiale elastico viscoso.

R. Gambino-Amato (Palermo)

Kaliski, Sylwester 4305

Magnetoelastic vibration of a perfectly conducting cylindrical shell in a constant magnetic field. (Polish and Russian summaries)

Arch. Mech. Stos. **15** (1963), 197–208.

From the author's introduction: "The author [Proc. Vibration Problems **3** (1962), 225–234] gave equations and certain solutions of the equations for flexural vibrations of a perfectly conducting plate in an originally constant magnetic field. Assuming as a starting point that the principle of plane sections is satisfied, we obtained generalized equations of motion of the plate expressed in terms of the function of the deflection surface, $w(x, y, t)$ and the Maxwell surface stress tensors expressing the action of the electromagnetic field (of the vacuum) below and above the plate. The investigation of such plates gives a number of qualitatively new effects. The object of the present paper is to build up analogous equations for perfectly conducting cylindrical shells in an originally constant magnetic field."

Šatašvili, S. H. 4306

A fundamental three-dimensional mixed problem in the theory of steady-state elastic vibrations. (Russian. Azerbaijani summary)

Akad. Nauk Azerbaĭdžan. SSR Dokl. **20** (1964), no. 4, 7–11.

For the differential equations of the steady-state elastic vibration, an elementary system of solutions is given using potentials. The problem is reduced to a system of singular integral equations. *N. Cristescu* (Bucharest)

Tomar, J. S. 4307

Flexural vibrations in uniform beams.

Bull. Calcutta Math. Soc. **54** (1962), 151–162.

Author's summary: "A numerical solution is given for the equations for flexural vibrations of a uniform cantilever beam which include the secondary effects of shear and rotatory inertia according to the Timoshenko theory. It is shown that the results similar to those of Anderson for a beam supported at both ends can be obtained by the method of numerical integration."

Vahl Olsen, T. 4308

Analysis of elastic structures on digital computers.

Nordisk Tidskr. Informations-Behandling **3** (1963), 257–272.

A brief survey of a general method for the analysis of elastic structures is given. The matrices involved are described for certain classes of structures in terms of basic data. Some methods for taking advantage of the general, simplifying pattern of the matrices when programming for an electronic computer are mentioned.

G. R. Verma (Kingston, R.I.)

Kukudžanov, S. N. 4309

Stability of a cylindrical shell under the simultaneous action of torsion and variable stress. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **35** (1964), 37–44.

The paper deals with the stability of a cylindrical shell under torsional moments acting at the edges, and axially-symmetric pressure acting on the side of the shell.

S. Drobot (Columbus, Ohio)

Šuležko, L. F. 4310

Increasing the stability of flexible plates and cylindrical panels by means of vibrations. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 10–14.

Author's summary: "The author considers nonlinear equations of the dynamic stability of flexible plates and cylindrical panels. The effect of compressive vibrational forces of high frequency on the stability of plates and panels is investigated."

Agalarov, D. G. 4311

Propagation of non-linear visco-elastic waves in rods. (Russian. Azerbaijani summary)

Izv. Akad. Nauk Azerbaĭdžan. SSR Ser. Fiz.-Mat. Tehn. Nauk **1963**, *no.* 6, 17–25.

The paper deals with the propagation of longitudinal waves when the sound velocity depends on deformation. Considering a semi-infinite cylindrical rod, a system of hyperbolic partial differential equations describing the motion is written. The case of longitudinal impact with constant velocity on a non-deformed rod is treated in detail. In particular, the elasticity modulus is taken to be constant for the strain in the interval $(0, e_1)$ and is decreasing for $e > e_1$. At the end, the method of determining experimentally the characteristic functions of the solution is given.

Şt. I. Gheorghiţă (Bucharest)

Chandrasekhar, S. 4312

Propagation of elastic waves in liquid crystals.

Proc. Roy. Soc. Ser. A **281** (1964), 92–98.

Author's summary: "The model proposed by Maxwell for a visco-elastic medium is generalized by combining the properties of a crystalline solid and an anisotropic fluid. The basic equations are derived taking the elastic constants and the coefficients of viscosity both to be tensors of rank four. Expressions are given for the complex velocity of propagation of elastic waves of small amplitude as a function of direction. The theory is then applied to the 'stepped drop' (goutte à gradins) discovered by Grandjean, which consists of a number of exactly parallel layers, each layer terminating in a sharp step. The layers slide over one another very easily, but in any other direction the drop is extremely viscous. It is shown that for vibrations of frequency less than $10^9 s^{-1}$, only the longitudinal wave is propagated normal to the layers, the two transverse waves being heavily damped. In the plane of the layers the longitudinal wave and one transverse component are propagated."

S. C. Hunter (Sevenoaks)

Chu, Boa-Teh 4313

Finite amplitude waves in incompressible perfectly elastic materials.

J. Mech. Phys. Solids **12** (1964), 45–57.

Author's summary: "Propagation of finite amplitude plane shear waves in an incompressible perfectly elastic material is considered. The material occupies the half-space, $X_2 \geq 0$, and is initially at rest. A translational motion is imparted to the surface $X_2 = 0$ by applying a spatially uniform and time-dependent shear stress on this surface. The subsequent deformation and change of state in the material are calculated. Two special problems are considered in detail. In the first, the material is in a stress-free state and the surface $X_2 = 0$ is given a uniform motion for time $t > 0$. In the second, the unloading problem following the removal of the applied stress is calculated. The loading and unloading processes in the material generally follow a distinctly different pattern. Depending on the nature of the material and the loading programme, the shear waves may or may not coalesce into a transverse shock wave. The influence of materials of different nature on the propagation of shear waves is also discussed."

Eason, G. 4314

On the torsional impulsive loading of an elastic half space.

Quart. J. Mech. Appl. Math. **17** (1964), 279–292.

Author's summary: "The axisymmetric torsional impulsive loading of an elastic half-space is considered. A general solution is obtained using integral transforms, and three particular types of surface loading are considered in detail. The displacement is determined at all points of the body, and numerical results for the surface displacement are presented in graphical form."

W. D. Collins (Manchester)

Payton, R. G. 4315

Shock-wave propagation in solid and compactible media.

J. Acoust. Soc. Amer. **35** (1963), 525–534.

Plane waves are considered in a medium which compacts under small pressure from density ρ_0/a to ρ_0, and then compresses according to the law: pressure $P = A[(\rho/\rho_0)^\gamma - 1]$, A and γ constants. Shock and characteristic relations are given, and the solution for a rectangular pulse of pressure on the surface of a semi-infinite body is determined. Change to characteristic variables gives linear partial differential equations which are solved numerically. For $\gamma = 3$, linear characteristics permit evaluation by quadratures.

The successive development of homogeneous regions, simple waves, and general waves has been given previously for a closely related solution by H. F. Bohnenblust. Euler coordinates are used in the present paper, and the reviewer suggests that Lagrange coordinates, giving a fixed value at the surface, might lead to a simpler solution.

E. H. Lee (Stanford, Calif.)

Berg, Charles A. 4316

Deformation of free boundaries in plane viscous creep. (German summary)

Z. Angew. Math. Phys. **15** (1964), 300–306.

The author considers the plane strain deformation under

conditions of linear viscoelasticity of a material with a hole (stress-free boundary), when stressed uniformly at infinity. The material plane is mapped via a suitable conformal transformation so that the hole is mapped into a circle. For an arbitrary hole the mapping function, which is developed following a procedure due to Muskhelishvili, in polynomial form, requires an infinity of terms. In an example the author considers the case of a mapping function with three terms, in which case the stress-free boundary may have the form of an almost equilateral triangle or of a slightly imperfect ellipse.

S. J. Tupper (Sevenoaks)

Biot, M. A.; Pohle, F. V. 4317

Validity of thin-plate theory in dynamic viscoelasticity.

J. Acoust. Soc. Amer. **36** (1964), 1110–1117.

Explicit solutions to the exact equations of three-dimensional dynamic elasticity can be found for a plate with normal loadings $\sigma_{yy} = \pm q \cos lx \exp(i\omega t)$ on its surfaces $y = \pm h$. The correspondence principle gives the solution for viscoelastic plates. Analogous approximate solutions can be found from thin plate theory, which theory also yields explicit solutions for a much wider class of surface loadings. The object of this paper is to estimate the accuracy of the latter theory by calculating the amplification factor at resonance, an important value in practice, from the two theories. For two examples of viscoelastic material, the authors find that the factors are identical for zero-thickness wavelength ratio, but that the approximate factor is larger as the ratio increases; it is of the order of 10 percent greater for a ratio $1/\pi$.

D. R. Bland (Manchester)

Distéfano, J. N. 4318

Sulla congruenza delle deformazioni in solidi viscoelastici non-omogenei.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **35** (1963), 285–290.

The author considers an isotropic solid subjected to external body forces. Part of its surface can be subjected to restrictions which hinder the motions in certain directions. In order that a state of total deformation be congruent (in the sense given by Colonetti) it is necessary that $\int_V \varepsilon_{ij}(t)\delta\sigma_{ij}\,dV = 0$, $\alpha\delta\sigma_{ij}$ being co-actions and V the body volume.

In the linear theory one may prove that in order that the elastic deformations be congruent it is necessary and sufficient that a state of co-actions should not exist and the viscous deformations be congruent. Then it is deduced that the necessary conditions for the congruence of elastic deformations in the absence of the co-actions are $f_1 = f_2$, $\alpha = \text{const}$, f_1, f_2 being the longitudinal and transverse creep functions. With the aid of these results a series of consequences are deduced. *M. Mişicu* (Bucharest)

Distéfano, J. N. 4319

Ancora sulla stabilità asintotica delle deflezioni di una trave viscoelastica.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **35** (1963), 504–508.

The author considers the asymptotic stability criterion of viscoelastic theory. Thus, by imposing the condition of asymptotic creep independently of any particularization of the creep functions, the ultimate load is $P/(1+E\gamma_0)$, where P is the elastic critical force, E the elasticity modulus and γ_0 the asymptotic value of the specific creep function $\ell_0(t_0, \tau)$ (for $\tau \to \infty$). *M. Mişicu* (Bucharest)

Kric'ka, S. S. [Krickaja, S. S.] 4320

Asymptotic behavior of eigenvalues and eigenfunctions of a boundary-value problem. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 866–871.

Author's summary: "The author considers the problem of impact on elasto-viscous pivots of variable cross-section. The solving of this problem by Fourier's method leads to a boundary-value problem of Sturm-Liouville equations containing a parameter. Asymptotic formulae are obtained for the eigenvalues and eigenfunctions of this problem. The formulae are derived by the iteration method."

Meyer, M. L. 4321

On spherical near fields and far fields in elastic and visco-elastic solids.

J. Mech. Phys. Solids **12** (1964), 77–111. (1 *plate*)

A harmonic, symmetric, normal excitation is applied to the surface of a spherical cavity imbedded in an elastic medium. This contribution is a review of a problem that has been treated by a number of authors over the past 30 years. The details of the field near the spherical surface and the extension to the traveling wave field at distance are presented. Some peculiarities of the solution for large Poisson's ratio and an extension to the visco-elastic case using complex moduli are also considered.

L. Knopoff (Pasadena, Calif.)

Migliacci, Antonio 4322

Soluzioni asintotiche di sistemi di equazioni integrali di Volterra relative a strutture iperstatiche con comportamento viscoso. Applicazione al caso delle travate continue con appoggi cedevoli in modo elasto-viscoso.

Ist. Lombardo Accad. Sci. Lett. Rend. A **97** (1963), 216–273.

In the first part of the paper systems of Volterra integral equations are considered in connection with the constitutive laws of the linear viscoelasticity. For different integral kernels, the known rheological models are analyzed. In the second part some considerations are elaborated in order to establish the asymptotic values of the solutions for different models. The results are applied to a continuum rod with viscoelastic supports.

The paper presents a systematization of certain theoretical and physical elements.

M. Mişicu (Bucharest)

Rogers, T. G.; Lee, E. H. 4323

The cylinder problem in viscoelastic stress analysis.

Quart. Appl. Math. **22** (1964), 117–131.

The problem of a rotating hollow linear viscoelastic circular cylinder, subjected to internal pressure and encased in an elastic shell, is treated on a quasi-static basis. Plane strain and cylindrical symmetry are assumed, but varying angular velocity and an ablating inner

surface are allowed. A general integral representation of the viscoelastic stress-strain relations is used, in contrast to the simple differential operator relations previously considered, and the problem is reduced to a pair of inter-related integral equations of convolution type. Solutions are obtained numerically by means of a high-speed computer, and accuracy assessed by comparison with a simple case solved analytically. The numerical method allows use of arbitrary viscoelastic functions, either from models or experimental data, as well as arbitrary angular velocity and applied pressure histories. Several examples are given to demonstrate the effects of compressibility, different viscoelastic behaviour, spin and pressure histories, and elastic casing properties.

L. W. Morland (Norwich)

Zelenskiĭ, V. B. 4324

An analogue of the Flaman problem for a pre-stressed visco-elastic medium. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1963**, *no.* 6 (37), 58–63.

Conditions of plane strain are produced in a visco-elastic half-space by applying a uniform normal line-load to the surface. The medium is assumed to have a homogeneous initial stress given by a non-zero direct component of stress in the direction parallel to the surface and perpendicular to the line-load. The problem is tackled using the equations of M. A. Biot [Quart. Appl. Math. **17** (1959), 185–204; MR **21** #5340], and the solution is particularized for a Voigt type of material. Since the material is time-dependent, and the surface load is taken as constant in time, the calculated stresses in the material approach infinity for large values of time. This would appear to place a severe handicap on the applicability of the results. *F. J. Lockett* (Sevenoaks)

Hodge, Philip G., Jr. 4325

Plastic plate theory.

Quart. Appl. Math. **22** (1964), 74–77.

An analysis is given for the determination of the mathematical type (i.e., according to the theory of characteristics) of the system of field and constitutive equations for the elementary bending theory of plane, isotropic, thin plates undergoing perfectly plastic flow. It is proved that, except for special types of yield conditions, the system is fully elliptic. The exceptions apply for piece-wise linear yield curves and also for corners of yield curves, and then the type is various (viz., elliptic or parabolic or hyperbolic), according to the precise circumstances; thus, for example, the results obtained earlier by the reviewer [Proc. Roy. Soc. London Ser. A **241** (1957), 153–179; MR **19**, 342] for Tresca's yield curve are recovered. The present analysis follows the general method of procedure given, for example, by R. Courant and K. O. Friedrichs [*Supersonic flow and shock waves*, § 22, Interscience, New York, 1948; MR **10**, 637] whereas the reviewer's [loc. cit.] analysis proceeds somewhat differently. The results obtained emphasize once again the mathematical advantages expected to accrue from the artifice of the deliberate choice of piece-wise linear yield curves (or surfaces) in the solution of plasticity problems.

H. G. Hopkins (Sevenoaks)

Drucker, D. C. 4326

On the postulate of stability of material in the mechanics of continua.

J. Mécanique **3** (1964), 235–249.

In view of certain criticism, the author once more takes up the problem of the stability postulate for time-independent and time-dependent materials. In particular, it is agreed that convexity of the yield surface and normality of the vector of permanent strain increment will, at least in principle, no longer be true for materials whose elastic moduli vary with plastic strain.

The severe restrictions imposed by the stability requirement on time-dependent materials are discussed.

H. Parkus (Vienna)

Malmeister, A. K. 4327

Deformation of an anisotropic elasto-plastic body. (Russian. Latvian and English summaries)

Latvijas PSR Zinātņu Akad. Vēstis Fiz. Tehn. Zinātņu Sēr. **1964**, *no.* 4, 79–83.

Author's summary: "A mode of applying the local deformation theory to the analysis of deformative anisotropy is described in the article. By the help of this mode the simultaneous growth and drop of the components of stress tensors in different coordinate systems can be determined."

Sewell, M. J. 4328

Inverse rigid/plastic constitutive equations. (French, German, Italian, and Russian summaries)

Internat. J. Engrg. Sci. **2** (1964), 317–325.

The non-unique inverse of Hill's rigid/plastic constitutive equations is expressed in the form of a dependence of stress-rate on strain-rate and an arbitrary symmetric tensor. In a region of a deforming body where the current yield surface normal is uniform the corresponding set of possible stress-rate and velocity fields is obtained. Equilibrated stress-rate fields in continuing plane strain are then deduced. *R. Hill* (Cambridge, England)

Singh, Manohar 4329

A linearized theory of nosing of shells.

Trans. ASME Ser. E. J. Appl. Mech. **31** (1964), 535–539.

Author's summary: "The problem of nosing of shells in finite plastic strain is analyzed by considering various methods of jointly linearizing yield condition and flow rule. The numerical results show that excellent approximations to the predictions of von Mises' theory can be obtained."

Chang, C. H.; Liu, C. K. 4330

Solutions of plane problems in dynamic thermoelasticity.

Trans. ASME Ser. E. J. Appl. Mech. **31** (1964), 543–546.

From the authors' introduction: "In the present work, solutions of the dynamic response of a two-dimensional problem concerning isotropic elastic solids subjected to a prescribed temperature field and time-dependent boundary conditions are presented. The latter may arise from the varying temperature and the prescribed pressure functions at the boundaries. The problem is solved by the classic

method of separation of variables in conjunction with the method of Mindlin and Goodman for the treatment of the time-dependent boundary conditions."

Datta, Subhendu K. 4331

Steady state thermal stresses in an elastic solid bounded by two cones. (German summary)

Z. Angew. Math. Phys. **15** (1964), 175–183.

The author applies similarity transformations to a steady-state thermoelastic problem of a solid bounded by two coaxial circular cones, where the surface tractions vanish identically and the distributions of the surface temperature vary according to arbitrary powers of the distance from the apex, and obtains stress formulas in terms of Legendre functions. Numerical results are presented for a case in which the two surfaces are maintained at different constant temperatures. {The author informed the reviewer that σ_R, σ_φ and $\tau_{R\varphi}$ in Figure 3 should be replaced by σ_R/β, σ_φ/β and $\tau_{R\varphi}/\beta$, where β is defined by (4.4) in the paper, and the numerical calculation is carried out for Poisson's ratio $\nu=1/4$ (which is not stated in the paper).}

R. Muki (Tokyo)

Herrmann, George 4332

On second-order thermoelastic effects. (German summary)

Z. Angew. Math. Phys. **15** (1964), 253–262.

In order to investigate temperature changes in a body subject to shear, the author extends Biot's second-order theory of elasticity to include thermal effects. Numerical results are given based on values of elastic constants measured by Poynting in his famous tests with steel piano wires. However, in the reviewer's opinion, these values should be accepted with caution since the elastic limit may have been exceeded in the tests. *H. Parkus* (Vienna)

FLUID MECHANICS, ACOUSTICS

See also 3760a–b, 3893, 4410, 4414, 4533, 4536, 4541, 4552, 4553.

Condiff, Duane W.; Dahler, John S. 4333

Fluid mechanical aspects of antisymmetric stress.

Phys. Fluids **7** (1964), 842–854.

Authors' summary: "Basic fluid mechanical concepts are reformulated in order to account for some structural aspects of fluid flow. A continuous spin field is assigned to the rotation or spin of molecular subunits. The interaction of internal spin with fluid flow is described by antisymmetric stress while couple stress accounts for viscous transport of internal angular momentum. With constitutive relations appropriate to a linear, isotropic fluid we obtain generalized Navier-Stokes equations for the velocity and spin fields. Physical arguments are advanced in support of several alternative boundary conditions for the spin field. From this mathematical apparatus we obtain formulas that explicitly exhibit the effects of molecular structure upon fluid flow. The interactions of polar fluids with electric fields are described by a body-torque density. The special case of a rapidly rotating electric field is examined in detail and the induction of fluid flow discussed. The effect of a rotating electric field upon an ionic solution is analyzed in terms of microscopically orbiting ions. This model demonstrates how antisymmetric stress and body torque can arise in 'structureless' fluids."

Ertel, H. 4334

Differentialoperationen der Hydrodynamik in Form Lagrangescher Klammersymbole.

Monatsb. Deutsch. Akad. Wiss. Berlin **5** (1963), 218–220.

The author considers various forms of the equations of motion of an incompressible fluid. By forming the curl of the equations of motion, expressed in terms of the velocity vector, and by using the continuity equation, the well-known relation for the vorticity vector is obtained. Multiplying both sides of this last equation by the generalized Kronecker tensor and then differentiating, the author obtains a relation between the skew-symmetric product of the first derivatives of the velocity and vorticity vectors (the Lagrange bracket) and the curl of the total time derivative of the vorticity vector. A modification of the vorticity for the compressible (isentropic) case is noted and is compared to the result of K. Oswatitsch [*Handbuch der Physik*, Bd. VIII/1, pp. 1–124, Springer, Berlin, 1959; MR **21** #6836a].

N. Coburn (Ann Arbor, Mich.)

Ertel, H. 4335

Äquivalente Wirbel-Differentialgleichungen der Hydrodynamik.

Monatsb. Deutsch. Akad. Wiss. Berlin **5** (1963), 689–692.

The study of the differential equations for the vorticity vector, initiated by the author in a previous paper [#4334 above] is continued for the case in which the external force has a potential and the fluid is incompressible or isentropic compressible. Three forms of the differential equation are obtained by use of (1) the generalized Kronecker tensor; (2) the deformation velocity tensor. Further, a symbolic scheme for writing one of these equations (in a form suitable for computation) is given. *N. Coburn* (Ann Arbor, Mich.)

Ertel, H. 4336

Implizite Darstellung der hydrodynamischen Wirbelgleichungen durch Projektions-Operatoren.

Monatsb. Deutsch. Akad. Wiss. Berlin **6** (1964), 339–341.

In the present paper, the author writes the equations of motion of a compressible isentropic fluid in a conservative force field discussed in his previous paper [#4335 above] in a new form involving the projections of the velocity and vorticity vectors on an arbitrary vector.

N. Coburn (Ann Arbor, Mich.)

Schwiderski, Ernst W.; Lugt, Hans J. 4337

Rotating flows of von Kármán and Bödewadt.

Phys. Fluids **7** (1964), 867–875.

The flow engendered by a rotating disc in a fluid otherwise at rest (von Kármán flow) and by a fixed disc in a fluid rotating at infinity (Bödewadt flow) are considered anew.

In each case the problem is reduced to the solution of

a set of ordinary differential equations containing a parameter R, obtained after an unusual procedure. The equations reduce to standard forms as $R\to\infty$. Numerical solutions of each problem are obtained for various values of R and it is pointed out that for Bödewadt's problem "when $R\geq 9$ no proper nonoscillating flows exist".

K. Stewartson (London)

Bostandzhiian, S. A. [Bostandžijan, S. A.] 4338

Homogeneous helical motion in a cone.

Prikl. Mat. Meh. **25** (1961), 140–145 (*Russian*); *translated as J. Appl. Math. Mech.* **25** (1961), 193–201.

The motion of a perfect incompressible fluid inside a finite right circular cone covered with a spherical cap is considered. Solutions are found for steady axisymmetric flows with azimuthal swirl when the fluid is injected along the perimeter of the cap as a source flow with finite azimuthal velocity, and removed through two sinks located at the apex and at the center of the cap. The partial differential equation satisfied by the stream function is solved by separation of variables, and contains two arbitrary constants, one of which specifies the vorticity in the flow. Since no effort is made to consider the effects of viscosity, it seems unrealistic to this reviewer to introduce any distributed vorticity unless the cone is infinite or regions of closed streamlines are anticipated. The result for potential motion is given, and a streamline pattern for non-zero vorticity is shown. Although the singularities of the separated solution on the spherical cap have not been studied, the mathematical analysis is interesting and should be of use in treating other flow problems in cones. {A number of misprints confuse k and κ.}

R. C. Ackerberg (Farmingdale, N.Y.)

Holodilin, A. N. 4339

Non-stationary theory of an airfoil with small aspect ratio. (Russian. English summary)

Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. **19** (1964), *no.* 1, 114–122.

This paper constitutes a development of Poljahov's theory concerning the lifting surface of the rectangular plane [Vestnik Leningrad. Univ. 14 (1959), no. 13, 93–110; MR **22** #3295] for the case of an airfoil of small aspect ratio. Using integration by successive approximations, the expression of the lifting force is obtained for the unsteady motion of a wing in an incompressible flow. Applications are given for the wing having vertical harmonic oscillations. Results are compared to those for the steady motion.

L. Dragos (Bucharest)

Greenberg, Michael D. 4340

An improved Glauert series for certain airfoil problems.

AIAA J. **2** (1964), 1666–1667.

Author's introduction: "In the presence of such complicating effects as unsteadiness, finite span, jet flap, and free-surface interference, for example, an exact solution of the thin airfoil problem is generally out of the question. As a result, it is often convenient to seek a solution for the unknown vortex strength in the form of a truncated Glauert series. In this note we consider the analytical and numerical merits of a slightly modified series solution."

Hyers, D. H. 4341

Some nonlinear integral equations of hydrodynamics.

Nonlinear Integral Equations (Proc. Advanced Seminar Conducted by Math. Research Center, U.S. Army, Univ. Wisconsin, Madison, Wis., 1963), pp. 319–344. Univ. Wisconsin Press, Madison, Wis., 1964.

A review of some of the recent more general existence proofs of various classical two-dimensional, permanent-type, irrotational surface waves. The two problems considered are the periodic deep water waves, and the finite amplitude solitary wave in shallow water. The recent applications of topological theories and general theories of nonlinear functional equations in Banach spaces are discussed.

E. V. Laitone (Berkeley, Calif.)

Holford, R. L. 4342

Short surface waves in the presence of a finite dock. I, II.

Proc. Cambridge Philos. Soc. **60** (1964), 957–983; *ibid.* **60** (1964), 985–1011.

From the author's summaries: "In the first part, a long thin plate of finite width (a finite dock) extends over part of the otherwise free surface of a semi-infinite body of water under gravity and is given a forced heaving motion of small amplitude about this position. The solution of the boundary-value problem for the velocity potential of the resulting forced (two-dimensional) surface wave motion may be reduced to that of an integral equation of the second kind for the value of the potential on the dock by means of an appropriate Green's function. There is an infinite number of choices for such a Green's function, and we show below how to construct one for which the kernel of the corresponding integral equation tends to zero with the ratio: wavelength/dock-width. The integral equation may then be solved by iteration to calculate the short-wave asymptotics of the original wave-making problem.

"In the second part, the generalized wave-making problem for the forced high-frequency oscillations of a finite dock is solved through use of the Green's function obtained in the first part of this paper. The specific cases of heave and roll are considered with particular reference to the forces on the dock and the amplitude and phase of the radiated waves. These results are then utilized to solve the problem of transmission of short waves under a fixed dock."

E. V. Laitone (Berkeley, Calif.)

Sretenskii, L. N. [Sretenskiĭ, L. N.] 4343

Periodic waves generated by a source located above a sloping bottom.

Prikl. Mat. Meh. **27** (1963), 1012–1025 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1557–1576.

The author investigates the two-dimensional linearized waves excited on the surface of a fluid by a periodic output source located arbitrarily on a constant slope bottom. The mathematical analysis is for bottom slopes which are integral fractions of $\pi/2$. Detailed applications are presented for slopes of $\pi/2$, $\pi/6$, and $\pi/8$. For a bottom slope of $\pi/4$ it is shown how to determine the depths at which a periodically oscillating source will not transmit progressive surface waves to infinity.

E. V. Laitone (Berkeley, Calif.)

Kotik, J.; Newman, D. J. 4344

A sequence of submerged dipole distributions whose wave resistance tends to zero.

J. Math. Mech. **13** (1964), 693–700.

The authors present an argument showing that there is no solution to the problem of minimizing the wave resistance of a submerged horizontal line dipole distribution that is parallel to the incident flow. They consider the linearized free surface boundary condition and a dipole line distribution that is absolutely integrable. In support of their argument they construct a sequence of dipole distributions whose linearized volume is fixed while their wave resistance becomes arbitrarily small.

E. V. Laitone (Berkeley, Calif.)

Chabert d'Hières, Gabriel; Gouyon, René; Kravtchenko, Julien 4345

Contribution à la théorie du clapotis plan.

J. Math. Pures Appl. (9) **43** (1964), 1–25.

The theory of steady plane free-surface flows of a heavy inviscid fluid which satisfy the exact free-surface condition has been the subject of a long series of papers in which the existence of periodic and other gravity waves has been established under a variety of conditions. The situation is quite different with standing waves. Even in the simplest situation no existence (or non-existence) proofs are available. The authors address themselves to this problem. Although they do not solve it, their treatment is clearly intended to provide the basis for an existence proof.

The problem is formulated for fluid of finite depth in terms of variables of Lagrangian type. It is assumed that the variables describing the particle motion and the period can be expanded in a power series in a small parameter of the same order as the wave amplitude. The equations to be satisfied by the nth-order coefficients are found explicitly and solved. What is still lacking, of course, is the proof that the series has a finite radius of convergence. However, assuming this, one can draw several conclusions: (1) The motion is irrotational; (2) the particle paths are arcs along which the particles move back and forth, all coming to rest simultaneously at each end. A final section gives more detailed results for solutions accurate to the third order. The relation of the results of this paper to earlier ones on the subject is also discussed. *J. V. Wehausen* (Berkeley, Calif.)

Nersisjan, È. M. 4346

Penetration of a cone into an incompressible non-uniform liquid. (Russian. Armenian summary)

Izv. Akad. Nauk Armjan. SSR Ser. Fiz.-Mat. Nauk **16** (1963), no. 5, 83–90.

The problem considered is that of a cone impinging on the free surface of a stratified fluid. The cone is assumed to have a small slope, and the linearized free surface condition is approximated by the condition of vanishing velocity potential. The fluid density is assumed to be an exponential function of the vertical coordinate. Analytic and numerical results are obtained for the potential, pressure, impulsive force during impact, and velocity of the cone. Comparison is made with the case of a uniform fluid density. *J. N. Newman* (Washington, D.C.)

Moiseev, N. N. 4347

Introduction to the theory of oscillations of liquid-containing bodies.

Advances in Applied Mechanics, Vol. 8, *pp.* 233–289. *Academic Press, New York,* 1964.

This review paper deals with a problem of rapidly increasing technological interest in a clear and authoritative manner. The treatment is restricted to linear oscillations of conservative systems and makes extensive use of the classical tools for such systems; in particular, Hamilton's principle forms the starting point, and the calculation of the Lagrangian the primary step, in the treatment of each of the individual problems. The first part of the paper surveys the special problems of: (1) free-surface oscillations in a fixed container; (2) a rigid body containing a cavity completely filled with liquid (the "Stokes-Zhukovskii problem"); (3) a pendulum that comprises a rigid rod and a rigid, open vessel of liquid; (4) a system that comprises a liquid-filled cavity with a free surface and has only a finite number of degrees of freedom in the absence of liquid motion; (5) torsional oscillations of a Bernoulli beam with a cavity that is either completely or partially filled; (6) flexural-torsional oscillations of the last configuration. Part II deals with the general properties of the equations that describe the foregoing systems. Perhaps the most interesting result is that a necessary and sufficient condition for the stability of the equilibrium of a system of solid bodies that contains liquid with a free surface is that an equivalent system of solid bodies be stable, even though the two systems cannot be dynamically equivalent in consequence of the free-surface oscillations [as originally established by both Stokes and Zhukovskii, dynamical equivalence is possible for problems of category (2) above].

J. W. Miles (Canberra)

Williams, W. E. 4348

Note on the scattering of long waves in a rotating system.

J. Fluid Mech. **20** (1964), 115–119.

Author's summary: "It is shown that the solution of the problem of scattering of long waves by a finite barrier in a rotating system may be obtained directly from the solution of the electromagnetic scattering problem for a finite strip. It is shown that the effect of rotation is to produce on both sides of the barrier, but away from the ends, a wave which is propagated without attenuation parallel to the barrier."

Airapetova, È. L. 4349

Formulas for the general solutions of the Oseen equations in cylindrical and spherical coordinates. (Russian)

Akad. Nauk Uzbek. SSR Trudy Inst. Mat. No. 23 (1961), 65–91.

The author points out that solutions for the pressure and the velocity in the case of Oseen's equations can be found in terms of ω, where ω satisfies

$$\left[\frac{\partial}{\partial t}+(\mathbf{c}\cdot\nabla)-\nu\nabla^2\right]\nabla^2\omega = 0.$$

Most of the pages in this paper are devoted to calculating various vector operations in cylindrical and polar coordinates. For instance, in spherical polars $(\mathbf{c}\cdot\nabla)\nabla^2\omega$ contains over 130 terms. In each coordinate

system a specimen general solution is constructed by taking the components of ω to be harmonic functions. As this amounts to saying that the vorticity is everywhere zero, these solutions are of little physical significance. A much more concise and illuminating theory is given in Lamb [*Hydrodynamics*, sixth edition, p. 610, Dover, New York, 1945], whilst explicit solutions are contained in Goldstein [Proc. Roy. Soc. London Ser. A **123** (1929), 225–235] and Tomotika and Aoi [Quart. J. Mech. Appl. Math. **3** (1950), 140–161; MR **12**, 59].
D. R. Breach (Toronto, Ont.)

Ranger, K. B. 4350
A Stokes flow problem between two non-concentric cylinders.
Mathematika **10** (1963), 147–156.
The Stokes flow is produced by a line source and an equal sink situated symmetrically on the outer cylinder. The problem is solved by converting it into a similar problem for two concentric circular cylinders by means of geometrical inversion. The action on the inner cylinder is determined, and the special cases (i) when the source and sink form a doublet, and (ii) when the fluid is inviscid, are included. *C. R. Illingworth* (Manchester)

Wood, William W. 4351
Stability of viscous flow between rotating cylinders. (German summary)
Z. Angew. Math. Phys. **15** (1964), 313–314.
The note gives an alternative proof of Rayleigh's stability criterion of the flow between two infinite, coaxial rotating cylinders based on an energy integral for the disturbed velocity. *L. N. Tao* (Chicago, Ill.)

Krueger, E. R.; Di Prima, R. C. 4352
The stability of a viscous fluid between rotating cylinders with an axial flow.
J. Fluid Mech. **19** (1964), 528–538.
The authors clear up certain discrepancies in recent papers and extend previous calculations to the case when the cylinders are rotating in opposite directions.
D. J. Benney (Cambridge, Mass.)

Pnueli, D. 4353
Lower bounds to thermal instability criteria of completely confined fluids inside cylinders of arbitrary cross section.
Trans. ASME Ser. E. J. Appl. Mech. **31** (1964), 376–379.
Author's summary: "A method is developed to compute the lower bounds for the thermal instability criterion (the critical Rayleigh number) for fluids completely confined inside cylinders of arbitrary cross-section; i.e., Rayleigh numbers below which no spontaneous flow may occur in spite of the density gradient being opposite to the body force direction."

Szablewski, Witold 4354
Querwirbelartige Störungen der laminaren Grundströmung im Innern und Äussern eines rotierenden Zylinders. (English summary)
Z. Angew. Math. Phys. **15** (1964), 107–115.
The author considers a novel variation of the classical linearized hydrodynamic stability problem for flows with curved streamlines. In particular, he examines the case of an infinite rotating circular cylinder both inside and outside the separating boundary. Moreover, the solutions for disturbances are explicitly evaluated without resort to the usual assumption of axisymmetry. The results of this work indicate the importance of transverse vortices of the type which are common to the parallel flow problem. Perturbations may also have a non-zero phase velocity. Finally, some special cases are discussed and depicted by graphs of the streamlines.
W. O. Criminale, Jr. (Princeton, N.J.)

Szablewski, W. [Szablewski, Witold] 4355
Längswirbel in ebener Poiseuillescher Strömung. (English and Russian summaries)
Z. Angew. Math. Mech. **44** (1964), 253–263.
Author's summary: "The mechanism between longitudinal vortices and a two-dimensional basic laminar flow of Poiseuille's type is explicitly evaluated in the linear as well as in the first nonlinear approach. Corresponding phenomena occurring in the transition to turbulence are pointed out."

O'Neill, M. E. 4356
A slow motion of viscous liquid caused by a slowly moving solid sphere.
Mathematika **11** (1964), 67–74.
The slow steady motion of a rigid sphere past an infinite flat plate in an incompressible viscous fluid is considered. Calculations of the forces and couples exerted on the sphere are made when the sphere has no rotation and for several different ratios of the sphere diameter to separation distance from the plate. The salient result is the demonstration that the effect of the plate is to diminish the order of magnitude of the stresses at large distances due to the motion of the fluid caused by the sphere.
W. O. Criminale, Jr. (Princeton, N.J.)

Osipov, V. Z. 4357
Two-dimensional non-stationary flow of a viscous fluid in a porous tube of circular annulus cross-section. (Russian. Georgian summary)
Soobšč. Akad. Nauk Gruzin. SSR **33** (1964), 535–542.
The two-dimensional, non-stationary, non-linear problem of laminar flow of a viscous incompressible fluid in a porous annular tube formed by two coaxial porous circular cylinders is solved. It is assumed that the suction through the pores is known and that the longitudinal velocity at the end of the tube is zero. The approach used permits the effective solution of some particular problems, and the result is obtained that the pressure difference in the longitudinal direction is independent of the radius.
M. D. Friedman (San Jose, Calif.)

Kanwal, R. P. 4358
Drag on an axially symmetric body vibrating slowly along its axis in a viscous fluid.
J. Fluid Mech. **19** (1964), 631–636.
The author considers the problem of an axially symmetric

body vibrating slowly along its axis in a viscous fluid, and by the use of expansion techniques, developed by Lagerstrom and Cole [J. Rational Mech. Anal. 4 (1955), 817–882; MR 17, 1021] and Proudman and Pearson [J. Fluid Mech. 2 (1957), 237–262; MR 19, 201] for discussing Stokes and Oseen flows, obtains an expression for the drag on the body in terms of the Stokes drag experienced by the same body. *W. D. Collins* (Manchester)

Kolton, G. A. 4359

Flow of a radiating gas in the vicinity of the critical point of a blunt-nosed body. (Russian. English summary)

Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. **19** (1964), *no.* 3, 103–109.

Author's summary: "The problem of motion of viscous radiative gas flow near the stagnation point of a blunt-nosed body is considered by Cherny-Freeman's method. The problem is reduced to resolving a single non-linear integro-differential equation for the gas enthalpy. This equation is transformed to an integral equation, the solution of which is suggested by iterations. The method permits one to obtain the surface body temperature."

Tao, L. N. 4360

On the variational principle and Lagrange equations in studies of gasdynamic lubrication.

Trans. ASME Ser. E. J. Appl. Mech. **31** (1964), 43–46.

The paper presents the variational formulation of hydrodynamic lubrication by taking into account the compressibility. Supposing a permanent motion, it is shown that the Reynolds equation is equivalent to a variational expression; then a set of partial differential equations in generalized coordinates is derived. Furthermore, from Hamilton's principle, the Lagrangian and dissipation-production densities are established, and the incompressible case is obtained as a particular problem. Thus the paper succeeds in giving a unified variational method for both gas and liquid lubrication.

{In spite of the attractive mathematical treatment, the reviewer's opinion is that for compressible lubricants the method is almost too difficult to use in numerical computations; for incompressible fluids some similar calculations were already performed by Weber [Z. Angew. Math. Mech. **30** (1950), 112–120; MR **11**, 751].}

N. Tipei (Bucharest)

v. Krzywoblocki, M. Z. 4361

On some mathematical aspects of the boundary layer theory. (Romanian and Russian summaries)

An. Şti. Univ. "Al. I. Cuza" Iaşi Secţ. I (N.S.) **7** (1961), 345–397.

A general discussion, mostly expository, of questions of existence of solutions to problems in the boundary layer theory. The author's basic theme is an association of partial differential equations with ordinary ones in accordance with his development of the theory of A. D. Michal [Proc. Nat. Acad. Sci. U.S.A. **37** (1951), 623–627; MR **13**, 251] and A. J. A. Morgan [Quart. J. Math. Oxford Ser. (2) **3** (1952), 250–259; MR **15**, 37]. Related results by the author may be found in Inter. Sympos. Nonlinear Differential Equations and Nonlinear Mechanics, pp. 221–240, Academic Press, New York, 1963 [MR **26** #6551] and in J. Reine Angew. Math. **206** (1961), 175–191; ibid. **207** (1961), 113–128 [MR **26** #981].

D. H. Hyers (Los Angeles, Calif.)

Leslie, F. M. 4362

The stability of Couette flow of certain anisotropic fluids.

Proc. Cambridge Philos. Soc. **60** (1964), 949–955.

The author discusses the stability of the flow of anisotropic fluids with preferred direction between concentric, rotating cylinders when the gap is small and the cylinders are rotating in the same direction. A small perturbation is given both in velocity and in preferred direction.

He makes the following assumptions in his analysis: (i) $\mu_1=0$; (ii) $\mu>0$, $\mu_2>0$, $\mu_3>0$, $2(3\mu+\mu_3)>\mu_2$; (iii) the gap is small; (iv) the angular velocity of the primary flow, which occurs in a coefficient of the differential equation in $v(r)$, is replaced by an average value, thereby reducing the equation to one with constant coefficients; (v) the function $v(x)$, $x=(r-R_1)/(R_2-R_1)$, R_1, R_2 being two radii, and its derivatives up to order six can be represented by Fourier series. He concludes that whatever be the sign of the parameter λ, the solution of the steady-state equations for which $N_1>0$, $N_2>0$ is always more stable than the one for which $N_1>0$, $N_2<0$, where $N_1^2=\frac{1}{2}(\lambda-1)/\lambda$, and $N_2^2=\frac{1}{2}(\lambda+1)/\lambda$. This does not, however, contradict the reviewer's results [Arch. Rational Mech. Anal. **10** (1962), 101–107; MR **26** #7247], which are obtained on the basis $\mu_1\neq 0$ and without making any one of the above assumptions. In fact, if $\mu_1=0$, the fluid does not behave like a Bingham plastic material. The reviewer's results show exactly the same thing obtained here, provided that $\mu_1=0$ is imposed on his solution. The author justifies taking $\mu_1=0$ by just quoting Green [Proc. Cambridge Philos. Soc. **60** (1964), 123–128; MR **29** #827], who has himself given constitutive equations for anisotropic fluids, which are, no doubt, slightly different from those of Ericksen [Arch. Rational Mech. Anal. **4** (1960), 231–237; MR **22** #1274], but which do not make stress reduce to hydrostatic pressure when the rate of strain vanishes. *P. D. S. Verma* (Kharagpur)

Mitchner, M.; Landshoff, R. K. M. 4363

Rayleigh-Taylor instability for compressible fluids.

Phys. Fluids **7** (1964), 862–866.

The stability problem of an accelerating contact surface separating two inviscid, compressible fluids is examined. The governing equations for the linearized perturbed motion are derived and then simplified by the assumption of a quasisteady state. The limiting case of small wave length disturbances shows that the interface is unstable when the slope of the unperturbed pressure decreases in the direction of increasing pressure and is stable otherwise. This agrees with the result of two incompressible fluids in a gravitational field by Taylor [Proc. Roy. Soc. London Ser. A **201** (1950), 192–196; MR **12**, 58]. Also, in the particular case of constant acceleration and static isothermal equilibrium, the result remains the same qualitatively as for two incompressible fluids.

L. N. Tao (Chicago, Ill.)

Pierce, John G. 4364

Stability of potential flow.

Phys. Fluids 7 (1964), 1109–1113.

Using Eckart's generalized coordinate expressions for the hydrodynamical equations [Phys. Fluids 6 (1963), 1037–1041; MR 27 #5418], the author obtains some general results concerning the stability of classes of steady two-dimensional potential flows. *L. A. Segel* (Troy, N.Y.)

Codegone, Cesare 4365

Sull'equazione termica della ventilazione nelle gallerie. (English summary)

Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. 98 (1963/64), 514–518.

Author's summary: "A thermal equation for ventilating flows in long tunnels is established and resolved, which gives as particular solutions the cases of transversal, or longitudinal, or mixed ventilation."

Nield, D. A. 4366

Surface tension and buoyancy effects in cellular convection.

J. Fluid Mech. 19 (1964), 341–352.

A Fourier series method is used to solve an eigenvalue problem relevant to the onset of convection in a layer of fluid heated from below, when surface tension variations with temperature and buoyancy act together to cause the instability. The physical significance of the results is discussed. *L. A. Segel* (Troy, N.Y.)

Palm, Enok; Øiann, Henry 4367

Contribution to the theory of cellular thermal convection.

J. Fluid Mech. 19 (1964), 353–365.

A motionless conductive state is possible in a horizontal fluid layer heated from below. Refining the work of Palm [same J. 8 (1960), 183–192], the reviewer and Stuart showed [ibid. 13 (1962), 289–306; MR 25 #3659] that the nonlinear interaction of two particular disturbance modes of the same horizontal wave-number could result in hexagonal convection cells if the difference between the actual and critical Rayleigh numbers was sufficiently small compared to a dimensionless measure of the viscosity variation with temperature. When only these two modes are considered, another stable configuration is possible, but the authors' elegant argument shows that it is unstable when a certain third mode is considered. Whether or not new stable equilibrium points occur when more than two disturbance modes are allowed is left open here (but is resolved in a forthcoming paper of the reviewer). The authors find the hexagonal cell to be stable to all infinitesimal disturbances of the same horizontal wave number. *L. A. Segel* (Troy, N.Y.)

Csanady, G. T. 4368

Turbulent diffusion in a stratified fluid.

J. Atmospheric Sci. 21 (1964), 439–447.

The author considers the dynamical effects on a particle drifting in a homogeneous turbulent fluid field which is stationary and uniform except for a constant vertical density gradient. The author makes several order-of-magnitude estimates regarding the characteristics of the turbulent field in which the diffusion takes place. In the model considered here, it is the initial hypothesis that the vertical gradient of mean potential temperature is constant, and according to the author, this model applies approximately to many practical situations, simple compared to a neutral atmosphere. Further, in the equation of energy there has been an ad hoc assumption regarding a diffusion term from dimensional reasoning to avoid complexity of the theory. The author finds adequate evidence to establish the qualitative correctness of his theoretical model. *K. M. Ghosh* (Belgharia)

Kraichnan, Robert H. 4369

Direct-interaction approximation for shear and thermally driven turbulence.

Phys. Fluids 7 (1964), 1048–1062.

Author's summary: "The direct-interaction approximation for turbulence is extended to shear flows and Boussinesq convection contained in regions of arbitrary size and shape. It is pointed out that the approximation provides a consistent generalization of mixing-length concepts which continues to be applicable when the scale of inhomogeneity is comparable to the effective mixing length. The degree of faithfulness to be expected of the approximations is discussed on the basis of inferences from numerical results on isotropic turbulence."

D. A. Lee (Dayton, Ohio)

Meecham, William C.; Siegel, Armand 4370

Wiener-Hermite expansion in model turbulence at large Reynolds numbers.

Phys. Fluids 7 (1964), 1178–1190.

Les auteurs se proposent de représenter les solutions aléatoires de certaines équations fonctionnelles non linéaires par des expressions exactes au lieu de se borner à calculer, suivant les méthodes classiques, quelques uns de leurs moments. Ils utilisent les développements en fonctionnelles de Wiener-Hermite. L'élément de base est la fonction aléatoire gaussienne $a(x)$, de moyenne nulle, ayant pour covariance $\langle a(x_1)a(x_2)\rangle = \delta(x_1 - x_2)$. On en déduit par convolution une fonction gaussienne régulière

$$f(x) = \int K(x-x')a(x')\,dx'.$$

On construit ensuite une famille statistiquement orthonormale de fonctions aléatoires

$$H^0(x) = 1, \qquad H^1(x) = a(x),$$
$$H^2(x_1, x_2) = a(x_1)a(x_2) - \delta(x_1 - x_2), \text{ etc.,}$$

et l'on représente la fonction cherchée $u(x)$ par une somme d'intégrales de convolution de H^0, H^1, H^2, par des noyaux $K^{(1)}$, $K^{(2)}$, $\cdots$. On note que cette forme de noyaux assure d'avance l'homogénéité. On cherche si $u(x)$, ainsi représentée, peut être solution de l'équation de Burgers (sans dimensions)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \frac{1}{R}\frac{\partial^2 u}{\partial x^2} = 0.$$

En substituant, et en ne prenant qu'un nombre fini de termes dans le développement de u, on obtient un ensemble d'équations intégro-différentielles, dont la première relie

les noyaux $K^{(1)}$, $K^{(2)}$ et conduit pour la fonction spectrale énergétique $E(k,t)$ à l'équation

$$\frac{\partial E}{\partial t} = -\frac{2}{R}k^2E - k^2\int [2E(k,t)E(k_1,t) - E(k-k_1,t)E(k_1,t)]\,dk_1.$$

Sa transformée de Fourier Q est telle que

$$\frac{\partial}{\partial t}q - \frac{\partial^2}{\partial r^2}q^2 = -\frac{2}{R}\frac{\partial^2}{\partial r^2}q(0,t),$$

avec $q = Q(0,t) - Q(r,t) + 1/R$. On vérifie que

$$\int_{-\infty}^{\infty} Q(r,t)\,dr = \text{const.}$$

Le résultat dimensionnel de Kolmogoroff ($E_k \sim k^{-5/3}$) n'est pas vérifié ici.

On sait obtenir des solutions de la forme $Q(r,t) = \Lambda(tr^{-2})$, mais elles sont d'interprétation peu physique. Le calcul de Q et de E a été fait par des méthodes numériques, pour $R = 100$ et successivement

$$Q(r,0) = e^{-r^2} \quad \text{et} \quad Q(r,0) = \frac{1}{1+\pi r^2},$$

d'où $E(0,t) = (4\pi)^{-1/2}$.

Les résultats sont très voisins, malgré la différence de forme des conditions initiales, et présentent une analogie sérieuse avec les résultats expérimentaux relatifs à la turbulence. On en conclut à l'existence d'une forme d'équilibre pour Q, et à la rapide convergence probable du développement de u. *J. Bass* (Paris)

Lumley, John L. 4371

Turbulence in non-Newtonian fluids.

Phys. Fluids 7 (1964), 335–337.

The motion and shearing properties of a Reiner-Rivlin body in turbulent regime are studied. Using the general form for the stresses, established by Truesdell, the dissipation function is written by means of the Fourier transformation and wave number vector. Then the case when the distribution of the turbulent shearing stresses is Gaussian is examined, and it is shown that the energy dissipation occurs as in a Newtonian fluid with an effective viscosity dependent on the mean-square shear.

If the effective viscosity is low, the dissipative terms sufficiently small and the wave number large, one obtains an inertial sub-range whose mean values are characterized by ε, the dissipation, and the wave number only. The behaviour for small shear and small dissipation is also analyzed. As a conclusion the viscoelastic effects are responsible for some anomalies observed in high-molecular-weight linear polymers, and the form of the probability density of the strain tensor is quite sensitive to the form of the stress-deformation relation.

The author presents some new interesting ideas and results in the field of turbulence in non-Newtonian fluids. Neither diagrams nor experimental data are given.

N. Tipei (Bucharest)

Nam Tum Po 4372

On the rotational motion of a sphere in a rarefied flow of high velocity. (Russian. English summary)

Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. 19 (1964), no. 3, 162–167.

The author considers the rotation motion of a sphere relative to its center of inertia in a rarefied gas flow having high velocity (in comparison with the thermal motion of molecules). It is assumed that the rarefied gas is such that the free path of molecules exceeds the dimensions of the body considered in such a way that the hydrodynamic methods are not applicable.

Several particular solutions are given for Euler's equations (solid with fixed body) in the case in which the center of inertia coincides with the geometric center of the body and in the case in which these centers do not coincide. *L. Dragos* (Bucharest)

Hosokawa, Iwao 4373

Unified formalism of the linearized compressible flow fields.

Quart. Appl. Math. 22 (1964), 133–142.

The author considers the linearized transonic small perturbation equation

$$(1-M_\infty^2)\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = K\varphi_x,$$

where K is a positive constant. For $K \to 0$ the equation reduces to the conventional linearized subsonic or supersonic potential equation. For $K \neq 0$ the equation is believed to yield approximate transonic solutions in the neighborhood of a suitable class of thin bodies. The author makes use of Schwartz's distribution theory in order to obtain the general solution to the equation, valid both in the elliptic ($M_\infty < 1$) and hyperbolic ($M_\infty > 1$) cases. As an example, the flow about a thin wing is treated in detail, and some new results for the lifting wing are found. Formulas describing operations with distributions are conveniently summarized in an appendix.

H. U. Thommen (San Diego, Calif.)

Rolando, Magda; Sarra, Mariangela 4374

Una funzione di corrente singolare per lo studio dei flussi transonici attorno a profili alari doppiamente simmetrici.

Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. 98 (1963/64), 383–390.

Les auteurs étudient l'écoulement transsonique autour d'un profil doublement symétrique dans l'approximation de Tricomi et en utilisant les variables uniformisantes de Nocilla. Pour des lignes de courant suffisamment proches du profil, la vitesse maximale dans la zone supersonique n'est pas atteinte sur l'axe de symétrie. Les auteurs semblent ignorer un travail de M. Liger [O.N.E.R.A. Publ. No. 64 (1953)] qui étudie le même type de profil dans une approximation dont le domaine de validité correspond à une gamme de vitesse plus large que celle de l'approximation de Tricomi et qui fournit le tracé du profil obtenu, ce qui n'est pas le cas dans la brève note considérée ici.

D. Vallée (Chatillon s/s Bagneux)

Babaev, D. A. 4375

Numerical solution of a supersonic flow problem past the lower surface of a delta wing. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. 2 (1962), 1086–1101.

This paper gives a numerical solution for the flow past the lower surface of a plane delta wing at an angle of attack. The case considered is for an attached shock at the

leading edge. In this case, the flow on the upper surface and that on the lower surface are independent, the flow near the wing being conical. The corresponding problem for the upper surface has been given by the author in an earlier paper [same Ž. 2 (1962), 278–289; MR 27 #1065].

C. K. Chu (New York)

Garvine, Richard W. 4376

Hypersonic viscous flow near a sharp leading edge.

AIAA J. **2** (1964), 1660–1661.

A great deal of re-thinking has gone into hypersonic flow problems since it was recognized that it does matter what happens near the leading edge of a body. Generally, it has been found that the reported experimental findings differ greatly from the values predicted by viscous strong interaction theory. The research note here is a theoretical effort which examines the departure under certain practical conditions. The findings indicate that a more complicated viscous flow is involved than that of the strong interaction region concept. *W. O. Criminale, Jr.* (Princeton, N.J.)

Maslen, S. H. 4377

Inviscid hypersonic flow past smooth symmetric bodies.

AIAA J. **2** (1964), 1055–1061.

This paper presents a simple, accurate method for solving the entire hypersonic flow fields over a blunt body. It is primarily an inverse method, and the concepts of thin shock layer are employed. The crucial step lies in the way of obtaining the pressure field approximately from the equation

$$\frac{\partial p(x, \psi)}{\partial \psi} \sim u,$$

where p is the pressure, x the distance along the shock, ψ the stream function, u the x-component of velocity. To determine the pressure, the present method takes $u = u_s(x)$, the u-component immediately behind the shock for the same value of x, while previous authors [W. D. Hayes and R. F. Probstein, *Hypersonic flow theory*, p. 180, Academic Press, New York, 1959; MR 21 #540] usually took $u = u_s(\psi)$, the u-component at the point of entry of the streamline being considered. The latter follows from the consistent approximation, yet becomes very inaccurate downstream near the body. Calculated examples using the present method show excellent agreement in comparison with more exact results. Extension to include the non-equilibrium effects is also discussed.

K. C. Wang (Baltimore, Md.)

Romberg, Günter 4378

Die Strömung im blockierten Kreiskanal. (English and French summaries)

Z. Flugwiss. **12** (1964), 323–339.

Author's summary: "The axisymmetric flow past a given slender body of revolution in a choked circular tunnel is investigated. Pressure distribution on the body, drag coefficient of the semi-body, limit between the elliptic and hyperbolic region and choking Mach number are determined as functions of the model shape and the tunnel radius. The numerical results state: The flow with free-stream Mach number one can be simulated in the vicinity of the body in a choked tunnel of relatively small diameter if the Mach waves reflected at the tunnel wall do not reach the body."

Bagdoev, A. G. 4379

On the law of motion of a shock wave in fluid. (Russian. Armenian summary)

Izv. Akad. Nauk Armjan. SSR Ser. Fiz.-Mat. Nauk **17** (1964), *no.* 3, 71–80.

The response of a fluid-filled half-space to an expanding ring pressure disturbance on its surface is considered. Specifically, the prescribed surface pressure on points at distance $R(t)$ from point 0 is $P_a(t)$. The fluid is assumed to obey a polytropic pressure-density relation. Approximate expressions for pressure on the shock front in the influence zone of 0 and far from 0 are derived.

D. A. Lee (Dayton, Ohio)

Pant, J. C.; Mishra, R. S. 4380

Shock waves of finite thickness in magneto-gas-dynamics.

Rend. Circ. Mat. Palermo (2) **12** (1963), 59–81.

Under the assumption of shock waves of finite thickness, the authors deduce the conservation equations (for normal components of the magnetic field, mass states, momentum states and energy) throughout the thickness of the shock wave. The conservation equations are also obtained when the diffusivities are so small that the discontinuities are not smoothed out very much. A particularization is made for the case in which the conductivity is infinite and the magnetic field is everywhere tangential to the shock surface.

The method employed is based on writing the magneto-hydrodynamic equation for a disk like the cylindrical volume in the thickness of the shock wave.

L. Dragos (Bucharest)

Jeffrey, Alan 4381

A note on the derivation of the discontinuity conditions across contact discontinuities, shocks and phase fronts. (German summary)

Z. Angew. Math. Phys. **15** (1964), 68–71.

The author studies the discontinuity theory associated with a system of conservation laws. By using an integral form of these laws, the associated Rankine-Hugoniot jump conditions are obtained. The theory is applied to the quasi-linear system of conservation laws determined by the equations of mass, momentum and energy of an inviscid, compressible, heat-conducting fluid in the equilibrium state. It is shown that, in addition to the well-known shock and contact waves, a phase front wave exists. This last wave is characterized by: (1) continuity of the density across the front; (2) the normal component of the velocity of the fluid differs from the normal component of the velocity of the wave front (as is the case for a shock wave); (3) pressure (as in a contact wave) and fluid velocity are continuous across the wave front. These effects are due to the inclusion of the heat conduction term in the energy relation. (Reviewer's note: A similar "integral form" approach to discontinuity theory was given by J. B. Keller [J. Appl. Phys. 25 (1954), 938–947;

MR 16, 761] and another "integral form" approach was given by the reviewer [Math. Mag. 27 (1954), 245–264; MR 15, 1000].) *N. Coburn* (Ann Arbor, Mich.)

Spence, D. A.; Woods, B. A. 4382

A review of theoretical treatments of shock-tube attenuation.

J. Fluid Mech. **19** (1964), 161–174.

The alteration of the ideal constant state flow in a shock tube, due to the viscous effects at the wall, is considered here for the case where the boundary layer is thin compared with the tube diameter. The transient problem in two space dimensions is reduced to a one-dimensional problem by applying small perturbation theory outside the boundary layer. After a further acoustic approximation, an explicit solution is obtained in terms of two arbitrary functions and a function, presumed known, of the boundary layer. It is shown that this solution contains the results of previous work by others (who used physical arguments) dealing with the shock speed and the running time. The finite opening time of the diaphragm and other factors are discussed briefly. The paper contains 32 references to theoretical and experimental work with shock-tube attenuation.

H. M. Sternberg (Silver Spring, Md.)

★Physical acoustics: Principles and methods. 4383

Vol. I—Part B: Methods and devices.

Edited by Warren P. Mason.

Academic Press, New York-London, 1964. xiv + 376 *pp.* $13.50.

The first volume in this series (Volume I, Part A) was reviewed earlier [MR 28 #4751]. The present volume is specialized to the use of high-amplitude waves in liquids and solids and to a new series of semiconductor devices now widely used in the measurement of pressures, forces and strains.

Müller, Karl-Heinz 4384

Zur Wellenausbreitung im heterogen-porösen Festkörper. (English summary)

Z. Angew. Math. Phys. **15** (1964), 57–61.

The Green's function is written for the coupled, second-order partial differential equations for wave propagation in a two-dimensional heterogeneous medium where the coupling coefficients and the wave numbers have a simple dependence upon the radial distance from each scatterer.

L. Knopoff (Pasadena, Calif.)

Frater, K. R. 4385

Flow of an elastico-viscous fluid between torsionally oscillating disks.

J. Fluid Mech. **19** (1964), 175–186.

In this paper the author discusses the flow of an incompressible elastico-viscous fluid characterized by the constitutive equations

$$S_{ik} = P_{ik} - Pg_{ik},$$

$$P^{ik} + \lambda_1 \frac{\delta P^{ik}}{\delta T} + \mu_0 P_j^{\ j} \cdot E^{ik} = 2\eta_0\left(E^{ik} + \lambda_2 \frac{\delta E^{ik}}{\delta T}\right)$$

given by Oldroyd [Proc. Roy. Soc. London Ser. A **200** (1950), 523–541; MR 11, 703], where S_{ik} and E_{ik} are, respectively, the stress tensor and the rate of strain tensor; $\delta/\delta T$ denotes the convective time-derivative; g_{ik} is the metric tensor; η_0 is a constant having dimensions of viscosity and λ_1 and λ_2 are the constants having the dimensions of time, between two parallel, infinite disks when one disk is held at rest and the other performs rotary oscillations about their common axis. It is found that the purely periodic primary motion has associated with it a secondary steady velocity distribution as well as a secondary periodic motion with twice the frequency of the primary motion, similar to that predicted by the reviewer and Rajeswari [J. Indian Inst. Sci. **44** (1962), 219–238] for the Rivlin-Ericksen fluids. The author finds that the main difference between the present investigation and that of the reviewer and Rajeswari is that the steady secondary motion always shows reversal above a certain critical Reynolds number in the case of Rivlin-Ericksen fluids, while in the case of incompressible elastico-viscous fluid discussed by the author it shows such a reversal if $\lambda_2/\lambda_1 < \frac{1}{2}$. This difference is attributed to the difference in the nature of these two types of fluids. For example, the Rivlin-Ericksen fluids studied by the reviewer and Rajeswari show retarded response to applied stress but do not show relaxation of stress at constant strain, and the normal stress differences do not correspond to a simple tension along the stream lines in contrast to the Oldroyd fluids discussed in the present paper.

P. L. Bhatnagar (Bangalore)

Kapur, J. N.; Gupta, R. C. [Gupta, R. K.] 4386

Two dimensional flow of visco-elastic fluids near a stagnation point with large suction. (Polish and Russian summaries)

Arch. Mech. Stos. **15** (1963), 711–717.

The paper deals with the two-dimensional steady flow of incompressible visco-elastic fluids near a stagnation point with large suction. Taking velocity components on the lines of Howarth, the authors substitute in the stress-strain relations $s_j^{\ i} + \tau \dot{s}_j^{\ i} = 2\mu d_j^{\ i}$ and the equations of motion. The final equation is solved by the method of successive approximations. The results, though plotted in the form of curves, are not discussed. Convergence of the series for ϕ for different numerical values of k deserves mention. Some of the basic equations written in tensor form are not quite correct. See, for instance, the equations (1.2) and (1.3),

$$(1.2)\qquad d_j^{\ i} = \tfrac{1}{2}(u_{i,j} + u_{j,i}),$$

$$(1.3)\qquad s_j^{\ i} + \tau \dot{s}_j^{\ i} = 2\mu d_j^{\ i},$$

where

$$\dot{s}_j^{\ i} = \frac{\partial}{\partial t} s_j^{\ i} + s^i_{j,k} v^k - s_k^{\ i} v_{,k}^{\ j} + s_j^{\ k} v_{,k}^{\ i} + s_j^{\ i} v_{,k}^{\ k}.$$

P. D. S. Verma (Kharagpur)

Sogawa, Wataru 4387

Rheological equations of generalized Maxwell model and Voigt model in three-dimensional, non-linear deformation.

Second-order Effects in Elasticity, Plasticity and Fluid Dynamics (Internat. Sympos., Haifa, 1962), *pp.* 756–763. *Jerusalem Academic Press, Jerusalem; Pergamon, Oxford*, 1964.

The author derives two sets of rheological equations for the generalised Maxwell and Voigt models in the case of three-dimensional, non-linear deformation, using the general relations between stress, strain and strain rate which were initially obtained by Reiner and Rivlin. Green's measure of strain in a strained state has been employed.

{The reviewer thinks that one or two illustrations ought to have been included.} *P. D. S. Verma* (Kharagpur)

Mohan Rao, D. K. 4388

Unsteady motion of a non-Newtonian fluid in the annular space between two porous concentric circular cylinders.

J. Indian Inst. Sci. **46** (1964), 57–64.

The unsteady motion of a Reiner-Rivlin fluid between two porous concentric cylinders is investigated when the outer cylinder is suddenly set in motion. The effect of "cross-viscosity" on initial and terminal axial velocity is discussed for the case where the constant rate of suction at the outer wall equals the rate of ejection at the inner wall. *P. P. Niiler* (Cambridge, Mass.)

Walters, K. 4389

Non-Newtonian effects in some general elastico-viscous liquids.

Second-order Effects in Elasticity, Plasticity and Fluid Dynamics (Internat. Sympos., Haifa, 1962), *pp.* 507–519. *Jerusalem Academic Press, Jerusalem; Pergamon, Oxford*, 1964.

This paper is concerned with the behaviour of elastico-viscous liquids at small rates of shear, based on Oldroyd's theory [Proc. Roy. Soc. London Ser. A **200** (1950), 523–541; MR **11**, 703; Quart. J. Mech. Appl. Math. **4** (1951), 271–282; MR **13**, 303; Proc. Roy. Soc. London Ser. A **245** (1958), 278–297; MR **20** #605]. The author attempts to construct two models of elastico-viscous liquids A' and B' based on a general linear equation of state. The liquids are specified by their respective distribution functions of relaxation times, known as relaxation spectra. Of the two models, the second one has been shown to exhibit all the essential features of observed steady non-Newtonian behaviour with the exception of the observed variation in apparent viscosity with rate of shear. The predicted normal stress effect in the case of flow between a horizontal flat plate and a cone of semi-angle very near to $\frac{1}{2}\pi$ has been shown to be in qualitative agreement with Roberts' experimental results on certain elastico-viscous liquids [Roberts, Proc. Second Internat. Congr. Rheology (Oxford, 1953), pp. 91–98, Academic Press, New York, 1954]. The reviewer agrees with the comment of Truesdell that the liquid models A' and B' constructed by the author are characterised after all by a linear equation of state in the sense of Noll, and hence the entire work of the present author will reduce to a much simpler analysis in accordance with the general solutions for viscometric flows by Coleman and Noll.

M. N. L. Narasimhan (Bombay)

Ziegler, H. 4390

Some limiting cases of non-Newtonian fluids.

Progress in Applied Mechanics, pp. 317–332. *Macmillan, New York*, 1963.

A continuation of the author's earlier paper [Progress in Solid Mechanics, Vol. IV, pp. 91–193, North-Holland, Amsterdam, 1963; MR **29** #772]. Four incompressible fluids are considered which are the Bingham material and its three inversions. For creeping motions in these materials the principle of least irreversible force is used to deduce two theorems concerning statically admissible stress fields and kinematically admissible velocity fields.

P. R. Paslay (Houston, Tex.)

Hoh, F. C. 4391

Instabilities due to resistivity gradients in a low-pressure plasma.

Phys. Fluids **7** (1964), 956–964.

Low-frequency, quasi-electrostatic modes are studied in the paper. Macroscopic particle equations are used. Displacement currents and terms of order m/M are discarded. Finite Larmor orbit corrections are included in the equations of motion, and heat conduction is permitted via a constant parallel heat conductivity only. The stability of quasi-static configurations (resistive diffusion is ignored) of a plasma embedded in a magnetic field is examined in slab and cylindrical geometries. Gravitational forces and forces due to curvature of magnetic lines are not included. By use of the low β approximation it is shown that the stability of the system may be determined from the eigenvalues of a second-order differential equation. In limiting cases the equation reproduces modes due to resistivity gradients which have been obtained previously. Analysis of the "rippling" mode shows, within the approximations of the paper, that both finite Larmor orbits and heat conduction reduce growth rates and that finite Larmor orbits introduce overstability. However, substantial growth rates are obtained for Stellarator conditions.

R. C. Mjolsness (Los Alamos, N.M.)

Carstoiu, John 4392

Sur les ondes magnétohydrodynamiques dans un fluide en rotation uniforme.

C. R. Acad. Sci. Paris **258** (1964), 2745–2747.

The author discusses wave motions in a perfectly conducting inviscid fluid in the presence of a magnetic field. The particular application is wave motions in the ionosphere, and the novel feature is the inclusion of the rotation of the earth, Ω_0, the direction of which is taken to coincide with the magnetic field (the z-direction). The equation for the z-component of vorticity is now coupled through Ω_0 to the equation of density variation in such a manner that, for small Ω_0, Alfvén waves predominate, while for large Ω_0, acoustic waves predominate, the propagation being without attenuation in the z-direction. It is suggested that this might offer an explanation of polar noise at times of extreme aurora activity.

The general equation for density variation is obtained and a wave number surface for plane waves is deduced. The equation for the wave number surface is of the sixth degree in frequency compared with the fourth degree for $\Omega_0 = 0$. *E. G. Broadbent* (Farnborough)

Clark, Alfred, Jr. 4393

Production and dissipation of magnetic energy by differential fluid motions.

Phys. Fluids **7** (1964), 1299–1305.

The magnetohydrodynamic equations, including viscosity, are solved as an initial-value problem, under the assumption of two-dimensional stagnation flow. The x and y components of fluid velocity are assumed to be Ax and $-Ay$, respectively. The magnetic field is chosen arbitrarily as a function of y in the x direction. The stagnation plane is the xz-plane, and all quantities are independent of z. The solution is examined with attention to two effects, often previously discussed in intuitive terms, but seldom studied quantitatively: (1) the increase of magnetic field energy due to the "stretching" of magnetic field lines; and (2) enhanced Joule heating by steepening the gradients of magnetic fields. Both effects are found in the solutions, and are exhibited for special cases. The possible role of the two processes in the transfer of energy in solar flares is discussed. *D. C. Montgomery* (Utrecht)

Dunwoody, N. T. 4394

Instability of a viscous fluid of variable density in a magnetic field.

J. Fluid Mech. **20** (1964), 103–113.

Author's summary: "The instability to small two-dimensional disturbances of an electrically conducting fluid of variable density is investigated. The viscous fluid is bounded between two vertical parallel planes normal to which a magnetic field of constant intensity is applied. Significant parameters upon which the behaviour of the Rayleigh number at neutral stability depends are the Hartmann number M and the wave-number α which is associated with a periodic disturbance with periodicity in the unbounded horizontal direction. The solution may be sought by considering basic disturbances which are either symmetric or antisymmetric about the median plane parallel to the boundary planes. It is found that for a given magnetic field strength the critical Rayleigh number governing stability is associated with an antisymmetric disturbance of zero wave-number. The least stable symmetric disturbance which arises when the wave-number is zero is less easily excited. This trend is seen again in the purely hydrodynamic case ($M=0$) where, corresponding to a finite wave-number value, the more unstable mode at neutral stability is found to be an antisymmetric one. The most unstable situation occurs when both the Hartmann number and the wave number are zero. In this case the result of Wooding [same J. 7 (1960), 501–515; MR **24** #B1417] that the minimum critical Rayleigh number is zero and is associated with a symmetric disturbance is reobtained." *C. S. Gardner* (New York)

Fogagnolo, Bruna 4395

Sulle linee di discontinuità in alcuni moti magnetofluidodinamici piani, generati da un vortice puntiforme.

Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **98** (1963/64), 613–624.

The author determines the characteristic lines for the propagation of cylindrically symmetric magnetohydrodynamic waves in a perfectly conducting fluid with a point-vortex at the central magnetic field line. He considers in detail the two special cases of constant magnetic field strength and constant Alfvén speed. The results are just what one would expect: the usual characteristic lines for a stationary fluid, which propagate radially outward with velocity $(v_s^2+v_a^2)^{1/2}$, are carried bodily into a spiral form by the assumed rotatory motion of the fluid. Here v_s denotes the speed of sound and v_a the Alfvén speed. *H. C. Kranzer* (Garden City, N.Y.)

Germeles, Apostolos E. 4396

Channel-flow of conducting fluids under an applied transverse magnetic field.

Trans. ASME Ser. E. J. Appl. Mech. **31** (1964), 165–169.

Author's summary: "The most general steady-state solution is derived for the laminar flow of an incompressible, viscous and electrically conducting fluid in a one-dimensional channel under an applied transverse magnetic field. The channel can act as an electromagnetic flowmeter or pump. The effect of the conductivity of the walls is included. The solution has two unknown constants and, by choosing them properly, it can be made to fit the solution of all two-dimensional channels whose geometry approaches in the limit that of the one-dimensional channel. This is done in detail for the two-dimensional channels with rectangular and annular cross-section."

Hantadze, A. G. 4397

On the rotation of a conducting fluid with a moving center. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **31** (1963), 543–549.

Proceeding from the works by Lord Rayleigh [Proc. Roy. Soc. London Ser. A **93** (1917), 148–154], Sir Napier Shaw [Meteorol. Office Geophys. Memoirs No. 12 (1918)] and others, the author studies the same problem of the rotation of a fluid, the fluid being considered compressible, however. *V. N. Constantinescu* (Bucharest)

Jeffrey, Alan 4398

Magnetoacoustic simple waves in a polytropic gas. (German summary)

Z. Angew. Math. Phys. **15** (1964), 217–227.

In magnetohydrodynamic flow small changes in the physical properties of the flow can be transmitted by magneto-acoustic simple waves. For one-dimensional flow these properties can be described by two quantities, α, β, related by a differential equation $d\alpha/d\beta=f(\alpha,\beta)$ [Friedrichs, Comm. Pure Appl. Math. **11** (1958), 333–418; MR **20** #7147]. In general, the magneto-acoustic waves have both a slow and a fast mode of propagation. By studying the integral curves in the (α,β)-plane the author shows here that no physical transition can occur between fast and slow waves. A numerical solution for one integral curve is given, together with a method for easily obtaining any other solution from it. *J. D. Jukes* (Culham)

Lykoudis, Paul S. 4399

Magnetofluidmechanic blast waves in a medium with finite electrical conductivity.

Phys. Fluids **7** (1964), 1372–1380.

The author extends the classical similarity solution of a blast wave in an ideal compressible gas to a medium with finite electrical conductivity σ and small magnetic Reynolds number. The presence of the magnetic field is found to decelerate the fluid behind the "shock wave" (a misnomer appearing in the paper as the wave does not have a finite structure) and to produce higher temperatures.

The validity and application of the solution is left obscure, for the similarity solution used is permitted to violate both Maxwell's equations and the "$\frac{3}{2}$" power law for the relation between σ and temperature. The important energy loss due to ionization is not taken into account. However, the work does represent an attempt to study a very difficult problem, even if the physics is oversimplified.

L. C. Woods (Berkeley, Calif.)

Power, Geoffrey; Walker, Dennis 4400

Some reciprocal relations in rotational magnetogasdynamic flow. (German summary)

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Les auteurs considèrent l'écoulement stationnaire, bidimensionnel d'un fluide compressible, non visqueux, non conducteur de la chaleur, parfaitement conducteur de l'électricité, soumis à un champ magnétique collinéaire à la vitesse. Après avoir rappelé l'existence d'une transformation ramenant l'étude de cet écoulement à celle de l'écoulement d'un fluide fictif non conducteur, les auteurs étudient cette analogie en étendant à la magnétodynamique des fluides une théorie déjà établie dans le cas de la dynamique des gaz [G. Power et P. Smith, J. Math. Mech. **10** (1961), 349–360; MR **23** #B1290]. Cette théorie permet alors de relier, grâce à un changement d'axes de coordonnées et à un certain nombre de relations réciproques, l'écoulement de magnétodynamique des fluides à une famille à quatre paramètres d'écoulements de dynamique des gaz, chacun d'eux correspondant à une loi d'état particulière. Les auteurs étudient alors les propriétés de ces écoulements en considérant aussi le cas d'une onde de choc.

R. Peyret (Montrouge)

Dragos, Lazar 4401

Theory of thin airfoils in magnetoaerodynamics.

AIAA J. **2** (1964), 1223–1229.

Author's summary: "This paper examines the motion of a compressible fluid with arbitrary finite electrical conductivity in the presence of a thin airfoil. At infinity, in the undisturbed stream, the magnetic fluid is assumed to be orthogonal to the direction of the fluid stream. The motion is irrotational. The velocity field and the magnetic field have an irrotational part, which coincides to that of the classical aerodynamics, and a rotational part, which is due to the field-motion interaction and which vanishes for $A \to \infty$. The general solution depends on five arbitrary functions. The boundary conditions determine the relations between these functions as well as the boundary problem for determining one of them. This one reduces to a Fredholm-type integral equation. Surface currents are not possible. The solution valid for arbitrary Mach number M coincides with the solution of the case of incompressible fluid for $M = 0$."

S. I. Pai (College Park, Md.)

Skorupski, A. 4402

Alfvén waves in inhomogeneous plasmas.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 329–334.

Author's summary: "The results of Gajewski and Winterberg concerning Alfvén waves in inhomogeneous magnetic fields are generalized for inhomogeneous plasmas and magnetic fields which are not external. Particularly the case of a plane and rotational symmetry of the magnetic field in equilibrium is considered and the corresponding differential equations for the vector potential are derived. A differential equation describing the Alfvén waves was reduced to a form convenient for approximations."

Tkalič, V. S. 4403

A stationary problem in magnetohydrodynamics where two coordinates are related; collision of jets of conducting fluid. (Russian)

Izv. Akad. Nauk SSSR Otd. Tehn. Nauk Meh. i Mašinostr. **1962**, *no.* 5, 32–38.

Steady flows of conducting fluids, in which one coordinate is quasicyclic, are studied. A coordinate is defined to be quasicyclic if the magnetic field **H**, the velocity **V**, and the metric tensor g_{kl} do not depend on it, while the total pressure P and the electric potential Φ depend on it linearly. General properties of these essentially two-dimensional flows of ideal fluids are given. For fluids with dissipation, a special class of flows is constructed for which the stream function obeys the Helmholtz equation, and an energy integral is obtained for these flows.

For near-potential flows, lift and moment formulas are given, these being generalizations of the Joukowsky formulas in ordinary fluid dynamics. As a special example, the collision of two conducting fluid jets is discussed.

C. K. Chu (New York)

Vacca, Maria Teresa 4404

Sul limite di Poincaré per una massa fluida di alta conduttività elettrica uniformemente rotante in cui si genera un campo magnetico.

Boll. Un. Mat. Ital. (3) **19** (1964), 127–137.

Poincaré has shown that, for a rotating mass of self-gravitating fluid of mean density ρ_m to be in stable equilibrium, its angular speed of rotation ω must lie below the limit $\omega_0 = (2\pi f \rho_m)^{1/2}$, where f denotes the universal constant of gravitational attraction. The author shows how this limiting angular velocity is changed by the introduction of a magnetic field, assuming the fluid to be a perfect conductor. She gives an explicit general formula for the new limiting velocity in terms of ω_0 and quantities derivable from the magnetic field.

The author considers in detail the case of a magnetic field $\mathbf{H} = \mathbf{H}_0 + \mathbf{h}$, where $\mathbf{H}_0$ is a constant magnetic field directed along the axis of rotation, while $|\mathbf{h}|$ is small compared to $|\mathbf{H}_0|$ and is cylindrically symmetric: $\mathbf{h} = \mathbf{h}(r, z)$ in cylindrical coordinates (r, φ, z). She shows, first, that the quantity $-\rho^{-1}(\operatorname{curl} \mathbf{h})_\varphi$ depends on r alone, and calls it $g(r)$; here ρ denotes the local fluid density. Then the condition for stable equilibrium becomes

$$\omega^2 < \omega_0^2 + \frac{\mu|\mathbf{H}_0|}{2\tau} \int_\sigma g(r) \frac{dr}{dn}\, d\sigma,$$

where σ is the outer fluid surface, with outward unit normal $\mathbf{n}$, μ is the magnetic permeability, and τ denotes the total volume of the fluid. Thus the stability limit may be either higher or lower than the Poincaré limit, depending on the predominant sign of $g(r)$ on σ.

H. C. Kranzer (Garden City, N.Y.)

Zeuli, Tino 4405

Un teorema di media in magnetofluidodinamica derivante dal teorema generalizzato del viriale per una massa fluida di conduttività elettrica infinita soggetta alla propria gravitazione.

Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **97** (1962/63), 1245–1253.

This paper contains the classical proof of the virial theorem in hydromagnetics [see, e.g., S. Chandrasekhar, *Hydrodynamic and hydromagnetic stability*, p. 577, Clarendon, Oxford, 1961; MR **23** #B1270], expressed in diadic notation. A few elementary applications are given.

B. Bertotti (Frascati)

OPTICS, ELECTROMAGNETIC THEORY, CIRCUITS

See also 3755, 4130, 4383, 4441, 4642.

Grimshaw, R. H. J. 4406

A note on the geometrical optics of diffraction by an interface.

Proc. Cambridge Philos. Soc. **60** (1964), 1013–1022.

It is known that solutions of the Cauchy problem for the wave equation represent disturbances obeying the laws of geometrical optics. Boundary conditions of the type representing reflection have been treated by Friedlander. Incident and reflected wavefronts exist and sometimes also a "shadow" where diffraction occurs. The diffracted wave fronts are normal to the reflecting surface, the corresponding rays travelling along the surface and leaving it tangentially. The present paper extends these results to refraction at an interface. The results are seen to be consistent with Keller's theory of diffracted rays.

E. W. Marchand (Rochester, N.Y.)

Pridmore-Brown, D. C. 4407

The radiation field of a magnetic dipole antenna in a conical sheath.

J. Math. and Phys. **43** (1964), 199–217.

The radiation field of a magnetic point source located on the axis interior to an axially symmetric, radially infinite, infinitesimally thin conical plasma sheath is considered. Boundary conditions at the sheath are those for an impedance surface. A potential function satisfying the Helmholtz equation is established, and Green's theorem is employed to obtain an integral equation for this potential function. An iteration procedure for determining the field in inverse powers of the square of the plasma frequency is then established, and with the aid of several approximations which have the effect of restricting the applicability to small cone angles the first term in the iteration is determined. Further terms are shown to be small. The pattern and some machine computations of eigenvalues and eigenfunction values (Legendre functions) for the 10° cone are included.

J. H. Harris (Culver City, Calif.)

Pozzolo, Vincenzo; Zich, Rodolfo 4408

Una dimostrazione semplice della formula per il calcolo dell'effetto di piccole perturbazioni geometriche sulla frequenza di risonanza di una cavità.

Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **97** (1962/63), 1056–1063.

Semplice dimostrazione della formula che esprime il cambiamento di frequenza determinato dall'introduzione, in una cavità elettromagnetica, di un piccolo corpo perfettamente conduttore.

D. Graffi (Bologna)

Boella, Mario; Zich, Rodolfo 4409

Sulla rappresentazione dei campi elettromagnetici in coordinate curvilinee ortogonali.

Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **97** (1962/63), 1043–1055.

Gli autori considerano un campo elettromagnetico **E** e **H** sinoidale e in un mezzo omogeneo isotropo. Definito un sistema di coordinate curvilinee ortogonali u_r $(r=1,2,3)$ dimostrano che **E** ed **H** possono esprimersi mediante un vettore Π con due sole componenti, non nulle, sulle coordinate, e che se **E** o **H** è normale a una u_r, Π ha solo la componente secondo u_r. Ritrovano così, con procedimento unitario, alcuni metodi usati per lo studio di diversi campi elettromagnetici.

D. Graffi (Bologna)

Jeffrey, A.; Taniuti, T. 4410

★Non-linear wave propagation. With applications to physics and magnetohydrodynamics.

Academic Press, New York-London, 1964. ix+369 *pp.* $12.00.

The first part of this book deals with the mathematical theory of hyperbolic partial differential equations (as well as methods of solution to these). The second portion is devoted to the application of these ideas to magnetohydrodynamics. The book is in general carefully written and well-explained.

In bringing new mathematical methods to attention and in dealing with magnetohydrodynamics this book is a valuable addition to Courant and Friedrichs [*Supersonic flow and shock waves*, Interscience, New York, 1948; MR **10**, 637] in the study of wave propagation in fluid mechanics. The chapter headings are: Part I, General Theory: Chapter 1, General Hyperbolic Equations; Chapter 2, The Method of Characteristics; Chapter 3, Conservation Laws and Weak Solutions. Part II, The Application to Magnetohydrodynamics: Chapter 4, The Fundamental Equations and Characteristics; Chapter 5, Simple Waves; Chapter 6, Magnetohydrodynamic Shocks; Chapter 7, Interaction of Hydromagnetic Waves; Chapter 8, Spatial Discontinuities; Appendices.

L. Sirovich (Providence, R.I.)

Wait, James R. 4411

Theory of radiation from sources immersed in anisotropic media.

J. Res. Nat. Bur. Standards Sect. B **68B** (1964), 119–136.

Author's summary and conclusion: "The electromagnetic fields produced by an electric dipole immersed in an anisotropic medium are considered. Various approaches to the problem are outlined with special reference to a cold plasma. An attempt is made to show the close relationship between previously published work on this subject. It is shown that information on the radiation field in an anisotropic media may be obtained directly from the shape of the refractive index surface.

"The present expository paper has been a self-contained treatment of the theory of radiation from sources in cold

The validity and application of the solution is left obscure, for the similarity solution used is permitted to violate both Maxwell's equations and the "$\frac{3}{2}$" power law for the relation between σ and temperature. The important energy loss due to ionization is not taken into account. However, the work does represent an attempt to study a very difficult problem, even if the physics is oversimplified.

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[illegible] 4405

Un teorema di media in magnetofluidodinamica derivante dal teorema generalizzato del viriale per una massa fluida di conduttività elettrica infinita soggetta alla propria gravitazione.

Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **97** (1962/63), 1245–1253.

This paper contains the classical proof of the virial theorem in hydromagnetics [see, e.g., S. Chandrasekhar, *Hydrodynamic and hydromagnetic stability*, p. 577, Clarendon, Oxford, 1961; MR **23** #B1270], expressed in diadic notation. A few elementary applications are given.

B. Bertotti (Frascati)

OPTICS, ELECTROMAGNETIC THEORY, CIRCUITS

See also 3755, 4130, 4383, 4441, 4642.

Grimshaw, R. H. J. 4406

A note on the geometrical optics of diffraction by an interface.

Proc. Cambridge Philos. Soc. **60** (1964), 1013–1022.

It is known that solutions of the Cauchy problem for the wave equation represent disturbances obeying the laws of geometrical optics. Boundary conditions of the type representing reflection have been treated by Friedlander. Incident and reflected wavefronts exist and sometimes also a "shadow" where diffraction occurs. The diffracted wave fronts are normal to the reflecting surface, the corresponding rays travelling along the surface and leaving it tangentially. The present paper extends these results to refraction at an interface. The results are seen to be consistent with Keller's theory of diffracted rays.

E. W. Marchand (Rochester, N.Y.)

Pridmore-Brown, D. C. 4407

The radiation field of a magnetic dipole antenna in a conical sheath.

J. Math. and Phys. **43** (1964), 199–217.

The radiation field of a magnetic point source located on the axis interior to an axially symmetric, radially infinite, infinitesimally thin conical plasma sheath is considered. Boundary conditions at the sheath are those for an impedance surface. A potential function satisfying the Helmholtz equation is established, and Green's theorem is employed to obtain an integral equation for this potential function. An iteration procedure for determining the field in inverse powers of the square of the plasma frequency is then established, and with the aid of several approximations which have the effect of restricting the applicability to small cone angles the first term in the iteration is determined. Further terms are shown to be small. The pattern and some machine computations of eigenvalues and eigenfunction values (Legendre functions) for the 10° cone are included. *J. H. Harris* (Culver City, Calif.)

Pozzolo, Vincenzo; Zich, Rodolfo 4408

Una dimostrazione semplice della formula per il calcolo dell'effetto di piccole perturbazioni geometriche sulla frequenza di risonanza di una cavità.

Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **97** (1962/63), 1056–1063.

Semplice dimostrazione della formula che esprime il cambiamento di frequenza determinato dall'introduzione, in una cavità elettromagnetica, di un piccolo corpo perfettamente conduttore. *D. Graffi* (Bologna)

Boella, Mario; Zich, Rodolfo 4409

Sulla rappresentazione dei campi elettromagnetici in coordinate curvilinee ortogonali.

Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **97** (1962/63), 1043–1055.

Gli autori considerano un campo elettromagnetico **E** e **H** sinoidale e in un mezzo omogeneo isotropo. Definito un sistema di coordinate curvilinee ortogonali u_r ($r=1, 2, 3$) dimostrano che **E** ed **H** possono esprimersi mediante un vettore **Π** con due sole componenti, non nulle, sulle coordinate, e che se **E** o **H** è normale a una u_r, **Π** ha solo la componente secondo u_r. Ritrovano così, con procedimento unitario, alcuni metodi usati per lo studio di diversi campi elettromagnetici. *D. Graffi* (Bologna)

Jeffrey, A.; Taniuti, T. 4410

★Non-linear wave propagation. With applications to physics and magnetohydrodynamics.

Academic Press, New York-London, 1964. ix + 369 *pp.* \$12.00.

The first part of this book deals with the mathematical theory of hyperbolic partial differential equations (as well as methods of solution to these). The second portion is devoted to the application of these ideas to magnetohydrodynamics. The book is in general carefully written and well-explained.

In bringing new mathematical methods to attention and in dealing with magnetohydrodynamics this book is a valuable addition to Courant and Friedrichs [*Supersonic flow and shock waves*, Interscience, New York, 1948; MR **10**, 637] in the study of wave propagation in fluid mechanics. The chapter headings are: Part I, General Theory: Chapter 1, General Hyperbolic Equations; Chapter 2, The Method of Characteristics; Chapter 3, Conservation Laws and Weak Solutions. Part II, The Application to Magnetohydrodynamics: Chapter 4, The Fundamental Equations and Characteristics; Chapter 5, Simple Waves; Chapter 6, Magnetohydrodynamic Shocks; Chapter 7, Interaction of Hydromagnetic Waves; Chapter 8, Spatial Discontinuities; Appendices.

L. Sirovich (Providence, R.I.)

Wait, James R. 4411

Theory of radiation from sources immersed in anisotropic media.

J. Res. Nat. Bur. Standards Sect. B **68B** (1964), 119–136.

Author's summary and conclusion: "The electromagnetic fields produced by an electric dipole immersed in an anisotropic medium are considered. Various approaches to the problem are outlined with special reference to a cold plasma. An attempt is made to show the close relationship between previously published work on this subject. It is shown that information on the radiation field in an anisotropic media may be obtained directly from the shape of the refractive index surface.

"The present expository paper has been a self-contained treatment of the theory of radiation from sources in cold

magnetoplasma media. In this review, no mention has been made of antenna impedance for anisotropic media. While this would seem to be a natural topic for this discussion, it has been deferred until a later time. One reason for this is the need to consider thermal effects and sheath phenomena, which are important near the source. Consideration resulting from the finite temperature leads to contributions from the acoustic-type waves not usually considered part of magneto-ionic theory."

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CLASSICAL THERMODYNAMICS, HEAT TRANSFER

See also 3765, 3766.

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J. Meixner (Aachen)

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QUANTUM MECHANICS

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P. J. Davis (Providence, R.I.)

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Wiener, Norbert; della Riccia, Giacomo 4420

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Die Verfasser stellen die Theorie der Brownschen Bewegung (unter Benutzung der Gibbs'schen Ensembles) mit Hilfe von Funktionalen (und funktionaler Integration) dar. In gewissem Sinne kann man für dieses Problem Eigenfunktionen einführen. Der formale Zusammenhang dieser Funktionen mit den Eigenfunktionen der wellenmechanischen Schrödingergleichung wird untersucht. Verfasser finden eine vollständige Analogie zwischen der Theorie der Brownschen Bewegung und der quantenmechanischen Bewegung eines Teilchens.

G. Heber (Leipzig)

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Der Verfasser konstruiert zunächst eine spezielle Darstellung des δ-Funktionals, der Verallgemeinerung der Dirac'schen Deltafunktion auf den Funktionenraum, durch ein Funktional-Integral. Die gewonnene Formel wird dann zur Ableitung von Umkehrformeln für funktionale Integrationsformationen verwendet. Verfasser diskutiert ferner die Frage der Eindeutigkeit der Lösungen gewisser funktionaler Differentialgleichungen. Schließlich verwendet Verfasser die Methode der Transformationen im Funktionenraum, um den Zusammenhang zwischen funktionalen Differentialgleichungen und Randwertproblemen für partielle Differentialgleichungen zu studieren.

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Tempered distributions in functional space.

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Der Verfasser berichtet über eine Ausdehnung des Begriffes der gemäßigten Distributionen auf Funktionale. Dazu wird zuerst die Theorie der gemäßigten Distributionen etwas umformuliert. Dies wird in enger Anlehnung an Begriffsbildungen der Quantenmechanik getan. Der Übergang zu unendlich großer Zahl von Variablen wird in enger Anlehnung an die Quantenfeldtheorie vorgenommen. Die Focksche Darstellung von Hilbertraum-Vektoren spielt dabei eine besondere Rolle. Zum Schluß wird der Formalismus auf die Translations-Operatoren angewandt. *G. Heber* (Leipzig)

Nelson, Edward 4425

Schrödinger particles interacting with a quantized scalar field.

Analysis in function space, pp. 87–120. *M.I.T. Press, Cambridge, Mass.*, 1964.

Die Quantentheorie der Wechselwirkung nichtrelativistischer Teilchen mit einem skalaren Felde wird mit Hilfe funktionaler Integrale dargestellt. Dabei spielen Markoff'sche Prozesse eine wesentliche Rolle. Ultraviolett-Divergenzen werden mit Hilfe einer speziellen Methode subtrahiert; nur ein unendlicher Phasenfaktor bleibt bestehen, welcher mit einer Selbstenergie verknüpft ist. Die Existenz der verwendeten funktionalen Integrale (erstreckt über Teilchen-Bahnen) wird streng bewiesen. Es handelt sich dabei zunächst um Wienersche Integrale; der Übergang zu Feynmanschen "path-integrals" wird durch analytische Fortsetzung bzgl. des Massen-Parameters vollzogen. *G. Heber* (Leipzig)

Salam, Abdus 4426

Developments in renormalization theory.

Analysis in function space, pp. 121–128. *M.I.T. Press, Cambridge, Mass.*, 1964.

Der Verfasser betrachtet die Wechselwirkung zwischen skalaren Teilchen und Photonen sowie diejenige zwischen Teilchen mit Spin 1 und Photonen. Sein Ausgangspunkt sind die bekannten Integralgleichungen für die Greenschen Funktionen der zugeordneten Felder. Außerdem spielt die Ward'sche Identität eine wesentliche Rolle. Der Verfasser schlägt eine spezielle, nicht-störungstheoretische Methode zur Lösung dieser Gleichungen vor, in welcher auch die gewöhnlichen Dispersionsrelationen verwendet werden. Es scheint sich zu ergeben, daß auch die Wechselwirkung zwischen Vektormesonen und Photonen bei Verwendung der vorgeschlagenen Näherung renormierbar wird, jedoch nur für spezielle Werte der Masse der Mesonen und der Kopplungskonstante. *G. Heber* (Leipzig)

Segal, Irving 4427

Quantum fields and analysis in the solution manifolds of differential equations.

Analysis in function space, pp. 129–153. *M.I.T. Press, Cambridge, Mass.*, 1964.

Der Verfasser befaßt sich recht allgemein mit dem Zusammenhang zwischen partiellen Differentialgleichungen, wie sie in der Quantenfeldtheorie auftreten, und Methoden der Analysis im Funktionenraume. Dabei beschränkt er sich wesentlich auf Bosonen-Felder, über Fermionen-Felder findet man am Schluß der Einleitung einige Bemerkungen. Zuerst betrachtet Verfasser "allgemeine lineare Felder", wobei er u.a. auf verschiedene Aspekte des Begriffes der Kausalität und Lokalität eingeht. Danach wendet er sich der wohl einfachsten Klasse nichtlinearer Felder zu, welche durch die Gleichung $(\Box - m^2)\varphi = g^2\varphi^p$ gekennzeichnet ist. Hier wird insbesondere der Fall $p = 3$ betrachtet. Der Verfasser schlägt vor, diese Gleichung leicht abzuändern, und zwar so, daß die neue Gleichung mathematisch wohldefiniert ist (die Ausgangsgleichung hat nicht diese Eigenschaft). *G. Heber* (Leipzig)

Siegert, A. J. F. 4428

Applications of function space integrals to problems in equilibrium statistical mechanics.

Analysis in function space, pp. 154–163. *M.I.T. Press, Cambridge, Mass.*, 1964.

Als Beispiel zur Verwendung der funktionalen Integration zur Berechnung einer Zustandssumme innerhalb der statistischen Mechanik wird das Ising-Modell behandelt. In einer bestimmten Näherung bzgl. des Integranden erhält man dabei die bekannte Molekularfeld-Näherung. Man kann aber auch die nächsten Korrekturen zu dieser Näherung berechnen. Möglichkeiten und Schwierigkeiten bei der Ausführung der hierzu erforderlichen Integrationen werden besprochen. *G. Heber* (Leipzig)

Edwards, S. F. 4429

Applications of functional integration to nonrelativistic physical problems.

Analysis in function space, pp. 167–178. *M.I.T. Press, Cambridge, Mass.*, 1964.

Der Verfasser geht davon aus, daß viele auf irgendeine Weise lösbaren Probleme der nichtrelativistischen theoretischen Physik auch durch Funktionalintegrale darstellbar sind. Oft ist es so, daß man diese Funktionalintegrale mit den heute bekannten Methoden nicht ausrechnen kann, ja daß noch nicht einmal Existenzbeweise für alle in der Physik auftretenden Integrale geführt sind. Da aber die Lösung anderweitig bereits bekannt ist, kann man indirekt auf Existenz und Eigenschaften vieler Funktional-Integrale schließen. Dies wird im vorliegenden Aufsatz für eine Reihe von konkreten Beispielen getan. Hierbei ergibt sich u.a. der sehr wichtige Schluß, daß ein Funktionalintegral, dessen Integrand von einem Parameter stetig abhängt, durchaus keine stetige Funktion dieses Parameters sein muß. *G. Heber* (Leipzig)

Kastler, D. 4430

A C^*-algebra approach to field theory.

Analysis in function space, pp. 179–191. *M.I.T. Press, Cambridge, Mass.*, 1964.

Gekürzte Fassung einer im Druck befindlichen Arbeit von R. Haag und D. Kastler. *G. Heber* (Leipzig)

Haag, R. 4431

Remarks on the mathematical structure of quantum field theory.

Analysis in function space, pp. 192–196. *M.I.T. Press, Cambridge, Mass.*, 1964.

Kurze Diskussion der mathematisch strengen Fassung der

grundlegenden Axiome (Lokalität, Lorentz-Invarianz und Stabilität) der Quantenfeldtheorie und einiger ihrer Konsequenzen. *G. Heber* (Leipzig)

Symanzik, K. 4432

Application of functional integrals to Euclidean quantum field theory.

Analysis in function space, pp. 197–206. *M.I.T. Press, Cambridge, Mass.*, 1964.

Der Verfasser zeigt, daß nach Übergang von einem 4-dimensionalen Minkowskischen zu einem 4-dimensionalen Euklidischen Raum und nach Einführung eines endlichen Volumens und eines Spektrums von Hilfsmassen die Funktional-Integral für die Greenschen Funktionen eines nichtlinearen, skalaren Feldes existieren. Er gibt auch Abschätzungen für diese Integrale an, kann sie aber nicht explizit ausrechnen. *G. Heber* (Leipzig)

Schwartz, Melvin 4433

Lagrangian and Hamiltonian formalisms with supplementary conditions.

J. Mathematical Phys. **5** (1964), 903–907.

A physical system with supplementary conditions can usually be treated by the Lagrangian multiplier method in classical physics. It is found, however, that the formulation of its quantum version is not as straightforward as it looks. A straightforward quantum mechanical treatment of such a system is to eliminate supplementary conditions by reducing the number of variables. However, this method sacrifies the symmetry among all the original variables. The reviewer formulated a Hamiltonian formalism when there is one supplementary condition, under the rule of our game, to construct a Hamiltonian formalism without discriminating any of the variables in such a manner that it can be quantized [the reviewer, Phys. Lett. **1** (1962), 278–279; MR **25** #4869]. The present paper deals with a system in which there are N independent supplementary conditions on the differentials of n coordinates. The relation of this formalism to the Lagrangian multiplier method is also discussed. Three examples are given; (i) particles on a sphere, (ii) vacuum electrodynamics in configuration space, and (iii) the same in momentum space. Compare with the similar approach by Skinner [Canad. J. Phys. **34** (1956), 901–913; MR **18**, 443]. *Y. Takahashi* (Dublin)

Kinoshita, T.; Loeffel, J. J.; Martin, A. 4434

New upper bound for the high-energy scattering amplitude at fixed angle.

Phys. Rev. Lett. **10** (1963), 460–462; *erratum, ibid.* **11** (1963), 138.

Froissart [Phys. Rev. (2) **123** (1961), 1053–1057] showed that (for equal mass scalar particles) the Mandelstam cut plane analyticity in $\cos\theta$, together with unitarity, provides the following upper bounds at high energy on the scattering amplitude:

$$(1)\qquad |f(s, \cos\theta)| < C_1 s(\log s)^2 \quad \text{for } \theta = 0 \text{ or } \pi,$$

$$(2)\qquad |f(s, \cos\theta)| < C_2 s^{3/4}(\log s)^{3/2} \quad \text{for } \theta \neq 0 \text{ or } \pi.$$

The reviewer and Low [ibid. (2) **124** (1961), 2047–2048; MR **24** #B1917] showed that the less restrictive analyticity in that Lehmann ellipse which follows from local quantum field theory, together with unitarity, leads to weaker high-energy bounds, and also pointed out that analyticity in the bigger Lehmann ellipse which touches the Mandelstam cuts leads to Froissart's bounds (1) and (2). This last result was later found independently by Martin [ibid. (2) **129** (1963), 1432–1436; MR **27** #2291]. In the present article, the authors assume analyticity in a domain D_s which includes the latter Lehmann ellipse and remains a fixed finite distance away from it, except near $\cos\theta = \pm 1$, where the Mandelstam cuts enter, for all values of s. From this larger domain of analyticity, together with unitarity, the authors find the high energy bound

$$(3)\qquad |f(s, \cos\theta)| < C' \frac{(\log s)^{3/2}}{\sin^2\theta},$$

uniformly in θ, which improves (2) above.
O. W. Greenberg (College Park, Md.)

Azimov, S. A.; Arušanov, G. G. 4435

Some formulae for the elastic scattering of particles of higher energy. (Russian. Uzbek summary)

Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk **1964**, *no.* 2, 69–76.

Considering the elastic scattering of spin-zero particles at high energies, the authors obtain some relations between scattering amplitudes or cross-sections by replacing the formulae

$$a_{l+1} - a_l = \frac{da_l}{dl}, \qquad \sum_{l=l_0}^{\infty} \to \int_{l_0}^{\infty}$$

by the exact series

$$a_{l+1} - a_l = \sum_{n=1}^{\infty} \frac{1}{n!} a_l^{(n)},$$

$$\sum_{l=l_0}^{L} f(l) = \int_{l_0}^{L} f(l)\,dl + \tfrac{1}{2}[f(l_0) + f(L)] + \sum_{k=1}^{n} \frac{B_{2k}}{(2k)!}[f^{(2k-1)}(L) - f^{(2k-1)}(l_0)] + R_n.$$

In particular, they consider the backward scattering and the scattering in the neighbourhood of 180°, the scattering at 90° and in the neighbourhood of 90°, and finally the scattering at angles not equal to 0 or π. They obtain some interesting results, e.g., that the cross-section $\sigma(\vartheta)$ for scattering at fixed non-zero angles must obey the inequality $\sigma(\vartheta) \lesssim 1/k^2$; this is a stronger statement than that obtained by T. Kinoshita, J. Loeffel, and A. Martin [see #4434 above] on the basis of analytical properties of the scattering amplitude. *M. Blažek* (Bratislava)

Bingel, Werner A. 4436

The behaviour of the first-order density matrix at the Coulomb singularities of the Schrödinger equation.

Z. Naturforsch. **18a** (1963), 1249–1253.

Author's summary: "The cusp conditions of Kato for a spinless n-electron wave function at the Coulomb singularities of the Schrödinger equation are used to derive corresponding conditions for the first-order density matrix. These results are applied to the united-atom expansion of the electronic energy of polyatomic molecules resulting in an exact relation between the coefficients of

the quadratic and cubic terms in this expansion. Finally it is shown how the cusp condition for the first-order density matrix is modified if the wave function includes electron spin in a proper way." *A. J. Coleman* (Kingston, Ont.)

Calogero, F. 4437

A novel approach to elementary scattering theory. (Italian summary)

Nuovo Cimento (10) **27** (1963), 261–302.

The conventional method to compute phase shifts is based on the regular solution of a second-order wave equation. On the other hand, in the method proposed a function $\delta(l, k; r)$ is introduced which satisfies a first-order nonlinear differential equation, and whose asymptotic value gives directly the value of the phase shifts. The phase function introduced, $\delta(l, k; r')$, has a very simple physical meaning: it is the phase shift produced by the potential $V(r)\theta(r'-r)$, where $\theta(r)$ is the step function, i.e., the phase shift that the potential $V(r)$ produces after it has been amputated of its part extending beyond r'. Also a closed expression for the amplitude of the Jost function, whose absolute value squared measures the effect of the potential on the wave function at the origin, is given in terms of this phase function.

For a numerical computation the method introduced is far superior to the conventional one. Also because of its simplicity and transparency, it allows one to derive general theorems on phase shifts in a much simpler way than would be possible through the standard method. The s-wave phase shift is worked out numerically for an attractive square-well. In particular, the validity of Levinson's theorem in the limit of zero energy is nicely demonstrated. Upper and lower bounds on the phase shifts and on their derivatives with respect to linear and angular momentum are established. For small phase shifts, e.g., in the case of high energy, it is possible to linearize the appropriate differential equation, and approximate expressions for the phase shifts are obtained. The zeroth approximation coincides with the first Born approximation and a less drastic approximation yields the so-called improved Born approximation. The latter turns out to coincide with the first term of an expansion recently derived by the reviewer and S. Tani [Phys. Rev. (2) **131** (1963), 396–406; MR **28** #3676]. Next, it is shown how, through the use of trial functions, approximate values of the phase shifts may be obtained. Under the assumption that the potential never changes sign and that the phase shift is smaller in magnitude than $\pi/2$, a formula is given which appears particularly suited for the evaluation of the tangent of the phase shift by means of trial functions, and which has the important advantage over the usual variational method that it provides information on the sign of the error involved. Finally, an interesting method is given for the determination of the energies of bound states, and two examples are briefly discussed.

S. Rosendorff (Haifa)

Calogero, F. 4438

A variational principle for scattering phase shifts. (Italian summary)

Nuovo Cimento (10) **27** (1963), 947–951.

A variational principle for the tangent of the phase shift and for the phase shift itself is established. The two restrictions (the potential never changes sign and the phase shift is smaller than $\pi/2$) required in the preceding paper [#4437] have been lifted; however, the information on the sign of the error is lost. *S. Rosendorff* (Haifa)

Calogero, F. 4439

The scattering of a Dirac particle on a central scalar potential. (Italian summary)

Nuovo Cimento (10) **27** (1963), 1007–1016.

The new approach to the theory of potential scattering developed in the two previous papers [#4437, #4438] for Schrödinger particles is extended to Dirac particles. This is achieved without transforming the Dirac problem into a Schrödinger problem. As a result, the equations that are obtained are simple and have a very direct physical interpretation. Variational formulas for the tangent of the phase shift and the phase shift itself are obtained, and approximate formulas are given. A theorem due to Parzen [Phys. Rev. (2) **80** (1950), 261–268; MR **12**, 571] concerning the asymptotic behavior of the phase shifts at large energies is easily proved. *S. Rosendorff* (Haifa)

Chan, H. H.; Razavy, M. 4440

Plane-wave approximation for the scattering amplitude.

Canad. J. Phys. **42** (1964), 1017–1029.

In potential scattering the first Born approximation is unsatisfactory for an approximate determination of the low-energy scattering parameters (s-wave effective range and scattering length) because it is non-unitary. It is possible to formulate a second Born approximation which is unitary in a number of different ways, depending on the exact equation to which approximate solutions are being sought. In this paper four such approximations are discussed and compared with the exact solution for a number of cases where this is readily obtainable. Although, of course, all of the approximations agree in the limit of vanishingly weak potentials, they differ considerably for stronger potentials. An approximation based on a variational technique due to Schwinger is consistently superior. *J. M. Charap* (London)

Fikioris, J. G.; Waterman, P. C. 4441

Multiple scattering of waves. II. "Hole corrections" in the scalar case.

J. Mathematical Phys. **5** (1964), 1413–1420.

Part I was under the authorship of Waterman and Truell [same J. **2** (1961), 512–537; MR **23** #B788]. The present paper takes up scattering due to a random distribution of spheres in the region $z \geq 0$, starting with the equation

$$(*)\quad \langle\psi^E(\mathbf{r}|\mathbf{r}_1)\rangle = \psi^{\text{inc}}(\mathbf{r}) + \int_\tau n(\mathbf{r}'|\mathbf{r}_1)T(\mathbf{r}')\langle\psi^E(\mathbf{r}|\mathbf{r}')\rangle\,d\tau'$$

for the configurational average of the exciting field $\psi^E(\mathbf{r}|\mathbf{r}_1)$ at $\mathbf{r}=(x, y, z)$ acting on a scatterer at $\mathbf{r}_1$; $n(\mathbf{r}|\mathbf{r}_1)$ is the conditional density of scatterers, which in the present case is taken to be n_0 for $|\mathbf{r}'-\mathbf{r}|>b$, $z'>0$, $z>0$, and 0 otherwise; $T(\mathbf{r}')$ is an operator, which when operating on $\langle\psi^E(\mathbf{r}|\mathbf{r}_1)\rangle$, yields the field scattered by the sphere at $\mathbf{r}_1$ due to the exciting field; $b \geq 2a$ is the least distance between centers of neighboring spheres, each of radius a. From the fact that the fields satisfy the Helmholtz

equation in k and $\mathbf{r}$, one obtains expansions that can be inserted into (*) and arrives at an infinite set of integral equations for the unknown coefficients in the expansion of $\langle\psi^{E}(\mathbf{r}|\mathbf{r}_1)\rangle$. This set is reduced to an infinite system of linear algebraic equations by means of certain heuristic assumptions; these give to the resulting solution the status of a zeroth order approximation in an iteration process. Successive approximations after the first become very complicated, and convergence has not been investigated.

In the low-frequency limit, the zero-order solution agrees with known results in the acoustic case. As a consequence of the deductions from this analysis, it is found to be necessary to qualify certain of the conclusions in Part I. *R. N. Goss* (San Diego, Calif.)

Gluckstern, R. L.; Lin, Shin-R 4442

Relativistic Coulomb scattering of electrons.

J. Mathematical Phys. **5** (1964), 1594–1602.

Authors' summary: "A simple and useful relation between the Coulomb amplitudes F and G (in Mott's notation) is derived and F and G are evaluated analytically up to α^4 terms for arbitrary $q=\alpha/\beta$. These results are valid for all angles, but are particularly useful at small angles. The general analytic behavior of F and G in the variable $x=\sin\frac{1}{2}\theta$ is discussed. The method is applicable to higher-order terms (α^6 and up). A double integral representation of F is also derived by using the Sommerfeld-Watson transformation. This integral representation exhibits the dependence on α, q, and θ separately."

Jordan, Thomas F. 4443

Lorentz invariant multichannel scattering formalism.

J. Mathematical Phys. **5** (1964), 1345–1360.

Author's summary: "A manifestly Lorentz invariant Hamiltonian formalism for multichannel scattering and production processes is developed by making two simple and natural extensions of the ordinary quantum mechanical formalism. The first is the asymptotic covariance postulate of Fong and Sucher [same J. **5** (1964), 456–470; MR **28** #5703] which is essentially a necessary and sufficient condition for Lorentz invariance of the scattering amplitudes. The second is a Lorentz invariant extension of the asymptotic condition. It is shown that the latter is in fact no extension at all in a case where the total momentum operators of the asymptotic (unperturbed) systems are the same as the total momentum operators of the interacting system. In such a case the ordinary multichannel scattering formalism is completely Lorentz invariant whenever asymptotic covariance is satisfied."

A. Peres (Haifa)

Otokozawa, J. 4444

A comment on the analytic continuation of scattering amplitudes in the external mass variable.

Nuovo Cimento (10) **32** (1964), 495–497.

The author investigates the analytic continuation in an external mass variable of a scattering amplitude when an unstable particle occurs in an intermediate state.

V. de Alfaro (Princeton, N.J.)

Pais, A.; Wu, Tai Tsun 4445

Scattering formalism for singular potential theory.

Phys. Rev. (2) **134** (1964), B1303–B1307.

The problem of determining the physically interesting quantities in the frame of static singular potentials is considered here. It is shown that the scattering amplitude in these cases exists as the limit of the ratio of two functions, neither of which, however, has a definite limit. For illustrative purposes some examples are given considering the various possibilities. A similar proposal for defining the S-matrix elements was earlier suggested by N. Limić [Nuovo Cimento (10) **26** (1962), 581–596; MR **26** #4662]. Alternative and different approaches have also been considered by several authors [N. N. Khuri and A. Pais, Rev. Modern Phys. **36** (1964), 590–595; MR **29** #973; G. Tiktopoulos and S. B. Treiman, Phys. Rev. (2) **134** (1964), B844–B847; MR **29** #3170; A. Pais and T. T. Wu, J. Mathematical Phys. **5** (1964), 799–804; MR **29** #974] by using cut-off and peratization-like procedures, and by H. Cornille and E. Predazzi [CERN preprint 9125/Th. 441] by connecting two limiting procedures.

E. Predazzi (Chicago, Ill.)

Cox, Joseph R. 4446

Many-channel Bargmann potentials.

J. Mathematical Phys. **5** (1964), 1065–1069.

The author discusses the analytic properties of the function $f(k_1,\cdots,k_n)$ of the energy, from which the S-matrix of an n-channel scattering system may be calculated. He also determines a set of generalized Bargmann potentials, with exponential dependence on distance, for which explicit construction of the S-matrix is possible. A two-channel example is treated in detail.

K. J. Le Couteur (Canberra)

Lebon, G. 4447

Perturbation des niveaux de résonance d'un potentiel central. (English summary)

Bull. Soc. Roy. Sci. Liège **33** (1964), 324–336.

Author's summary: "The general solution of the perturbation problem of resonant states is given to the first order for a spherically symmetric potential. In the particular case of the perturbation of a critical potential, higher orders are given and as illustrative example, a square-well potential is finally considered."

Lütken, Hans; Winther, Aage 4448

Coulomb excitation in deformed nuclei.

Mat.-Fys. Skr. Danske Vid. Selsk. **2**, no. 6, 32 pp. (1964).

Authors' summary: "In multiple Coulomb excitations of deformed nuclei one may observe also a weaker excitation of rotational bands which are associated with states of different intrinsic structure. In the present work, the excitation amplitudes of such states have been computed in the approximation where one neglects the energy differences between the states of a rotational band. For the case of dipole, quadrupole and octupole excitations, the results are given in the form of tables. They show that the relative population of the states within a band

depends strongly on the spin and K quantum numbers, as well as on admixtures in the wave function of components from the ground-state band. A detailed investigation of the dependence of the cross-section on scattering angles is presented. One finds here appreciable deviations from the so-called $\chi(\vartheta)$-approximation, especially concerning the excitation of the unnatural parity states."

V. de Alfaro (Princeton, N.J.)

Dirac, P. A. M. 4449

Hamiltonian methods and quantum mechanics. (Larmor Lecture)

Proc. Roy. Irish Acad. Sect. A **63**, 49–59 (1964).

The author surveys the Hamiltonian form of dynamics starting with classical nonrelativistic mechanics and working up through relativity and quantization to quantum field theory and also to general relativity. In relativistic theories, the role of the Hamiltonian is played by the ten generators of the inhomogeneous Lorentz group, and the consistency conditions on these "Hamiltonians" is nothing else than the commutation (or Poisson bracket) relations of the Lie algebra of the generators of this group. The lecture concludes with some remarks on difficulties encountered in nonlocal theories. *R. Ingraham* (University Park, N.M.)

Grossmann, Alexander 4450

Nested Hilbert spaces in quantum mechanics. I.

J. Mathematical Phys. **5** (1964), 1025–1037.

Let A be a self-adjoint operator defined on a Hilbert space H. The resolvent $R(z)=(z-A)^{-1}$ of A is then a bounded normal operator-valued function of z, analytic for all values of z in the upper half-plane, but singular on the spectrum of A. It may happen, however, that for certain vectors f in H, the function $(R(z)f,f)$ remains bounded as z approaches the spectrum of A, and even admits an analytic continuation across the spectrum into the lower half-plane. In many physical applications it is of considerable interest to locate and describe such vectors in terms of the physically relevant available data. For an extended discussion of this situation, see the survey article by C. L. Dolph [Bull. Amer. Math. Soc. **67** (1961), 1–69; MR **25** #5612].

The author's contribution to the problem boils down to the observation that it is sometimes possible to renorm the Hilbert space in such a way that the new norm is finite only on vectors of the required form. This means only that there sometimes exists a bounded positive operator P with zero null space, such that $PR(z)P$ remains bounded as z approaches the spectrum of A and even admits an analytic continuation across this spectrum. He gives several concrete examples of operators A for which there does exist an operator P which "removes" the continuous singularities of $R(z)$ (e.g., $H=\mathscr{L}^2(E_3)$, $A=-\nabla^2$, and P is multiplication by a bounded square-integrable function $V(x)$), and proves that the isolated singularities of $R(z)$ cannot be removed by any choice of P.

The analytic continuation of $R(z)$ so obtained, however, depends on the choice of P. The problem of providing a prescription for making this choice in particular cases remains unsolved. *R. T. Prosser* (Lexington, Mass.)

Bailey, Paul B. 4451

Exact quantization rules for the one-dimensional Schrödinger equation with turning points.

J. Mathematical Phys. **5** (1964), 1293–1297.

Author's summary: "It is pointed out that, for a number of problems, exact quantization rules exist which closely resemble those of Bohr-Wilson-Sommerfeld. In some cases it is shown how these rules may be derived mathematically from the Schrödinger equation."

I. Bialynicki-Birula (Warsaw)

Spector, Richard M. 4452

Exact solution of the Schrödinger equation for inverse fourth-power potential.

J. Mathematical Phys. **5** (1964), 1185–1189.

The problem of constructing the various physically relevant quantities for the $1/r^4$ potential is considered. The Schrödinger equation is explicitly solved in terms of Mathieu functions (for both the cases of attractive and repulsive potentials) in the nonzero energy case. The S matrix is also briefly discussed in the repulsive case. The advantage of this approach lies in the fact that it reduces the problem to the use of mathematically well-known functions. It should, however, be noticed that the method cannot usually be extended to more general kinds of singular interactions because the $1/r^4$ case is the most singular one for which the Schrödinger equation can be solved exactly in terms of known functions in the non-zero energy limit. This is due to the appearance of more and more essential singularities as it increases the kind of singularity of the interactions. Furthermore, the $1/r^4$ case is very symmetrical in the sense that the solutions of the Schrödinger equation have in this case the same kind of (essential) singularity both at infinity and at the origin. This can be most simply understood by making the change of variable $r=1/x$ which shows that the energy and the potential term exchange their role. If one wants to calculate the physical quantities in the non-zero energy limit for more general classes of singular interactions, other approaches need to be considered. This problem has been recently investigated and solved by several authors and with different techniques. *E. Predazzi* (Chicago, Ill.)

Segal, Irving 4453

Interprétation et solution d'équations non linéaires quantifiées.

C. R. Acad. Sci. Paris **259** (1964), 301–303.

The author rewrites a typical non-linear local field equation, such as $\Box\phi=m^2\phi+g\phi^3$ together with the canonical commutation rules, in terms of commutator bracket expressions for the 3-dimensionally smeared fields $\Phi(f,t)=\int\phi(x,t)f(x)\,d^3x$ and their first and second time derivatives. He claims that the resulting set of multiple commutator brackets is a well-defined set of relations equivalent to the original poorly defined local field equation. He also claims to give an explicit solution of these commutator bracket equations. {To the reviewer at least some of the divergence difficulties arising in the local field equation must persist in the commutator bracket equations. This is because the wave function renormalisation constants would seem to enter into the commutator bracket equations for renormalised fields.}

John G. Taylor (New Brunswick, N.J.)

Čistjakov, A. L. 4454

The scattering operator in the second quantization space. (Russian)

Dokl. Akad. Nauk SSSR **158** (1964), 66–69.

It is proved that there exists an elastic part of the scattering operator for the energy operators of quantum mechanical systems with a variable number of particles. Systems with a variable number of particles are described by means of second quantization methods. The author also treats the Bose- and the Fermi-fields separately.

The existence of the wave operators, by analogy with the usual quantum mechanics [J. M. Jauch and I. I. Zinnes, Nuovo Cimento (10) **11** (1959), 553–567; MR **21** #6223], is asserted on the basis of the fact that the curves described in Hilbert space by the vectors $U_0(t)\phi$ when the time t varies from 0 to $\pm\infty$ are finite. One has $U_0(t)=\exp(iHt)\exp(-iH_0t)$, where H (and H_0) is the Hamiltonian of the interacting (and non-interacting) system and ϕ is the state vector from which one forms the Hilbert space considered. *M. Blažek* (Bratislava)

Domokos, G. 4455

On algebraic problems in the theory of complex angular momentum.

Nuclear Phys. **47** (1963), 124–128.

Author's summary: "A method is given for reducing 'algebraic' operations on complex angular momenta (addition, recoupling, etc.) to ordinary angular momentum algebra, by expanding the matrix elements $\mathscr{D}^j_{mm'}(R)$ for complex j into an orthogonal series. The generalization of vector coupling coefficients and their sum rule for complex total angular momenta are treated as examples."

Efimov, G. V. 4456

On local quantum field theory without ultra-violet divergences. Third order. (Italian summary)

Nuovo Cimento (10) **32** (1964), 1046–1058.

This article continues the investigation of a self-coupled scalar field theory, where the coupling term grows less rapidly for large field strengths than the free field term. In an earlier investigation [Ž. Èksper. Teoret. Fiz. **44** (1963), 2107–2117; MR **28** #2805] it was shown that such couplings, if also obeying some additional conditions, will not give rise to ultraviolet divergences in second-order perturbation theory. In this article the third order in perturbation theory is investigated, and it is shown by the same methods that under the same conditions no ultraviolet divergences appear, and unitarity is satisfied. *W. M. Frank* (Rehovot)

Flamm, D.; Freund, P. G. O. 4457

Requirements of self-consistency in quantum field theories. (Italian summary)

Nuovo Cimento (10) **32** (1964), 486–492.

This interesting paper analyzes the self-consistency of the known renormalizable quantum field theories. In this context self-consistency consists in the finiteness of solutions of the Dyson-Schwinger system of integral equations for propagators and vertex functions, that is, finite values for all renormalization constants, along the lines of Baker and Johnson [K. Johnson, M. Baker and R. S. Willey, Phys. Rev. Lett. **11** (1963), 518–520; MR **29** #3157]. The answer is that none of the known renormalizable theories, other than electrodynamics of spin $\frac{1}{2}$ particles, is self-consistent. *V. de Alfaro* (Princeton, N.J.)

Frantz, L. M. 4458

High-intensity quantum electrodynamics. I. Intensity dependences for Feynman diagrams.

Phys. Rev. (2) **134** (1964), B1419–B1429.

Author's summary: "The prescriptions for evaluating Feynman diagrams in quantum electrodynamics are extended to the regime of incident photon beams having nonzero intensity. This is done both for the time-dependent formulation of the S-matrix perturbation theory, and for the stationary-state Brillouin-Wigner perturbation development. The results are applicable to a theory of any boson field, as they depend only on the Bose-Einstein commutation properties of the field operators; no restrictions are made on the nature of the interaction Hamiltonian. It is shown that, as far as radiative corrections are concerned, the presence of the incident beam is to be ignored; it is to be manifested by a weighting factor which depends on the count of only external photon lines connected to the scatterer line. The distribution of the incident photons over modes of the radiation field is completely arbitrary, and the results are valid to all orders in the perturbation theory." *F. Calogero* (Rome)

Chow, Yutze 4459

On some topological properties of Feynman graphs and their application to formulas related to the Feynman amplitudes.

J. Mathematical Phys. **5** (1964), 1255–1260.

The topological notion of tree set is introduced. It is then used to prove two theorems about Feynman integrals in parametric form. The first theorem proves the equivalence of the usual integral representation to that used by Symanzik. The second is the rederivation of a result of Nakanishi. *F. R. Halpern* (La Jolla, Calif.)

Hepp, Klaus 4460

Lorentz invariant analytic S-matrix amplitudes.

Helv. Phys. Acta **37** (1964), 55–73.

As a continuation of a previous work [same Acta **36** (1963), 355–375; MR **27** #5504], the author investigates a decomposition of analytic relativistic S-matrix elements $T_\alpha(p)$ into a sum of Lorentz covariant polynomials $Q_\alpha{}^\lambda(p)$ with invariant analytic amplitudes $T_\lambda(p)$ as coefficients. Instead of a domain in C^{4n} in the previous work, the author is now concerned with a domain (R,π,M) over the mass shell $M=\{p;\sum p_i=0, p_i{}^2=m_i{}^2\}$ of n complex momenta $p=\{p_i\}$ with a locally topological mapping π from R into M (in short, a non-schlicht domain with the mass shell restriction). The masses m_i are assumed positive.

$I_{(+)}$ denotes the mapping from p to $L_{(+)}(C)$ invariants of p. ($I_{(+)}$ stands for "I or I_+" and the same for $L_{(+)}(C)$.) A domain (R,π,M) is $I_{(+)}$ saturated if there exists a schlicht covering $\{V_\mu\}$ of R such that $I_{(+)}{}^{-1}I_{(+)}\pi(V_\mu)=\pi(V_\mu)$. The largest $I_{(+)}$ saturated subdomain of (R,π,M) is its $I_{(+)}$ saturated kernel $(R^{\kappa+},\pi,M)$.

It is proved (Theorem 1) that $I_{(+)}M$ is a normal algebraic set and that every $L_{(+)}(C)$ invariant amplitude $T(p)$, holomorphic on an $L_{(+)}(C)$ invariant domain, can be

written as $T = \hat{T}_{(+)} I_{(+)}$ on $(R^{n(+)}, \pi, M)$ with a unique holomorphic function $\hat{T}_{(+)}$ of $L_{(+)}(C)$ invariants. Further, it is proved (Theorem 2) that every covariant $T_\alpha(p)$, holomorphic in a common domain (R, π, M), can be decomposed (globally) as $T_\alpha = \sum_\lambda T_\lambda Q_\alpha^{\ \lambda}$ with T_λ holomorphic in a given holomorphically convex $L_{(+)}(C)$ invariant subdomain $(R^{H(+)}, \pi, M)$ of $(R^{n(+)}, \pi, M)$. The decomposition is unique for $n \leqq 4$.

The author investigates the restrictions due to Π-, T- and C-invariance and due to the Pauli principle. The results are exemplified for the "spin $\frac{1}{2}$, spin $\frac{1}{2}$" scattering amplitudes in Appendix B. *H. Araki* (Kyoto)

Stepanov, B. M. 4461

The construction of S-matrices.

Dokl. Akad. Nauk SSSR **151** (1963), 84–86 (*Russian*); *translated as Soviet Physics Dokl.* 8 (1964), 663–665.

A recursive method is presented for determining the test-function spaces over which the successive terms in the perturbation expansion of the S-matrix can be defined as generalized functions. The Bogolyubov scheme for construction of the successive terms of the S-matrix via causality and unitarity is utilized.

W. M. Frank (Rehovot)

Jin, Y. S.; Martin, A. 4462

Remarks on the polynomial boundedness in the Mandelstam representation.

J. Mathematical Phys. 5 (1964), 1406–1412.

Authors' summary: "The Mandelstam representation is a statement about the region of analyticity and asymptotic behavior (polynomial boundedness) of a scattering amplitude. In virtue of the unitarity condition, however, these two are not completely independent. Some physical consequences, e.g., uniqueness, polynomial boundedness of the total cross-section, etc., which have been already derived from the Mandelstam representation, are shown to be preserved, even if the polynomial boundedness is replaced by a somewhat weaker assumption. By making use of unitarity, analyticity and crossing symmetry, the following type of scattering amplitude $F = E + M$, where E is an entire function in both variables s and t, while M denotes a Mandelstam-type function with finite number of subtractions, is shown to be ruled out. Similarly, $F = EM$ is also ruled out if one imposes the additional restriction that E should increase less fast than an exponential in one variable while the other is finite."

F. R. Halpern (La Jolla, Calif.)

Kamefuchi, S.; Umezawa, H. 4463

Bose fields and inequivalent representations. (Italian summary)

Nuovo Cimento (10) **31** (1964), 429–446.

In this paper, the "self-consistent method", which was developed for fermion fields in a previous paper [H. Umezawa, the reviewer and S. Kamefuchi, Ann. Physics **26** (1964), 336–363], is improved slightly and applied to boson fields. In particular, the Goldstone model characterized by the Lagrangian

$$\mathscr{L} = -\partial_\mu \phi^+ \partial_\mu \phi - \frac{\lambda}{6} (\phi^+ \phi)^2$$

is discussed in terms of this method to show that the Goldstone conjecture on the massless boson [J. Goldstone, Nuovo Cimento (10) **19** (1961), 154–164; MR **22** #B1417; J. Goldstone, A. Salam and S. Weinberg, Phys. Rev. (2) **127** (1962), 965–970; MR **28** #3712] does not always hold true. See also a note by P. W. Higgs [Phys. Lett. **12** (1964), 132–133]. *Y. Takahashi* (Dublin)

Krass, Allan S. 4464

Relativistic model field theory.

Phys. Rev. (2) **134** (1964), B605–B611.

Author's summary: "A relativistic generalization of the Lee model is constructed and solved in the first sector. The solubility is achieved with an indefinite metric and a redefinition of antiparticle operators, which amounts to a selection rule prohibiting pair creation. The renormalization of the V-particle results in three dressed states, one of which is a ghost. The properties of these states and their dependence on cutoff and coupling strength are discussed. The effects of various modifications of the interaction are analyzed. Thirdly, the N-θ problem is solved exactly." *W. M. Frank* (Rehovot)

Nakanishi, Noboru 4465

Fundamental properties of perturbation-theoretical integral representations. III.

J. Mathematical Phys. 5 (1964), 1458–1473.

Part II appeared in same J. **4** (1963), 1385–1392 [MR **27** #6524]. In this continuation of the author's investigation of perturbation-form integral representations, their asymptotic behavior is discussed. A number of theorems are proven for the two-variable representation, upon imposing certain strong restrictions on the asymptotic behavior and singularities of the weight function. A heuristic discussion of more general cases is also given, relating asymptotic increase of the amplitude in certain directions to the singularities of the weight function for unsubtracted representations, and giving some conjectures in the subtracted case. *S. Deser* (Waltham, Mass.)

Petrina, D. Ja. 4466

On the principle of maximal analyticity over the complex orbital moment. (Russian)

Ukrain. Mat. Ž. **16** (1964), 502–512.

The properties of the partial scattering amplitude in the (total) orbital momentum complex plane, both in the nonrelativistic quantum mechanics and in the quantum field theory, were studied by various authors. In this paper the investigation is performed by means of the regularization methods developed for the diverging integrals of the form $\int_0^1 x^\lambda \varphi(x)\, dx$ (Re $\lambda < -1$, $\varphi(x)$ regular for $x \to 0$) [in accordance with the work of I. M. Gel'fand and G. E. Šilov, *Generalized functions* (Russian), Part 1, Fizmatgiz, Moscow, 1958; MR **20** #4182].

For instance, in the nonrelativistic scattering when one deals with the Schrödinger equation

$$\psi''(z) + k^2\psi(z) - \frac{l(l+1)}{z^2}\psi(z) - v(z)\psi(z) = 0$$

and with the Yukawa potential $v(z) = [\exp(-\mu z)]/z$, one

uses the following analytic continuation in the regular solution obtained by means of the iteration procedure

$$\frac{1}{z^{\lambda-1/2}}\int_0^z z_1^{2\lambda+1}[v(z_1)-k^2]\,dz_1 =$$

$$\frac{1}{z^{\lambda-1/2}}\left\{\int_0^z z_1^{2\lambda}\left[V(z_1)-V(0)-\cdots-\frac{z_1^m}{m!}V^{(m)}(0)\right]dz_1\right.$$

$$\left.+\frac{z^{2\lambda+1}}{2\lambda+1}V(0)+\cdots+\frac{z^{2\lambda+m+1}}{(2\lambda+m+1)m!}V^{(m)}(0)\right\},$$

$$\lambda = l+\tfrac{1}{2},$$

where $V(z)=z[v(z)-k^2]$ for $-1<m+1+2\,\mathrm{Re}\,\lambda$ (m is arbitrary). The author uses a similar continuation for the relativistic case (Mandelstam representation).

M. Blažek (Bratislava)

Rastogi, N. C. 4467

Linear and antilinear transformations of the Dirac wave function for massless particles under conformal coordinate transformations.

Proc. Nat. Inst. Sci. India Part A **29** (1963), 520–527.

Der Vektor (x_ν) des reellen Lorentzraumes möge gemäss

$$y_\nu = (x_\nu - \tfrac{1}{2}x^2a_\nu)/(1-(ax)+\tfrac{1}{4}a^2x^2)$$

mit festem Vektor (a_ν) einer konformen Transformation unterzogen werden. Wie lauten die zugehörigen Transformationsgleichungen für die 4-Spinoren der Diracgleichung? Der Verfasser gibt zwei Matrizen S und T, die explizite Funktionen der a_ν sind, an derart, dass sich die Transformationsgleichungen der Spinoren linear mit S und T ausdrücken lassen.

J. F. Weier (Bonn)

Rayski, Jerzy 4468

Resonant states of nucleon and bilocal field theory.

Helv. Phys. Acta **36** (1963), 1081–1094.

The author presents a modified formulation of Yukawa's bilocal field theory [Phys. Rev. (2) **77** (1950), 219–226; MR **11**, 567; ibid. (2) **80** (1950), 1047–1052; MR **12**, 571] in order to get a description of certain families of resonant states occurring in nature. These states, of increasing spins and alternating parities (such as the second and third isobars of the nucleon) cannot be correctly described by current local field theory, because the divergences which occur cannot be eliminated by renormalization. Indeed, previous studies [Fierz, Helv. Phys. Acta **23** (1950), 412–416; MR **12**, 67] had shown that, in the case of free fields, a bilocal field is equivalent to an infinite set of local fields of increasing spins and alternating parities. In the present work, an interaction is introduced, and an example is taken of a bilocal scalar field (the "nucleon") coupled with a local scalar field (the "pion"). It is shown that all the fields composing the bilocal nucleon field must have a nonlocal coupling with the pion, i.e., some "form-factors" have to be introduced in the interaction. The behaviour of such form factors is studied in detail, and two possible solutions are given, which preserve macrocausality, at least at low energies. The most interesting conclusion from this analysis is that in the limit where the characteristic length defining the nonlocality of the interaction goes to zero, all the form factors of the higher-spin fields vanish identically, and only the lowest-spin state survives. In other words, the existence of resonant states is intrinsically connected to the nonlocal character of the theory. Finally, the problem of quantization is discussed, and it is shown to be easily solved. It still remains an open question whether the families of particles described by this theory can be identified with those observed in nature.

E. Ferrari (Rome)

Rosen, Mervine; Yennie, Donald R. 4469

A modified WKB approximation for phase shifts.

J. Mathematical Phys. **5** (1964), 1505–1515.

Authors' summary: "Extending an idea of Good, a modified WKB approximation using radial wavefunctions having the form of free-particle solutions to the radial wave equation rather than an exponential form is developed. The lowest-order phase shifts are the same as those of the usual WKB approximation, but are improved by the contribution of the next order. The method is applied to two examples: the radial Dirac equation in the high-energy limit and the radial Schrödinger equation."

Schroer, B. 4470

The concept of nonlocalizable fields and its connection with nonrenormalizable field theories.

J. Mathematical Phys. **5** (1964), 1361–1367.

A field with a very singular behavior at small distance is investigated. Wightman functions of a field with a test function in $\mathscr{D}_z$ has a short distance behavior not worse than some inverse power of the distance. A field which does not have this property is called nonlocalizable by the author. Any infinite Wick power series is an example. The following examples are studied in detail. (1) $B(x)=\,:e^{gA(x)}:$. The two-point function is $\exp[ig^2\Delta^{(+)}(\xi)]$ which is analytic in a cut plane and has an essential singularity at the light cone. (2) $B(x)=\,:e^{gA(x)^2}:$. The two-point function is $i\Delta^{(+)}(\xi)\{1+[2g\Delta^{(+)}(\xi)]^2\}^{-1}$ and is singular at spacelike ξ satisfying $2gi\Delta^{(+)}(\xi)=1$. In these examples, the Lehmann weight has an exponential increase (of the order $\frac{1}{2}$ for (1) and worse for (2)) and the Green's function cannot be defined.

The author feels that nonrenormalizable field is nonlocalizable and supports this belief by an example of a spinor field ψ having a coupling $\bar\psi\gamma^\mu\psi\partial_\mu A$ with a zero mass scalar field A. Taking only those self-energy diagrams with uncrossed multiple bosons, the author obtains the two-point function $-2\gamma^\mu\xi_\mu\{I_2[g\sqrt{3}/\pi\sqrt{z}]/3g^2z\}$, $z=-\xi^2$ for mass zero ψ, which has an essential singularity at the light cone. Alternatively, the replacement of A with a free field yields the free two-point function of ψ multiplied by $\exp[ig^2\Delta^{(+)}(\xi)]$, in which the contributions from graphs with crossed boson lines increase the bad behavior.

The nonexistence of Green functions for a nonlocalizable theory is not necessarily an obstacle for the scattering theory. The author exhibits the following computational prescription for elastic scattering amplitude: $\langle p'|\tilde{A}(q)\times \tilde{j}(-k)|p\rangle$ has the form $i[P/(q^2-m^2)]F-i\pi\delta(q^2-m^2)G+H$ near $q^2=m^2$ and the scattering amplitude is given by $T=(G+iF)/(2\pi)^3$.

Finally, the author suggests the possibility that in cases of practical interest, nonlocalizability appears only for nonobservable fields like spinor fields or charged fields and that local observables are well-defined.

H. Araki (Kyoto)

Gruber, B.; O'Raifeartaigh, L. 4471

Uniqueness of the harmonic oscillator commutation relation.

Proc. Roy. Irish Acad. Sect. A **63**, 69–73 (1964).

Starting with the generalized harmonic oscillator Hamiltonian $H=\mu aa^+ + (1-\mu)a^+a$, μ any real number, the commutation relation $[H, a] = -a$, and the assumption that H has a point spectrum and a least eigenvalue, a short proof of the following results is given. (a) There exists an operator A satisfying the above relations for a with $\mu=0$ (except for a trivial shift of the zero point energy); (b) the only commutation relations for A allowed are $[A, A^+]=1$, $[A, A^+]_+=1$, or that corresponding to para-Fermi statistics.

R. Ingraham (University Park, N.M.)

Scharfstein, H. 4472

Trilinear commutation relations between fields. (Italian summary)

Nuovo Cimento (10) **30** (1963), 740–761.

Author's summary: "A set of trilinear equal-time commutation relations is derived from Schwinger's action principle under the assumption that for equal times the field variations commute with bilinears of field operators. Some consequences of these trilinear equal-time commutation relations are investigated and representations of the operators which satisfy them are considered."

Morimoto, Tetsuzo 4473

Markov processes and the H-theorem.

J. Phys. Soc. Japan **18** (1963), 328–331.

Author's summary: "The H-theorem is investigated in view of Markov processes. The proof is valid even in fields other than physics, since no physical relations, such as the principle of microscopic reversibility, the unitarity of the S-matrix and so on, are utilized. The well-known 'Wiederkehreinwand' is also re-examined from this point of view and a reconciliation is regained between the H-theorem and the recurrence in the sense of probability theory."

Ingraham, R. L. 4474

Consequences of a fundamental length in very-high-energy scattering. (Italian summary)

Nuovo Cimento (10) **32** (1964), 323–355.

Author's summary: "We show that a fundamental length provides the mechanism in quantum field theory to produce the effects noticed in very-high-energy scattering. In the very-high-energy large momentum-transfer (s, $|t| \gg \lambda^{-2}$, $\lambda \equiv$ fundamental length) approximation already semi-quantitative agreement with the following features is obtained. (1) Detailed shape of the elastic scattering curves, including: very sharp forward (and sometimes backward) 'diffraction' peaks with universal exponential tails in t (or u), the 'flattening' and huge experimental 'shrinking of the diffraction peak' with energy of the p-p curves, the weak 'flattening' and absence of 'shrinking' of the π-p curves. (2) Asymptotic constancy of total cross-sections, asymptotic equality of $\sigma_{tt}(\pi^- \text{-}p)$ and $\sigma_{tt}(\pi^+ \text{-}p)$, the rough relation $\sigma_{tt}(\pi\text{-}p) \approx \frac{2}{3}\sigma_{tt}(p\text{-}p)$. One characteristic of $\lambda \neq 0$ is the rapid vanishing of higher-order corrections at high energies and of radiative corrections at large momentum transfers. With $\lambda = 0$, this is not so, the sharp peaking and exponential tails disappear, and the 'shrinking' of the p-p peak is far too small. An essential 'dynamical' assumption is that a neutral vector meson may be exchanged. These experiments indicate $\lambda \approx 10^{-14}$ cm; the vector meson mass is not yet clear. The fundamental-length field theory used requires an explicit dependence of the S-operator on the measuring frame (frame in which the experiment is performed); it is believed that any relativistically invariant, etc., fundamental-length theory requires this generalization of field theory. This 'kinematical' fundamental-length-theoretic and the 'dynamical' Regge pole explanations of very-high-energy scattering are compared point by point; they are clearly distinguishable at ultra-high momentum transfers. A $\lambda \neq 0$ implies changed asymptotic properties of analytic amplitudes (essential singularities at infinity), thus changes in the physical conclusions derived from dispersion relations."

F. Calogero (Rome)

★**Dispersion relations and their connection with causality.** 4475

Proceedings of the International School of Physics "Enrico Fermi", Course XXIX, Varenna on Lake Como, Villa Monastero, 15th July–3rd August 1963. Edited by E. P. Wigner.

Academic Press, New York-London, 1964. xvi + 256 *pp.* (1 *plate*) $12.00.

Seven lecture courses on current topics in dispersion relation techniques and a preface on invariance principles are collected in this book. These articles will be reviewed separately.

P. Roman (Boston, Mass.)

Wigner, E. P. 4476

The role of invariance principles in natural philosophy.

Dispersion Relations and Their Connection with Causality (*Proc. Internat. School of Phys. "Enrico Fermi", Course XXIX, Varenna*, 1963), *pp.* ix–xvi. *Academic Press, New York*, 1964.

The role and place of invariance principles in physics; the nature, development, inter-relation of invariance principles; their possible epistemological value in the future are given a short, but delightful discussion.

P. Roman (Boston, Mass.)

Froissart, M. 4477

The proof of dispersion relations.

Dispersion Relations and Their Connection with Causality (*Proc. Internat. School of Phys. "Enrico Fermi", Course XXIX, Varenna*, 1963), *pp.* 1–39. *Academic Press, New York*, 1964.

The basic features and methods of deriving specific dispersion relations in the framework of axiomatic quantum field theory are presented. The reduction formulae and the Jost-Lehmann-Dyson formulae, forward dispersion relations, the Lehmann ellipse, non-forward dispersion relations, and a critical evaluation of the results is the content of this review.

P. Roman (Boston, Mass.)

Wigner, E. P. 4478

Causality, *R*-matrix, and collision matrix.

Dispersion Relations and Their Connection with Causality (Proc. Internat. School of Phys. "Enrico Fermi", Course XXIX, Varenna, 1963), pp. 40–67. Academic Press, New York, 1964.

The analytic properties of the R-matrix are proved, first directly, then from the principle of macroscopic (phenomenological) causality. Then the collision matrix is expressed in terms of the R-matrix and its analytic properties are determined. Many-channel processes are considered. *P. Roman* (Boston, Mass.)

Wong, D. Y. 4479

Dispersion relations and applications.

Dispersion Relations and Their Connection with Causality (Proc. Internat. School of Phys. "Enrico Fermi", Course XXIX, Varenna, 1963), pp. 68–96. Academic Press, New York, 1964.

After a brief recapitulation of the relation between causality and analyticity, elementary perturbation theory is used to illustrate poles of amplitudes, unitarity, and the crossing relations. The N/D method is introduced, and the difference between elementary and bound-state (and ghost) poles is demonstrated. Then the N/D method is used to discuss the elastic pion-nucleon scattering, with the main result that the 3,3 resonance and the nucleon pole can be treated on the same dynamical basis. A similar consideration is given to nucleon-nucleon scattering. Finally, electromagnetic form factors and the inclusion of strange particles are briefly discussed.
P. Roman (Boston, Mass.)

Newton, R. G. 4480

The nonrelativistic angular-momentum plane.

Dispersion Relations and Their Connection with Causality (Proc. Internat. School of Phys. "Enrico Fermi", Course XXIX, Varenna, 1963), pp. 97–134. Academic Press, New York, 1964.

A very thorough discussion of the analytic behavior in the complex angular momentum plane is given relative to the solutions of the non-relativistic Schrödinger equation. Simple models of particles with spin and of many-channel processes, as well as a model of the three-body problem, are also considered.
P. Roman (Boston, Mass.)

Landshoff, P. V. 4481

Production amplitudes.

Dispersion Relations and Their Connection with Causality (Proc. Internat. School of Phys. "Enrico Fermi", Course XXIX, Varenna, 1963), pp. 135–166. Academic Press, New York, 1964.

This review is concerned with the difficult properties of production amplitudes, as far as they can be discerned from perturbation theory. The discussion starts with the consideration of the unitarity condition, and then summarizes the analytic properties of production amplitudes. A discussion of partial wave amplitudes is then given. This is followed by a brief review of general higher amplitudes, i.e., the discussions are extended to other than five-point functions. The review ends with comments on the asymptotic behavior. *P. Roman* (Boston, Mass.)

Oehme, R. 4482

High-energy scattering and dispersion theory.

Dispersion Relations and Their Connection with Causality (Proc. Internat. School of Phys. "Enrico Fermi", Course XXIX, Varenna, 1963), pp. 167–256. Academic Press, New York, 1964.

The first section of this lucid review establishes, by way of plausibility arguments, various connections between high-energy limits of scattering amplitudes in a given channel and the single-particle states associated with the crossed channels. The second section introduces the complex angular momentum plane in a systematic manner, starting with the dispersion relations in momentum transfer and establishing the interpolation function. In the following, the connection between singularities of the interpolation function in the angular momentum plane and the high-energy properties of the scattering amplitude in the crossed channels is studied, and the properties of Regge poles are established. Then the possibility of elementary poles is explored. In Section 5 the nature of singularities in the angular momentum plane is further studied, and distinction between allowed and forbidden singularities is made. Section 6 discusses the asymptotic effects of the Sommerfeld-Watson representation and also places restrictions on possible "elementary" vector meson poles. In Section 7 the factorization of Regge poles and the Pomeranchuk theorem are discussed. Next, the possibility of pole condensation and of moving branch points is studied. Section 9 reviews the experimental situation and presents models for diffraction scattering. Finally, in Section 10 perturbation theory is used to show how bound-state Regge poles can be obtained by summing ladder graphs, and also the Regge treatment of single-particle poles is studied. *P. Roman* (Boston, Mass.)

★**Dispersion and absorption of sound by molecular processes.** 4483

Proceedings of the International School of Physics "Enrico Fermi", Course XXVII, Varenna on Lake Como, Villa Monastero, August 6–August 18, 1962. Edited by D. Sette.

Academic Press, New York-London, 1963. xi + 443 *pp.* (1 *plate*) $17.00.

Those papers of mathematical interest will be reviewed individually.

Jones, C. Edward 4484

Consistency of the strip approximation.

Phys. Rev. (2) **135** (1964), B214–B219.

Author's summary: "An investigation is made to determine those properties demanded of the Regge-pole formulas used by Chew and Jones in order to ensure the consistency of the strip approximation. A study of the asymptotic behavior of the Regge parameters based on the dynamical equations is also made in relation to the same question. The conclusion is drawn that the dynamical equations appear capable of producing solutions that are essentially self-consistent with the strip approximation that was used as an input. The Chew-Jones Regge-pole formula is also compared with one suggested earlier by Khuri." *E. Predazzi* (Turin)

Chew, Geoffrey F.; Jones, C. Edward 4485
New form of strip approximation.
Phys. Rev. (2) **135** (1964), B208–B213.
Authors' summary: "A detailed set of 'bootstrap' equations is formulated for zero-spin 'external' particles based on a combination of the N/D method with the superposition of top-ranking Regge poles in all three reactions of a four-line connected part. The contribution from each pole arises from a distinct strip in the Mandelstam representation so that double counting is avoided. Only real values of l with $l \leq 1$ need be considered in the bootstrap calculation. The amplitude emerging from our N/D equations is meromorphic in the right-half l plane, and the Regge poles approach high-energy limits that are dynamically determined and which in some cases may lie to the right of $l=0$. The reduced residues vanish in the high-energy limit." *E. Predazzi* (Turin)

Kim, Y. S. 4486
Anomalous thresholds and three-particle unitarity integral.
Phys. Rev. (2) **132** (1963), 927–929.
Author's summary: "Anomalous thresholds of the square diagram and their effect on the two-particle unitarity integral are discussed. It is shown for a certain diagram that the three-particle unitarity integral does not have counter terms which cancel the singularity from the two-particle contribution. Consistency with the crossed-channel unitarity is also discussed."

Logunov, A. A.; 4487
Nguyen Van-Hieu [Nguyen Van Hieu];
Tavkhelidze, A. N. [Tavhelidze, A. N.];
Khrustalev, O. A. [Hrustalev, O. A.]
Regge poles and perturbation theory.
Nuclear Phys. **49** (1963), 170–176.
Authors' summary: "The contributions of the cuts in the l-plane to the scattering amplitude are investigated. A way of separating the pole terms is discussed. These terms should be understood as the main ones in the expansion in powers of $e^2m(m^2-E^2)^{-1/2}$."

Minguzzi, A. 4488
One-variable dispersion relations. (Italian summary)
Nuovo Cimento (10) **32** (1964), 198–211.
In the scattering process $A+B \to C+D$, with the masses of the particles A, B, C and D unequal, there is an unphysical region which makes the proof of dispersion relations difficult. In the Bogoliubov method one first establishes the dispersion relation for a negative value of the mean mass $\rho = 1/2(p_B^2+p_D^2)$, where p_B, p_D are the 4-momenta of particles B and D [see H. Lehmann, Nuovo Cimento (10) **14** (1959), suppl., 153–176; MR **23** #B824]. The "unphysical" dispersion relation is then

$$(1)\quad T(\omega,\rho) = \frac{1}{\pi}\int_{s_0}^{\infty} \frac{A_1(s,\rho)\,ds}{s-\rho-\mu^2-2\Delta^2-2\omega E_\Delta} + \int_{s_0}^{\infty} \frac{A(u,\rho)\,du}{u-\rho-\mu^2-2\Delta^2+2\omega E_\Delta},$$

where Δ is the momentum transfer and

$$2E_\Delta = (2M_A^2+2M_C^2+4\Delta^2)^{1/2}.$$

One hopes to continue the absorptive parts $A(s,\rho)$ as an analytic function of ρ to its physical value $\rho^{ph} = 1/2(M_B^2+M_D^2)$.

One theorem of the present paper shows that if s, the square of the centre-of-mass energy, is above its physical threshold s^{ph}, then it is relatively easy to prove that $A(s,\rho)$ is an analytic function of ρ in the desired region, namely, Re $\rho \leq \rho^{PH}$ and $|\mathrm{Im}\,\rho| < \delta$, for some $\delta > 0$ independent of s. The method uses the Jost-Lehmann-Dyson representation as applied by Lehmann [see the above reference and also A. S. Wightman, Relations de Dispersion et Particules Élémentaires (Grenoble, 1960), pp. 227–315, esp. pp. 219–307, Hermann, Paris, 1960; MR **26** #3927]. This is not sufficient to prove a dispersion relation since in equation (1) we have no knowledge of the analytic properties of $A(s,\rho)$ in the unphysical region $s_0 \leq s \leq s^{ph}$.

The author remarks that the contribution G to T in (1) due to $s \geq s^{ph}$ and $u \geq u^{ph}$ (i.e., the physical ranges) is, because of the above-mentioned theorem, an analytic function of (ω,ρ) in two wedge-shaped regions joined at the edge. Causality implies that T itself also has this property, and a calculation shows that the boundary values of $T-G=F$ taken from the two regions of analyticity are equal in the physical region. Thus F satisfies the conditions of the edge of the wedge theorem, and so is real analytic in a neighbourhood of the physical region in both ω and ρ. This shows that T is an analytic function of ρ in the physical region, and the physical amplitude is the boundary value of a function analytic in ω.
R. F. Streater (London)

Newton, Roger G. 4489
Nonrelativistic S-matrix poles for complex angular momenta.
J. Mathematical Phys. **3** (1962), 867–882.
Author's summary: "Regge's introduction of complex angular momenta is studied in more detail. The shape and number of trajectories of S-matrix poles as functions of the energy is investigated, with particular attention to the way they leave the real axis, and to their ends at $E \to \pm\infty$. The conditions are found under which the S-matrix is meromorphic even in Re $l < -\frac{1}{2}$. Some properties of the S-matrix in the left half-plane are discussed and so are its symmetry between the left and right half-planes, its branch point at $E=0$, and the residues at its poles."
T. Regge (Turin)

Newton, Roger G. 4490
Errata: "Nonrelativistic S-matrix poles for complex angular momenta".
J. Mathematical Phys. **4** (1963), 1342.
In this paper some mistakes relative to the above paper [#4489] are corrected by the author. *T. Regge* (Turin)

Zichichi, A. (Editor) 4491
★Strong, electromagnetic, and weak interactions.
1963 International School of Physics "Ettore Majorana". Contributors: J. S. Bell, S. Berman, N. Cabibbo, H. Harari, G. Puppi, T. Regge, and L. Van Hove.
W. A. Benjamin, Inc., New York-Amsterdam, **1964.** vi+248 *pp.* **$9.00.**

This lecture-note volume contains seven theoretical reviews on current topics in elementary particle physics and five illustrative seminar notes on associated experimental topics.

The first chapter gives a general introduction to the whole set of lectures. It discusses, in an easily understandable and introductory exposition, the S-matrix idea, elements of quantum field theory, and the basic features of strong, electromagnetic, and weak interactions. The second chapter is a survey of various topics in non-relativistic potential scattering, including the highlights of Regge theory. The next chapter discusses, by means of phenomenological analysis, inelastic collisions and shadow scattering of strongly interacting particles at high energy. This is followed by an equally brief introduction to higher symmetry schemes, in particular, SU_3. The next chapter discusses various pion resonances, both from the dynamical and from the symmetry point of view. Next, a rather detailed review of weak interaction physics is given which emphasizes more the highlights than the details. The role of SU_3 symmetry in the leptonic decays of strongly interacting particles is the topic of the last theoretical chapter.

The experimental seminars deal with proton-antiproton collisions, proton-proton and pion-proton high energy elastic scattering, questions related to the η-meson μ-pair production, and radiative-capture.

P. Roman (Boston, Mass.)

Sakurai, J. J. 4492

★Invariance principles and elementary particles.
Investigations in Physics, No. 10.
Princeton University Press, Princeton, N.J., 1964. x + 326 *pp.* \$8.50.

This monograph appears to have a multiple purpose: (a) to give a first and selective introduction to students in theoretical physics concerning the most fundamental features of elementary particle theory, and (b) to acquaint experimentalists, in a relatively painless manner, with theoretical issues relevant in their work. In addition, the more advanced student in elementary particle theory will also profit from the treatise, since many concepts and subtleties which are given only a superficial and inadequate treatment in the existing literature have been clarified and illuminated by the present author.

The book starts with a brief general characterization of continuous space-time transformations. This is followed by chapters on parity, time reversal, charge conjugation and the CPT theorem. Chapter 7 discusses γ_5 invariance in weak interactions. Chapter 8 is devoted to gauge transformations and "Number Laws". The next two chapters describe isospin invariance and many ramifications thereof. The last chapter, entitled "Unsolved Problems", calls attention to a number of puzzles and is sprinkled with "notes added in proof".

The style and presentation is, in general, clear and sometimes terse. Occasionally, however, a somewhat less condensed treatment or a more systematic approach might have been advisable.

{A comment is in place concerning editorial policies. This book is based on a graduate course which the author gave in early 1959. Although he clearly made a strong effort to incorporate further developments, it appears that the manuscript has reached the publisher not later than 1961. The most exciting developments did not get into the book; for example, unitary symmetry is mentioned in two sentences only which were added in proof. It appears most puzzling why it took the publisher two years or more to bring out the book, precisely when the field in question is in such an unbelievably rapid development.}

P. Roman (Boston, Mass.)

Schmutzer, E.; Weber, G. 4493

Zur nichtrelativistischen zweiten Näherung der Diracschen Theorie des Elektrons. (English summary)
Annuaire Univ. Sofia Fac. Sci. Phys. Math. Livre 2 Phys. **55** (1960/61), 105–115 (1962).

It is shown that the same nonrelativistic Schrödinger equation can be obtained by eliminating the two "small components" from the Dirac equation, followed by a two-parameter family of transformations to a Hermitian Hamiltonian, as can be obtained by directly decoupling the two two-component spinors by a unitary Foldy-Wouthuysen transformation of the Dirac equation.

R. Ingraham (University Park, N.M.)

Garrido, L. M.; Sesma, J. 4494

Observables of relativistic particles. (Spanish. English summary)
Collect. Math. **14** (1962), 279–286.

Authors' summary: "We study the most general unitary transformation that transforms the Hamiltonians of particles of spins 0, 1/2 or 1, into Hamiltonians containing even or odd matrices only. We present also the expressions for the position operators for each transformation that are valid for the three kinds of particles mentioned above."

Hagen, C. R. 4495

Resonances in multichannel systems.
Phys. Rev. Lett. **12** (1964), 153–155.

The scattering matrix is considered in a model problem that has two coupled s-wave channels. The movement of bound states and resonances on the sheets of the Riemann surfaces are discussed. It is shown that crossings from one sheet to another require special assumptions on the energy dependence of the matrix elements. The purpose of this note is to refute arguments that suggest that the unitary symmetry model may be suspect, because the resonances would have to change sheets as the symmetry-violating interactions varied from zero strength to their actual values.

F. R. Halpern (La Jolla, Calif.)

Hagen, C. R.; Macfarlane, A. J. 4496

Triality type and its generalization in unitary symmetry theories.
J. Mathematical Phys. **5** (1964), 1335–1339.

Authors' summary: "Within the context of an extension of the SU_3-symmetry theory recently suggested by Gell-Mann and further developed by the authors, certain aspects of the theory of the special unitary groups are examined. The plurality type of a given representation is introduced as the generalization of the triality concept to SU_{n+1} and is shown to be associated with a multiplicative conservation law. Theorems for the reduction of representations of SU_{n+1} with respect to SU_n, $U_1^{(n)}$ are derived

which are subsequently used to relate plurality type to the existence of fractional eigenvalues for the generator $Y_1^{(n)}$ of $U_1^{(n)}$." *G. Heber* (Leipzig)

Gerstein, I. S. 4497

Virtual particles and the baryon-baryon system. (Italian summary)

Nuovo Cimento (10) **32** (1964), 1707–1714.

Author's summary: "The ${}^1S_0$ virtual bound state of the nucleon-nucleon system is shown to belong to the 27-dimensional representation of the symmetry scheme based on SU(3), known as the eight-fold way. The other states of this multiplet are derived. Experimental evidence bearing on this state is discussed and it is suggested that if the many complications of baryon number-2 states can be disentangled they can provide a fertile field for studying the validity of SU(3)."

{Reviewer's remarks: The (conjectured) approximate symmetry of strong interactions under SU(3) should not be expected to characterize the low-energy behavior of the two-baryon system. The far longer range of pion-mediated forces precludes the relevance, in this regime, of any symmetry scheme mixing pions and kaons.}

S. L. Glashow (Berkeley, Calif.)

Fairbairn, W. M.; Fulton, T.; Klink, W. H. 4498

Finite and disconnected subgroups of SU_3 and their application to the elementary-particle spectrum.

J. Mathematical Phys. **5** (1964), 1038–1051.

From the authors' summary: "The techniques previously employed by Case, Karplus, and Yang [Phys. Rev. (2) **101** (1956), 874–876; MR **17**, 923] for the application of finite subgroups of SU(2) to isotopic spin are extended to the (finite and disconnected) subgroups of SU(3). As a first step, character tables, irreducible representations, and other relevant properties are derived for these subgroups. Next, the classification of elementary particles is made on the basis of the representations of these subgroups. The structure of SU(3) is utilized to suggest how charge and hypercharge operators are to be assigned in the subgroups. The results obtained are similar to those of SU(2) and isotopic spin. In particular, there exist finite subgroups of SU(3) which can accommodate the eight baryons in one of the irreducible representations. In scattering problems, however, use of the finite subgroups gives charge or hypercharge conservation only modulo an integer determined by the subgroup. Charge independence is also lost. Exact charge and hypercharge conservation can be maintained for the disconnected subgroups, but the maximum dimension of the irreducible representations is six, and only charge symmetry, not charge independence, is satisfied. A short discussion of the representations of the group $SU(3)/C$ (C = center) is included in the appendix."

R. T. Prosser (Lexington, Mass.)

Gürsey, F.; Radicati, L. A. 4499

Spin and unitary spin independence of strong interactions.

Phys. Rev. Lett. **13** (1964), 173–175.

It is claimed that the group SU(4), introduced by Wigner to classify nuclear states, can be extended to the relativistic domain and that it may be relevant to elementary particle physics. Containing both the 3-dimensional spatial rotation group and the isotopic-spin group as sub-groups, Wigner's group expresses the spin and isotopic spin independence of nuclear forces. Associated with the adjoint representation of SU(4) are the mesons ρ, ω, and π (15 distinct particles with spin and isotopic spin taken into account). The nucleons and the $\frac{3}{2}$-$\frac{3}{2}$ pion-nucleon resonance are accommodated in a 20-dimensional irreducible representation. By replacing the isotopic spin group by SU(3), a symmetry scheme involving SU(6) is obtained. The adjoint representation is 35-dimensional and it describes the octet of pseudoscalar mesons and a nonet of vector mesons. The baryon octuplet and the $j=\frac{3}{2}^+$ decuplet of meson-baryon resonances are accommodated in a 56-dimensional representation. The existence of some spin-dependent forces is invoked in order to explain the mass splittings between vector and pseudoscalar mesons and between baryons and resonances. It is by no means evident {and, this reviewer denies} that the models discussed do have the invariance properties under SU(4) or SU(6) that are claimed. In this connection, the authors refer to a forthcoming paper of theirs.

S. L. Glashow (Berkeley, Calif.)

Pais, A. 4500

Implications of spin-unitary spin independence.

Phys. Rev. Lett. **13** (1964), 175–177.

The SU(6) model of strong interaction symmetry [F. Gürsey and L. Radicati, #4499] is further developed. A conjectured mass formula for the ($j=0^-, 1^-$) mesons, and one for the ($j=\frac{1}{2}^+, \frac{3}{2}^+$) baryons, are approximately satisfied by experimental masses. The form of the Yukawa meson-baryon coupling is unique and involves both symmetric "D-type" and antisymmetric "F-type" interactions with a determined ratio. Meson-baryon resonances are considered in the light of this symmetry scheme, and the possible existence of a 70-plet of $j=\frac{3}{2}^-$ states is suggested. *S. L. Glashow* (Berkeley, Calif.)

Lomont, J. S.; Moses, H. E. 4501

Representations of the inhomogeneous Lorentz group in terms of an angular momentum basis: Derivation for the cases of nonzero mass and zero mass, discrete spin.

J. Mathematical Phys. **5** (1964), 1438–1457.

From the authors' summary "In a previous paper [same J. **5** (1964), 294–298; MR **28** #2842] the authors showed how the infinitesimal generators of the proper, orthochronous, inhomogeneous Lorentz group acted in a basis in which the square of the angular momentum, the z component of the angular momentum, the helicity and the energy were diagonal for the ireducible representations which correspond to the cases of non-zero and zero mass, discrete spin. In that paper no derivation of the results were given. It was possible, however, to verify them directly. In the present paper we carry out the derivation." *R. T. Prosser* (Lexington, Mass.)

Alvazjan, Ju. M. 4502

Differences of masses in isotopic multiplets. (Russian. Armenian summary)

Izv. Akad. Nauk Armjan. SSR Ser. Fiz.-Mat. Nauk **17** (1964), no. 2, 115–117.

The author calculates the mass-splitting in isotopic multiplets of particles following from the irreducible representations $D^8(1, 1)$ and $D^{10}(3, 0)$ of the SU_3 group.
D. ter Haar (Oxford)

Bell, J. S. 4503
Nuclear optical model for virtual pions.
Phys. Rev. Lett. **12** (1964), 57–59.
This paper contains an evaluation of the cross-section of virtual pions on large nuclei. It is of possible use for the explanation of certain reactions (e.g., lepton production by neutrinos on nuclei, which can be explained by the one-pion-exchange model). The author shows that the optical model, used for collisions of real pions on nuclei, can be generalized to virtual pions. However, the effect of a new term in the equations, that vanishes for real pions, makes the absorption rate proportional to the volume of the nucleus. Consequently, this absorption rate is linear in the atomic number A, whereas real-pion absorption goes roughly like $A^{2/3}$. The contribution of this linear term to the cross-section is less than the sum of the cross-sections on the individual nucleons, and decreases with increasing virtual-pion energy.
E. Ferrari (Rome)

Oehme, Reinhard 4504
Weak currents and broken unitary symmetry.
Phys. Rev. Lett. **12** (1964), 550–552.
This is an attempt to compute the relative strength of the strangeness-changing and the strangeness-conserving currents entering into semi-leptonic weak interactions. The result is invalid because of a serious error in equation (4). Correcting this equation, we find that equation (7), which the author obtains as a formula for the relative strength, reduces to a tautology in this quantity.
S. L. Glashow (Berkeley, Calif.)

Pruski, S. 4505
On an approximate second-order fermion density matrix. Addendum.
Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 227–229.
The author's previous note [same Bull. **11** (1963), 615–619; MR **28** #3785] dealt with the relation of the exact 1-matrix, μ, of a fermion system to an approximation, τ, of the 2-matrix proposed by the reviewer. The author shows here that a necessary and sufficient condition that μ equals the first trace of τ is that the largest eigenvalue of μ be degenerate. *A. J. Coleman* (Kingston, Ont.)

Rylov, Ju. A. 4506
On a universal six-dimensional space of events. (Russian)
Vestnik Moskov. Univ. Ser. III Fiz. Astronom. **1963**, *no.* 6, 23–34.
The sixth dimension is added to the five-dimensional theory of Kaluza and Klein in order to include a hypothetical vector gauge-field coupled to baryonic charge. The 6-space is closed in the fifth and the sixth dimension to give quantized electric and baryonic charges and to prevent ordinary space-time from mixing with two added coordinates. Classical field equations in this theory are proposed. *I. Białynicki-Birula* (Warsaw)

Rylov, Ju. A. 4507
Particles with half-integer spin in a universal 6-dimensional space. (Russian)
Vestnik Moskov. Univ. Ser. III Fiz. Astronom. **1964**, *no.* 3, 3–14.
The present paper, as a continuation of the preceding [#4506 above], deals with the spinor equation in the 6-dimensional space (the sense of the last two coordinates is stated through the corresponding momenta which represent the electric and baryonic charge, respectively; all physical quantities are periodic functions of these two coordinates). The introduction of spinors in 6-space allows one to consider all elementary particles as the various states of one 6-dimensional Fermi-field. However, for instance, the mass spectrum or other interesting properties of interacting particles can be obtained after the second quantization of these fields.
M. Blažek (Bratislava)

Vachaspati; Punhani, Sudarshan L. 4508
Electromagnetic field of a charged particle in uniformly accelerated motion.
Proc. Nat. Inst. Sci. India Part A **29** (1963), 501–519.
Authors' summary: "A uniformly accelerated charged particle is peculiar in that it gives rise, at any given space-time point, either to a retarded or to an advanced electromagnetic field, but not to both. On general grounds one expects both kinds of fields to exist at all space-time points. The mathematical reason why only one kind of field exists in the case of uniform acceleration is pointed out and a limiting procedure is introduced to overcome the difficulty. It is then possible to obtain both kinds of fields at all points excepting the boundaries of some regions. The field at these boundaries can be determined by requiring that it should eliminate the presence of any unphysical charges. The solutions at these boundaries are not unique. Explicit expressions for the potentials and the fields are given. Our results partly confirm those of Bondi and Gold [Proc. Roy. Soc. London Ser. A **229** (1955), 416–424; MR **16**, 1166]."
F. Rohrlich (Syracuse, N.Y.)

Takahashi, Yasushi 4509
The structure of the nucleon core by the Hartree approximation.
Nuclear Phys. **26** (1961), 658–669.
Author's summary: "A method is proposed to investigate the structure of the nucleon core. A set of equations is derived to define the nucleon core and the meson cloud simultaneously. The equations are formulated by a variational method which enables us to find an approximate solution. The size of the nucleon core is estimated for a non-relativistic nucleon interacting with a neutral scalar meson. The coupling constant between nucleon and meson is given by the ratio of the sizes of the core and the cloud. It is shown that for $f^2/4\pi \approx 1$, the core size may be about half that of the meson cloud, where the number of mesons around the nucleon is about one. The generalization to a more realistic case is also suggested. The renormalization is not considered in this paper."

Bahtadze, A. K. 4510

On the solution of the equations of one-dimensional cascade theory of electron-photon showers with arbitrary boundary conditions and in the form of a source function. (Russian)

Vestnik Moskov. Univ. Ser. III Fiz. Astronom. **1964**, *no.* 3, 28–37.

In this paper the author obtains approximate solutions of equations of one-dimensional cascade theory of showers in the case of arbitrary boundary conditions and form of source functions. The method used permits one to obtain the general form of the distribution function of the number of particles without taking any particular form of the boundary conditions and source functions. This makes it convenient to compare the theory with experimental results. A new form of a source function is considered which is applicable to broad atmospheric showers. *P. K. Ghosh* (Calcutta)

Dietrich, Klaus; Mang, Hans J.; Pradal, Jean H. 4511

Conservation of particle number in the nuclear pairing model.

Phys. Rev. (2) **135** (1964), B22–B34.

A trial wave function of pairing type which conserves the number of particles is used in a variational calculation with a δ-function interaction. The results are compared with results from a BCS wave function (particle non-conserving), and from using a BCS wave function with subsequent projection onto the given particle number. Individual level occupation probabilities are altered by up to 60% in unfavourable cases (weak pairing interaction, low occupation probability), but are usually correct to within 10–20%. On the other hand, neglect of particle conservation frequently results in a qualitatively different spectrum of excitation energies of a nucleus. *J. M. Blatt* (Kensington)

Perlin, Yu. E. [Perlin, Ju. E.] 4512

Modern methods in the theory of many-phonon processes.

Uspehi Fiz. Nauk **80** (1963), 553–595 (*Russian*); *translated as Soviet Physics Uspekhi* **6** (1964), 542–565.

The author reviews modern theories of many-phonon processes in solids, particularly those associated with localized electronic states. The emphasis is put on the theoretical methods rather than the physical facts. It includes the method of using generating functions, the method of moments, Green's function formulation, etc. *R. Kubo* (Tokyo)

Sen, D. K. 4513

A theoretical basis for two neutrinos. (Italian summary)

Nuovo Cimento (10) **31** (1964), 660–669.

The object of this paper is to show how an antisymmetric tensor of rank two can be split up in a covariant manner into the sum of a self-dual term and an anti-self-dual term, each giving rise to a form of the two-component Weyl equation for a neutrino. The two neutrinos will behave identically when they are free but their interactions with other particles may be different.

The second of Maxwell's equations is shown to be satisfied by this tensor. This implies that such a combination of two neutrinos can be considered as an example of an electromagnetic field, though it is by no means obvious that such a field corresponds to a physical photon. *I. M. Barbour* (Rome)

Kónya, A. 4514

Über die Reihenfolge der Besetzung der Quantenzustände in Atomen. (Russian summary)

Acta Phys. Acad. Sci. Hungar. **13**, 219–231 (1961).

Jensen and Luttinger [Phys. Rev. (2) **86** (1952), 907–911] have found a good agreement between the experimental dependence of average square of the total angular momentum of atoms on the atomic number Z and the theoretical dependence calculated by means of statistical atomic theory. In the present paper the same agreement is obtained for the average total angular momentum, $\langle L\rangle$, and its average z-component, $\langle L_z\rangle$. The corresponding statistical formulae for $\langle L\rangle$ and $\langle L_z\rangle$ are given.

On the other hand, the results of the statistical model are compared with those obtained by simple summations of the azimuthal quantum numbers of the atoms where Z is assumed to be great. A very good agreement between both calculations is attained if one assumes that the electrons occupy the quantum levels in the sequence close to this which is known from experiment. *S. Olszewski* (Warsaw)

McLone, R. R.; Power, E. A. 4515

On the interaction between two identical neutral dipole systems, one in an excited state and the other in the ground state.

Mathematika **11** (1964), 91–94.

It is found that the virtual exchange of transverse photons diminishes the interaction between two identical dipoles when one of them is in an excited state. *N. S. Wall* (College Park, Md.)

Glick, Arnold J. 4516

The linear response function of a many-body system.

The Many-Body Problem (Lecture Notes, First Bergen Internat. School of Physics, 1961), *pp.* 261–285. *W. A. Benjamin, New York*, 1962.

The first part of this lecture elucidates clearly the definition and properties of the linear "response function" in a many-body system. When an external field is turned on, a system in a state ψ_n will undergo transitions to other states exchanging energy with the field. In the linear approximation (suitable when the external field is weak) we use first-order perturbation theory, and the net rate of absorption from the field is given by

$$\frac{dE}{dt} = 2\pi\omega\left|\frac{V(k,\omega)}{2\pi\Omega}\right|^2[R_n(k,\omega)-R_n(k,-\omega)],$$

where $V(k,\omega)$ is one typical Fourier component of a field $V(r,t)$. The response of the system is characterised by the response function $R_n(k,\omega)$, which depends on the internal structure of the system and is given by

$$R_n(k,\omega) = \sum_m \delta(E_m-E_n-\hbar\omega)\left|\left\langle\psi_m\right|\sum_i e^{ik\cdot r_i}\left|\psi_n\right\rangle\right|^2.$$

In Sections 3 and 4 two integral rules for $R_n(k,\omega)$ are derived in a neat fashion, and it is shown that any two-body operator F of the form $F=\sum_{ij}F(r_i-r_j)$ can be

averaged over the system to yield $\langle F\rangle$, which can be related to the integral of the response function over ω.

This idea is used in relating scattering cross-sections in the Born approximation for a fast external particle to be scattered by a system losing momentum k, to the pair correlation function in the system through integrals over $R_n(k, \omega)$, thus making contact with the theory of van Hove for such processes. It is proved that the real part of the correlation between values of a one-body operator X at different times can be related to integral over response functions. Also, the dissipation of energy in a system in the ground state in interaction with an external field of the form $V(r, t) = f(t)X(r)$ (where $X(r)$ is a one-body operator) can be related to the integral over response functions. Therefore the fluctuations in X in equilibrium can be easily shown to be closely connected to the dissipation in energy, thus leading to the fluctuation-dissipation theorem. The response function itself is written in a more suggestive form

$$R_n(k, \omega) = \frac{1}{\pi}\operatorname{Re}\int_0^\infty ds e^{i\omega s}\langle\psi_n|e^{iHs}\rho_k{}^+e^{-iHs}\rho_k|\psi_n\rangle.$$

Mention is made of the simple linear relation of the response function with various other physical quantities like susceptibility, dielectric constant, etc. It is desirable that a direct connection between Green's functions, dielectric constants, etc., and the response functions might have been demonstrated in detail in the lecture.

From Section 7 onwards the lecture is devoted to the calculation of response functions in a Fermi system. To this end one has to calculate the matrix element $\langle\psi_0|e^{iHS}\rho_k{}^+e^{-iHS}\rho_k|\psi_0\rangle$ for $S>0$. In the second quantized form ρ_k is the operator which destroys a particle in a momentum eigenstate and creates another with momentum increased by k, and it has a role similar to an external field and is represented as such graphically. Feynman diagrams are drawn as in perturbation theory calculations in many-body theory introduced by Hubbard to represent time translation operators, etc., and the rules for drawing each graph are given in detail. In Section 8, the simple case of a non-interacting Fermi gas is investigated and $R_0(k, \omega)$ is derived. In Section 9, interacting systems are considered and $\Delta R(k, \omega)$, the correction to $R_0(k, \omega)$, is calculated in different orders of perturbation theory. It is shown that however small the strength of the interaction, provided it is not a delta function interaction and provided it is not zero, then perturbation theory is not valid for some k and ω. In Section 10, a distinction is made between the graphs which cannot be separated into two parts by cutting a single interaction line and those which can be separated by cutting a single interaction line. Attention is drawn to the entities $(1/\tau)B(x, x')$ which represents the polarization bubble parts between x and x' and $\nu(x, x')$, the effective interaction between x and x', which represents the sum of a class of elementary graphs which begin and end with simple interaction lines. Integral equations are written for these, by grouping certain classes of terms, and an expression for $R(k, \omega)$ in terms of Fourier transforms of $B(x, x')$ and $\nu(x, x')$ is obtained:

$$R(k, \omega) = \frac{\Omega}{2\pi^2}\operatorname{Im}\int_0^\infty ds\int d\omega' e^{i(\omega-\omega')s}\frac{B(k, \omega')}{1+\nu(k)B(k, \omega')}.$$

In Section 11, $B(k, \omega)$ is obtained by summing over the lowest-order graphs. For a system with repulsive interactions, $R(k, \omega)$ exhibits a strong peak for the vanishing of the denominator in the integrand of the above expression for $R(k, \omega)$ below a critical wave number k. For a system with Coulomb interactions this resonance is associated with plasma oscillations, and for finite range forces with zero sound. But for spinless fermions with a δ force, the interaction $R(k, \omega)$ is given by its value for the non-interacting case. Finally, the case of the electron gas (spin $\frac{1}{2}$ fermions with interaction $V(k) = 4\pi e^2/k^2$) is mentioned and the inadequacy of the approximation (the RPA) in which the response function is calculated using B^0, the uncorrected bubble contribution, is brought out.

R. Vasudevan (Madras)

Wagner, Herbert 4517

Two-particle approximation for Fermion systems with pair correlation.

Z. Physik **178** (1964), 146–158.

The Green's function method of a previous work [W. Brenig and H. Wagner, same Z. **173** (1963), 484–489; MR **27** #4598] is used to study collective excitations of a superfluid Fermion system. A connection between particle-particle excitations and particle-hole excitations is pointed out. An approximation for treating hard-core potentials together with pairing is given. *K. K. Gupta* (Bombay)

Svidzinskii, A. V. 4518

A study of the equation for a gap in the energy spectrum of a superconductor. (Russian)

Ukrain. Mat. Ž. **16** (1964), 544–550.

In this paper N. N. Bogoliubov's equation for gaps in the spectrum of a unilaterally stimulated superconductor has been studied by considering a very general model, viz., considering dependence of the kernel on energy variables. The investigation encompasses some results obtained by others previously. It is shown that the universal inequality for the jump in heat capacity remains valid.

P. K. Ghosh (Calcutta)

STATISTICAL PHYSICS, STRUCTURE OF MATTER

See also 3706, 4195, 4205, 4407, 4410, 4411, 4420, 4428, 4518.

Brout, R.; Carruthers, P. 4519

★Lectures on the many-electron problem.

Interscience Monographs and Texts in Physics and Astronomy, Vol. X.

Interscience Publishers [*John Wiley & Sons*], *New York-London-Sydney*, 1963. viii + 204 *pp*. $9.50.

The field of the many-body problem is one of the most rapidly growing in present-day physics; this is quite natural if one considers that practically all physical problems can be viewed as particular cases of the many-body problem. Before this problem could be treated adequately, appropriate mathematical techniques had to be developed, and actually this field has adopted the great advances made recently in field theory. In its modern form the many-body problem is therefore a rather recent field, and the number of monographs published to date is still rather limited. We can therefore only welcome

the publication of this book which fills a gap in the literature. Indeed, this book is, in our opinion, the best existing text which would enable a student to learn the problem from the beginning up to the advanced level, and which treats a sufficient number of examples to give a good idea of the field of applicability of the methods.

Chapter 1 reviews the more important results of classical equilibrium statistical mechanics, such as the cluster expansion, the theory of plasma oscillations, the "random-phase approximation", etc. The presentation is simpler and more intuitive (although rigorous) than in usual texts. The stress is laid on the peculiarities introduced by the long-range forces.

Chapter 2 contains a very detailed exposition of the techniques of calculation, such as the perturbation expansion of the ground-state energy and the finite temperature expansion of the free energy.

Chapter 3 is devoted to a detailed calculation of the correlation energy of an electron gas in RPA. The two equivalent, but formally different, methods used here are the one due to Gell-Mann and Brueckner, based on summation of ring diagrams, and Sawada's method based on the exact solution of a simplified "pseudo-Hamiltonian". It is shown that the RPA correctly gives the high-density limit of the thermodynamic properties.

Chapter 4 is an exposition of the dielectric-constant approach to the many-body problem, originally due to Nozières and Pines. In this very elegant formalism it can be shown that all thermodynamic properties of a system in equilibrium can be expressed in terms of the dielectric constant of the system. The situation of the RPA in this formalism is discussed, as well as its links with the self-consistent field method.

Chapter 5 treats a number of applications of the formalism to the theory of metals. Among the topics discussed are the specific heat, the magnetic susceptibility, the characteristic energy loss, as well as a few indications on superconductivity.

All the calculations are done in great detail, with useful remarks stressing the delicate points. This feature, combined with a great clarity of the style, gives the book a significant pedagogical value. We strongly recommend these lectures to graduate students and to all research physicists in the fields of solid-state physics, statistical mechanics, plasma physics, nuclear physics, etc.

R. Balescu (Brussels)

Sandri, G. 4520

Global master equation. I. (Italian summary)
Nuovo Cimento (10) **32** (1964), 985–990.

The author derives a master equation for a classical system of interacting particles, using similar techniques as in his previous papers [Phys. Rev. Lett. 11 (1963), 178–179; MR 25 #4620; Ann. Physics 24 (1963), 332–379; ibid. 24 (1963), 380–418]. The basic smallness assumption is introduced in the following way. Let F be the normalized distribution function of the system in phase space, and Φ its configurational part; then it is assumed that (1) $(F/\Phi)-1\sim O(\varepsilon)$, and that Φ and F are of the form (2) $\Phi(t,\varepsilon t)=\Phi^{(0)}+\varepsilon\Phi'$; $F(t,\varepsilon t)=\Phi^{(0)}+\varepsilon F'$.

{The author interprets the condition (1) as characterizing the strength of the statistical correlation between the momentum and position degrees of freedom of the system. Moreover, he states that the validity of the resulting equation is independent of the details of the two-body potential. Both these remarks are obscure to the reviewer. First, assumption (1) need not be satisfied, in general, even in thermal equilibrium, in spite of the fact that then there is no correlation between positions and momenta; second, even in thermal equilibrium, the satisfaction of this assumption depends on the nature of the two-body potential.} *N. L. Balazs* (Stony Brook, N.Y.)

da Costa, R. C. T. 4521

On fluctuations in statistical mechanics. (Italian summary)
Nuovo Cimento (10) **32** (1964), 654–678.

A general discussion is given of the theory of fluctuations in microcanonical, canonical, and grand canonical ensembles. *D. ter Haar* (Oxford)

Vetchinkin, S. I. [Vetčinkin, S. I.] 4522

Conditions for optimal choice of approximate wavefunctions and the hypervirial theorem.
J. Chem. Phys. **41** (1964), 1991–1995.

Author's summary: "The hypervirial theorem for the operator L does not guarantee the independence of the expectation value L from all essential errors of the approximate wavefunction. The more complete system of conditions includes the matrix elements of higher commutators (and anticommutators) L with the Hamiltonian. These conditions prevent the effect of approximate wavefunction errors and may be useful for evaluations of expectation values and transition probabilities. The problem is considered in terms of dynamics or statics. These treatments are shown to be equivalent only for the exact solution."

Wichmann, Eyvind H. 4523

Density matrices arising from incomplete measurements.
J. Mathematical Phys. **4** (1963), 884–896.

Author's summary: "The problem of how to associate a statistical ensemble of a quantum mechanical system with an incomplete set of measured ensemble averages is discussed. The view adhered to in this note is that the most chaotic ensemble consistent with the measured ensemble averages is a reasonable choice of ensemble. The properties of this ensemble are studied rigorously for the case when the quantum mechanical system is associated with a finite-dimensional Hilbert space. The results are extended heuristically to some particular ensembles of many-boson and many-fermion systems."

Yaris, Robert 4524

Linked-cluster theorem and unitarity.
J. Chem. Phys. **41** (1964), 2419–2421.

Author's summary: "It is shown that the linked-cluster theorem for nonperturbational treatments (previously demonstrated) follows from the principle of unitarity (conservation of probability). The reason why unlinked clusters appear in CI calculations, but can be cancelled out of the results, is also briefly discussed."

Ninham, B. W. 4525

Specific heat of a degenerate electron gas.

Ann. Physics **28** (1964), 220–224.

Author's summary: "Computational errors in a calculation of the specific heat of a degenerate electron gas as computed by Du Bois [Ann. Physics 7 (1959), 174–237; MR 24 #B1567; ibid. 8 (1959), 24–77] are corrected."

Sherman, S. 4526

Product property and cluster property equivalence.

J. Mathematical Phys. **5** (1964), 1137–1139.

From the author's summary: "Uhlenbeck and Ford in their presentation of the Ursell development express a sequence of symmetric functions W_N in terms of another sequence of symmetric functions U_N. They invert this sequence of equations and remark that from the first set of equations it follows that the product property of the W_N sequence is equivalent to the cluster property for the U_N sequence. They ask for a simple proof of the equivalence. Here a modification of an algebra developed by Bohnenblust to prove Spitzer's formula on the fluctuations of the sums of independent, identically distributed, random variables is used systematically to invert the relations and to prove the equivalence under lighter assumptions than previously used."

F. L. Spitzer (Ithaca, N.Y.)

Distefano, E.; Fraidenraich, N. 4527

Electrical conductivity of partially ionized gases. (Spanish. English summary)

An. Soc. Ci. Argentina **176** (1963), 45–55.

Authors' summary: "The authors calculate the conductivity of a partially ionized gas by solving Boltzmann's equations. Collision terms are found by expressing the velocity of a particle after the collision has happened as a function of the velocity before it, and for that the properties of the vector $\mathbf{K}=\mathbf{C}'-\mathbf{C}$ are used. The resultant conductivity has the same form as the conductivity calculated by means of the theory of mean free paths, but its terms are affected by correction factors."

Gold, L. 4528

On the wave mechanics of gases. (Italian summary)

Nuovo Cimento (10) **32** (1964), 1622–1628.

Some approximate expressions are derived for the thermodynamic functions and for the equation of state of a system of non-interacting particles obeying Boltzmann statistics. The transition from wave mechanics to kinetic theory is discussed. *H. B. Rosenstock* (Washington, D.C.)

Louck, J. D.; DeVault, G. P. 4529

Eigenfunctions of the Boltzmann collision operator.

Phys. Fluids **7** (1964), 1388–1390.

This is a simple and elegant derivation of the known fact that the oscillator wavefunctions are eigenfunctions of the Boltzmann collision operator in the case of Maxwellian interactions between the particles. *R. Balescu* (Brussels)

Naze, Jacqueline 4530

★Sur l'équation de Boltzmann des gaz faiblement ionisés.

Préface de Henri Cabannes.

Publ. Sci. Tech. Ministère de l'Air, No. 387, *Paris,* 1962. x + 92 *pp.*

Table of Contents: Chapitre I, Équation de Boltzmann; Chapitre II, Opérateurs collisionnels linéaires; Chapitre III, Opérateurs collisionnels particuliers. Gaz de Lorentz parfait et imparfait; Chapitre IV, Calcul exact de la conductivité électrique et de la température d'un gaz faiblement ionisé; Chapitre V, Grandeurs macroscopiques d'un gaz faiblement ionisé en deuxième approximation; Appendice I, Fonctions sphériques. Calcul de $\mathscr{C}_{l_0}(c)$ et de $(\partial/\partial c)\mathscr{C}_{l_0}(c)$; Appendice II, Propriétés des valeurs propres de l'opérateur K pour la loi en $1/r^5$; Appendice III, Variation de la conductivité électrique et de la température en fonction du champ magnétique et de la masse des particules.

Rose, Marian H. 4531

Drag on an object in nearly-free molecular flow.

Phys. Fluids **7** (1964), 1262–1269.

The steady linearized single relaxation kinetic model is considered for rarefied flow past a sphere. Denoting the mean-free-path by L, the sphere radius by a, then the lowest-order solution is found for distances R such that $a \ll R \leqq L$. This solution is then used to find the drag correction on the sphere. *L. Sirovich* (Providence, R.I.)

Ruelle, David 4532

Cluster property of the correlation functions of classical gases.

Rev. Modern Phys. **36** (1964), 580–584.

An algebraic formalism is set up to study the relationship connecting the various correlation and cluster functions used in the theory of fluids. This is used to obtain upper bounds on the integrals of the absolute values of the Ursell functions at zero density, and also at finite densities provided the fugacity is less than a specified quantity. The "cluster property" referred to in the title is the finiteness of these integrals. *O. Penrose* (London)

Struminskiĭ, V. V. 4533

On Hilbert's method of solving the kinetic Boltzmann equation. (Russian)

Dokl. Akad. Nauk SSSR **158** (1964), 70–73.

It is shown that the Hilbert method of solution of Boltzmann's equation, based on an expansion in powers of a small parameter, is not appropriate for a dissipative fluid. In particular, it is shown that the Hilbert solution (contrary to the Chapman-Enskog solution) leads to equations of motion which are essentially different in structure from those describing a viscous fluid.

R. Balescu (Brussels)

Su, C. H. 4534

Kinetic theory of a weakly coupled gas.

J. Mathematical Phys. **5** (1964), 1273–1280.

The author considers the problem of the kinetic theory of gases by use of the multiple time scale method, first

invented by Poincaré in dealing with the perturbation of the orbits of celestial bodies, and applied here to the BBKGY hierarchy of coupled equations governing n-body distribution functionals. After an initial discussion of the physical problem and the relevant parameters, the author considers the weak coupling case and then higher-order approximations. Because the expansion considered is not valid for the case of small relative velocities between particles, a new expansion valid for this case is formulated to first order only. A discussion and analysis of the kinetic equations of the one-particle function which then follows concerns itself with the rate of relaxation of a system to the state governed by the Fokker-Planck equation and with the termination of the time scales in accord with the phenomena under investigation. A lengthy section concerning the effects of initial correlations shows that they are smoothed out during a time interval of the order of the collision time if certain simple conditions usually met are obtained. The paper concludes by discussing irreversibility and by pointing out that the irreversible nature of the kinetic equation has been introduced by requiring the solution to be nonsecular in the limit of large times. *Sylvan Katz* (Newport Beach, Calif.)

Cheng, Ping 4535

Two-dimensional radiating gas flow by a moment method.

AIAA J. **2** (1964), 1662–1664.

Starting from the radiation transport equation (in terms of the specific intensity) for a gray gas in local thermodynamic equilibrium, the author develops an infinite sequence of moment equations which are closed by assuming isotropy of radiation (Milne-Eddington approximation). The partial differential equation for the first moment (the radiative heat flux) is thus derived. The one-dimensional form has been derived earlier by others and, in fact, the three-dimensional form is noted by S. C. Traugott [Proc. 1963 Heat Transfer and Fluid Mech. Inst. (Calif. Inst. Tech., Pasadena, Calif., 1963), pp. 1–13, Stanford Univ. Press, Stanford, Calif., 1963; MR **29** #5622]. Cheng's development seems to parallel the earlier one of Traugott. The proposed general expansion in the complete set of spherical harmonics is of little value unless anisotropy is "small", in which case the harmonic expansion becomes an asymptotic one. Otherwise no rational truncation scheme is evident. The radiative heat flux equation is then combined with the usual mass, momentum, and energy conservation equations for inviscid flow of a thermally and calorically perfect gas. This system of equations is linearized about some uniform state and specialized to two-dimensional steady and one-dimensional unsteady flow. The application of this formulation to wave propagation and flow over a wavy wall is promised for a later paper.

I. M. Cohen (Providence, R.I.)

Abonyi, I. 4536

Basic equations of magnetofluido-dynamics in anisotropic media. (Russian summary)

Acta Phys. Acad. Sci. Hungar. **17** (1964), 91–95.

Zuerst werden die aus den Maxwellschen Differentialgleichungen und den Navier-Stokesschen Gleichungen folgenden bekannten Grundgleichungen der Magnetohydrodynamik explizit angegeben und dann wird die Frage besprochen, wie diese Gleichungen abzuändern sind, wenn das fragliche Medium anisotrop wird, die elektrische und die magnetische Permeabilität (ε und μ) und die Leitfähigkeit σ also zu Tensoren werden. Der Verfasser beruft sich dabei auf die Tatsache, dass dieser Fall tatsächlich auftreten kann, wenn das magnetische Feld recht gross wird, weil dann die Zyklotronbewegung der Elektronen diese Anisotropie verursacht.

Zur tatsächlichen Durchführung der Rechnung werden die zu μ_{ij} und σ_{ij} reziproken Tensoren eingeführt und dann werden alle Grössen durch die mechanischen Variablen p, ρ, und v_i und die Feldgrössen H und B ausgedrückt. Die erhaltenen Resultate werden am Ende der Arbeit zusammengefasst und kritisch besprochen.

T. Neugebauer (Budapest)

Balescu, R.; de Gottal, Ph. 4537

Effet des corrélations sur les coefficients de transport d'un plasma. (English summary)

Acad. Roy. Belg. Bull. Cl. Sci. (5) **47** (1961), 245–258.

Authors' summary: "A closed formula is obtained for the long living correlations in an inhomogeneous plasma; it is expressed in terms of the one-particle distribution function. This forms an appropriate starting point for a rigorous theory of transport phenomena in plasmas, including the effect of molecular correlations. An expression is obtained for the thermal conductivity."

Dupree, Thomas H. 4538

Theory of radiation emission and absorption in plasma.

Phys. Fluids **7** (1964), 923–940.

The author gives a consistent treatment of emission and absorption of radiation on the basis of kinetic theory for plasma situations where the collective effects are dominant. For the radiation field this means the low-frequency range that coincides for most plasmas of laboratory or astrophysical origin with the radio frequency spectrum. In this range, the basic radiation mechanisms are Bremsstrahlung and cyclotron radiation; however, due to the collective interaction of the plasma electrons, longitudinal and radiating modes are coupled. General formulas are worked out that cover various coupling possibilities in addition to explicit expressions for Bremsstrahlung and cyclotron radiation.

L. F. Oster (New Haven, Conn.)

Fante, Ronald 4539

Kinetic equations for a plasma including radiation in the presence of an external magnetic field.

Phys. Fluids **7** (1964), 940–948.

The interaction of electromagnetic radiation with a one-component plasma in an external magnetic field is investigated. Canonical position and momentum variables for the radiation field are introduced by replacing the field by a system of oscillators. Following Simon and Harris [Phys. Fluids **3** (1960), 245–254; MR **22** #7320], cluster expansions for both oscillator and particle distribution functions are introduced. A note on the dimensionless expansion parameters of this perturbation solution is appended.

An equation of radiative transfer and a Fokker-Planck

equation are obtained for describing relaxation to equilibrium. As is to be expected, the final expressions are quite cumbersome. The limiting case of vanishing magnetic field is also presented. Time scales for the relaxation process are briefly discussed. The calculation is a lengthy one but is clearly summarized.

G. L. Lamb, Jr. (E. Hartford, Conn.)

Gogosov, V. V. 4540

On the resolution of an arbitrary discontinuity in magnetohydrodynamics.

Prikl. Mat. Meh. **26** (1962), 88–95 (*Russian*); *translated as J. Appl. Math. Mech.* **26** (1962), 118–129.

Riemann's plane problem in a perfectly conducting adiabatic gas is discussed in the presence of a magnetic field which is set normally to the plane of discontinuity at $x=0$ at the instant $t=0$. (The author asserts that the limiting process from the more general case [the author, Prikl. Mat. Meh. **25** (1961), 108–124; MR **27** #5462] is rather complicated.) Possible types of resolution of the initial discontinuity into some combinations of various discontinuities are determined by use of $P-H_y$ diagrams and are shown as domains in the $(\delta u, \delta v)$-plane, where P is the square of pure sound speed divided by initial Alfvén speed, H_y the induced transversal magnetic field and $(\delta u, \delta v)$ is the initial velocity jump across the discontinuity. Six different diagrams are found to be possible for the $(\delta u, \delta v)$-plane depending upon the initial values of P in $x>0$ and $x<0$ referred to in the first case.

H. Hasimoto (Baltimore, Md.)

Johnson, Eric G., Jr. 4541

Use of the index of refraction as a means for study of plasma configurations.

Phys. Fluids 7 (1964), 1551–1552.

An analysis is given of a laboratory plasma which is in an equilibrium configuration but subject to small perturbations as a result of the action of an external A.C. signal. It is shown on the basis of fairly general assumptions that the off-diagonal elements of the dielectric tensor cannot all vanish. *S. Gartenhaus* (Lafayette, Ind.)

Jukes, J. D. 4542

Gravitational resistive instabilities in plasma with finite Larmor radius.

Phys. Fluids 7 (1964), 52–58.

Author's summary: "The linearized equations of a gravity driven, resistive instability of a plasma supported by a planar, sheared magnetic field are shown to be characterized by a variational principle from which an eigenvalue corresponding to a periodic growth is obtained. The effect of a finite ion Larmor radius is included through off-diagonal terms in the pressure tensor in the momentum fluid equation. When the Larmor radius is zero, a numerical solution for the growth rate is found using a one-parameter trial function which indicates that the correct function maximizes this rate. When the Larmor radius becomes significant, viz., with decreasing wavelength, the instability becomes an overstability with a reduced growth rate becoming proportional to the square root of the ratio of the resistivity to the wavelength. Whilst numerical results are given beyond this transition, the Larmor radius has become comparable with the thickness of an internal resistive layer and the equations cease to be reliable."

H. Weitzner (New York)

Jukes, J. D. 4543

Micro-instabilities in magnetically confined, inhomogeneous plasma.

Phys. Fluids 7 (1964), 1468–1474.

Author's summary: "Plasma with a density gradient supported in a strong, nearly uniform magnetic field is liable to resistive, gravitationally (g-) driven instability and short-wave drift instability. Waves associated with both propagate almost perpendicular to the field lines. Only a low g, and low density plasma are considered, for which these flutelike perturbations remain quasi-electrostatic, so that dispersion relations are readily found. When the magnetic-field lines are sheared, the localized perturbations which arise are treated by a phase-integral approximation and growth rates appear as eigenvalues of certain standard equations. The ions are treated throughout by the Vlasov equation so that fine-scale features such as finite ion Larmor radius a_i and Hall current are correctly included. The results for the resistive, gravitational instability are compared with those based on fluid equations and are valid if the perturbation is not localized within a_i."

Kuehl, Hans H. 4544

The kinetic theory of waves in a warm plasma excited by a current source.

J. Math. and Phys. **43** (1964), 218–226.

The author considers the waves in a warm, homogeneous and unbounded plasma excited by an electric dipole by using Boltzmann's equation. He determines the expression for the electric field and has shown that, except for Landau damping, the hydrodynamic and kinetic theories give the same result for the longitudinal waves for frequencies near the plasma frequency only. However, in the case of transverse waves, the two approaches give identical results for all frequencies greater than plasma frequency. *F. C. Auluck* (Delhi)

Lamb, G. L., Jr.; Burdick, B. 4545

Exact solution of the differential equation for the nondivergent correlation function in a one-component plasma.

Phys. Fluids 7 (1964), 1087–1088.

Frieman and Book [Phys. Fluids **6** (1963), 1700–1706; MR **28** #3821] derived a differential equation for the position-dependent factor in the two-particle equilibrium correlation function for an electron plasma balanced by infinite mass ions. By taking account of the mutual Coulomb force between the two electrons (usually ignored because it is small except when they are very close) they avoided a singularity for the interparticle distance 0, thus achieving a "nondivergent" correlation function for this special plasma.

The present paper refines the work of Frieman and Book by showing that this correlation function can be expressed exactly in terms of the solution of the associated Mathieu equation (with the independent variable t replaced by iz).

E. Pinney (Berkeley, Calif.)

Laval, Guy; Pellat, René 4546

Méthode d'étude de la stabilité de certaines solutions de l'équation de Vlasov.

C. R. Acad. Sci. Paris **259** (1964), 1706–1709.

Authors' summary: "On étudie la stabilité de systèmes plasma-champ électromagnétique en négligeant le rayonnement, c'est-à-dire le flux du vecteur de Poynting. Le champ électromagnétique est décrit par les équations de Maxwell et le plasma par l'équation de Vlasov. La méthode consiste à rechercher les perturbations linéaires, qui conservent l'énergie et l'impulsion du système champ-particules."

Lenard, Andrew 4547

On Bogoliubov's kinetic equation for a spatially homogeneous plasma.

Ann. Physics **10** (1960), 390–400.

Author's summary: "An integral equation, proposed by Bogoliubov, to determine the time development of the velocity distribution of a spatially homogeneous plasma is considered. It is shown that the time derivative of the velocity distribution can be expressed explicitly in terms of the distribution itself. The resulting kinetic equation has all the required properties, in particular, it satisfies the $\mathscr{H}$-theorem. By means of an excellent approximation it can be reduced to that derived by Landau."

A similar result has been achieved by R. Balescu [Phys. Fluids **3** (1960), 52–63; MR **23** #B1959].

Matsuura, Kenya 4548

Interaction between plasma oscillation and radiation field. I. Radiation damping of plasma oscillation.

J. Phys. Soc. Japan 18 (1963), 1319–1325.

This paper discusses an interaction between plasma oscillations and the radiation field in a high temperature plasma, free from density and temperature gradients and magnetic fields, but not in thermal equilibrium. For a system of plasmons and photons in thermal equilibrium it is clear that the longitudinal and transverse modes are independent, but for a non-equilibrium system coupling can take place.

A quantum-mechanical treatment is used to derive an expression for the radiation damping of the plasma oscillations; however, this expression is simply the classical limit. This result is then compared (in the case of a classical plasma, $T=10^6$ °K, $n=10^{14}$ particles cm^{-3}) with Landau damping, mode-coupling damping and collision damping for a range of $k\lambda_D$ (k is the wave number, λ_D the Debye length). The radiation damping is non-vanishing for $k\to 0$ and dominates Landau damping for $k\lambda_D \lesssim 1/7$ but is several orders of magnitude smaller than mode-coupling damping over the range of k.

T. J. M. Boyd (Culham)

Matsuura, Kenya 4549

Interaction between plasma oscillation and radiation field. II. Radiation from plasma oscillation.

J. Phys. Soc. Japan **18** (1963), 1649–1656.

This paper again considers radiation from plasma oscillations when there is a non-equilibrium distribution of (i) plasmons and (ii) electrons in velocity space. Radiation intensities are evaluated. *T. J. M. Boyd* (Culham)

Oppenheim, I.; van Kampen, N. G. 4550

Field correlation functions in a plasma.

Phys. Fluids 7 (1964), 813–815.

Authors' summary: "The correlation function between the electrostatic field at point a and the electrostatic field at point b in a fully ionized hydrogen plasma is computed. The correlation function is shown to be a long-range function of the distance between points a and b. This result has significance in the treatment of transport properties in a plasma." *H. Weitzner* (New York)

Rand, S. 4551

Quantum effects on the conductivity of a dense plasma.

Phys. Fluids 7 (1964), 64–71.

The interaction of a radiation field with the collective modes of a plasma becomes important when the plasma frequency approaches the radiation frequency. The particle interaction must be treated quantum mechanically when electron plasma wave-phonon energies are in the neighborhood of, or greater than, the average random electron energies ($\bar{E}$). Both these conditions are established when optical or infrared radiation interacts with a dense plasma. The values for conductivity and opacity are computed (for the high frequencies) in the limit of a fully ionized high-density plasma. Finally, it is proved that, when the limit of the photon energy is much greater than the average random energy, and when $\omega_p \gg \omega_{\text{rad}}$, then the free-free contribution to the opacity is reduced by a factor of $\bar{E}\omega_{\text{rad}}^2/\hbar\omega_p^4$.

W. K. R. Watson (Riverside, Calif.)

Sanfeld, Albert 4552

Répartition des pressions dans un fluide chargé et polarisé.

Acad. Roy. Belg. Bull. Cl. Sci. (5) **50** (1964), 339–357.

Author's summary: "Ce travail expose quelques modes de calcul de la répartition de la pression dans des couches chargées et polarisées, à température uniforme, où la constante diélectrique dépend de la température, des concentrations des divers constituants et du champ électrique. Une expression très simple a été obtenue pour la couche plane en équilibre mécanique. Pour des couches de forme sphérique, cylindrique ou quelconque, le calcul est théoriquement possible, à condition de connaître la répartition du champ et de la constante diélectrique en tout point. Une méthode thermodynamique permet un calcul approché de la répartition des pressions à l'équilibre de diffusion. Enfin, quelques estimations d'ordres de grandeurs sont données."

Seto, Y. J.; Dougal, Arwin A. 4553

The wave equation and the Green's dyadic for bounded magnetoplasmas.

J. Mathematical Phys. 5 (1964), 1326–1334.

Authors' summary: "In studies of electromagnetic wave propagation and radiation in magnetoplasmas, the wave equation takes the form of a dyadic-vector Helmholtz equation. The investigation here shows that the dyadic-vector Helmholtz equation is solvable by the separation method in four cylindrical coordinate systems. Solutions in the form of complete sets of eigenfunctions are possible when boundary surfaces are present. For problems

involving current sources in the plasma, the Green's dyadics for finite or semifinite domains can be constructed from the complete sets of eigenfunctions which are solutions to the homogeneous equation. The Green's dyadic for infinite domain is also shown to be obtained from that for a semifinite domain through a limiting process."

Tidman, D. A.; Guernsey, R. L.; Montgomery, D. 4554
"Test particle" problem for an equilibrium plasma.
Phys. Fluids 7 (1964), 1089–1091.

The authors derive a Fokker-Planck equation for a "test particle" distribution from the many-species Balescu-Lenard equation. The friction and dispersion coefficients are reduced to simple expressions for fast test particles, and compared with those previously calculated by Rostoker and Rosenbluth [Phys. Fluids **3** (1960), 1–14; MR **22** #4340]. The calculation has the advantages of being much more straightforward than earlier ones, and of yielding a Fokker-Planck equation for which there is no limitation in time and which predicts a relaxation to a Maxwell distribution. *H. Weitzner* (New York)

Trübenbacher, E. 4555
Die Thermodiffusion für ein binäres Gemisch von Gasen aus rauhen Kugeln gleicher Massen und gleicher Durchmesser.
Z. Naturforsch. **17a** (1962), 539–549.

In this doctoral dissertation the Boltzmann equation is formulated for a model of rough spherical molecules in isotropic approximation of the angular velocities by means of corresponding scattering cross-section, and for this the validity of "detailed-balance equation" presented. The thermal diffusion factor for a binary gas mixture of rough spheres is then computed with this scattering cross-section by means of the general formal theory. The derived formulae are used to determine the constants of thermal diffusion and thermal conductivity. While strong similarities in the applied mathematical model yield theoretical agreements with S. Chapman's findings, the great difference between the theoretical and experimental value of the thermal diffusion factor (1:15) makes the application of the chosen mathematical model questionable since even in a D_2 and H_2 gas the molecular behavior strongly deviates from that of a spherical molecule. Many authors expressed doubts lately concerning the validity of the Boltzmann equation, emphasizing the necessity for a deeper analysis of the collision mechanism. The Boltzmann description, even in the formulation of a rough spherical molecule, corresponds to a hypothetical physical situation with many independent random reflections, and the distribution function can be defined only by specifying the collision mechanism. The author also admits that greater accuracy is not to be expected from the "unphysical model of rough spheres". Besides, in a thermal diffusion process the irreversibilities in the collision mechanism cannot be neglected. The problem at hand demands a more sophisticated approach.
W. A. Werner (Syracuse, N.Y.)

Warsi, N. A. 4556
On steady plane flows of magnetogasdynamics.
Tensor (N.S.) **15** (1964), 275–281.

This paper is concerned with the intrinsic equations of steady plane magnetohydrodynamic flows. Various relations involving the constancy of flow and field quantities along the tangential as well as the normal direction to the stream and field lines are discussed. The author is apparently not aware that the intrinsic relations have been given for general three-dimensional gas flows [the reviewer, Z. Angew. Math. Mech. **41** (1961), 462–463].
R. P. Kanwal (University Park, Pa.)

Woods, L. C. 4557
On the boundary conditions at an insulating wall for hydromagnetic waves in a cylindrical plasma.
J. Fluid Mech. **18** (1964), 401–408.

The dispersion relation for the propagation of hydrodynamic waves along a magnetized cylindrical plasma falls by a factor of ten as the plasma resistivity tends to zero. This results in the two required boundary conditions at the walls being reduced to a single condition due to the presence of a current sheet. Furthermore, if the ratio of the wave frequency to the ion cyclotron frequency (Ω) is assumed to tend to zero, then the nature of this single boundary condition for the slow hydromagnetic wave depends on the quantity $\sigma^{1/2}\Omega^2$ (where $\Omega^2=\omega^2/\omega_{c_i}^2$). Similarly, if $\Omega\gg 1$, the fast wave is determined by the boundary conditions associated with the limiting value $\Omega\sigma^{-1/2}$. Finally, it is shown that "resistive waves" accompany both fast and slow waves. (These are required in order to satisfy small but finite values of σ^{-1}.) The effects of these waves are studied and their contribution to the wave damping is determined.
W. K. R. Watson (Riverside, Calif.)

Alder, B. J. 4558
Studies in molecular dynamics. III. A mixture of hard spheres.
J. Chem. Phys. **40** (1964), 2724–2730.

Part II (by the author and Wainwright) appeared in same J. **33** (1960), 1439–1451 [MR **22** #6116].

Author's summary: "For an equimolar mixture of hard spheres differing in radius by a factor of 3 it is shown that the numerical method gives the thermodynamic properties within 1% accuracy in the fluid region but generates metastable states at solid densities. The Percus-Yevick theory for mixtures is shown to be as accurate as for pure systems, which makes it plausible that no two-fluid phase region exists for mixtures of spheres. The volume of the mixture is found to be always smaller than the volume of the two pure fluid components, however, at high pressures when a solid and fluid must be mixed to give a fluid mixture, the excess volume is positive. The excess properties are small so that the assumption of ideal mixing (no excess volume) leads to an accurate predication of the properties of hard sphere mixtures. Furthermore, the conclusion can be drawn that the excess properties in mixing ordinary liquids are primarily determined by differences in the attractive potentials rather than solely by the molecular sizes of the pure components."
G. R. Stell (New York)

Callaway, Joseph 4559
Theory of scattering in solids.
J. Mathematical Phys. **5** (1964), 783–798.

Author's summary: "The general theory of the scattering of excitations in solids by localized imperfections is discussed. The solid-state analogue of the usual partial-wave expansion of the scattering amplitude is derived. In an appendix, the applicability of the general theory to phonons and spin waves as well as electrons is demonstrated." *W. Franzen* (Boston, Mass.)

Callaway, Joseph; Boyd, Robert 4560

Scattering of spin waves by magnetic defects.

Phys. Rev. (2) **134** (1964), A1655–A1662.

Authors' summary: "A previous calculation of the scattering amplitude for the scattering of spin waves by magnetic defects in a simple cubic lattice is simplified and extended to body-centered and face-centered cubic lattices. Expressions are given for the mean free path, and the thermal resistivity due to defect scattering is calculated by a method which takes some account of spin-wave interactions." *W. Franzen* (Boston, Mass.)

Lloyd, P.; O'Dwyer, J. J. 4561

Boundary conditions and the anharmonic contributions to the free energy of a lattice.

J. Chem. Phys. **41** (1964), 2416–2418.

Authors' summary: "In the evaluation of the free energy of a lattice in the harmonic approximation it is well known that the results are, to thermodynamic accuracy, independent of the boundary conditions used. It is shown here, by way of a simple example, that this is not so in the evaluation of the anharmonic contributions to the free energy. Physically the difference corresponds to whether or not the lattice is free to expand, or is constrained to a fixed volume, for the differing temperatures at which the partition function is evaluated."

Chambré, Paul L. 4562

On the energy moderation of neutrons in a heavy monatomic gas.

Rev. Un. Mat. Argentina **21**, 138–143 (1963).

Author's summary: "The subject of this investigation is the space and energy dependent neutron spectrum in a heavy monatomic gas due to a point source as described by the Wilkins differential equation for the case of constant neutron scattering cross-sections. Utilizing Laplace transformation methods the solution (Green's function) is given in closed analytical form."

RELATIVITY

See also 3579, 4008, 4230, 4234, 4598.

Bacry, H. 4563

Les moments multipolaires en relativité restreinte. Application à la diffusion des rayons X par une particule douée d'un moment magnétique quelconque.

Ann. Physique (13) **8** (1963), 197–237.

This is the essence of the author's thesis [Univ. Aix-Marseille, Masson, Paris, 1963; MR **29** #1874].

Dolci, Alba 4564

Brevi appunti di elettromagnetismo e relatività ristretta.

Rend. Sem. Fac. Sci. Univ. Cagliari **34** (1964), 19–81.

Lovelock, David 4565

On variational principles in which the Lagrangian function involves third order derivatives.

Ann. Mat. Pura Appl. (4) **64** (1964), 77–97.

Many of the theories of relativistic particles endowed with spin are based on variational principles whose Lagrangians involve second-order derivatives of the position vector (principles of the second kind). A detailed discussion of such variational problems subject to conditions involving parameter-invariance was given by the reviewer [same Ann. (4) **55** (1961), 77–104; MR **25** #1469], while an entirely new theory of relativistic spin particles based on a variational principle of the third kind (i.e., one whose Lagrangian involves third-order derivatives) was recently suggested by Z. Borelowski [Acta Phys. Polon. **21** (1962), 609–635]. The present paper is devoted to an extension of the above-mentioned parameter-invariant theory to third-order problems. The conditions to be satisfied by the Lagrangian as a result of parameter-invariance are discussed in detail. Further requirements, prompted by physical considerations, are used to narrow down the choice of suitable Lagrangians. One of these leads to a special class of the equations of motion obtained by the reviewer [Ann. Physik (7) **7** (1961), 17–27; MR **22** #10725] from first-order variational principles associated with a conservation of total angular momentum law, while an alternative stipulation gives rise to the equations of motion of Z. Borelowski [loc. cit.]. *H. Rund* (Pretoria)

Romain, Jacques E. 4566

Remarks on a coordinate transformation to an accelerated frame of reference.

Amer. J. Phys. **32** (1964), 279–285.

A coordinate transformation from an inertial frame $R(x, y, z, t)$ to a non-inertial frame $R'(X, Y, Z, T)$ accelerating along the x-axis was recently derived by H. Lass [same J. **31** (1963), 274–276; MR **26** #7389]. The present paper analyses the assumptions on which the derivation of this transformation is based. The properties, and their mutual implications, involved in Lass's transformation, are studied in great detail. A particularly important implication of these properties is that R' moves with respect to R as a rigid body in the Born sense, and that each point of R' has a constant proper acceleration. *P. S. Florides* (Dublin)

Valqui, Holger 4567

Derivation of the transformation equations in the special theory of relativity. (Spanish)

Univ. Nac. Ingen. Inst. Mat. Puras Apl. Notas Mat. **1** (1962/63), 41–48.

Weinstock, Robert 4568

Derivation of the Lorentz-transformation equations without a linearity assumption.

Amer. J. Phys. **32** (1964), 260–264.

The author derives the Lorentz equations from four

assumptions which are, briefly: (A) Inertial frame β moves at velocity $\mathbf{v}$ relative to inertial frame α along their common xx' axis; α moves at velocity $-\mathbf{v}$ with respect to β; (B) Two lines, each parallel to $\mathbf{v}$ in α, are also parallel lines in β; the distance between them is the same in both frames; (C) At a point where $x'=x$, t and t' are equal; (D) The velocity of light is the same for all reference frames and all directions.

The discussion is based on a thought experiment in which a flash of light is emitted at the origin of α and received at the origin of β. Most of the derivation is a proof that the Lorentz transformation is linear; this proof requires only (A) and (D) and the assumption that the transformation is continuously differentiable. Assumption (B) gives $y=y'$, $z=z'$, and assumption (C) is used to obtain the constant $\gamma=(1-v^2/c^2)^{-1/2}$.

Mary L. Boas (Chicago, Ill.)

Droz-Vincent, Philippe 4569

Suppression des contraintes du champ de gravitation par l'introduction d'une source vectorielle.

C. R. Acad. Sci. Paris **259** (1964), 515–518.

It is well known [P. G. Bergmann, Phys. Rev. (2) **75** (1949), 680–685; MR **10**, 408] that the Einstein gravitational equations are subject to four constraints, due to the invariance of the field Lagrangian under the group of four-dimensional coordinate transformations.

The author proposes to remove these constraints (and also to destroy the invariance of the Lagrangian) by introducing an "external" vector source, which is given a priori as a function of the coordinates. *A. Peres* (Haifa)

Efinger, Helmut J. 4570

Der kugelsymmetrische Fall einer statischen Ladungsverteilung bei stark komprimierter Materie in der allgemeinen Relativitätstheorie.

Acta Phys. Austriaca **17** (1963/64), 347–350.

The author derives the static, spherically symmetric form of Einstein's field equations for a charged fluid in the presence of an electric field. The equations are integrated under the assumption of an ultra-relativistic equation of state, corresponding to vanishing trace of the energy tensor. This generalizes results found by O. Klein [Ark. Mat. Astronom. Fys. **34A** (1948), no. 19; MR **9**, 627] for the case of no electric field. *W. Israel* (Edmonton, Alta.)

Graiff, Franca 4571

Sull'uso di coordinate armoniche in relatività generale.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **35** (1963), 312–320.

The author discusses the advantages of writing various expressions which involve the Christoffel symbols and their derivatives as tensorial expressions in which each term is a tensor by replacing the Christoffel symbols $\{_j{}^i{}_k\}$ by three-index tensors $\rho^i{}_{jk}$ which equal the difference of the Christoffel symbols of the V_4 and an E_4. Thus any special choice of coordinates defined by special values of the $\{_j{}^i{}_k\}$ and their derivatives leads to covariant equations for the $\rho^i{}_{jk}$ satisfied in any frame. As an example, he shows that if the "cartesian" coordinates (x^i) associated with the most general static, spatially spherically symmetric solution of the Einstein field equations are required to be harmonic $(g^{rs}\{_r{}^i{}_s\}'=0,\ i=1,\cdots,4)$, then the metric specializes to that of Schwarzschild.

R. Ingraham (University Park, N.M.)

Klauder, John R. 4572

Linear representation of spinor fields by antisymmetric tensors.

J. Mathematical Phys. **5** (1964), 1204–1214.

Let γ^μ be the four Dirac matrices and $\gamma^{\mu\nu}\equiv\frac{1}{2}(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)$; let v be any non-zero spinor for which $\gamma_5 v\neq\pm iv$. An arbitrary spinor field $\psi(x)$ can then be expressed by $\psi(x)=\frac{1}{2}f_{\mu\nu}(x)\gamma^{\mu\nu}v$. This relation provides a linear mapping of antisymmetric tensors $f_{\mu\nu}(x)$ onto spinors $\psi(x)$ for any given v. The mapping involves two classes of transformations (gauges); (1) the class of all $f_{\mu\nu}$ which lead to the same $\psi(x)$ for fixed v, called the c-gauge, and (2) the class of all transformations of $f_{\mu\nu}$ which are induced by transformations of v leaving ψ invariant, $\psi=\frac{1}{2}f_{\mu\nu}{}^F\gamma^{\mu\nu}v_F$, called F-gauges. One can then impose two Lorentz-invariant and gauge-covariant subsidiary conditions which permit tensors to behave like spinors. The Dirac equation is cast into tensorial form. The whole study is then generalized to general coordinate transformations leading to coordinate-dependent Dirac matrices. The F-gauge transformations then require a term in addition to the Christoffel term in the linear connection. A corresponding generalization of the Dirac equation is also proposed.

F. Rohrlich (Syracuse, N.Y.)

Komar, Arthur 4573

Commutators on characteristic surfaces.

Phys. Rev. (2) **134** (1964), B1430–B1440.

The author discusses derivation of commutation relations for Klein-Gordon, Maxwell and linearized gravitational fields from a Lagrangian formalism. His reason for this is the suitability of Penrose's scalar ψ [R. Penrose, Ann. Physics **10** (1960), 171–201; MR **22** #6563; Phys. Rev. Lett. **10** (1963), 66–68; MR **26** #7397] for a description of gravitational radiation. In particular, he considers the commutators on null surfaces at infinity, deriving the relations for Bondi's "news function" [Bondi, van der Burg, and Metzner, Proc. Roy. Soc. Ser. A **269** (1962), 21–52; MR **26** #4793]. He considers also the failure of the quantization methods in general relativity and indicates a way in which his approach may remedy the situation at least for asymptotically flat spaces.

A. H. Klotz (Liverpool)

v. Krzywoblocki, M. Z. 4574

On the equations of motion of a body with variable mass in the Einstein general theory of relativity.

Acta Phys. Austriaca **17** (1963/64), 159–185.

This work consists of three parts: I, II, III (Parts II and III are reviewed below [#4575, #4576]). It deals with the equations of motion, within the framework of general relativity, of a system of (non-rotating) bodies of which one body, A say, loses mass. The ejection of mass from A gives rise to a reaction force (exhaust force) on A. Consideration of the mass variation of A and the resulting exhaust force on A is what makes the present work different from all previous works on the problem of motion in general relativity.

The method of approach is that of Fock [*The theory of space, time and gravitation*, Pergamon, New York, 1959; MR **21** #7042] and Papapetrou [Proc. Phys. Soc. Sect. A **64** (1951), 57–75; MR **12**, 546; ibid. **64** (1951), 302–310; MR **13**, 695]. All their physical and mathematical assumptions on this problem, namely, (1) slow motion of the bodies (compared with c, the speed of light), (2) weakness of the gravitational field, (3) smallness of the dimensions of the bodies as compared with their mutual separation, (4) the harmonic coordinate condition, (5) the expansion of the metric tensor in powers of $1/c$, are here assumed unaltered. A further assumption made is that the lost mass of the body A, once it has left A, does not influence the gravitational field of the system of bodies. Also, the requirement that the body A loses mass necessitates an energy tensor which is different from the energy tensor of a perfect fluid used by Fock and Papapetrou.

The metric tensor, the energy tensor and the equations of motion are derived up to (and including) the second approximation. When the mass variation of A is set equal to zero, the results reduce to those of Fock and Papapetrou.

Part I reviews the approaches, to the problem of motion, of Einstein and his collaborators, and of Fock and Papapetrou, derives the energy tensor incorporating the mass variation of A, and sets up the field equations ($G_\mu{}^\nu\sqrt{-g}=T_\mu{}^\nu\sqrt{-g}$) and the dynamical equations ($(T_\mu{}^\nu\sqrt{-g})_{|\nu}=0$) of the problem.

P. S. Florides (Dublin)

v. Krzywoblocki, M. Z. 4575

On the equations of motion of a body with variable mass in the Einstein general theory of relativity. II.

Acta Phys. Austriaca **17** (1963/64), 271–300.

In this second part [see #4574 above for Part I] the energy tensor, the metric tensor and the equations of motion in their first approximation are derived. By comparing the equations of motion in their first approximation with the classical rocket equation (Section 4.2) it is shown that the exhaust force on A depends on the exhaust mass and its exhaust velocity.

Part II concludes with the calculation of the energy tensor and the metric tensor in their second approximation.

P. S. Florides (Dublin)

v. Krzywoblocki, M. Z. 4576

On the equations of motion of a body with variable mass in the Einstein general theory of relativity. III.

Acta Phys. Austriaca **17** (1963/64), 392–413.

In this final part [for Parts I and II, see #4574 and #4575 above] the equations of motion in their second approximation are derived. It is shown that when the mass variation of A is set equal to zero, the equations reduce to those of Fock and Papapetrou. The paper concludes with an extensive bibliography and an appendix where various results of the text are proved.

P. S. Florides (Dublin)

Mikhail, F. I. 4577

Unified field theory. (Italian summary)

Nuovo Cimento (10) **32** (1964), 886–894.

The author continues the investigation of a unitary field theory proposed earlier by W. H. McCrea and the author [Proc. Roy. Soc. London Ser. A **235** (1956), 11–22; MR **17** #1144]. The structural elements are a tetrad field, along with an asymmetric affine connection so chosen that the tetrad field is covariantly constant. The metric is symmetric and determined by the requirement that the tetrad field be orthonormal. Using methods analogous to those by Einstein and Mayer [S.-B. Preuss. Akad. Wiss. Phys.-Math. Kl. **1930**, 110–120], the author shows that the only spherically symmetric solution of his field equations (all contractions of the Riemann-Christoffel tensor are to vanish) possesses the geometry of Schwarzschild's solution.

P. G. Bergmann (New York)

Penney, R. 4578

Duality invariance and Riemannian geometry.

J. Mathematical Phys. **5** (1964), 1431–1437.

It is the purpose of the paper to show how one is led to field equations relating the electromagnetic field to space-time geometry solely by algebraical considerations based on the symmetry properties of the Maxwell-Einstein field. The underlying hypotheses are the following: (I) Space-time is a Riemannian space V_4; (II) Over V_4 is defined a skew-symmetric tensor $F_{\mu\nu}$ which contains all information concerning the electromagnetic field; (III) The field equations relating the geometry of V_4 to $F_{\mu\nu}$ must be algebraic; (IV) The laws of physics must be either first-order or (linear) second-order differential equations.

The author's starting point is what he calls "the duality invariance" of the Maxwell field, by which is meant the invariance of the field under the transformation

$$\begin{bmatrix} F'_{\mu\nu} \\ {}^*F'_{\mu\nu} \end{bmatrix} = M \begin{bmatrix} F_{\mu\nu} \\ {}^*F_{\mu\nu} \end{bmatrix},$$

where M is a unitary matrix and ${}^*F_{\mu\nu}=\frac{1}{2}\eta_{\mu\nu\lambda\rho}F^{\lambda\rho}$ the dual of $F_{\mu\nu}$ (here $\eta_{\mu\nu\lambda\rho}=-(-g)^{1/2}\epsilon_{\mu\nu\lambda\rho}$ is the permutation tensor). The properties of the tensors $t_\mu{}^\alpha=F_{\mu\nu}F^{\alpha\nu}+{}^*F_{\mu\nu}{}^*F^{\alpha\nu}$ and $T_{\mu\nu\alpha\rho}=F_{\mu\nu}F_{\alpha\rho}+{}^*F_{\mu\nu}{}^*F_{\alpha\rho}$ are studied with particular reference to the right dual ${}^*T_{\mu\nu\alpha\rho}=\frac{1}{2}\eta_{\mu\nu\epsilon\gamma}T^{\epsilon\gamma}{}_{\alpha\rho}$ and the left dual $T^*_{\mu\nu\alpha\rho}=\frac{1}{2}\eta_{\alpha\rho\epsilon\gamma}T_{\mu\nu}{}^{\epsilon\gamma}$. It is found that $T_{\mu\nu\alpha\rho}$ has exactly the same symmetry properties as the curvature tensor $R_{\mu\nu\alpha\rho}$ of V_4. By means of an investigation of all the possibilities resulting from an application of the contraction operations and the dual operation, the above postulates yield the field equations $R_{\mu\nu\alpha\rho}+{}^*R^*_{\mu\nu\alpha\rho}=kT_{\mu\nu\alpha\rho}$. These are found to be reducible to the set of nine equations $R_{\mu\nu}-\frac{1}{4}Rg_{\mu\nu}=-kt_{\mu\nu}$. The relations $F_{\mu\nu,\rho}+F_{\rho\mu,\nu}+F_{\nu\rho,\mu}=0$ are then introduced as an additional gauge condition. To this is adjoined the set of equations ${}^*F^{\mu\nu}{}_{|\mu}=0$ (which are now assumed to hold), which renders the system determinate.

H. Rund (Pretoria)

Rylov, Yu. [Rylov, Ju. A.] 4579

Relative gravitational field and conservation laws in general relativity.

Ann. Physik (7) **12** (1963/64), 329–353.

Author's summary: "It is shown that by a description of the gravitational field within the limits of Einstein's theory only a relative gravitational field, i.e., the gravitational field at a point x with respect to one at a point x', is physically essential. A reflection of curved space-time into a continuum of flat spaces $E_{x'}$ depending on coordinates

of an arbitrary point x' is made. The relative gravitational field is described by tensor potentials in terms of the two-metric formalism. The relative gravitational field depends on two points: on a current point x and a base point x'. This allows one to localize the gravitational field without violating the equivalence principle. Integral conservation laws for energy momentum and angular momentum are obtained, the energy-momentum tensor being a true relative tensor, i.e., a tensor depending on two points: x and x'. All values connected with a gravitational field are relative to what is interpreted as the presence of some general relativity in the gravitational field."

M. Blažek (Bratislava)

Schöpf, Hans-Georg 4580

Allgemeinrelativistische Prinzipien der Kontinuumsmechanik.

Ann. Physik (7) **12** (1963/64), 377–395.

Starting with a comparison between general relativistic field theory and continuum mechanics, so-called conservative systems are considered. These are defined by assuming the existence of an elastic potential Φ and the absence of heat flow. For these systems field equations taking entropy into account are derived from a variational principle which involves variation of the world-lines of the medium. Eckart's interpretation [Phys. Rev. (2) **58** (1940), 919–924] of the energy tensor is used, of which the spatial part (stress tensor) is equated to the derivatives of Φ with respect to the elements of the strain tensor as defined by the reviewer [Proc. Roy. Soc. Ser. A **272** (1963), 44–53; MR **26** #7414]. The entropy balance equation is formulated for non-conservative systems using ideas of Kluitenberg and others [Kluitenberg, Mazur, and de Groot, Physica **19** (1953), 689–704; MR **15**, 490; ibid. **19** (1953), 1079–1094; MR **16**, 185; Kluitenberg and de Groot, ibid. **20** (1954), 199–209; MR **16**, 186; ibid. **21** (1955), 148–168; MR **16**, 1188]. *C. B. Rayner* (Brighton)

Staruszkiewicz, A. 4581

Gravitation theory in three-dimensional space.

Acta Phys. Polon. **24** (1963), 735–740.

Author's summary: "Gravitation theory in three-dimensional space (with two space-like and one time-like dimensions) is considered. In particular, a general solution of the problem of motion of point particles with finite mass is proposed: the particles move along straight lines in Euclidean space, whence by cutting out certain fragments and connecting the edges of the cutting one obtains the space produced by the bodies."

C. Gilbert (Newcastle upon Tyne)

Świątak, H. 4582

On the algebraic structure of gravitational fields admitting of 5- and 6-parameter groups of motions.

Acta Phys. Polon. **25** (1964), 161–168.

Résumé de l'auteur: "In this paper the algebraic structure of gravitational fields admitting five- and six-parameter groups of motions is investigated, special note being taken of the gravitational fields produced by electromagnetic fields. Applying the modified criterion of integrability of Killing's equations, relations between the algebraic structure of the energy-momentum tensor and that of Weyl's tensor are derived. These show, in particular, that a zero electromagnetic field produces a gravitational field of type N or O, whereas a non-zero electromagnetic field gives rise to a gravitational field of type D or O. It is proved too that in either case Weyl's tensor and the energy-momentum tensor of the electromagnetic field assume the canonical form simultaneously."

J. Charles-Renaudie (Montpellier)

Watanabe, Shôji 4583

A remark on the theory of general geometrodynamics.

J. Math. Mech. **13** (1964), 547.

This note adds a further result to those obtained by the author in a previous paper [same J. **12** (1963), 831–846; MR **28** #1947]. *G. F. R. Ellis* (Cambridge, England)

Hamoui, Adnan 4584

Sur le théorème de Birkhoff et la solution "radiative" de Petrov.

C. R. Acad. Sci. Paris **258** (1964), 6085–6087.

Birkhoff's theorem states that all spherically symmetric solutions of the equations $R_{ik}=0$ can be transformed into the Schwarzschild solution and are therefore essentially static. According to Petrov [Ž. Èksper. Teoret. Fiz. **44** (1963), 1525–1533; MR **28** #1965], this theorem is true only under certain unnecessarily restrictive assumptions concerning the differentiability of the g_{ik}.

The author refutes the work of Petrov by showing that Birkhoff's theorem is valid if the g_{ik} satisfy the conditions of Lichnerowicz. He shows, moreover, that Petrov's alleged counterexample to Birkhoff's theorem is, in fact, equivalent to the Schwarzschild solution.

W. B. Bonnor (London)

Papapetrou, Achille 4585

Champs gravitationnels à symétrie sphérique avec rayonnement électromagnétique.

C. R. Acad. Sci. Paris **258** (1964), 6081–6084.

The author writes the Schwarzschild field in the form

$$(1)\quad ds^2 = -\left(1+\frac{m}{r}\right)dr^2+\frac{2m}{r}\,dr dt + \left(1-\frac{m}{r}\right)dt^2 - r^2(d\theta^2+\sin^2\theta\, d\phi^2),$$

where m is a constant. He then takes m to be a function of $r-t$,

$$(2)\qquad m = f(r-t),$$

and shows that the metric (1) yields an energy tensor representing an outgoing radial flux of photons. He therefore interprets (1) together with (2) as the gravitational field of a star emitting electromagnetic radiation.

The relation of the solution to those of Synge [Proc. Roy. Irish Acad. Sect. A **59** (1957), 1–13; MR **20** #734] and Israel [Proc. Roy. Soc. London Ser. A **248** (1958), 404–414; MR **21** #4807] is discussed.

W. B. Bonnor (London)

ASTRONOMY

See also 3850, 4243.

Allen, William A.; Knolle, Werner E. 4586
Differential corrections applied to the Izsak equations of artificial satellite motion.
Astronom. J. **69** (1964), 393–401.

Authors' summary: "Differential coefficients are derived for use with the Izsak solution of the Vinti dynamical problem. These coefficients are then applied to a typical problem—the least-squares fit of the Izsak equations to a comparison ephemeris. The comparison ephemeris was generated by numerical integration of the differential equations of motion for a satellite in a force field that includes the J_2, J_3, J_4 and J_{22} terms. The Izsak and Borchers second-order equations were determined to be accurate to eight decimal places for a near-earth satellite of low eccentricity in a Vinti gravitational potential. The Izsak equations constitute an intermediary orbit for the one-body problem of celestial mechanics. The Izsak orbit is compared with another well-known intermediary orbit—the rotating-precessing ellipse representation. The Izsak orbit, specified by only six parameters, is approximately twice as accurate as the moving ellipse orbit which requires eight parameters."

J. M. A. Danby (New Haven, Conn.)

Chebotarev, G. A. [Čebotarev, G. A.] 4587
On orbits with small eccentricities. (Russian summary)
The Use of Artificial Satellites for Geodesy (*Proc. First Internat. Sympos., Washington, D.C.*, 1962), *pp.* 8–11. *North-Holland, Amsterdam*, 1963.

Perturbations due to an oblate planet with potential of the form

$$(MG/R)(1-JP_2(\cos\delta)/R^2)$$

are calculated for orbits of small eccentricity, e, to the first order of J and e. The formulas given are contained in another paper which was published in English [AIAA J. **2** (1964), 203–208; MR **28** #3861]. The latter account is to be preferred since more details are provided. But formulas for Ω in the latter, and λ in the former, contain misprints.

J. M. A. Danby (New Haven, Conn.)

Čobanov, Ivan 4588
On Bertrand's problem. (Bulgarian. German summary)
Annuaire Univ. Sofia Fac. Sci. Phys. Math. Livre 1 *Math.* **56** (1961/62), 7–21 (1963).

The author discusses the solution of Bertrand's problem, i.e., the determination of the central force under the action of which the particle's path is a conic. The Darboux and Halphen solutions are shown to be equivalent and another mathematically possible solution (the particle's path passes through the center of force) is found. A further conclusion of the author is that from the solutions of Bertrand's problem it does not necessarily follow that the forces of gravity in the universe are only the Newtonian ones.

Z. Janković (Zagreb)

Contopoulos, George 4589
On the existence of a third integral of motion.
Astronom. J. **68** (1963), 1–14.

This is a survey of results, recent and classical, concerning integrals of motion other than the energy and angular momentum integrals, with particular attention to adelphic integrals. There is a discussion of existence, convergence, and ergodic properties, and examples are given of motions in symmetric and asymmetric stellar systems.

R. G. Langebartel (Urbana, Ill.)

Robe, H. 4590
Note sur les oscillations du sphéroïde de Jeans. (English summary)
Acad. Roy. Belg. Bull. Cl. Sci. (5) **49** (1963), 1148–1155.

Author's summary: "The small oscillations of an homogeneous and incompressible mass distorted by the tidal effects of a secondary are investigated in detail on the basis of the small perturbation method. General expressions for the characteristic frequencies, together with the corresponding displacements belonging to any harmonic, are determined which allow a complete discussion of stability."

J. M. A. Danby (New Haven, Conn.)

Ural'skaja, V. S. 4591
Polar orbits of artificial celestial bodies. (Russian)
Vestnik Moskov. Univ. Ser. III Fiz. Astronom. **1964**, *no.* 4, 34–44.

Polar orbits of artificial satellites are very useful for establishing the parameters of the tri-axiality of the earth and for finding how to avoid radiation belts. In the earth's potential considered by the author certain terms of second order of smallness with respect to the earth's polar flattening are neglected. This permits solving the problem by quadratures. The author takes into account that the earth is not symmetric with respect to the equator and keeps in the expression for the potential the zonal harmonics of odd order which characterize the asymmetry of the earth with respect to the equator. The author investigates twelve types of polar orbits belonging to three main classes: ballistic orbits which intersect the earth's surface, satellite orbits, and receding orbits. Satellite orbits are investigated in detail. For this case the author obtains closed formulas for the satellite coordinates as functions of an intermediate variable which in turn is time dependent. Since these formulas contain inconvenient elliptic functions, the author expands the satellite coordinates in trigonometric series, where terms of order higher than two with respect to the earth's polar flattening are neglected. The author finds that the earth's asymmetry with respect to the equator does not introduce secular terms in all the satellite coordinates, but causes very considerable periodic disturbances.

T. Leser (Aberdeen, Md.)

Vinti, J. P. 4592
The spheroidal method for satellite orbits. (Russian summary)
The Use of Artificial Satellites for Geodesy (*Proc. First Internat. Sympos., Washington, D.C.*, 1962), *pp.* 12–16. *North-Holland, Amsterdam*, 1963.

A method is suggested for the elimination of non-gravitational effects in the motion of an artificial satellite, so that the field of the Earth can be accurately studied.

J. M. A. Danby (New Haven, Conn.)

Ahmad, H. H. 4593

Perturbations of hyperbolic elements. (Russian)

Vestnik Moskov. Univ. Ser. III Fiz. Astronom. **1964**, *no.* 3, 15–27.

The paper is concerned with the construction of an analytic theory of perturbations for the hyperbolic orbital elements. First the differential equations for the hyperbolic osculating elements are derived by using Lagrange's bracket expressions. Then, following R. A. Ljah [Bjull. Inst. Teoret. Astronom. 7 (1959), 422–440], the expansion of the perturbation function in terms of powers of the ratio of the semi-axes is given for the circular restricted three-body problem. This is done for the inner as well as the outer case, i.e., for $r \lessgtr a$, where r denotes the distance between the perturbed and central bodies and a that between the perturbing and central bodies [cf. also N. B. Elenevskaja, ibid. 6 (1957), 434–465; MR 19, 616].

E. Leimanis (Vancouver, B.C.)

Bartlett, James H. 4594

The restricted problem of three bodies.

Mat.-Fys. Skr. Danske Vid. Selsk. **2**, *no.* 7, 48 *pp.* (1964).

From the author's summary: "The case of two equal finite masses has been studied extensively, using Thiele variables E, F, a modified Runge-Kutta method, and an electronic computer. The main classes of Strömgren have been traced continuously from beginning to end, and seven new classes are reported. For convenience, a class is represented by an eigensurface in E, F, K-space, where K is the Jacobi constant. General methods of locating periodic solutions, in particular, asymmetric ones, are discussed. Initial and final conditions, in the form of tables and curves, are given for more than 800 periodic orbits."

E. Leimanis (Vancouver, B.C.)

Pitman, George R., Jr. (Editor) 4595

★Inertial guidance.

University of California Engineering and Physical Sciences Extension Series.

John Wiley & Sons, Inc., New York-London, 1962. xiii + 481 *pp.* $18.50.

A good book in "inertial navigation" has been needed for a long time. Until recently most of the material on inertial navigation has been withheld due to security reasons. With the declassification of some material, an excellent text was organized and born. The book is based upon a collection of notes given originally in an intensive course on inertial guidance at UCLA. These notes have since been extended, edited and compiled in book form.

The book is divided into three main sections. The first section, after a short introduction and historical resumé of inertial navigation, contains a comprehensive discussion of the principal inertial sensing instruments and components of a modern inertial guidance system. The analysis consists of vector analysis notations describing the motion of a system in space. The applied mathematician will immediately recognize the theory as that previously studied in analytical mechanics texts. The main chapters of this section discuss the various aspects of gyros (single and two-degree-of-freedom) and internal sensing accelerometers. A discussion of various problems, namely, vibration, mass unbalance, anisoelasticity and random gyro drift, is given. Each of these errors is expressed in a mathematical form. Chapter IV is concerned with the proper testing of the inertial components. Chapter V is an excellent chapter on "stable platforms", which is considered the heart of the inertial system. The platform is replete with mathematical analysis and relies entirely on feedback control analysis. Unfortunately, non-linear effects creep in and cause instability problems. This is a very fertile area for the applied mathematician to be engaged.

The second section is concerned mainly with the problems of designing and mechanizing inertial navigation systems for cruise vehicles, i.e., vehicles requiring navigational information over long periods of time. Chapter VI discusses guidance kinematics and its relation to the coordinate systems and their respective mathematics. Chapters VII and VIII are excellent contributions to the various errors associated with the platform and means of error rectification. This is extremely important for cruise type vehicles. Chapter IX discusses augmented types of inertial systems, namely, "stellar monitored", "Doppler inertial" and damped systems.

The last section discusses guidance systems for rocket vehicles. This is most important for boost vehicles, namely, IRBM, ICBM, Saturn. The flight is short with respect to the cruise type, but the requirements are demanding and exacting. The various schemes are explained and their interactions with the elastic body effects are discussed ever so lightly. The problem of fuel sloshing is explained briefly, but this is a major problem in large-size boosters. The chapter on error is an applied mathematician's haven. The error determination must be precise since the orbiting of satellites is very exacting.

The book ends with an application of inertial guidance to space missions. Celestial mechanics is the heart of the guidance schemes and contains the well-known Kepler equations, Hohmann transfer between orbits and the discussion of elliptical orbits. The chapter ends with the description of the successful space navigation system for the Ranger spacecraft.

In conclusion, the book is an excellent source of information dealing with both practical aspects and theoretical aspects of inertial navigation systems. Persons interested in the applied mechanics of inertial navigation systems will find a wealth of potential problems as well as a number of mathematical solutions to previous ones. Inertial navigation is dynamic in both methods of analysis and existence. More improvements are necessary for space vehicles of the future. The reviewer believes that the book would be enhanced by further additions on aeroelastic effects (wind shears and gusts acting on the vehicle), fuel sloshing effects and autopilot-structural mode coupling effects. The book does not discuss "Pontryagin's Principle" (optimal control) or adaptive control systems. These are becoming extremely important. Any further revision to the book should include these topics.

H. Saunders (Philadelphia, Pa.)

Finzi, Arrigo 4596

On the validity of Newton's law at a long distance.

Monthly Notices Roy. Astronom. Soc. **127** (1963), 21–30.

The author suggests that at great distances Newton's law of attraction should be changed to

$$F = (Gm_1m_2/\rho^2)(\rho/r)^{3/2}, \qquad r \gg \rho,$$

where ρ is some characteristic length (taken here to be half a kiloparsec). This is given no theoretical basis, but is suggested as an ad hoc hypothesis.

It is shown that it is then possible to resolve certain paradoxes arising from apparent contradictions when some problems are attacked by different methods. The problems considered here include the stability of clusters of galaxies, and the relations between galactic rotation and galactic mass.

J. M. A. Danby (New Haven, Conn.)

Stanyukovich, K. P. [Stanjukovič, K. P.] 4597

Generalization of models of the Friedman universe.

Dokl. Akad. Nauk SSSR **151** (1963), 546–549 (*Russian*); *translated as Soviet Physics Dokl.* **8** (1964), 697–700.

Krause, Helmut G. L. 4598

Relativistic perturbation theory of an artificial satellite in an arbitrary orbit about the rotating oblated earth spheroid and the time dilatation effect for this satellite. (Russian summary)

The Use of Artificial Satellites for Geodesy (*Proc. First Internat. Sympos., Washington, D.C.*, 1962), *pp.* 69–107. *North-Holland, Amsterdam*, 1963.

A 39-page exposition of the well-known astronomical consequences of general relativity.

G. M. Clemence (New Haven, Conn.)

GEOPHYSICS

See also 4368.

Tsirul'skiĭ, A. V. [Cirul'skiĭ, A. V.]; Sirotin, M. I. 4599

On the analytic continuation of a logarithmic potential.

Izv. Akad. Nauk SSSR Ser. Geofiz. **1964**, 105–109 (*Russian*); *translated as Bull.* (*Izv.*) *Acad. Sci. USSR Geophys. Ser.* **1964**, 59–61.

In discussing the interpretation of magnetic and gravimetric anomalies in the case of a two-dimensional problem with the aid of methods based on finding by analytic continuation the singularities of a potential, the authors prove that the location of the sources of the potential (which is the true problem of interpretation) and finding the singularities are not, in general, identical problems because of the multivalued nature of the logarithmic potential. An example illustrates very well this important point.

E. Kogbetliantz (New York)

Siebert, Manfred 4600

Ein Verfahren zur unmittelbaren Bestimmung der vertikalen Leitfähigkeitsverteilung im Rahmen der erdmagnetischen Tiefensondierung. (English summary)

Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II **1964**, 25–35.

Author's summary: "A method is given by means of which the distribution of electric conductivity with depth can directly be determined from observations of geomagnetic variations on the earth's surface; no assumption of a conductivity-model is needed. The method can be applied if the inducing magnetic fields are two-dimensional and quasi-stationary in a wide frequency domain. In its present state the method is not appropriate to investigate horizontal conductivity anomalies. A few concluding remarks concern the practical use of the formalism."

E. Kogbetliantz (New York)

Strakhov, V. N. [Strahov, V. N.] 4601

The problem of obtaining the best numerical method for the transformation of potential fields. I.

Izv. Akad. Nauk SSSR Ser. Geofiz. **1963**, 1780–1797 (*Russian*); *translated as Bull.* (*Izv.*) *Acad. Sci. USSR Geophys. Ser.* **1963**, 1081–1090.

The special properties of harmonic functions involved in the theory of geological interpretation of gravimetric and magnetic anomaly maps are studied and the basic ideas concerning the best numerical schemes and methods of interpretation are formulated.

Essentially these methods (not yet defined in this first part) will be based on the theory of Fourier transforms and, in particular, on Parseval and translation theorems.

The numerical schemes—to be defined in the following parts of this paper [#4602]—will depend on the depth H (presumably as computed from the mapped potential field) of perturbing masses nearest to the ground, but not on their particular shape.

This new and systematic approach to the problem of interpretation seems to be a very promising one.

E. Kogbetliantz (New York)

Strakhov, V. N. [Strahov, V. N.] 4602

The problem of obtaining the best numerical methods for the transformation of potential fields. II, III.

Izv. Akad. Nauk SSSR Ser. Geofiz. **1964**, 55–67; *ibid.* **1964**, 68–81 (*Russian*); *translated as Bull.* (*Izv.*) *Acad. Sci. USSR Geophys. Ser.* **1964**, 30–35; *ibid.* **1964**, 36–42.

The basic ideas of a new interpretation method formulated by the author in his previous paper [#4601] are developed in detail in these two papers. The first one deals with the two-dimensional problem while the second treats the general three-dimensional case of interpretation.

In both cases the optimum numerical coefficients of the final approximate formula are obtained minimizing either the integral of the square of error (global error) or the upper bound of the maximum error. In the first case the positive definite symmetric matrix of the system of linear equations which determine the said coefficients has positive entries only. In the second case only rough approximations can be obtained.

E. Kogbetliantz (New York)

ECONOMICS, OPERATIONS RESEARCH, GAMES

See also 4076, 4116.

Neuts, Marcel F. 4603

An inventory model with an optional time lag.

J. Soc. Indust. Appl. Math. **12** (1964), 179–185.

The author investigates a dynamic one-dimensional inventory model with independent, identically distributed demands and known probability density function. Two

types of replenishment, at equally spaced times, are available: (1) with immediate delivery and unitary cost k; (2) with one period lag in supply and unitary cost l. The cost of storage is a convex increasing and twice differentiable function of the stock in store, the penalty is proportional to the under-supply, and no other costs or revenues are considered. If x denotes the initial stock, $y-x$ the amount ordered in conditions (1), and u the quantity ordered in conditions (2), it is shown that the reordering policy which minimizes the total discounted cost of a process of unbounded duration is the following:

$$\begin{aligned} y &= x^*, & u &= u^*, & &\text{for } 0 \leq x \leq x^*, \\ y &= x, & u &= \bar{u}(x), & &\text{for } x^* \leq x \leq \bar{x}, \\ y &= x, & u &= 0, & &\text{for } x \geq \bar{x}, \end{aligned}$$

in which $\bar{u}(x)$ is non-negative, continuous and monotone decreasing in x and x^*, $\bar{x}$ are critical levels of the stock. Some degenerate cases are mentioned. For instance, when $l \geq k$, $u^* = 0$ and $x^* = \bar{x}$. The proof is by induction using a sequence $\{f_n(x)\}$ of successive approximations to $f(x)$, the minimum cost of an infinite process starting with x. This sequence is monotone because its first element is obtained by an approximation in the policy space. The arguments are similar to those of R. Bellman [*Dynamic Programming*, pp. 159–164, Princeton Univ. Press, Princeton, N.J., 1957; MR **19**, 820] and S. Karlin [*Studies in the mathematical theory of inventory and production*, Stanford Univ. Press, Stanford, Calif., 1958; MR **20** #767], which are given as references. *B. Bereanu* (Bucharest)

Slagle, James R. 4604

An efficient algorithm for finding certain minimum-cost procedures for making binary decisions.

J. Assoc. Comput. Mach. **11** (1964), 253–264.

A procedure, in this paper, is a method for making a true-false decision based on the outcome of a sequence of elementary binary tests. These tests are represented by random variables x having probability p of being true and $1-p$ of being false. With each test x there is associated a cost c. A procedure over a finite set of tests X is defined inductively on the cardinality of X. A procedure over 0 tests is true or false. A procedure over $n > 0$ tests is a triple $[x, P_0, P_1]$, where $x \in X$ and P_0, P_1 are procedures over subsets of $X - \{x\}$. The value of a procedure is likewise defined inductively. The value of a procedure over 0 tests is the procedure itself, i.e., true or false. The value of $[x, P_0, P_1]$ is defined as the value of P_0 if x is true and P_1 if x is false. Also the average, or expected, cost of a procedure is defined by $c(\text{true}) = c(\text{false}) = 0$

$$c([x, P_0, P_1]) = c + pc(P_0) + (1-p)c(P_1).$$

The general problem is, given a procedure P, to efficiently find a procedure P' of least average cost such that the value of P is the same as the value of P'. In general, a solution can be found by exhaustive search, so that the essential difficulty is to find an efficient method for solution. The author develops such a method for certain restricted classes of procedures, and shows by illustration how his results can be used in the more general case. The presentation is highly readable, beginning with examples from computer programming, geometry and personnel selection. *L. Hodes* (Yorktown Heights, N.Y.)

De Finetti, Bruno [de Finetti, Bruno] 4605

Probabilità composte e teoria delle decisioni.

Rend. Mat. e Appl. (5) **23** (1964), 128–134.

Let P be subjective probability. Consider two events, E and H. We have: $P(H\cdot E) = P(H)\cdot P(E|H)$. Let us set $x = P(E|H)$, $y = P(H\cdot E)$, $z = P(H)$. In a decision problem we have the quadratic loss function: $L = H\cdot(E-x)^2 + (H\cdot E - y)^2 + (H-z)^2$. Decisions for which $y = zx$ are admissible, i.e., not dominated by other decisions. The author gives a discussion in terms of three-dimensional geometry in the space of x, y, z. *G. Tintner* (Los Angeles, Calif.)

Bereanu, Bernard 4606

Programme de risque minimal en programmation linéaire stochastique.

C. R. Acad. Sci. Paris **259** (1964), 981–983.

It is unfortunate that access to Western literature is still so difficult in some parts of Europe. The results are a very special instance of those of the reviewer and W. W. Cooper [Operations Res. **11** (1963), 18–39; MR **27** #3448]. Specifically, the author treats the P-model with zero-order decision rule and no chance-constraints. *A. Charnes* (Evanston, Ill.)

Hanson, M. A. 4607

Duality and self-duality in mathematical programming.

J. Soc. Indust. Appl. Math. **12** (1964), 446–449.

A scalar function $\varphi(x)$ is functionally convex over a set $S \subset E_n$ if for all $x_1, x_2 \in S$, $\varphi(x_1) < \varphi(x_2)$ implies

$$\varphi[\theta x_1 + (1-\theta)x_2] < \varphi(x_2)$$

for all values of θ in the open interval $(0, 1)$.

The author establishes the following duality theorem: If $\varphi(x)$ is a differentiable scalar functionally convex function of the n-vector x, if $f(x)$ is a differentiable m-dimensional functionally concave function, and if x_0 minimizes $\varphi(x)$ over the set $S = \{x : f(x) \geq 0\}$, then there exists an m-vector $v_0 \geq 0$ such that (x_0, v_0) maximizes $\chi(x, v) \equiv \varphi(x) - v'f(x)$ over the set $W = \{(x, v) : \nabla\chi = 0, v \geq 0\}$; and the minimal value $\varphi(x_0)$ of the primal is equal to the maximal value $\chi(x_0, v_0)$ of the dual.

A self-dual convex programming problem is one whose dual is equivalent to it. The following is a paraphrase of the author's self-duality theorem. Let $f: R^n \to R^n$ be a concave mapping and suppose $\varphi(x) = x'f(x)$ is convex on the set $T = \{x : f(x) \geq 0, x \geq 0\}$. If x_0 minimizes $\varphi(x)$ over T, then the problem is self-dual, $\varphi(x_0) = 0$, and x_0 is an extreme point of T. *R. W. Cottle* (Holmdel, N.J.)

Bondareva, O. N. 4608

Some applications of the methods of linear programming to the theory of cooperative games. (Russian)

Problemy Kibernet. No. 10 (1963), 119–139.

The methods and results involve linear inequalities but not extrema. The main result is a distinct improvement of Gillies' theorem [*Contributions to the theory of games*, Vol. IV, pp. 47–85, Princeton Univ. Press, Princeton, N.J., 1959; MR **21** #4850] that a positive fraction of all games have unique solutions. There are a number of other results on necessary and/or sufficient conditions for

existence and nature of cores or k-quotas, and for a 1-dimensional core to be a solution. Systematic and effective use is made of the (finite) set of all expressions of $(1, 1, \cdots, 1)$ as a non-negative combination of independent 0-1 vectors. *J. R. Isbell* (New Orleans, La.)

Huyberechts, Simone 4609

Vers une classification des jeux sur le carré-unité.

Cahiers Centre Études Recherche Opér. **4** (1962), 179–200.

Author's summary: "Dans les paragraphes I et II, nous rappelons quelques définitions et résultats généraux relatifs aux jeux de somme nulle à deux joueurs et aux jeux sur le carré-unité. Nous procédons ensuite à l'analyse de quelques classes importantes de jeux sur le carré-unité. Au cours de cette analyse, nous mettons fréquemment l'accent sur la notion de bonne stratégie égalisante, notion qui, dans de nombreux cas, nous permet d'obtenir des conditions suffisantes pour que les bonnes stratégies de l'un ou l'autre des joueurs soient équivalentes."

Janovskaja, E. B. 4610

Minimax theorems for games on the unit square. (Russian. English summary)

Teor. Verojatnost. i Primenen. **9** (1964), 554–555.

Author's summary: "We consider a class of infinite games with unbounded kernels and establish the existence of their value. It is shown that a game with the kernel

$$\begin{aligned} K(x, y) &= L(x, y), & x < y, \\ &= \varphi(x), & x = y, \\ &= M(x, y), & x > y, \end{aligned}$$

where the functions L and M are defined and continuous on the triangles $0 \leq x \leq y \leq 1$, $0 \leq y \leq x \leq 1$ respectively, the function φ is arbitrary and $L(0, 0) \geq M(0, 0)$, $L(1, 1) \leq M(1, 1)$, is a game with value."

W. H. Fleming (Providence, R.I.)

BIOLOGY AND BEHAVIORAL SCIENCES

Criswell, Joan H.; Solomon, Herbert; Suppes, Patrick (Editors) 4611

★Mathematical methods in small group processes.

Stanford Mathematical Studies in the Social Sciences, VIII.

Stanford University Press, Stanford, Calif., 1962. viii + 361 *pp.* $9.75.

This volume contains the papers presented at a symposium on mathematical methods in small group processes held at Stanford University, 20–23 June 1961. The introduction by Joan H. Criswell (pp. 1–10) gives a lucid summary of the progress in this field.

The volume contains the following papers. (1) Logic, Norms, and Roles (Alan R. Anderson); (2) Choice Behavior and Monetary Payoff: Strong and Weak Conditioning (Richard C. Atkinson); (3) Can Subjects be Human and Humans be Subjects? (Kurt W. Back); (4) Two-Person Interactive Learning: A Progress Report (C. J. Burke); (5) The Application of Learning Models to Interactive Behavior (Robert R. Bush); (6) Application of a Markov Learning Model to a Simple Detection Situation Involving Social Pressure (E. C. Carterette and M. J. Wyman); (7) The Process of Choosing a Reference Group (Bernard P. Cohen); (8) Reward Structures and the Allocation of Effort (James S. Coleman); (9) Theoretical Treatments of Differential Reward in Multiple-Choice Learning and Two-Person Interactions (W. K. Estes); (10) Le Théorème de Bayes et les Processus d'Influence Sociale (Claude Flament); (11) The Structure of Interpersonal Behavior in the Dyad (Uriel G. Foa); (12) Two-Alternative Learning in Interdependent Dyads (Robert L. Hall); (13) Some Aspects of Power in Small Groups (Georg Karlsson); (14) Experimental Studies of Conflict in some Two-Person and Three-Person Games (Bernhardt Lieberman); (15) Group and Individual Behavior in Free-Recall Verbal Learning (Irving Lorge and Herbert Solomon); (16) Some Puzzling Aspects of Social Interaction (Omar K. Moore and Alan R. Anderson); (17) Speed and Accuracy of Cognitive Achievement in Small Groups (Frank Restle); (18) Subtask Phasing in Small Groups (Thornton B. Roby); (19) Two-Person Interactions in a Continuous-Response Task (Seymour Rosenberg); (20) A Topological Approach to the Measurement of Social Phenomena (Maynard W. Shelly); (21) On the Reliability of Group Judgments and Decisions (William H. Smoke and Robert B. Zajonc); (22) Analysis of Social Conformity in Terms of Generalized Conditioning Models (Patrick Suppes and Madeleine Schlag-Rey).

INFORMATION, COMMUNICATION, CONTROL

See also 3375–3379, 3405, 3406, 3410, 3591, 3778, 4096, 4123, 4171, 4227, 4595.

Colonnese, Giulio 4612

Sulla decomposizione di segnali informativi statistici in serie di sequenze periodiche.

Ricerca (Napoli) (2) **13** (1962), *settembre–dicembre*, 26–41.

Winograd, S.; Cowan, J. D. 4613

★Reliable computation in the presence of noise.

The M.I.T. Press, Cambridge, Mass., 1963. xiv + 96 *pp.* $5.00.

Ever since von Neumann's fundamental study of the problem of the construction of reliable automata from unreliable components [von Neumann, *Automata studies*, pp. 43–98, Princeton Univ. Press, Princeton, N.J., 1956; MR **17**, 1040], a way to define computational capacity and to generalize Shannon's fundamental theorem for information transmission in the presence of noise to information processing in the presence of noise has been sought. A solution is presented in this monograph. It is a natural generalization that reduces to Shannon's result in the trivial case of transmission with no processing, and to von Neumann's system in the case in which the processing is limited to a single automaton. The method is based on the use of an error-correcting code with k information symbols applied to k copies of the automation which are not necessarily processing the same information. The probability of error can be made to approach zero by increasing the length of the code in the redundant automaton. The percentage of redundant logic modules is

essentially the percentage redundancy of the code. However, the complexity of the modules increases as the code length increases. The result appears, therefore, to be of considerable theoretical interest but presently not of practical value. *W. W. Peterson* (Honolulu, Hawaii)

Ambarcumjan, R. V. 4614

Signal detection in a pulse flow. (Russian. Armenian summary)

Akad. Nauk Armjan. SSR Dokl. **38** (1964), 71–76.

For the signal detection in a pulse flow one considers the stationary flow $\Pi = S + N$, where S is the signal flow and N the noise flow, when the moments $t_1, t_2, \cdots, t_n \in (0, t)$ at which the pulses of Π appear are known; using the statistical properties of S and N we can obtain the distribution of the pulses of Π in S and N.

P. Constantinescu (Bucharest)

Jelinek, Frederick 4615

Coding for and decomposition of two-way channels.

IEEE Trans. Information Theory **IT-10** (1964), 5–17.

This paper summarizes a study of two-way discrete memoryless channels. The regions of possible rates of forward and reverse transmission are discussed and investigated. Also, it is found that under rather general conditions a two-way discrete memoryless channel is equivalent to a pair of oppositely oriented interconnected memoryless one-way channels connected in cascade to special channels that are noiseless whenever the signal transmitted in the opposite direction is an appropriate one. In particular, if the transition probabilities are symmetrical, the nature of the decomposition suggests that maximum forward transmission rate for a fixed reverse rate occurs with sources of independent symbols.

W. W. Peterson (Honolulu, Hawaii)

Arimoto, Suguru 4616

On a non-binary error-correcting code.

Information Processing in Japan **2** (1962), 22–23.

Let p be a prime number, $p > n$, and let T be the $n \times 2e$ matrix with ith row $(1, i, i^2, \cdots, i^{2e-1})$, $i = 1, \cdots, n$, with entries from GF(p). The author's code is the null space of T. It is an $(n, n-2e)$ code over GF(p) with minimum distance $d = 2e + 1$. Thus it is optimal with respect to the bound $d \leq n - k' + 1$ for linear (n, k') codes.

To decode, let X be sent and Y received. The error-vector is then $Y - X = A$. Let $C = (c_0, c_1, \cdots, c_{2e-1})$ denote $AT = YT$. By looking at the $i \times i$ matrix $C^{(i)}$ whose jth row is $c_j, c_{j+1}, \cdots, c_{j+i-1}$ $(j = 0, 1, \cdots, i-1)$, the author can prove that if the Hamming weight k of A is at most e, then $\det C^{(i)} \neq 0$ for $i \leq k$ and $\det C^{(i)} = 0$ for $i > k$. He can then determine the values $n_1, \cdots, n_k$ (considered in GF(p)) such that $A = (a_0, \cdots, a_{n-1})$ has $a_{n_i} \neq 0$ as the roots of a polynomial over GF(p) of degree k which is closely related to $C^{(k)}$. At this point he can solve k linear equations, say $\sum a_{n_i} n_i^{\,j} = c_j$, $0 \leq j < k$, arising from $AT = C$, for the k unknown coordinates of A. The author concludes with: "The error-correcting method is systematic and easily programmed for a digital computer and requires a small storage capacity."

Misprint: For $C^{(i)}$, use the definition in this review.

This paper is stated to have first appeared in Japanese in the Journal of the Information Processing Society of Japan **2** (1961), no. 6, 320–325. The present short version contains no proofs, but in a letter solicited by the reviewer the author gives nice proofs of his results. This review contains the substance of the paper.

Whether or not the decoding procedure is as simple as the author claims, the codes he has constructed are of theoretical interest and might well repay further study.

H. F. Mattson (Waltham, Mass.)

Neumann, Peter G. 4617

On a class of cyclically permutable error-correcting codes.

IEEE Trans. Information Theory **IT-10** (1964), 75–78.

Author's summary: "Cyclically permutable codes have error-correcting properties which are invariant under arbitrary cyclic permutation of any of their code words. This paper summarizes the results of an empirical investigation of certain of these codes, which have parameters not covered by a previous paper of E. N. Gilbert [same Trans. **9** (1963), 175–182]. These codes are thought to be nearly optimal. Estimates of the obtainable number of code words are given. The codes may be suitable for use in certain asynchronous multiplex communication systems." *W. W. Peterson* (Honolulu, Hawaii)

Elias, Peter 4618

Information theory and decoding computations.

Proc. Sympos. Appl. Math., Vol. XV, pp. 51–58. *Amer. Math. Soc., Providence, R.I.*, 1963.

Expository paper.

A. V. Balakrishnan (Los Angeles, Calif.)

Kasami, Tadao 4619

Optimum shortened cyclic codes for burst-error correction.

IEEE Trans. Information Theory **IT-9** (1963), 105–109.

Author's summary: "This paper is concerned with the construction of the most efficient shortened cyclic (pseudo-cyclic) codes that can correct every burst-error of length b or less. These codes have the maximum number of information digits k among all shortened cyclic burst-b codes with a given number of check digits r. The search procedure described is readily programmable for computer execution and is efficient particularly for the case where r is close to the theoretical minimum of $2b$ check digits. For $2 \leq b \leq 10$, several optimum shortened cyclic codes in the above-mentioned sense have been found. Their code-lengths and generators are tabulated in this paper."

W. W. Peterson (Honolulu, Hawaii)

Metzner, John J. 4620

Burst-error correction for randomly-chosen binary group codes.

IEEE Trans. Information Theory **IT-9** (1963), 281–285.

A lower bound is found for the fraction of codes of a given length that are capable of correcting all possible burst errors up to a given length. In the cases where this fraction is not too small, a burst-error-correcting code could be found with high probability by randomly choosing several

codes and examining them. A decoding procedure that could be used with such a code is described.

W. W. Peterson (Honolulu, Hawaii)

Leont'ev, V. K. 4621

On a problem of close-packed codes. (Russian)

Diskret. Analiz. *No.* 2 (1964), 56–58.

In the paper by H. S. Shapiro and D. L. Slotnick [IBM J. Res. Develop. **3** (1959), 25–34; MR **20** #5092] it has been shown that if a close-packed $(n, 2l+1)$-code exists and l is of the form $4r+1$, then $(n+1)$ is a divisor of $l!$.

In the note under review a result is obtained for the more general case of an odd l, which states that in this case $(n+1) < l!B \ln l$, where B is the largest odd divisor of $l!$.

Also it is shown that for values of n greater than some number x_0 (whose existence depends upon a particular factorization of the polynomial $l!(1+C_x^1+\cdots+C_x^l)=\varphi(x)$ and whose definition, given in terms of the divisors of and the coefficients of polynomials arising from this specific factorization, is too involved to be quoted in a short review) no close-packed $(n, 2l+1)$-code exists.

No numerical examples are given.

S. Kotz (Toronto, Ont.)

Varšamov, R. R. 4622

On some singularities of linear codes which correct non-symmetric errors. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 546–548.

One considers the non-symmetric channels for which the probability of the passage $1 \to 0$ is essentially less than for $0 \to 1$ or conversely. If we have the codes $x=(x_1, \cdots, x_n)$ of length n we propose to find M codes stable with respect with r non-symmetric errors. We obtain the necessary and sufficient condition for this and interesting evaluations.

P. Constantinescu (Bucharest)

Wolf, J. K.; Elspas, B. 4623

Error-locating codes—a new concept in error control.

IEEE Trans. Information Theory **IT-9** (1963), 113–117.

Authors' summary: "A new coding technique is proposed lying midway between error-detection and error-correction coding. The block of received digits is regarded as subdivided into mutually exclusive sub-blocks. Errors occurring within particular sub-blocks are detected at the receiver and, in addition, the receiver is able to determine, by using the code redundancy, which particular sub-blocks contain errors. Such error-locating codes permit the location of digit errors to within a sub-block of the received message block without, in general, permitting the precise determination of erroneous digit positions. Two families of such codes are described, both of them limited to locating a single erroneous sub-block. One family permits the detection of up to $t-1$ errors in any one sub-block of length t. The second family locates up to two errors per sub-block, but is generally more efficient in its use of redundancy than the first family. Upper and lower bounds are given for the number of check digits required with any error-locating code. Codes meeting the lower bound exactly are termed optimum error-locating codes. All the codes of the second family, as well as some isolated examples of $t-1$ error-locating codes are optimum in this sense. The amount of redundancy required for such codes does not appear excessive and error location may provide an attractive alternative to conventional error detection in decision feedback communications."

W. W. Peterson (Honolulu, Hawaii)

Čulík, K. 4624

On equivalent and similar grammars of ALGOL-like languages. (Preliminary communication)

Comment. Math. Univ. Carolinae **5** (1964), 93–95.

This brief note considers ways to produce a new grammar from a context-free grammar $G=(T, N, R, S)$ by two operations called extension and reduction, which are defined in terms of certain conditions imposed on the symbols of T and N and the rules of R. The author calls two grammars strongly similar or weakly similar if they have extensions or reductions, respectively, which are equivalent, i.e., if they can be mapped homomorphically onto the same grammar. An example is given to illustrate the theorem that strong similarity implies weak similarity, but not conversely.

H. P. Edmundson (Pacific Palisades, Calif.)

Grignetti, Mario C. 4625

A note on the entropy of words in printed English.

Information and Control 7 (1964), 304–306.

Author's summary: "Shannon estimates the entropy of the set of words in printed English as 11.82 bits per word. As this figure seems inconsistent with some results deduced from several encoding procedures, the entropy was recalculated and found to be roughly 9.8 bits per word."

Gross, Maurice 4626

Inherent ambiguity of minimal linear grammars.

Information and Control 7 (1964), 366–368.

Author's summary: "We give an example of minimal linear language, all of whose minimal linear grammars are ambiguous; this language is not ambiguous in the class of linear context-free languages."

Haines, L. H. 4627

Note on the complement of a (minimal) linear language.

Information and Control 7 (1964), 307–314.

Author's summary: "We exhibit a 'pathological' minimal linear language and prove that its complement is not context-free. This settles two open problems posed by Chomsky."

Borisenok, I. T. 4628

A control system with doubling control devices in the presence of dry friction. (Russian. English summary)

Vestnik Moskov. Univ. Ser. I Mat. Meh. **1964**, *no.* **3**, 75–86.

Der Verfasser betrachtet ein Regelungssystem mit zwei Stellgliedern, in denen trockene Reibung vorhanden sei; diese werden eingeführt, um in der Arbeitsweise des Systems eine größere Zuverlässigkeit zu gewährleisten. Die zugrundeliegenden Differentialgleichungen lauten:

$$T D^2\varphi + q\rho + r\delta = \chi(t),$$

$$(VD^2+\tau D+n_1)\rho - s(\sigma D+1)\varphi = -k\,\mathrm{sgn}(D\rho),$$

$$(VD^2+\tau D+n_2)\delta - s(\sigma D+1)\varphi = -k\,\mathrm{sgn}(D\delta);$$

hierbei ist φ die Koordinate der Regelstrecke, $\chi(t)$ die Fremderregung, φ und δ sind die Koordinaten der Stellglieder; die Koeffizienten werden als positiv vorausgesetzt. Zuerst untersucht Verfasser das zugehörige lineare System ($k=0$), vor allem in einigen Sonderfällen, wo periodische Regime auftreten, und zieht sodann die nichtlinearen Glieder in Betracht, indem er sich auf Näherungsrechnungen stützt. Diese sollen den Einfluß der Nichtlinearität auf die möglichen freien Schwingungen sowie die Abhängigkeit der Stabilitätseigenschaft von verschiedenen Systemparametern klären. Einige Beispiele werden angeführt. *R. Reissig* (Berlin)

Hamza, M. H. 4629

Synthesis of control systems seeking extrema. (Russian. English summary)

Avtomat. i Telemeh. **25** (1964), 1156–1161.

Author's summary: "A method is presented for designing high performance extremum-seeking regulators. The method is based on injecting test signals—or alternately reversing the polarity of the input—and measuring the resultant discontinuities in an appropriate derivative of the system's output. Examples are given, and also a sample of analog computer results."

Imaz, C.; Vorel, Z. 4630

On domains of controllability of proper and normal systems.

Bol. Soc. Mat. Mexicana (2) 8 (1963), 79–88.

Consider the linear control system (1) $\dot{x}(t)=Ax(t)+Bu(t)$, $x(0)=0$, A, B constant matrices ($n\times n$ and $n\times s$, respectively). Define Ω as the set of measurable control functions u with components bounded in absolute value by one; $S_T\subset E^n$ as the set of points attainable at time T by solutions of (1) with $u\in\Omega$; $\mathscr{A}_T=e^{-AT}S_T$. For (1) normal it is known [LaSalle, *Contributions to the theory of nonlinear oscillations*, Vol. V, pp. 1–24, Princeton Univ. Press, Princeton, N.J., 1960; MR **26** #2704] that if η belongs to the unit $(n-1)$-sphere, $u(t,\eta)\equiv\operatorname{sgn}[\eta' e^{-At}B]$, then $x_\eta=\int_0^T e^{-At}Bu(t,\eta)\,dt$ belongs to the frontier of $\mathscr{A}_T(\operatorname{Fr}\mathscr{A}_T)$. The authors show that this map $\eta\to x_\eta$ is continuous. This is then used to obtain an approximation to $\operatorname{Fr}\mathscr{A}_T$ by considering a polyhedron P_k generated by $x_{\eta_1},\cdots,x_{\eta_k}$. It is shown that if $\{\eta_k\}$ is dense on the unit sphere, then the closure of $\bigcup_{k=1}^\infty P_k$ is $\mathscr{A}_T$.

The main theorem then characterizes S_T as being essentially a linear map of a set $\mathscr{A}_\infty$ which is independent of T. Specifically, with appropriate conditions on (1), the main theorem states that there exist convex compact sets $\tilde{\mathscr{A}}_T$, $\mathscr{A}_\infty$ symmetrical with respect to the origin and containing a neighborhood of the origin, with $\mathscr{A}_\infty$ independent of T and the Hausdorff distance between $\tilde{\mathscr{A}}_T$, $\mathscr{A}_\infty\to 0$ as $T\to\infty$, such that $x\in\operatorname{Fr}S_T$ if and only if $x=e^{DT}D_T\omega$, $\omega\in\operatorname{Fr}\tilde{\mathscr{A}}_T$, where D is the Jordan matrix similar to A and D_T is a diagonal matrix whose elements are natural powers of T. It is also shown that $\mathscr{A}_\infty$ depends on at most several rows of B.

H. Hermes (Providence, R.I.)

Pavlov, V. V. 4631

On the multi-invariance of combined non-linear automatic systems. (Ukrainian)

Avtomatika **1964**, *no.* 4, 87–88.

A set of non-linear difference equations describing automatic control systems is studied. Necessary conditions are given for the simultaneous invariance of certain quantities in such systems.

H. P. Thielman (Alexandria, Va.)

Gabasov, R.; Kirillova, F. M. 4632

The solution of certain problems in the theory of optimal processes. (Russian. English summary)

Avtomat. i Telemeh. **25** (1964), 1058–1066.

Consider the differential equation

$$(1)\qquad dx/dt=A(t)x+B(t)u(t),$$

where $A(t)$, $B(t)$ are real n-by-n and n-by-m matrices, respectively, $t\in[0,T]$, and $u(t)$ belongs to some linear normed space U of m-vectors (controls). Given $\rho>0$ and two compact, convex sets C_0, C_1 of the euclidean n-space E^n, it is required to find a $u\in U$, and correspondingly a solution $x(t;u)$ of (1) such that (2) $|u|_U\leq\rho$, $x(0;u)\in C_0$, $x(T;u)\in C_1$. Describing C_0 and C_1 in terms of their distance functions, it is also required to find min diam C_0, or min diam C_1, such that (2) hold. The problem is dealt with by using an inequality involving the dual space $(E^n)^*$; this is an extension of similar inequalities by W. T. Reid [Duke Math. J. **29** (1962), 591–606; MR **27** #424] when C_0 and C_1 are single-point sets, and by H. A. Antosiewicz [Arch. Rational Mech. Anal. **12** (1963), 313–324; MR **26** #4017] when C_0 is a single-point set and C_1 is a sphere. The inequality is obtained by Antosiewicz's approach, i.e., by means of the strict separation theorem for convex sets in E^n. (However, the assumptions on the nature of the space U should be made more precise in order to conclude that the set $\alpha(L)$ on page 1064 is closed. The same remark applies to the set $\alpha(L,\Delta)$ on page 1065.) Some variants, as well as a problem of tracking, are also dealt with along the same lines. *R. Conti* (Florence)

Kuržanskiĭ, A. B. 4633

Construction of an optimal control by the method of moments to minimize the mean-square error. (Russian. English summary)

Avtomat. i Telemeh. **25** (1964), 624–630.

Author's summary: "An optimum control problem is considered for a linear system with quadratic functional minimization at the finite time interval under the condition of transient damping."

Romiti, Ario 4634

Un metodo per la soluzione dei problemi di ottimo nei processi controllati con tempi morti.

Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **98** (1963/64), 279–298.

Among the generalizations of optimal control problems solved by Pontryagin's maximum principle, the following, which seems to be important in some applications, is solved here. Let (1) $\dot{x}=f(x,u,v)$ be the differential equation with x, u, v vectors of dimensions n, r and s, respectively. u is subjected to taking values in a compact set $U\subset R^r$. The components v_i of v are assumed to take only the values $\pm V_i$, except when they change from $+V_i$ to $-V_i$ or vice versa, in which case $v_i=0$ during a

"dead time" of length Δt (condition (*)). In order to take condition (*) into account, the following expanded system is posed: (2) $\dot{x}_i = f_i(x, u, v)$ $(i = 1, 2, \cdots, n)$; and $\dot{y}_i = w_i$, $(y_i^2 - V_i^2)(w_i^2 - W_i^2) = 0$, $(y_i - v_i)(w_i^2 - W_i^2) = 0$, $v_i w_i = 0$, $i = 1, 2, \cdots, s$. Here the y_i are auxiliary state variables such that $y_i = v_i = \pm V_i$ except for the "dead time" Δt, when $v_i = 0$. The constants $W_i = 2V_i/t$ assure that the duration of this commuting time from $+V_i$ to $-V_i$ (or vice versa) is Δt.

The last three equations are algebraic constraints and the standard method of Lagrange's multipliers is applied, together with Pontryagin's principle. The obtained result is the following: defining ψ, H by $\dot{\psi} = -\psi \cdot f_x$, $H = \psi \cdot f$, the optimal solution satisfies (3) $M(\psi, x) = \sup H(\psi, x, u, v) =$ const, $u \in U$, $v_i = \pm V_i$ $(i = 1, 2, \cdots, s)$, $M(t) = H(t)$ or (in the dead time intervals) (3′) $M'(\psi, x) = \sup H(\psi, x, u, v) =$ const $\leqq M$, $u \in U$, $v_i = \pm V_i$ $(i \neq k)$, $v_k = 0$, $M'(t) = H(t)$.

The switching intervals $(t_k, t_k + \Delta t)$ satisfy the relation

$$\psi(t_k)[f(t_k)_- - f(t_k)_+] = -\psi(t_k + \Delta t)[f(t_k + \Delta t)_- - f(t_k + \Delta t)_+],$$

where the subindices + and − refer to the limit values to the right and left, respectively.

The more general case, when $v \in V$ but the components v_i have a dead time Δt with $v_i = 0$ when changing sign, is also treated like the preceding case. Finally, some applications, especially to the linear case, are given.

In the reviewer's opinion, this result is of importance in many applications, where the dead time may be treated as a small perturbation, i.e., it does not change qualitatively the existence and behaviour of the optimal solution. But mathematically there is some ambiguity in the definition of allowed solutions. In the non-linear case, indeed, the value $v_i = 0$ may give a better performance than either $v_i = \pm V_i$, and this during a time longer than Δt; in this case equation (1) with condition (*) does not apply exactly. System (2), on the other hand, is mathematically satisfying, but (3) and (3′) give merely a necessary condition for optimality, without defining the solution progressively (as Pontryagin's principle does in the now classical case), because H is not maximum in the dead time intervals. *E. O. Roxin* (Baltimore, Md.)

Romiti, Ario 4635

Problemi di ottimo nei processi a più stadi e con parametri di controllo; applicazione ai missili pluristadio.

Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. 98 (1963/64), 445–464.

This is a discussion of necessary conditions, with reference to the approaches of both Bellman and Pontryagin, for an extremum of $\sum_1^n c_i x_i(T)$, T the terminal time, subject to side-conditions $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{w}, t)$, in which $\mathbf{w}$ is a parameter, and to various end-conditions. It is shown, as an example, that a necessary condition on thrust altitude $u(t)$, for an idealized planar trajectory of maximal horizontal range for a two-stage missile with fixed mass-function $m(t)$, is $u(t) =$ const. This condition, in an almost everywhere sense, is actually necessary under more general hypotheses and, with the constant properly chosen, such a u furnishes a global maximum for the range problem [the reviewer, Notices Amer. Math. Soc. 9 (1962), 404].

G. M. Ewing (Norman, Okla.)

Romiti, Ario 4636

Una nuova estensione dell'applicazione del principio di massimo nei problemi di ottimo controllo.

Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. 98 (1963/64), 698–706.

The paper is concerned with necessary conditions for a problem $\mathbf{c} \cdot \mathbf{x}(T) =$ extremum, where T is the terminal time, having side-conditions that involve possible discontinuities and intermediate values of state vectors. The problem is similar to that of C. H. Denbow [*Contributions to the calculus of variations 1933–37*, pp. 449–484, Univ. Chicago Press, Chicago, Ill., 1937].

G. M. Ewing (Norman, Okla.)

Pšeničnyi, B. N. 4637

A numerical method of solving certain optimal control problems. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. 4 (1964), 292–305.

In the first section of the paper the author constructs an algorithm for solving optimal control problems for objects whose phase coordinates are explicitly expressed by the control. The method is then compared with that described in the author's earlier paper [same Ž. 4 (1964), 52–60; MR 29 #1101]. In the second part some new necessary conditions of optimality are given for linear control systems with phase conditions. *M. Kuczma* (Katowice)

★**Computing methods in optimization problems.** 4638

Proceedings of a conference held at University of California, Los Angeles, January 30–31, 1964. Edited by A. V. Balakrishnan and Lucien W. Neustadt.

Academic Press, New York-London, 1964. x + 327 pp. $7.50.

The papers in this volume will be reviewed individually.

Amer, R. A.-R.; Schwarz, H. R. 4639

Contributions to the approximation problem of electrical filters.

Mitt. Inst. Angew. Math. Zürich No. 9 (1964), 99 pp.

Two papers treating the construction of transfer functions for filters with multiple pass bands. The filters are effectively required to have equal maxima of loss in all the pass bands and equal minima of loss in all the stop bands. The problem is solved in terms of the so-called characteristic function of filter theory, an odd or even rational function of frequency.

The part by Amer considers the general problem of a filter with any number of pass bands and stop bands and at arbitrary locations. A special method for the restricted case of one pass band and two stop bands is developed as an eigenvalue problem and is solved by a simple iteration. The more general method involves finding, by linear-programming techniques, the local optimal solutions for all possible pole-zero distributions among the bands; the best local solution is then selected by inspection.

The part by Schwartz deals only with the one-pass-band, two-stop-band case (the general asymmetrical band-pass filter) and follows more closely the classical pattern using hyperelliptic integrals.

H. J. Orchard (San Carlos, Calif.)

Beneš, V. E. 4640
Optimal rearrangeable multistage connecting networks.
Bell System Tech. J. **43** (1964), 1641–1656.

This paper, which continues work by the same author [same J. **43** (1964), 1619–1640; MR **29** #6985] and extends work by C. E. Shannon [ibid. **29** (1950), 343–349; MR **12**, 35], makes good use of elementary number theory (specifically unique factorization into rational primes) in order to design certain classes of rearrangeable connecting networks having a minimum number of cross-points (e.g., relay contacts).

Recall that Shannon's scheme minimizes the number of memory elements used, but, in general, it requires considerable spring load on certain relays. Contrariwise, the author's scheme minimizes the number of cross-points without regard to the number of relays used. The author's principal theorem, which deals with the cost of design, makes crucial use of a number-theoretic function D defined as follows: $D(n)$ is the sum of the prime divisors of n counting repetitions according to their multiplicity.

A. A. Mullin (Livermore, Calif.)

Buharaev, R. G. 4641
On calculating the reliability of switching circuits. (Russian. English summary)
Avtomat. i Telemeh. **25** (1964), 1210–1215.

Author's summary: "Formulae for the calculation of reliability characteristics of arbitrary switching circuits are obtained, taking into accout the dynamics of control relays operation and large assumptions for the contact reliability."

Hammond, J. L., Jr.; Johnson, R. S. 4642
A review of orthogonal square-wave functions and their application to linear networks.
J. Franklin Inst. **273** (1962), 211–225.

Authors' summary: "Attention is called to the Haar and Walsh functions as being potentially useful in engineering applications. The Walsh functions have a single magnitude with either a positive or negative sign in the range of definition. Each member of the Haar family also has a single magnitude although this magnitude is different for different members of the set. This interesting property makes the Haar and Walsh functions easy to generate as physical signals and enables analog multiplication by these functions to be performed by an appropriate sequence of sign changes.

"In this paper the well-known properties of the Haar and Walsh functions are summarized. Equations are derived which give the output of linear electric networks in terms of Haar and Walsh functions. Expressions relating the Haar and Walsh 'spectra' to the complex Fourier spectrum are presented, and practical applications of the results are suggested. An example illustrating the properties of the Haar and Walsh functions is worked out in detail."

Barzdin', Ja. M. 4643
Universal pulsating elements. (Russian)
Dokl. Akad. Nauk SSSR **157** (1964), 291–294.

The concept of a logical net is here extended to that of a network of pulsating elements. In such a network the connections between elements may change in the course of time and the behavior of each element at time $t+1$ depends on the states of its neighbors at time t. For these circuits, the author solves problems related to the problem of completeness of basis for ordinary logical nets. In particular, the existence of a universal pulsating element is announced. Using blocks built of copies of this element, one can construct a network whose behavior in T units of time simulates that of a given network of pulsating elements in one unit of time.

G. N. Raney (Storrs, Conn.)

Barzdin', Ja. M. 4644
Problems of universality in the theory of growing automata. (Russian)
Dokl. Akad. Nauk SSSR **157** (1964), 542–545.

In the preceding paper [#4643] the author has extended the concept of a logical net to that of a network of pulsating elements (P.E.), where not only the state of an element but also its connections with its neighbors at time $t+1$ are dependent upon the states of its neighbors at time t. In the present article the author goes a step further, considering networks of reproducing elements (R.E.) which, besides being able to change their states and connections, are able to reproduce copies of themselves and to form connections with these copies. The definitions of simulation and universality developed in the paper on networks of P.E. are here extended to networks of R.E. The author announces the existence of an R.E., A^0, which is universal for the class of all R.E. Roughly speaking, this means that for every R.E. A, there is a block B, a network built of copies of A^0, whose inputs, outputs and states correspond in a definite manner with those of A, and a natural number T such that for every network L built of copies of A, there is a network built of copies of B whose changes of state in T units of time correspond to the changes of state which L undergoes in one unit of time. Also considered are generalized reproducing elements (G.R.E.), whose behavior depends upon certain global properties of the net as well as on the states of their neighbors. The existence of a G.R.E. which is universal for the class of all G.R.E. is announced. By declaring some of the elements of a net to be input elements and others to be output elements, one may regard a network as performing a sequence-to-sequence transformation. It is stated that the class of such transformations realized by nets of R.E. coincides with the class of transformations realized by nets of G.R.E.

G. N. Raney (Storrs, Conn.)

Krylov, N. V. 4645
Representation of coded regular events in abstract finite automata. (Russian)
Ukrain. Mat. Ž. **16** (1964), 385–389.

Es bezeichne F die freie Halbgruppe mit dem Erzeugendensystem $X=\{x_1, \cdots, x_n\}$ und G die freie Halbgruppe mit dem Erzeugendensystem $Y=\{y_1, \cdots, y_m\}$. Unter einer unitären Kodierung von F in G wird ein Homomorphismus φ von F in G verstanden, bei dem für kein $i\neq j$ das Wort $\varphi(x_j)$ ein Anfang von $\varphi(x_i)$ ist $(i,j=1, \cdots, n)$. Gegeben sei ein System $A_1, \cdots, A_k$ von regulären Ereignissen in F und es sei $\mathfrak{A}$ ein endlicher initialer

Automat mit dem Eingabealphabet X, in dem die Ereignisse $A_1, \cdots, A_k$ durch gewisse Teilmengen $S_1, \cdots, S_k$ der Zustandsmenge dargestellt werden. Es sei ferner $\mathfrak{B}$ ein endlicher initialer Automat mit dem Eingabealphabet Y, in dem die (einelementigen) Ereignisse $\{\varphi(x_1)\}, \cdots, \{\varphi(x_n)\}$ $(\subseteq G)$ durch gewisse Zustände $\alpha_1, \cdots, \alpha_k$ dargestellt werden. Es wird ein endlicher Automat $\mathfrak{A}(\mathfrak{B})$ mit dem Eingabealphabet Y (Superposition von $\mathfrak{A}$ und $\mathfrak{B}$) konstruiert, so daß in $\mathfrak{A}(\mathfrak{B})$ die Ereignisse $\varphi A_1, \cdots, \varphi A_k$ $(\subseteq G)$ durch bestimmte Teilmengen der Zustandsmenge von $\mathfrak{A}(\mathfrak{B})$ dargestellt werden können. Die Konstruktion läßt sich auf den Fall verallgemeinern, daß statt $\{\varphi(x_1)\}, \cdots, \{\varphi(x_n)\}$ beliebige reguläre Ereignisse in G genommen werden, sofern diese in $\mathfrak{B}$ durch paarweise disjunkte Zustandsmengen dargestellt werden (in $\mathfrak{A}(\mathfrak{B})$ wird dann die Superposition von regulären Ereignissen dargestellt (vgl. V. G. Bodnarčuk [dasselbe Ž. **14** (1962), 351–361; MR **26** #3557])). *G. Asser* (Greifswald)

Oehmke, Robert H. 4646

On the structures of an automaton and its input semi-group.

J. Assoc. Comput. Mach. **10** (1963), 521–525.

Let S be a semi-group (input signals). A right-congruence on S is an equivalence such that $x \sim y$ implies $xu \sim yu$. An S-automaton is defined to be a system $(A, /)$, A a set (states), $/$ a function from $A \times S$ into A such that $a/(xy) = (a/x)/y$ for $a \in A$ and $x, y \in S$ (transition operator). There is a natural relationship between S-automata and right-congruences on S. The paper contains some rather trivial remarks concerning these matters.

J. R. Büchi (Columbus, Ohio)

Ritchie, Robert W. 4647

Finite automata and the set of squares.

J. Assoc. Comput. Mach. **10** (1963), 528–531.

It is shown that the set of squares (in binary notation) is not the behavior of any finite automaton. This has previously been found as a biproduct of results on logical systems. The present proof proceeds directly from the definition of finite automata, and hinges on the following lemma: If $y^2 = (2^n - 1)2^l + 1$ with y, n, l positive and $n + l$ even, then $l = n + 2$ and $y = 2^{n+1} - 1$. It is noted that closely related results on Diophantine equations require much more powerful methods.

J. R. Büchi (Columbus, Ohio)

AUTHOR INDEX

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GENERAL

Lusin, N. N. [Лузин, Н. Н.] 4648

★Collected works. Vol. III: Papers on various problems in mathematics [Собрание сочинений. Том III: Работы по различным вопросам математики].

Izdat. Akad. Nauk SSSR, Moscow, 1959. 507 *pp.* 2.53 *r.*

Volume I [1953; MR 15, 591] and Volume II [1958; MR 27 #3487] have already been reviewed. The papers of the present volume are divided into five sections as follows: Partial differential equations and differential geometry (I); Differential equations (II); Miscellaneous papers on analysis (III); History of mathematics (IV); Associated papers (some written by other mathematicians) (V).

★Proceedings of the Fourth All-Union Mathematical Congress. Vol. II: Sectional Lectures [Труды Четвертого Всесоюзного Математического Съезда. Том II: Секционные Доклады]. 4649

Leningrad, 3–12 July 1961.

Izdat. "Nauka", Leningrad, 1964. 706 *pp.* 4.70 *r.*

The first volume [Izdat. Akad. Nauk SSSR, Leningrad, 1963] was reviewed earlier [MR 27 #5666]. The articles in this volume are grouped into thirteen areas of mathematics; some of them are merely abstracts of results, while others are extended lectures. Papers in the latter category containing original results will be reviewed individually.

★Reports of the Third Siberian Conference on Mathematics and Mechanics [Доклады Третьей Сибирской Конференции по Математике и Механике]. 4650

8–13 September 1964.

Izdat. Tomsk. Univ., Tomsk, 1964. 407 *pp.* 2.53 *r.*

This volume contains the reports given at the conference described in the heading; these reports consist almost entirely of abstracts of papers which are to be published elsewhere. The conference was divided into sections on function theory, differential equations, geometry, algebra, numerical analysis and control theory, theoretical mechanics, and the teaching of mathematics.

★Differentialgeometrie und Topologie. 4651

Internationales Kolloquium, Zürich, 1960. Monographies de L'Enseignement Mathématique, No. 11.

L'Enseignement Mathématique, Université, Genève, 1962. 159 *pp.* *Frs.* 22.00.

This volume, in addition to a memorial tribute to J. H. C. Whitehead [P. J. Hilton, Enseignement Math. (2) 7 (1961), 107–124; MR 25 #2943], contains papers (which have already been reviewed) by R. Bott [ibid. (2) 7 (1961), 125–138; MR 25 #2615], H. Busemann [ibid. (2) 7 (1961), 139–152; MR 25 #3421], S. S. Chern [ibid. (2) 7 (1961), 179–187; MR 25 #229], B. Eckmann [ibid. (2) 8 (1962), 209–217; MR 27 #1958], M. F. Atiyah and F. Hirzebruch [ibid. (2) 7 (1961), 188–213; MR 27 #4243], A. Lichnerowicz [ibid. (2) 8 (1962), 1–15; MR 26 #717], J. Milnor [ibid. (2) 8 (1962), 16–23; MR 27 #2989], N. E. Steenrod [ibid. (2) 7 (1961), 153–178; MR 28 #3422], and R. Thom [ibid. (2) 8 (1962), 24–33; MR 26 #5588].

★Séminaire de mécanique analytique et de mécanique céleste, dirigé par Maurice Janet, 5ᵉ année: 1961/62. 4652

Secrétariat mathématique, Paris, 1963. iii + 164 *pp.*

Table of Contents: Exp. 1, Léon Brillouin, Discussion du rôle effectif des discontinuités de Poincaré; Exp. 2, Pierre Pigeaud, Sur les équations du mouvement en théorie de Jordan-Thiry; Exp. 3, Eliane Blancheton, Les équations aux variations de la relativité générale; Exp. 4, Joseph Klein, Notion de tenseur-force en mécanique classique et relativiste; Exp. 5, Pierre Pigeaud, Le schéma fluide parfait généralisé en théorie de Jordan-Thiry; Exp. 6, Jules Geheniau, Sur les complexes d'impulsion-énergie; Exp. 7, Eliane Blancheton, Problèmes de stabilité en relativité générale; Exp. 8–9, F. A. E. Pirani and Gareth Williams, Rigid motion in a gravitational field; Exp. 10, Marie-Antoinette Tonnelat, Théorie euclidienne de la gravitation et vérifications expérimentales; Exp. 11, Cécile Dewitt-Morette, Fonctions de Green dans un espace de Riemann; Exp. 12, Cécile Dewitt-Morette, Quantification des champs classiques admettant une invariance de groupe à un nombre infini de dimensions, d'après les travaux de H. Van Dam; Exp. 13, Jean Kovalevsky, Aspects analytiques du problème du mouvement d'un satellite artificiel; Exp. 14, Jean-Claude Blaive, Le dernier théorème de Poincaré et le problème restreint des trois corps (Mécanique newtonienne classique); Exp. 15, Fernand Nahon, Le théorème de Jeans et la rotation des amas globulaires; Exp. 16, André Avez, Modèle d'univers stationnaire sans section d'espace globale.

★Structures feuilletées. 4653

Grenoble, 25–30 Juillet 1963. Colloques Internationaux du Centre National de la Recherche Scientifique, No. 125.

Éditions du Centre National de la Recherche Scientifique, Paris, 1964. viii + 268 *pp.* 35 *NF.*

This is a bound copy of the proceedings of the international colloquium on global properties of completely integrable Pfaffian systems held at Grenoble, 25–30 July 1963. The individual papers have already been published in Ann. Inst. Fourier (Grenoble) **14** (1964), fasc. 1, and have been (or are being) reviewed individually.

★Deuxième Congrès Mathématique Hongrois, Budapest, 24.–31. August 1960. I, II. 4654
Akadémiai Kiadó, Budapest, 1961. *I:* 361 *pp.* (*not consecutively numbered*); *II:* 324 *pp.* (*not consecutively numbered*)

★American Mathematical Society Translations. Series 2, Vol. 42: 15 papers on differential equations. 4655
American Mathematical Society, Providence, R.I., 1964. iv + 288 *pp.* $4.20.

This volume contains translations of four papers by B. L. Roždestvenskiĭ, two by O. A. Oleĭnik, and one each by A. N. Tihonov and A. A. Samarskiĭ, A. M. Il'in and O. A. Oleĭnik, S. D. Èĭdel'man, A. A. Dezin, G. E. Šilov, M. S. Šneerson, A. F. Filippov, N. V. Azbelev and Z. B. Caljuk, and V. A. Jakubovič.

★Recent Soviet contributions to mathematics. 4656
Edited by J. P. LaSalle and S. Lefschetz.
The Macmillan Co., New York, 1962. viii + 324 *pp.* $8.75.

A disappointingly superficial account of recent Soviet progress in mathematics in view of the promise of the title and the impression given on p. v of the Preface that "... it is not certain that our general scientific community quite realizes the intense scientific activity that prevails in the Soviet Union".

The present volume is an outgrowth of an essay of the editors ["Recent Soviet Contributions to Ordinary Differential Equations and Nonlinear Mechanics", RIAS Tech. Rep. 59-3, 1959], to which have been added similar chapters. Several major fields of Soviet mathematical activity have been neglected, however, and the development of those topics included is very uneven in character. Some sections consist of little more than names and references (which are neither accurate nor orderly), while other parts seem to be intended for someone with no knowledge of mathematics (for example, pp. 59–66 contain nothing concerning Soviet mathematics, but rather an exposition of the inverse-square law of attraction and Newton's second law, together with an attempt to define a derivative and a differential equation). The material on pp. 19–20 concerning the solution of Hilbert's fifth problem is misleading, to cite another example.

Soviet sources have appeared regularly with comprehensive and well-integrated accounts of Soviet developments in all fields (cf., e.g., *Forty years of mathematics in the USSR: 1917–1959* [Fizmatgiz, Moscow, 1959; MR **22** #6672], which contains a bibliography of some 22,000 items by 3600 mathematicians; too infrequently was any real use made of this exhaustive survey); the volume reviewed below [#4657] is one of a series of more recent Soviet accounts of modern developments in mathematics.

A. J. Lohwater (Providence, R.I.)

★Algebra. Topology (1962) [Алгебра. Топология (1962)]. 4657
Itogi Nauki.
Akad. Nauk SSSR Inst. Naučn. Informacii, Moscow, 1963. 224 *pp.* 1.14 *r.*

This volume contains nine articles, largely expository in nature, describing recent developments in Lie groups and homogeneous spaces, semigroups, rings, modules, categories, homotopy theory, differential topology, manifold theory, and graph theory. Each article contains an extensive bibliography of recent work, both Russian and non-Russian; the papers will be given individual reviews.

★Mathematisches Wörterbuch. Mit Einbeziehung der theoretischen Physik. Band I: A–K. Band II: L–Z. 4658
Bearbeitet und herausgegeben unter Mitwirkung zahlreicher Fachgelehrter von Joseph Naas und Hermann Ludwig Schmid.
Akademie-Verlag, Berlin; B. G. Teubner Verlagsgesellschaft, Stuttgart, 1961. *Band I:* xii + 1043 *pp.*; *Band II:* viii + 952 *pp.* *Beide Bände: DM* 450.00.

These two volumes constitute an encyclopedia of most of the terminology of modern mathematics, and give not only definitions (usually with expository accounts) of the terms listed, but also references to the basic literature, whenever needed. Terms or expressions which are used in the definition of a given expression and which are listed elsewhere in the *Wörterbuch* are indicated in the text in a clear but unobtrusive way. At the end of Band II is an extensive list of symbols used in mathematics and theoretical physics. One must know the German expression or word to find its definition, for no word lists in other languages are given; a mathematical dictionary of this type, and not a bilingual or trilingual glossary, was the objective of the editors, who have succeeded in their task.

The *Wörterbuch* is the result of an extensive collaboration of 127 mathematicians, and is a tour de force of editing. The development of mathematics and theoretical physics is such, however, that new editions must be published regularly, as is the case of the *Iwanami mathematical dictionary* (Japanese), revised and enlarged [Iwanami Shoten, Tokyo, 1960; MR **24** #A644], whose first edition was published in 1954 [MR **15**, 1011]. The two dictionaries are similar in structure, the Iwanami dictionary having the disadvantage (for most of us) that its text is in Japanese, although a third edition is in preparation with a parallel English version. It appears that the Iwanami dictionary is considerably more comprehensive, with more current references to the literature, and the material is indexed in English, French and German for the use of Japanese mathematicians. The Iwanami dictionary contains several numerical tables of special functions at the end; however, it seems to the reviewer that the exclusion of such numerical tables from the *Wörterbuch* was a wise decision of its editors, particularly in view of its size.

Misprints are always inevitable in any first edition of a work of this magnitude; the number of these is rather large, although most of them appear in the references to the literature cited at the end of various topics. Typical of those appearing in the text, on the other hand, are the following. "Kloosterman" is spelled correctly on page 926 of Band I, but incorrectly on page 940 of

Band II; "Cesàro" is misspelled on page 926 of Band I, as is "Jaśkowski" on page 833 and "Kerékjártó" on page 730, and "Erdélyi" seems to be misspelled wherever it appears. The bold-face entry "Godefrey Herold Hardy" on page 691 was obviously meant to be "Godfrey Harold Hardy". The nature of these misprints shows, of course, that they are trivial points which detract nothing from the extremely valuable character of the work.

HISTORY AND BIOGRAPHY

See also 5531.

Adamo, Marco 4659

Saggi sulla struttura del sistema aritmetico del Regno-Medio egiziano.

Rend. Sem. Fac. Sci. Univ. Cagliari **32** (1962), 101–183.

An Hand von Texten des mittleren Reiches wird der Inhalt der ägyptischen Arithmetik (Zahlbegriff, Rechenoperationen, Bruchrechnung, Maßsysteme) analysiert. Dabei legt der Verfasser seinen Überlegungen eine Gruppe von Axiomen zugrunde, die er—nach einer Übersicht über die Quellen—am Anfang seiner Untersuchungen formuliert (S. 107). Diesen freilich nicht ausgesprochenen Axiomen (S. 152: "taciti postulati") habe der Ägypter seine Regeln und Operationen untergeordnet. Auch die Existenz des logischen Beweises (für kleine Zahlen) und der Begriff der Induktion (für große Zahlen) wird für den Ägypter in Betracht gezogen. *K. Vogel* (Munich)

★Correspondance du P. Marin Mersenne, religieux minime. VIII: Août 1638–Décembre 1639. 4660

Publiée et annotée par Cornelis de Waard. Édition entreprise sur l'initiative de Mme Paul Tannery et continuée par le Centre National de la Recherche Scientifique.

Éditions du Centre National de la Recherche Scientifique, Paris, 1963 [1964]. ix + 787 *pp.* (4 *plates*) 100 *F.*

Dieser Band umfasst die Briefe Nr. 691–803, ferner anhangsweise zahlentheoretische Studien Frenicles und Mersennes Schreiben an G. Naudé über den Magneten. Wir befinden uns kurz nach Erscheinen des *Discours de la méthode* von Descartes (1637) und mitten in den Auseinandersetzungen zwischen Fermat und Descartes um die Extremwert und Tangentenmethode. Mersenne hat den 2. Teil der *Harmonie universelle* (1637) und die *Harmonicon libri* XII (1638) veröffentlicht; gerade entstehen die anonym erscheinenden *Nouvelles pensées de Galilei* (1639), eine Übertragung der *Discorsi e dimostrazioni* (1638). Die einschlägigen Teile der Korrespondenz mit Galilei, Descartes, Fermat, Frenicle und Debeaune sind grösstenteils bekannt; neu treten zahlreiche weitere Stücke der immer mehr in die Breite gehenden Korrespondenz hinzu, die hier erstmals ediert sind. Die wertvollen biographischen, bibliographischen und sachlichen Erläuterungen zu den einzelnen Stücken enthalten eine Fülle bisher unbekannter Einzelheiten; vorzüglich führt das ausgezeichnete Namenregister. *J. E. Hofmann* (Ichenhausen)

Euleri, Leonhardi [Euler, Leonard] 4661

★Commentationes astronomicae ad praecessionem et nutationem pertinentes.

Edited by Leo Courvoisier. Leonhardi Euleri Opera Omnia (Series Secunda, Opera Mechanica et Astronomica, Vol. Tricesimum), Vol. XXX.

Orell Füssli, Zürich, 1964. lxxix + 352 *pp.* *sFr.* 100.00.

Band XXIX (1961) wurde schon in MR **27** #1344 rezensiert. Band XXX *Leonhardi Euleri Opera Omnia* ist für den Astronomen und Mathematiker eine Fundgrube interessanter Untersuchungen. Aus dem Gebiet der sphärischen Astronomie behandelt Euler zwei Aufgaben (lateinisch) sowie eine Anweisung zur Bestimmung der Mittagsgleichung der Sonne (lateinisch), die Berechnung von Neu- und Vollmonden aus Mondfinsternissen (französisch) und einen kurzen Hinweis zum Problem der Sternbedeckungen (französisch). Der Berechnung von Sonnen- und Mondfinsternissen sind drei Arbeiten gewidmet (französisch und lateinisch). Die Ableitung der Mondparallaxe wird für den Fall einer ellipsoidischen Erde diskutiert (französisch); eine weitere Betrachtung beschäftigt sich mit der geozentrischen Monddistanz von einem Fixstern (lateinisch).

Die umfangreichste Arbeit gilt dem Venusdurchgang vom 3. Juni 1769 und der am gleichen Tag stattfindenden totalen Sonnenfinsternis. Euler behandelt hier die Berechnung der Sonnenparallaxe aus Beobachtungen des Venusdurchganges. Im zweiten Teil diskutiert er die Bestimmung der Elemente der Mondbewegung sowie der Längen von Erdorten, von denen aus man Beobachtungen der Sonnenfinsternis anstellte (lateinisch). Der Herausgeber fügte einen Anhang bei (deutsch).

In einer historischen Untersuchung gibt Euler eine Regel an, aus der sich der Beginn irgendeines indischen Jahres berechnen läßt (lateinisch). Die Arbeit wird von O. Neugebauer (Brown University, USA) kommentiert (deutsch). Ein kurzer Brief Eulers an H. W. Clemm (Berlin) über die Bestimmung der Jahreslängen in historischen Zeiten ergänzt diese Betrachtung (lateinisch).

Weitere Arbeiten gelten der Bewegung der Knoten der Planetenbahnen (lateinisch), der Figur und Neigung der Saturnsringe (lateinisch), der Theorie der Fixsternparallaxen (lateinisch), der Vermessung von Meridian-Gradbögen auf der Erde (französisch) und einer Minimumsaufgabe zur Berechnung der kürzesten Dämmerung (französisch). Erwähnt sei noch ein kurzer Brief Eulers über die Sternbilder (deutsch).

Der Band enthält noch vier Arbeiten von Eulers Sohn Johann Albrecht, der ein geometrisches Verfahren zur Bestimmung der Sonnenrotation aus Fleckenbeobachtungen vorschlägt (lateinisch) und auf die Ableitung der Sonnenparallaxe nach den Methoden seines Vaters kurz eingeht (englisch). J. A. Euler stellt ferner recht aufschlußreiche Rechnungen über eine magnetische Sonnenuhr an (deutsch) und behandelt in einer weiteren Arbeit die Figur der Erde (deutsch).

Der Herausgeber Dr. Leo Courvoisier hat es in seiner 69 Seiten langen Einleitung (deutsch) verstanden, die Arbeiten Eulers nicht nur vorzüglich zu erläutern, sondern sie auch einer gründlichen Durchsicht und Korrektur unterzogen. *Rolf Müller* (Degerndorf)

Kopp, V. G.; Laptev, B. L.; Širokov, A. P.; Sulikovskiĭ, V. I. 4662

Aleksandr Petrovič Norden (on his sixtieth birthday). (Russian)

Uspehi Mat. Nauk **19** (1964), *no.* 5 (119), 171–179. (1 *plate*)

A brief account of the life and professional career of Norden. There is a bibliography of 80 items, together with a list of 10 books edited by Norden.

Lomadze, G. A.; Čogošvili, G. S. 4663
Arnol'd Zel'manovič Val'fiš [Arnold Walfisz]. (Russian)
Uspehi Mat. Nauk 18 (1963), *no.* 4 (112), 119–128. (1 *plate*)

A description of the contributions of Val'fiš, together with a bibliography of 100 items; see also MR **20** #1025a–c.

Scorza Dragoni, Giuseppe 4664
Renato Caccioppoli (20 gennaio 1904–8 maggio 1959).
Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. Appendice **1963**, 85–93.

An account of Caccioppoli's life and career, together with a bibliography of 80 items.

Segre, Beniamino 4665
Francesco Severi (13 aprile 1879–8 dicembre 1961).
Ann. Mat. Pura Appl. (4) **61** (1963), i-xxxvi.

An extensive account of Severi's life and scientific contributions, together with a bibliography of 404 items extending from 1898 to 1963.

LOGIC AND FOUNDATIONS

See also 4716, 5406, 5411, 5690.

Curry, Haskell B. 4666
Logic as a mathematical science.
Ann. Japan Assoc. Philos. Sci. **2** (1963), 131–143.

This is the text of an invited address at the annual meeting of the Mathematical Society of Japan at the University of Tokyo on 1 June 1962.

Hasenjaeger, Gisbert 4667
★Einführung in die Grundbegriffe und Probleme der modernen Logik.
Verlag Karl Alber, Freiburg-Munich, 1962. 202 *pp. DM* 21.50.

The author's intention was not to write an elementary textbook of logic but to show the origin of the modern logical notions and to discuss some problems selected among the most typical. The notions and problems with a common origin form a particular aspect of the logic. Every chapter of the book is devoted to another aspect of logic, although different chapters are bound by deductive interconnections.

Chapter I contains the known interpretation of the ancient theory of syllogisms in the modern predicate calculus. In Chapter II the logic is treated as a kind of ontology, i.e., as a theory of being. The world can be described by the catalogue of properties of things and by the catalogues of properties of n-tuples of things. The author refers to some possible philosophical points of view: the realistic, idealistic, fictionalistic, and the skeptic one. The truth tables and the quantifiers are discussed here. In Chapter III, "Logic as Theory of the Language", the standard language of the predicate calculus is described. More general remarks concern the relation of inference and the main semantical notions. The rules of inference are discussed in Chapter IV, "Logic as Methodology". Gentzen's calculus of sequents is the subject of that discussion. "From ... one can derive ..." is written as "$\ldots \vdash_p \ldots$"; hence, "$\vdash_p \ldots$" means "... is provable in the predicate calculus". Thus the different aspects of the calculus of predicates is known to the reader of the first part of the book.

The second part deals with some extensions of the logical language. Its first subjects are the theory of types and some examples of antinomies. The distinction between decidable and undecidable theories is discussed in Chapter VII, "Logic and the Criticism of Reason". The sequences Z of symbols of a theory can be represented as terms t_Z of a fixed symbolism K with a predicate B. A set M of symbol sequences is called regular (as well as enumerable) if a formula A in the symbolism K exists such that for every symbol sequence Z: Z belongs to M if and only if $\vdash_p (\mathbf{A} \to Bt_Z)$. Chapter VIII deals with the logical problems of probability. The author distinguishes between logical, subjective, and statistical probability. Shannon's information measure is defined. The inductive inference is based on incomplete information. The rules of plausible inference of Pólya [*Patterns of plausible reasoning*, Princeton Univ. Press, Princeton N.J., 1954; MR **16**, 556] are quoted.

S. Jaśkowski (Toruń)

Robinson, Abraham 4668
Recent developments in model theory.
Logic, Methodology and Philosophy of Science (*Proc.* 1960 *Internat. Congr.*), *pp.* 60–79. *Stanford Univ. Press, Stanford, Calif.*, 1962.

This article gives a very suggestive survey of some important aspects of model theory of ordinary first-order logic. The "principle of localization" (compactness theorem, completeness theorem) is, of course, given a central role. The well-known results of Łoś and Tarski concerning universal classes are discussed and refined, an interesting interpolation theorem being given in this connection. Lyndon's interpolation theorem is also generalized. The ultraproduct construction is discussed. The author gives a novel proof of the localization principle and the basic property of ultraproducts, using a set-theoretical valuation lemma.

All of this material is given a concrete application by considering all along a theorem of Mal'cev about normal chains of subgroups of a group. The final section of the paper describes some of the author's work about model-completeness, with applications to the theories of algebraically closed fields, real closed fields, and differentially closed fields.

{Important errata: p. 61, beginning of Section 2: the set K_p does not seem to play any role later. p. 62^9, $\sim E(b_\mu, b_\nu) \ldots$. p. 63, third paragraph from bottom: since, according to p. 61, functions may be allowed, it should be assumed that R is closed under all the functions. p. 66^2, "positive sentence Z". p. 66, "Y, Y', Y''" for the three occurrences of "Y".}

D. Monk (Boulder, Colo.)

Menger, Karl; Schultz, Martin 4669
Postulates for the substitutive algebra of the 2-place functors in the 2-valued calculus of propositions.
Notre Dame J. Formal Logic **4** (1963), 188–192.

This paper is concerned with postulates for a substitutive algebra. They show that the algebra can be derived from an easily obtainable set of transformations.

B. A. Galler (Ann Arbor, Mich.)

Nishimura, Iwao 4670

On a classification of intermediate propositional calculi. (Japanese. English summary)

Sci. Rep. Fac. Lib. Arts Ed. Gifu Univ. Natur. Sci. **2** (1962), 5–9.

Let Σ be the lattice of all the intermediate propositional calculi and $\Pi = \{LP_i(X)\}$ be the lattice of all the propositional calculi $LP_i(X)$ which can be defined by a single basic formula $P_i(X)$ of one variable X. The author defines a homomorphism Σ onto Π by the fact that, for every calculus L in Σ, the strongest basic formula provable in L under the basic formulae $P_i(X)$ can be decided.

K. Ono (Nagoya)

Skolem, Th. 4671

Interpretation of mathematical theories in the first order predicate calculus.

Essays on the foundations of mathematics, pp. 218–225. *Magnes Press, Hebrew Univ., Jerusalem*, 1961.

The author discusses the possibility of developing, in general, ordinary mathematical theories within the first-order predicate calculus (with just constant predicates and finitely many axioms). The author observes that the faith in classical set theory may explain why the distinction between predicate calculi of different orders has been considered essential, asserts that there is no justification for assuming that the notion of subset in axiomatic set theory has Cantor's absolute meaning, and asks if the above distinction is not just a formal one. He recalls that Gilmore [*Summaries of talks presented at the Summer Institute for Symbolic Logic, Cornell Univ.*, 1957, pp. 309–312, second edition, Comm. Res. Div., Inst. Defense Analyses, Princeton, N.J., 1960] has shown how to transform a theory equivalent to the simple theory of types into one formulated in the first-order predicate calculus with axiom schemes and so that provable statements correspond.

The author's method [Dialectica **12** (1958), 443–450; MR **21** #3325] of transforming a theory formulated with axiom schemes (free predicate variables) into one with only ordinary axioms and having at least the same deductive power is very briefly sketched. (That any formula provable in this theory is a transform of a provable formula has not yet been shown.) This sketch is preceded by a description of the steps to be taken so as to properly formulate an ordinary mathematical first-order theory: treatment of the identity relation and elimination of functors in favor of predicates, with an illustration of how the rule allowing for the substitution of individual variables by functors becomes a valid rule of inference. Finally, the author emphasizes the preference for treatment, if possible, within free variable theories.

H. Ribeiro (University Park, Pa.)

Žuravlev, Ju. I. 4672

Set-theoretic methods in a Boolean algebra. (Russian)

Problemy Kibernet. No. 8 (1962), 5–44.

This paper contains a thorough and detailed presentation of the construction of algorithms for computing minimal disjunctive normal forms (dnf's) for Boolean functions. Definitions and basic results are given in the first chapter. The second deals with basic conjunctions that occur in irreducible dnf's. A strengthened form of Quine's algorithm is defined for obtaining the disjunction of all of these from reduced dnf's. A specific algorithm for simplifying dnf's is investigated in Chapter 3, and in Chapter 4 such algorithms are treated as devices for processing information about basic conjunctions that occur in dnf's. In Chapter 5 algorithms of finite order are considered and it is shown that with this class of algorithms the question of passing from a simplified dnf to a dnf whose basic conjunctions each occur in at least one minimal dnf is unsolvable.

E. J. Cogan (Bronxville, N.Y.)

Summersbee, S.; Walters, A. 4673

Programming the functions of formal logic. II. Multi-valued logics.

Notre Dame J. Formal Logic **4** (1963), 293–305.

Part I appeared in same J. **3** (1962), 133–141 [MR **27** #1374]. The authors are interested in programming the functions of multi-valued logic. The method is illustrated by means of a class-scheduling problem. The strategy is to consider the number of lectures given by a particular professor to a particular class to be a function on a multi-valued logic, with the collection of functions subject to conditions imposed by various practical considerations.

B. A. Galler (Ann Arbor, Mich.)

Gnidenko, V. M. 4674

Determination of orders of pre-complete classes in three-valued logic. (Russian)

Problemy Kibernet. No. 8 (1962), 341–346.

A class A of functors in a propositional calculus P is called complete if any functor of this calculus can be expressed by means of functors of A; a class B is pre-complete if it is not complete but if $B \cup \{f\}$, where f is any functor which is not in B, is complete. A class A of functors in P is closed if it is closed with respect to substitution of variables and functors of A for variables. The basis of a class A is a subset $A' \subset A$ such that all other functors in A can be expressed by means of elements of A'. The order of a closed class A is the least non-negative integer p such that there exists a basis A' of A whose functors do not contain more than p arguments.

The author proves that in the three-valued propositional calculus P_3 all pre-complete classes of functors are of order 2. This makes a difference in the case of classical calculus P_2, where this order can be greater than 2.

The proof employs the analysis of pre-complete classes, elaborated by Jablonskiĭ [Trudy Mat. Inst. Steklov. **51** (1958), 5–142; MR **21** #3331].

V. Vučković (Notre Dame, Ind.)

Partis, M. T. 4675

Commutative partially ordered recursive arithmetics.

Math. Scand. **12** (1963), 199–216.

The "numerals" in the commutative partially ordered recursive arithmetic (in short, in V) of the reviewer [Math.

Scand. 7 (1959), 305–320; MR 22 #7939; Bŭlgar. Akad. Nauk. Izv. Mat. Inst. 6 (1962), 15–25; MR 26 #1250] can be conceived, in the case of a finite number n of successors, as n-dimensional "vectors" with the basis $S_1, S_2, \cdots, S_n$ and with non-negative integers as coefficients: every "numeral" in V can be represented in the form $\sum_{i=1}^{n} \xi_i S_i$, where the ξ_i are non-negative integers.

The author applies systematically this point of view and shows that every primitive recursive function f in V can be represented by $f(\sum_{i=1}^{n} \xi_i S_i) = \sum_{i=1}^{n} f_i(\xi_1, \xi_2, \cdots, \xi_n) S_i$, where every f_i is a primitive recursive numerical function uniquely determined by f. The result is proved by establishing an isomorphism between V and an appropriate arithmetic C of n-tuples $(f_1, f_2, \cdots, f_n)$ of primitive recursive numerical functions.

Moreover, the author establishes a deductive isomorphism between V and C in the following sense: from the proof of an equation in V one can obtain the proof of the corresponding equation (between n-tuples) in C, and vice versa. This makes it possible to reduce some involved proofs in V to known proofs in recursive number theory. Also, a very simple investigation of the lattice structure of V is now made possible.

The method of the author (suggested by R. L. Goodstein) seems to provide a powerful tool for the examination of the structure of V.

V. Vučković (Notre Dame, Ind.)

Skordev, D. 4676

Rekursiv vollständige arithmetische Operationen. (Russian summary)

C. R. Acad. Bulgare Sci. 16 (1963), 465–467.

Let α be a binary partial operation on natural numbers whose value for arguments x, y, if it exists, is $x\alpha y$; in iterations of this function parentheses will be omitted according to the rule of association to the left. Then α is called recursively complete when there are natural numbers P, Q, R, S such that the following hold for all natural numbers x, y, z (with existence of either side of the equation entailing that of the other): (I) $P\alpha x = x+1$; (II) $Q\alpha(x+1) = x$; (III) $R\alpha 0\alpha y\alpha z = y$ and $R\alpha(x+1)\alpha y\alpha z = z$; (IV) $S\alpha x\alpha y$ is always defined and $S\alpha x\alpha y\alpha z = x\alpha y\alpha(x\alpha z)$. The principal result of this paper is that if α is such an operation, then every partial recursive function ϕ of n arguments is representable in the form $f\alpha x_1 \alpha x_2 \alpha \cdots \alpha x_n$, where f is a fixed natural number (for fixed ϕ) and $f\alpha x_1 \alpha \cdots \alpha x_{n-1}$ is defined for all $x_1, \cdots, x_{n-1}$. This result is obtained by noticing that the α-operation can be taken as an interpretation of the application operation of combinatory logic; that the combinator S can be interpreted as the number S, and the combinator K as $R\alpha 0$; and then using techniques similar to those which show that all partial recursive numerical functions can be defined in terms of combinators. From the main result it follows that a binary numerical function is a principal function [in the sense of Uspenskiĭ, *Lectures on computable functions* (Russian), Fizmatgiz, Moscow, 1960; MR 22 #12043] for partial recursive functions of one argument just when it is recursively complete and partial recursive; and a recursively complete function is partial recursive just when it has a certain minimal property. It is claimed that these results indicate a possible method for the axiomatization of the theory of partial recursive functions.

H. B. Curry (University Park, Pa.)

Tugué, Tosiyuki 4677

On the partial recursive functions of ordinal numbers.

J. Math. Soc. Japan 16 (1964), 1–31.

In this review functions and functionals take ordinals as values, functions have ordinals, functionals have functions and/or ordinals as arguments. Takeuti and Kino [same J. 14 (1962), 199–232; MR 27 #4747, hereinafter "TK"], developed a theory of primitive recursive (Pr.R) and general recursive functionals; this can be easily extended to include partial recursive functionals (Pa.R). {Reviewer's remark: the definition of Pr.R in TK and the one used by the author are unnecessarily elaborate. More specifically, Pr.R (relative, perhaps, to a class C of functions) is the least class which contains the identity and successor functions, $\lambda x\omega$, $\lambda xy(x \cdot y)$, $\lambda x(1 \dot{-} x)$ (and the functions of C), and which is closed with respect to substitution, the bounded μ-operator, and the recursion schema $\phi(x) = \chi(\phi^x, x)$, where parameters have been omitted, and $\phi^x(y) = \phi(y)$ $(y < x)$, $\phi^x(y) = 0$ $(y \geq x)$. Pa.R is obtained by allowing the unbounded, instead of the bounded, μ-operator.} The author points out that these definitions can be widened (from the "strict" to the "classical" sense) by permitting all constant functions. This distinction has also been drawn by Levy [Notices Amer. Math. Soc. 10 (1963), 286] who refers to "finitarily computable" and "computable" functionals.

The chief aim of the paper under review is to develop an analogue of Kleene's concept of "formally calculable". For this purpose the author adapts work by Machover [Bull. Amer. Math. Soc. 67 (1961), 575–578; MR 26 #17] and sets up an equational calculus in which $\mathrm{Sup}_{z<x} f(z) = y$ may be inferred for appropriate y from the (infinite) set of equations which evaluate f for all ordinals less than x. It is shown that the partial functionals which are formally calculable (from C) in this calculus are exactly those in Pa.R (relative to C). For the strict sense, primitive symbols for ordinals other than 0 and ω must not appear in the defining equations (although, of course, they do appear in the deductions). A Gödel numbering is set up; here the axiom of constructibility is assumed, so that a single ordinal may be used to encode the infinite sequence of ordinals which is necessary to describe a given deduction. In this way a Pr.R analogue of Kleene's T predicate is obtained; this is used to give new proofs of the enumeration and hierarchy theorems of TK. The enumeration of the predicates of a given quantifier form is by natural numbers when the scope of the quantifiers is general recursive in the strict sense; by ordinals if it is general recursive in the classical sense.

{Reviewer's comments: (1) The term "primitive recursive" to describe the class Pr.R is not well-chosen, for Pr.R is a very narrow class. It is not hard to show that if $x < \omega^\omega$ and ϕ is in Pr.R, then $\phi(x) < \omega^\omega$; thus $\lambda x(x^\omega)$ is not in Pr.R. A class which is more properly described as "primitive recursive" is obtained if ϕ^x in the recursion schema is replaced by $\mathrm{sup}_{y<x}\phi(y)$; this does not alter the extent of Pa.R. (2) The assumption of the axiom of constructibility is not necessary. For the sequences of ordinals needed to describe the deduction of a value of a function in Pa.R (or Pa.R relative to C) are themselves general recursive, and are therefore constructible (or constructible from the restrictions to some ordinal of a finite number of functions in C). A similar remark applies to the use of the axiom in TK.}

R. O. Gandy (Manchester)

Péter, Rózsa 4678

Über die Rekursivität der Begriffe der mathematischen Grammatiken. (Russian summary)

Magyar. Tud. Akad. Mat. Kutató Int. Közl. 8 (1963), 213–228.

As an application of her version of the recursive arithmetic of words the author proves that the notion of "grammatically true sentence" in the categorial and phase-structure grammars of Chomsky and Bar-Hillel is primitive recursive in a suitable arithmetic of words, and that the "grammatically true sentences" in the combinatorial grammar of Chomsky form a recursively enumerable set.

The theorems cannot be considered as new (in the first case, a representation by finite automata is possible, and in the second case the recursive enumerability follows by known theorems about combinatorial systems), but the method of proof deserves attention by showing an elegant application of the recursive arithmetic of words to problems of grammatical structures.

V. Vučković (Notre Dame, Ind.)

Tennenbaum, S. 4679

Degree of unsolvability and the rate of growth of functions.

Proc. Sympos. Math. Theory of Automata (New York, 1962), pp. 71–73. Polytechnic Press of Polytechnic Inst. of Brooklyn, Brooklyn, N.Y., 1963.

A recursively enumerable (r.e.) set M is a "maximal simple" set if, though its complement M' is infinite, every r.e. superset N of M has the property that either $N-M$ or $M'-N$ is finite. The author proves the theorem: If π is the enumerating function of a maximal simple set M and g is any general recursive function, then $\pi(n)>g(n)$ for all $n>k$.

A. P. Ershov (Novosibirsk)

Belnap, N. D., Jr.; Leblanc, H.; Thomason, R. H. 4680

On not strengthening intuitionistic logic.

Notre Dame J. Formal Logic 4 (1963), 313–320.

The paper is concerned with Leblanc's conjecture [Leblanc and Belnap, same J. 3 (1962), 79–82; MR 27 #1367; Belnap and Thomason, ibid. 4 (1963), 39–43; MR 27 #28] to the effect that any rule for "v" which holds in CPC (Gentzen formulation of the classical propositional calculus) also holds in IPC (intuitionistic propositional calculus), from which it follows that the only way of turning standard Gentzen rules of inference for IPC into rules for CPC is to strengthen rules for "$\sim$", "$\supset$", or "$\equiv$". Leblanc's formulation, however, left room for unintended interpretations under which the conjecture proved to be false [cf. Vesley, ibid. 4 (1963), 80; MR 27 #29]. By interpreting a rule for "v" as one with the property that every inference condoned by it can be obtained by substitution from one condoned by it that exhibits no connective but "v" and by interpreting "holding in a system" as derivability of the conclusion of a rule by means of the axioms and primitive rules of the system from the premises, the conjecture thus obtained is shown to be true. A number of alternative interpretations are considered which help to clarify the situation.

B. van Rootselaar (Amsterdam)

Thomas, Ivo 4681

S1° and Brouwerian axioms.

Notre Dame J. Formal Logic 4 (1963), 151–152.

Let B_n be the modal axioms $C'pL^nMp$ (C' for strict implication, L^n for L iterated n times). We say B_n is sufficient if its addition to Feys's $S1°$ yields $S5$. The author proves that (i) no set of axioms of the form B_{2k+1} ($k\geq 0$) is sufficient; (ii) any pair B_1, B_{2k} ($k\geq 1$) is sufficient; (iii) for all m, n ($n>2$), if m and n are co-prime, then B_{m-1}, B_{n-1} are sufficient.

E. J. Lemmon (Claremont, Calif.)

Thomas, Ivo 4682

S1° and generalized S5-axioms.

Notre Dame J. Formal Logic 4 (1963), 153–154.

Let $A_{j,k}$ be the modal axioms $C'M^jpL^kMp$ ($j,k\geq 1$) (C' for strict implication); then $A_{1,1}$ is the usual special $S5$-axiom. We say $A_{j,k}$ is sufficient if its addition to Feys's $S1°$ yields $S5$. The author's main results are: (i) if $j+k$ is odd, $A_{j,k}$ is not sufficient; (ii) if $j=k$, $A_{j,k}$ is sufficient; (iii) if $j=k+2$, $A_{j,k}$ is sufficient.

E. J. Lemmon (Claremont, Calif.)

Thomas, Ivo 4683

A final note on S1° and the Brouwerian axioms.

Notre Dame J. Formal Logic 4 (1963), 231–232.

Let B_n be the modal axioms $LCpL^nMp$ ($n\geq 1$). The author shows that, if any single B_n is added to $T°$ (Feys's $S1°$ together with the rule from α to infer $L\alpha$), the resulting system lacks $CpMp$ and so does not contain T.

E. J. Lemmon (Claremont, Calif.)

Sobociński, Bolesław 4684

A note on modal systems.

Notre Dame J. Formal Logic 4 (1963), 155–157.

It is known that the system $S1°$ of Feys affords the rule: If $\vdash C'M\alpha L\beta$, then $\vdash C'\alpha\beta$. The author states several results concerning interconnexions between various systems which result from $S1°$ or $S2°$ by adding as axioms theses related to this rule (such as $CCMpLqC'pq$, $CCMpLqCpq$).

E. J. Lemmon (Claremont, Calif.)

Zeman, J. Jay 4685

Bases for S4 and S4.2 without added axioms.

Notre Dame J. Formal Logic 4 (1963), 227–230.

Consider the two modal rules: RL1: from $\vdash C\alpha\beta$ to infer $\vdash CL\alpha\beta$; RL2: from $\vdash C\alpha\beta$ to infer $\vdash C\alpha L\beta$, if α is completely modalized. It is known that RL1 and RL2, subjoined to a propositional calculus base with detachment, give $S5$, where α is completely modalized if and only if all occurrences of propositional variables in α lie within the scope of a modal operator. The author shows that the same two rules, subjoined to the same base, give $S4$ and the Dummett-Lemmon $S4.2$ under different interpretations of complete modalization. Thus, for $S4$, we define: α is completely modalized if and only if either $\vdash_{S4}\alpha$ [and α is full modalized in the $S5$ sense] or α is of the form $KL\gamma KL\delta\cdots L\nu$ with $\alpha=L\gamma$ as limiting case. (Actually, the clause in square brackets is redundant.) The $S4.2$ definition of complete modalization requires a third clause: "or of the form $\phi\gamma$, where ϕ is composed of alternating 'L' and 'N', beginning with 'N', and containing at least two 'L's'".

E. J. Lemmon (Claremont, Calif.)

Putnam, Hilary 4686

Uniqueness ordinals in higher constructive number classes.

Essays on the foundations of mathematics, pp. 190–206. *Magnes Press, Hebrew Univ., Jerusalem*, 1961.

The author extends Kleene's definition of H_a to all numbers a in the Kreider-Rogers system C of notations for ordinals. He then extends Spector's uniqueness theorem, which says that the degree of H_a is determined by the ordinal for which a is a notation. The proof consists of an exceedingly complicated application of the recursion theorem. The reviewer feels that the concepts studied here cannot be of much use in further investigations until more manageable ways of dealing with them are developed.

J. R. Shoenfield (Durham, N.C.)

Sedol, Ja. Ja. 4687

The free product of associative calculi with joined subalphabets and certain related questions. (Russian)

Dokl. Akad. Nauk SSSR **158** (1964), 1034–1037.

An associative calculus $\mathfrak{A}$ in the alphabet A is the semigroup of words in A (under concatenation), together with a finite system of generating relations $\alpha_i \leftrightarrow \beta_i$, $i = 0, 1, \cdots, r$, which defines, in the usual way, the equivalence $\mathfrak{A}: \alpha \amalg \beta$ of words in A. $\mathfrak{A}$ is inversive if to every word α corresponds its inverse α^{-1} such that $\mathfrak{A}: \alpha\alpha^{-1} \amalg \alpha^{-1}\alpha \amalg \Lambda$, where Λ denotes the empty word.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be associative calculi in the alphabets A and B, respectively. Let $C = A \cup B$ and $D = A \cap B$. Suppose that the following two conditions are satisfied. (1) The group condition: If M is a word in letters of D, then there exists an inverse M^{-1} (in letters of D) such that $\mathfrak{A}: MM^{-1} \amalg M^{-1}M \amalg \Lambda$; (2) the isomorphism condition: If M and N are words in the alphabet D, then $\mathfrak{A}: M \amalg N$ implies $\mathfrak{B}: M \amalg N$, and vice versa.

The associative calculus $\mathfrak{C}$ in the alphabet C, obtained from $\mathfrak{A}$ and $\mathfrak{B}$ by the union of their systems of generating relations, is called the free product of $\mathfrak{A}$ and $\mathfrak{B}$ with the common subalphabet D.

The author quotes the following theorems. (1) $\mathfrak{C}: P \amalg Q$ implies $\mathfrak{A}: P \amalg Q$ and (2) $\mathfrak{C}: P \amalg Q$ (P in A, Q in B) implies the existence of Q_1 in A and Q_2 in B such that $\mathfrak{A}: P \amalg Q_1$ and $\mathfrak{B}: Q \amalg Q_2$.

Some more complicated theorems, involving notions of generators, isomorphism, stationary letters, special letters (which cannot be explained here) are quoted, leading to the theorem that one can construct an inversive associative calculus in which the problem of the equivalence is not constructively solvable. This should give a relatively simple constructive proof of Novikov's theorem on the word problem in groups.

The author asserts that the plan of proof of these theorems is borrowed from the paper of J. L. Britton [Ann. of Math. (2) **77** (1963), 16–32; MR **29** #5891].

V. Vučković (Notre Dame, Ind.)

Wang, Hao 4688

Dominoes and the AEA case of the decision problem.

Proc. Sympos. Math. Theory of Automata (New York, 1962), pp. 23–55. *Polytechnic Press of Polytechnic Inst. of Brooklyn, Brooklyn, N.Y.*, 1963.

Author's introduction: "It has recently been established that the AEA case is unsolvable and forms a reduction class. Several people have looked into possible directions along which the result can be strengthened, using in part earlier methods developed by Büchi for the E and AEA case. There are three different aspects. First, unsolvable AEA subcases such as restrictions on the number of dyadic predicates, on the form of the quantifier-free component, on the complexity of the models (e.g., finite, essentially periodic, etc.). Second, solvable AEA subcases. Third, the detailed structure of the reduction of the general case to the AEA case. A survey of these questions is presented."

A. P. Ershov (Novosibirsk)

Kahr, Andrew S. 4689

Improved reductions of the Entscheidungsproblem to subclasses of AEA formulas.

Proc. Sympos. Math. Theory of Automata (New York, 1962), pp. 57–70. *Polytechnic Press of Polytechnic Inst. of Brooklyn, Brooklyn, N.Y.*, 1963.

The author is interested, for each positive integer k, in whether the class Δ_k of AEA formulas in the predicate calculus involving k dyadic predicates (in addition to monadic predicates) is a reduction class.

The author establishes reduction to the classes Δ_7 and Δ_5, and discusses for some particular cases a procedure of reduction of a class of diagonal-constrained domino problems to formulas in Δ_1.

A. P. Ershov (Novosibirsk)

Dobrev, Dimitr 4690

Finite automata. (Bulgarian)

Fiz.-Mat. Spis. Bŭlgar. Akad. Nauk. **7** (**40**) (1964), 43–47.

Expository paper, giving the elements of the theory of Mealey automata.

V. Vučković (Notre Dame, Ind.)

Gluschkow, W. M. [Gluškov, V. M.] 4691

★Theorie der abstrakten Automaten.

VEB Deutscher Verlag der Wissenschaften, Berlin, 1963. 103 pp. DM 16.00.

This is a translation of the author's paper on the abstract theory of automata [Uspehi Mat. Nauk **16** (1961), no. 5 (101), 3–62; MR **25** #1976], together with a translation of a paper on problems of design of automata [Ž. Vyčisl. Mat. i Mat. Fiz. **1** (1961), 371–411].

Red'ko, V. N. 4692

On the commutative closure of events. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 1156–1159.

Let Y be an event (a set of words) over an alphabet X. By the commutative closure of Y one means the set $\bar{Y}$ of all words which are permutations of words in Y. Let C_X denote the commutative free semigroup over X. There is an obvious correspondence between the commutative closure $\bar{Y}$ and the image of Y in C_X. The author defines a normal commutative event as any set in C_X which may be obtained from finitely many powers of elements in X by finitely many disjunctions and (semigroup) multiplications. His main result is then that the commutative closure $\bar{Y}$ of an event Y is regular (in the sense of Kleene) if and only if the image of Y in C_X is a normal commutative event.

The paper is not self-contained: notably missing is a definition of Medvedev machine (which is invoked to support the proof in one direction).

{Cf. also a recent note by the same author on the same subject [Ukrain. Mat. Ž. 16 (1964), 185–195; MR 29 #16].}

R. M. Baer (Berkeley, Calif.)

SET THEORY

See also 4677, 5215.

Linés Escardó, E. 4693

On the structure of the set of natural numbers. (Spanish)

Collect. Math. **14** (1962), 287–300.

This is a smooth exposition of the construction of the natural numbers and of the definition of the notion of finite set, starting (à la Dedekind) with a one-one function from a set into a proper subset of itself.

E. Mendelson (Flushing, N.Y.)

Piccard, Sophie 4694

Quelques propriétés des constituantes des ensembles de Souslin linéaires et de leurs complémentaires.

Publ. Sém. Géom. Univ. Neuchâtel (1) *Fasc.* 3 (1963), 43–58.

E étant un ensemble souslinien de R, la donnée d'un crible permet de décomposer canoniquement E et son complémentaire E' en une infinité d'ensembles E_α $[E_\alpha']$ deux à deux disjoints et indexés par les nombres ordinaux de deuxième classe. De même, la donnée d'une famille d'ensembles $E_{n_1,\cdots,n_k,k}$ (système déterminant de E) dont E est déduit par l'opération de Souslin, permet une décomposition de même nature que la précédente de E et de E'.

Ceci étant, l'auteur montre: (1) qu'à tout crible élémentaire de E on peut associer un système déterminant régulier tel que les deux décompositions canoniques coïncident; (2) qu'il est possible que les ensembles E_α $[E_\alpha']$ soient mesurables-B et vides lorsque α est de deuxième espèce; (3) que pour un même ensemble E, il peut exister deux cribles C et C^* tels que pour une partie infinie A de $[0, \Omega]$, les E_α et E_α' relatifs à C et d'indices appartenant à A soient vides, tandis que les ensembles de mêmes indices relatifs à C^* ne le soient pas.

A. Revuz (Poitiers)

Erdős, Paul; Tarski, Alfred 4695

On some problems involving inaccessible cardinals.

Essays on the foundations of mathematics, pp. 50–82. *Magnes Press, Hebrew Univ., Jerusalem*, 1961.

The authors investigate six properties of infinite cardinals. λ has the property P_1 if there is a set A of power λ which is simply ordered by a relation $\leq$ such that every subset of A well-ordered by $\leq$ or $\geq$ has power $<\lambda$; the property P_2 if the complete graph on a set of power λ can be divided into two subgraphs, neither of which includes a complete graph on a set of power λ; the property P_3 if, in the set algebra of all subsets of a set of power λ, every λ-complete prime ideal is principal; the property P_4 if there is a λ-complete and λ-distributive Boolean algebra $\mathfrak{B}$ not isomorphic to any λ-complete set algebra; the property Q if, in P_4, $\mathfrak{B}$ is restricted to be λ-generated by λ elements; and the property R if there is a ramification system (tree) $\langle A, \leq\rangle$ of order λ such that for each $\xi<\lambda$, the set of elements of A of order ξ has power $<\lambda$, and every subset of A well-ordered by $\leq$ has power $<\lambda$. It is shown that P_1 implies P_2, P_2 implies P_3, P_3 implies P_4, P_2 implies Q and R implies Q for inaccessible λ, Q implies P_3, and R implies P_3. Furthermore, the properties P_1, P_2, P_3, and P_4 are shown to hold for all accessible cardinals; all the properties fail for ω. Not all of the results mentioned are new; exact references to earlier works are found in the paper, in particular, to an earlier paper of the authors in which most of the results are announced. Additional results are found in the papers of Keisler and Tarski [Fund. Math. **53** (1963/64), 225–308; MR **29** #3385], Hanf [ibid. **53** (1963/64), 325–334; MR **28** #3944], and the reviewer and Scott [ibid. **53** (1963/64), 335–343; MR **29** #3386]. Several open problems are mentioned. The paper is largely self-contained. {Errata: On p. 66 it is erroneously stated that every λ-distributive Boolean algebra is isomorphic to a quotient algebra $\mathfrak{A}/I$, where $\mathfrak{A}$ is a λ-complete set algebra and I is a λ-complete ideal; see the Keisler and Tarski paper [loc. cit.], footnote (9), for the correction.}

D. Monk (Boulder, Colo.)

COMBINATORIAL ANALYSIS

See also 4914, 5157, 5657.

Abramson, Morton 4696

Explicit expressions for a class of permutation problems.

Canad. Math. Bull. **7** (1964), 345–350.

The author rederives the usual expression for selecting k indistinguishable elements from a linear array of n elements with at most j consecutive elements being selected. This result is then applied to counting the number of selections with the maximum number of consecutive elements being j, computing the probability that the success run of maximum length is j elements long, computing the probability the maximum length of success run is at most j, and counting the number of ways of selecting k elements so that no pair of elements x_i, x_{i+2}, $i=1, 2, \cdots, n-2$, is selected. This last result is applied to computing the probability that in a random permutation none of the $2(n-1)$ events $1\to2$, $2\to1$, $3\to2$, $2\to3$, $\cdots$ occur.

Bernard Harris (Madison, Wis.)

Cheema, M. S. 4697

Vector partitions and combinatorial identities.

Math. Comp. **18** (1964), 414–420.

Let S be a given set of vectors (x, y) whose components are non-negative integers. Let $p(m, n\,|\,S)$ be the number of partitions $(m, n)=\sum (x_i, y_i)$, where the parts (x_i, y_i) are all in S. Let $q_e(m, n\,|\,S)$ $[q_0(m, n\,|\,S)]$ be the number of such partitions into an even [odd] number of distinct parts. Put $A=\{(a,a)\,|\,a\in\mathrm{N}\}$, $B=\{(b,b-1)\,|\,b\in\mathrm{N}\}$, $C=\{(c-1,c)\,|\,c\in\mathrm{N}\}$, $D=\{(2d+1,\ 2d-1)\,|\,d\in\mathrm{N}\}$, $E=\{(2e-1,\ 2e+1)\,|\,e\in\mathrm{N}\}$. It is shown that $q_e(m, n\,|\,A\cup B\cup C)-q_0(m, n\,|\,A\cup B\cup C)=(-1)^r$ or 0 according as $(m, n)=(\frac{1}{2}r(r+1), \frac{1}{2}r(r-1))$ (where $r\in\mathrm{Z}$) or not. Recurrence relations are obtained for $p(m, n\,|\,D\cup E)$ and $p(m, n\,|\,A\cup B\cup C\cup D\cup E)$, but these are incorrect since equations (1.5) [1.8] must be divided by $(1-xy^{-1})$ $[1-x^{-1}y]$ before the reciprocals of

their left-hand sides can be interpreted combinatorially. The correct recurrence for $C(m,n)=p(m,n\,|\,A\cup B\cup C\cup D\cup E)$, for example, is: $C(0,0)=1$, and if $(m,n)\neq(0,0)$,

$$C(m,n) = \sum_{r=1}^{\infty}\left[\sum_{s=0}^{3r-2} C\left(m-\frac{3r^2+r}{2}+s+1,\, n-\frac{3r^2-5r}{2}-s-1\right)\right.$$
$$\left.-\sum_{s=0}^{3r} C\left(m-\frac{3r^2+5r}{2}+s,\, n-\frac{3r^2-r}{2}-s\right)\right].$$

Next, it is shown that the partitions of $(n_1,\cdots,n_s)$ into vectors with at least one odd component are equinumerous with those into distinct vectors. Putting

$$f(x) = \prod_{n=1}^{\infty}(1-x^n)^{-1},\quad F(x) = f(x^2)f(x^3)f(x)^{-2}f(x^6)^{-1},$$
$$G(x) = f(x^2)f(x^3)f(x^{12})f(x)^{-1}f(x^4)^{-1}f(x^6)^{-2},$$

the author evaluates

$$\int_0^1 x^{-1}(G(x)-1)\,dx \quad\text{and}\quad \int_0^1 x^{-1}(F(x)-1)\,dx.$$

The second of these contains an error; the correct result is $2-2\pi/\sqrt{3}$. The paper concludes with a table of the number of partitions of $(49, 49)$ into at most r parts with non-negative components $(1\leq r\leq 98)$.

B. Gordon (Los Angeles, Calif.)

Conway, J. H. 4698

Mrs Perkins's quilt.

Proc. Cambridge Philos. Soc. **60** (1964), 363–368.

Let a square with integral side N be subdivided into squares of integral sides. Let $f(N)$ be the least number of such squares in any proper subdivision with 1 as the g.c.d. of the lengths of the sides. It is shown that $f(N)\leq 6\sqrt[3]{N}+1$. The method is to consider the subdivision of the square into selected rectangles and to note that if $\langle a_1, a_2, \cdots, a_k\rangle$ is the simple continued-fraction decomposition of p/q, $(p,q)=1$, $p<q$, the rectangle of size p by q is divisible into $a_1+a_2+\cdots+a_k$ squares in a simple manner. It is also shown, by network methods, that $f(N)\geq\log_2(N)$. The logarithm is the "natural order" of f. A table of $f(N)$ for $N\leq 100$ is given, and the author conjectures that $f(N)\sim c\log N$, where $c\leq 7/\log(\frac{1}{2}(7+\sqrt{33}))$.

J. D. Swift (Los Angeles, Calif.)

Dočev, Kiril 4699

The solution of a difference equation and a combinatorial problem related to it. (Bulgarian)

Fiz.-Mat. Spis. Bŭlgar. Akad. Nauk. **6** (**39**) (1963), 284–287.

Another solution of the problem considered by Sendov in an earlier paper [C. R. Acad. Bulgare Sci. **15** (1962), 5–8; MR **27** #55].

B. S. Popov (Skopje)

Erdős, P. 4700

On a combinatorial problem. II.

Acta Math. Acad. Sci. Hungar. **15** (1964), 445–447.

Part I appeared in Nordisk Mat. Tidskr. **11** (1963), 5–10 [MR **28** #6061]. Let M be a set and F a family of its subsets. F is said to have property B if there exists a subset K of M so that no set of the family F is contained either in K or in $C(K)$, the complement of K in M. One of the unsolved problems is to find the smallest integer $m(n)$ for which there exists a family of sets $A_1,\cdots,A_{m(n)}$, each having n elements, which does not have property B.

The earlier bounds, $m(n)\leq\binom{2n-1}{n}$, and $m(n)>2^{n-1}$ for all n, $m(n)>(1-\epsilon)2^n\log 2$ for $n>n_0(\epsilon)$, have been improved to $m(n)<n^2 2^{n+1}$ in this paper, and to

$$m(n) > 2^n(1+4n^{-1})^{-1}$$

by Schmidt [#4701 below].

M. S. Cheema (Tucson, Ariz.)

Schmidt, W. M. 4701

Ein kombinatorisches Problem von P. Erdős und A. Hajnal.

Acta Math. Acad. Sci. Hungar. **15** (1964), 373–374.

We use the same notation as in the preceding review [#4700]. The author proves that $m(n)>2^n(1+4n^{-1})^{-1}$, and up to date this is the best lower bound known for $m(n)$.

M. S. Cheema (Tucson, Ariz.)

Melzak, Z. A. 4702

Partition functions and spiralling in plane random walk.

Canad. Math. Bull. **6** (1963), 231–237.

A spiral of n steps on the square lattice in the plane is a path starting at the origin which always turns clockwise or counter-clockwise, never occupies the same position twice, and can be continued to an m-step spiral for every $m>n$. The number of spirals in plane symmetric random walk is calculated. In particular, the author shows that if $N(n)$ is the number of different n-step spirals, then

$$\sum_{n=1}^{\infty} N(n)x^n = [x/(1-x)]\sum_{k=1}^{\infty} x^{k^2}\Big/\prod_{i=1}^{k}(1-x^i)^2.$$

G. Baxter (Minneapolis, Minn.)

Shen, Mok-Kong 4703

On the generation of permutations and combinations.

Nordisk Tidskr. Informations-Behandling **2** (1962), 228–231.

The author suggests a procedure for generating the $n!$ permutations on n distinct marks in lexicographical order. Taking the marks to be the integers $1(1)n$, suppose that, starting with the permutation $(1, 2, 3, \cdots, n)$, we have arrived at any stage at the permutation $(k_1, k_2, \cdots, k_n)$. The next permutation is produced from this one by the following procedure. (1) Find the largest i such that $k_{i-1}<k_i$; (2) Find the largest j such that $k_{i-1}<k_j$; (3) Interchange k_{i-1} and k_j; (4) Reverse the order of $k_i, k_{i+1}, \cdots, k_n$. This method compares favorably with a rather inefficient additive method proposed by Howell [Math. Comp. **16** (1962), 243–244; MR **26** #2369]. For generating the combinations of n things taken m at a time in lexicographical order the author proposes the following procedure. If the n things are the integers $1(1)n$, and if the current selection is $(k_1, k_2, \cdots, k_m)$ with $k_i<k_{i+1}$ $(i=1(1)m-1)$, the next combination is found as follows. (1) Find the largest i such that $k_i<n-m+i$; (2) Replace k_i by $1+k_i$; (3) Replace k_{i+j} by k_i+j $(j=1(1)m-i)$.

D. H. Lehmer (Berkeley, Calif.)

Harrison, Michael A. 4704

On the classification of Boolean functions by the general linear and affine groups.

J. Soc. Indust. Appl. Math. **12** (1964), 285–299.

Author's summary: "The general linear group and the affine group of transformations are defined over the field of two elements and are applied as transformation groups to Boolean functions. Algorithms for counting the number of classes under both groups are derived. The concept of equivalence is extended by allowing complementation of the range of the functions, and the number of such classes is also obtained. The number of classes of invertible Boolean functions is calculated when these groups are considered. Some other problems are stated and solved."

A. Lehman (Washington, D.C.)

Tallini, Giuseppe 4705

Un'applicazione delle geometrie di Galois a questioni di statistica.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **35** (1963), 479–485.

Denote by $M_q^{(r,u)}$ ($u \geq r$, $r \geq 3$) an $r \times u$ matrix over GF(q) such that all subdeterminants of order r and $r-1$ are distinct from zero. The author shows that such matrices exist if and only if $q \geq 3$.

For $q \geq 3$ let $m(r, q)$ denote the largest possible value of u. The author finds the value of $m(r, q)$ for $q-1 \leq r$, for $r=3$ and arbitrary q and for $r>3$ if q is odd and large with respect to r. For arbitrary r and q the author finds lower and upper bounds for $m(r, q)$.

H. B. Mann (Madison, Wis.)

Hanani, Haim 4706

On covering of balanced incomplete block designs.

Canad. J. Math. **16** (1964), 615–625.

Given a set E of v elements and given positive integers k ($k \leq v$) and λ, we understand by balanced incomplete block designs (BIBD) $B[k, \lambda, v]$ a system of blocks (subsets of E) having k elements each such that every pair of elements of E is contained in exactly λ blocks. A set $F \subset E$ covers a given BIBD $B[k, \lambda, v]$ if the intersection of F with every block of $B[k, \lambda, v]$ is nonempty. A set F covering a given BIBD $B[k, \lambda, v]$ is denoted by $F(B[k, \lambda, v])$, or briefly by $F(B)$.

The author considers the following problem raised by H. J. Ryser. Given integers v, k and λ for which some $B[k, \lambda, v]$ exists, find the greatest integer $f(k, \lambda, v)$ such that for any BIBD $B[k, \lambda, v]$ every $F(B)$ has at least $f(k, \lambda, v)$ elements. A set $F(B[k, \lambda, v])$ having $f(k, \lambda, v)$ elements is called minimal.

We say that $v \in B(k, \lambda)$ if there exists a BIBD $B(k, \lambda, v)$. Some of the important results obtained are the following. Theorem 1: Let $v \in B(k, \lambda)$; then $f(k, \lambda, v) \geq (v-1)/(k-1)$. Theorem 2: Let $qu+1 \in B(q+1, \lambda)$. A necessary condition for $f(q+1, \lambda, qu+1) = u$ is the existence of a BIBD $B[q+1, \lambda, u]$. Theorem 3: If there exists a projective plane of order p, then a necessary and sufficient condition for $f(p+1, \lambda, pu+1) = u$ is the existence of a BIBD $B[p+1, \lambda, u]$. These results contain as a particular case the results of Ryser and Fulkerson for $k=3$, $\lambda=1$ [same J. **15** (1963), 371–396; MR **26** #6068]. The complete solution to Ryser's problem for $k=3$ is given by the following theorem. Theorem 7: For every λ, if $v \in B(3, \lambda)$, then

$$\begin{aligned} f(3, \lambda, v) &= \tfrac{1}{2}(v-1) \text{ if } v \equiv 3 \pmod 4 \text{ or if } v \equiv 1 \pmod 4 \\ &\qquad\qquad \text{and } \lambda \equiv 0 \pmod 2, \\ &= \tfrac{1}{2}(v+1) \text{ if } v \equiv 1 \pmod 4 \text{ and } \lambda \equiv 1 \pmod 2, \\ &= \tfrac{1}{2}v \text{ if } v \equiv 0 \pmod 2). \end{aligned}$$

The author makes extensive use of the construction of designs given by himself in [Ann. Math. Statist. **32** (1961), 361–386; MR **29** #4161].

S. S. Shrikhande (Bombay)

Weisner, Louis 4707

A Room design of order 10.

Canad. Math. Bull. **7** (1964), 377–378.

A Room design is in appearance a Greco-Latin square reduced to $(2n-1)^2$ cells by removing all entries xx, and with a blank replacing one out of each pair xy, yx. This paper gives a construction for such a square of order 10 ($n=5$). The whole square is determined by its first row.

T. G. Room (Sydney)

Carlitz, L. 4708

A binomial identity arising from a sorting problem.

SIAM Rev. **6** (1964), 20–30.

The author obtains generating functions and recursion relations for sums of the form

$$\sum \binom{i_0+i_1}{i_1}\binom{i_1+i_2}{i_2}\cdots\binom{i_{2r-1}+i_0}{i_0}$$

(summation on i_s satisfying $i_s + i_{r+s} = n_s$, $s = 0, 1, \cdots, r-1$), yielding elementary proofs and generalizations of a problem discussed by Brock, Slepian and Klamkin [same Rev. **4** (1962), 396–398, Problem 60–2].

L. K. Durst (Houston, Tex.)

Gould, H. W. 4709

The operator $(a^x\Delta)^n$ and Stirling numbers of the first kind.

Amer. Math. Monthly **71** (1964), 850–858.

The author proves the following operational formulas:

$$(*) \quad \left(a^x \underset{x,h}{\Delta}\right)^n f(x) = \frac{a^{nx}}{h^n}\sum_{j=0}^{n}(-1)^{n-j}\begin{bmatrix} n \\ j\end{bmatrix} q^{j(j-1)/2} f(x+jh),$$

where

$$\underset{x,h}{\Delta} f(x) = \frac{f(x+h)-f(x)}{h}$$

and

$$\begin{bmatrix} n \\ k\end{bmatrix} = \prod_{j=1}^{k}\frac{q^{n-j+1}-1}{q^j-1} \qquad (q = a^h);$$

$$(**) \quad \left(a^x \underset{x,h}{\Delta}\right)^n f(x) = $$

$$\frac{a^{nx}}{h^n}\sum_{k=0}^{n} h^k \underset{x,h}{\Delta^k} f(x) \sum_{j=k}^{n}(-1)^{n-j}\begin{bmatrix} n \\ j\end{bmatrix}\binom{j}{k} q^{j(j-1)/2}.$$

When $h \to \infty$, (**) yields

$$(a^x D_x)^n f(x) = a^{nx}\sum_{k=0}^{n}(-1)^{n-k} s(n,k)(\log a)^{n-k} D_x^k f(x),$$

where $s(n, k)$ is a Stirling number of the first kind; this in turn implies

$$(z^2D_z)^n g(z) = z^n \sum_{k=0}^{n} (-1)^{n-k} s(n, k)(zD_z)^k g(z).$$

Finally, a formula inverse to (*) is obtained.

L. Carlitz (Durham, N.C.)

ORDER, LATTICES

See also 4758, 4774, 4830, 4833, 5096, 5211.

Novák, Vítězslav 4710

A note on a problem of T. Hiraguchi. (Czech. Russian and English summaries)

Spisy Přírod. Fak. Univ. Brno **1962**, 147–149.

The author shows that if G is a non-empty partially-ordered set, with a a maximal element in G, and if there exists in G an element b such that $b<a$, and such that $x<a$ implies $x \leq b$, $x \in G$, then the element a is d-removable. Here an element $a \in G$ is said to be d-removable if $\dim(G-a)=\dim(G)$. The author gives two examples to show that the converse does not hold, and asserts that his results solve a problem of T. Hiraguchi [Sci. Rep. Kanazawa Univ. 4 (1955), no. 1, 1–20; MR **17**, 1045; errata, MR **19**, 1431].

Blyth, T. S. 4711

Matrices over ordered algebraic structures.

J. London Math. Soc. **39** (1964), 427–432.

Three generalisations of a result due to R. D. Luce [Proc. Amer. Math. Soc. **3** (1952), 382–388; MR **14**, 347] are first established. The following is typical. Let $\mathfrak{M}(\mathscr{G})$ be the set of matrices of order $n \times n$ with elements from an ordered groupoid $\mathscr{G}$ which is also a lattice with respect to the ordering, and define $A \leq B \Leftrightarrow a_{ij} \leq b_{ij}$. $AB=C \Leftrightarrow \bigcup_j a_{ij}b_{jk}=c_{ik}$; then $\mathfrak{M}(\mathscr{G})$ is residuated if and only if $\mathscr{G}$ is residuated.

Secondly, the following generalisation of results due to J. H. M. Wedderburn [Ann. of Math. (2) **35** (1934), 185–194] and the reviewer [Proc. Glasgow Math. Assoc. **6** (1963), 49–53; MR **26** #6092] is obtained. If $\mathscr{H}$ is a groupoid which is a union semi-lattice and in which multiplication, though not necessarily commutative, is distributive with respect to unions and which has universal and null elements which are, respectively, a multiplicative unit and a zero, then the set $\mathscr{C}$ of multiplicatively complemented elements ($a \cup a'=1, aa'=0$) in $\mathscr{H}$ forms a Boolean algebra. If I is the Kronecker matrix and $A, B \in \mathfrak{M}(\mathscr{H})$ such that $AB=I=B^TA^T$, then $B=A^T$ and $BA=I$. The necessary and sufficient conditions that A have an inverse are that $A \in \mathfrak{M}(\mathscr{C})$ and $AA^T=I$.

D. E. Rutherford (St. Andrews)

Insel, Arnold J. 4712

A relationship between the complete topology and the order topology of a lattice.

Proc. Amer. Math. Soc. **15** (1964), 847–850.

A complete subset of a lattice L is a non-empty subset $C \subset L$ such that each non-empty subset of C possesses a supremum s and an infimum t in L and $s, t \in C$. The smallest topology for L in which the complete subsets of L are closed is called the complete topology and is designated by $K(L)$. $O(L)$ is the order topology for L. A net (i.e., a directed set of elements) in a set X is called universal if it is eventually in or eventually outside of any subset of X [J. L. Kelley, *General topology*, Van Nostrand, Toronto, Ont., 1955; MR **16**, 1136]. L has the complete separation property if for any two distinct elements x, y of L there exists a finite collection $\mathbf{C}$ of complete subsets of L such that $\mathbf{C}$ covers L and for each $C \in \mathbf{C}$, $x \in C \Rightarrow y \notin C$.

The main result: For any lattice L the following are equivalent. (1) $K(L)$ is Hausdorff; (2) Every universal net in L is order-convergent; (3) Every net in L has an order-convergent subnet; (4) L has the complete separation property. If $K(L)$ is Hausdorff, then $O(L)=K(L)$, and $O(L)$ is compact.

B. M. Schein (Šaĭn) (Saratov)

Halmos, Paul R. 4713

★Lectures on Boolean algebras.

Van Nostrand Mathematical Studies, No. 1.

D. Van Nostrand Co., Inc., Princeton, N.J., **1963**. v+147 *pp.* $2.95.

This is a very good introduction to the theory of Boolean algebras. It contains a discussion of all fundamental notions like subalgebras, homomorphisms, ideals, filters, complete algebras, σ-complete algebras, etc. α-complete algebras for $\alpha > \aleph_0$ are not discussed. The book contains almost no references to the literature.

Contents: Boolean rings. Boolean algebras. Fields of sets. Regular open sets. Elementary relations. Order. Infinite operations. Subalgebras. Homomorphisms. Free algebras. Ideals and filters. The homomorphism theorem. Boolean σ-algebras. The countable chain condition. Measure algebras. Atoms. Boolean spaces. The representation theorem. Duality for ideals. Duality for homomorphisms. Completion. Boolean σ-spaces. The representation of σ-algebras. Boolean measure spaces. Incomplete algebras. Products of algebras. Sums of algebras. Isomorphisms of factors. Isomorphisms of countable factors. Retracts. Projective algebras. Injective algebras.

R. Sikorski (Warsaw)

Mori, Tôru 4714

On existence of some subalgebra of a given Boolean algebra which is countably infinite.

Yokohama Math. J. **11** (1963), 41–50.

The author proves the following known results. (1) Every countably infinite Boolean algebra contains an infinite atomic subalgebra. (2) Every infinite complete Boolean algebra has a subalgebra isomorphic to the field of all subsets of a countably infinite set.

F. M. Yaqub (Davis, Calif.)

Scognamiglio, Gennaro 4715

Filze di funzioni algebriche booleane e matrici associate.

Ricerca (Napoli) (2) **14** (1963), *gennaio-aprile*, 22–35.

A method is given for computing the composite of functions of a certain type on Boolean algebras.

F. D. Veldkamp (New Haven, Conn.)

GENERAL MATHEMATICAL SYSTEMS

Grätzer, G. [Grätzer, György] 4716

Free algebras over first order axiom systems. (Russian summary)

Magyar Tud. Akad. Mat. Kutató Int. Közl. **8** (1963), 193–199.

The author clarifies and generalizes the notion of a free algebra of a given kind with β generators, and derives some theorems about them, in particular, existence theorems. The algebras have a not necessarily finite number of finitary operations. A Σ-algebra has operations which satisfy an axiom system Σ. The axioms of Σ are expressible as sentences of the first-order predicate calculus and may involve existential quantifiers as well as universal quantifiers.

If axioms with existential quantifiers occur, the usual definition of a homomorphism of one Σ-algebra into another is replaced by that of a Σ-homomorphism, which is required to preserve all inverses of a set of elements relative to such an operation, in addition to preserving the results of the operation. The free Σ-algebra $F(\beta)$ is then defined to be a Σ-algebra with a set of β generators $\{x_\gamma\}$ such that if A is any Σ-algebra, then any mapping of the generators $\{x_\gamma\}$ onto elements $\{a_\gamma\}$ of A can be extended to a Σ-homomorphism of $F(\beta)$ into A. Such homomorphisms need not be unique. If they are unique, the algebra is called free in the stronger sense.

Some of the theorems about free Σ-algebras $F(\beta)$ are the following. If $F(\beta)$ exists, it is unique up to isomorphism. If $F(\beta)$ exists, so does $F(\delta)$ for all $\delta<\beta$. If $F(n)$ exists for all $n<\omega$, so does $F(\omega)$. If $F(\omega)$ exists, so does $F(\beta)$ for every cardinal β. The author gives a necessary and sufficient condition for the existence of the free Σ-algebra $F(\beta)$ which is too complicated to give here. It requires that there exist a finite upper bound to the number of inverses of a finite set of elements relative to a given compound operation.

If K is a class of algebras with operations of the same type, one can define the free K-algebra $F(\beta)$ with β generators in the usual way without using the notion of inverse. The author considers the relationship between these two notions of free algebra in the case where K is the class of all Σ-algebras. To do this, he introduces the notion of an axiom system with the inverse-preserving property. Finally, he gives a simplified necessary and sufficient condition for the existence of the free Σ-algebra $F(\beta)$ in the classical case where no existential quantifiers occur in the axiom system Σ.

This paper contains no proofs. Proofs of the results stated are to appear in a later publication.

O. Frink (University Park, Pa.)

Grätzer, György 4717

On the Jordan-Hölder theorem for universal algebras. (Russian summary)

Magyar Tud. Akad. Mat. Kutató Int. Közl. **8** (1963), 397–406 (1964).

A. W. Goldie [Proc. London Math. Soc. (2) **52** (1950), 107–131; MR **12**, 238] derived forms of the Jordan-Hölder-Schreier theorems for normal series and principal series of subalgebras of an algebra with finitary operations. Here the author simplifies Goldie's proofs. This allows him to prove various generalizations of the theorem.

A normal series $A=A_0\supset A_1\supset\cdots\supset A_n$ of subalgebras of A has associated with it a series $\theta_0,\theta_1,\cdots,\theta_n$ of congruence relations, θ_i being a relation on A_i. The first form of the theorem states that two normal series have isomorphic refinements if their congruence relations are weakly associable over A_n. In a principal series, each A_i is a normal subalgebra, and θ_i is a congruence relation over A, rather than over A_i. The theorem for principal series has the same form as that for normal series.

Noting that the operations of A play little role in the proof, the author considers more generally a class of subsets of A to take the place of the subalgebras, and of mappings between these subsets called homomorphisms. The subsets and homomorphisms satisfy a set of nine axioms. With the proper definitions in terms of these notions, the Jordan-Hölder theorems for normal and principal series are then derived. Only eight axioms are required for normal series.

The generalized theorems are then applied to multialgebras, in which the operations are multivalued, sending ordered sets into subsets rather than elements. They are also applied to ideals in semigroups and groupoids, to multigroups, and to lattices with congruence relations determined by standard ideals.

O. Frink (University Park, Pa.)

THEORY OF NUMBERS

See also 4761.

Brun, Viggo 4718

Euclidean algorithms and musical theory.

Enseignement Math. (2) **10** (1964), 125–137.

Andrew, A. L. 4719

Two properties of integers.

Math. Gaz. **48** (1964), 285–287.

The set of integers $n_0<n_1<\cdots<n_N$, $n_0>1$, is said to have property P if every x, $0<x\leq 1$, can be written as $\sum_1^\infty(1/m_i)$, m_i an integer, and m_{i+1}/m_i belongs to the set.

Generalizing a problem of the Miklós Schweitzer Competition of 1957, the author shows that P holds if and only if, for some k, $n_i=2^{i+1}$, $i=0,1,\cdots,k-1$, and $n_k\leq 2^k$.

P. Ungar (New York)

de Bruijn, N. G. 4720

Some direct decompositions of the set of integers.

Math. Comp. **18** (1964), 537–546.

Let S be the set of non-negative integers whose representation in the scale of 4 does not contain the digits 2 or 3. A pair (M,N) of non-negative integers is called "good" if every non-negative integer can be represented uniquely in the form Ms_1-Ns_2, where each s_i is in S. In an earlier paper [Publ. Math. Debrecen **1** (1950), 232–242; MR **12**, 590] the author listed all good pairs for $1\leq M\leq N\leq 100$. New numerical data obtained with the aid of an IBM 620 computer suggested the existence of three infinite sequences of good pairs, namely $(11,22\cdot 4^k+1)$, $(1,10\cdot 4^k-3)$, and $(2^{2k+2}-3,\ 2^{2k+1}-1)$, where $k=0,1,2,\cdots$. Using

oriented graphs whose vertices are the integers, the author proves that each such pair is indeed good.
T. M. Apostol (Pasadena, Calif.)

Carlitz, L. 4721
Functions and polynomials (mod p^n).
Acta Arith. **9** (1964), 67–78.
Let p be a prime and n an integer $\geqq 1$. Let Z_n denote the ring of integers (mod p^n). A function f over Z_n is a mapping $Z_n \to Z_n$; two such functions, f and g, are equal if $f(a) \equiv g(a)$ (mod p^n) for all $a \in Z_n$. When $n=1$, it is well known that every function is equal to a (not necessarily unique) polynomial, but this is not true for $n>1$.

The author gives the following necessary and sufficient conditions for a function to be equal to a polynomial over Z_n for general n. Theorem 2: $\Delta^r f(0) \equiv 0$ (mod $p^{\nu(r)}$) for $0 \leqq r < p^n$, where $\nu(r) = \min(n, \mu(r))$, $p^{\mu(r)} \| r!$. Theorem 3: $f(x+kp) \equiv \sum_{j=0}^{n-1} (kp)^j f_j(x)$ (mod p^n), where k is an arbitrary integer and the f_j are functions over Z_n. Theorem 4: $\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} f(c+sp) \equiv 0$ (mod $p^{\nu(rp)}$) for $0 \leqq r < p^{n-1}$ and $0 \leqq c < p$. Examples are given, and corresponding theorems for functions of two variables are stated without proof.
W. J. LeVeque (Ann Arbor, Mich.)

Chowla, S. 4722
On the inequality $|x^2-y^2-2xyk| \geqq 2k$ (x, y, k odd).
Norske Vid. Selsk. Forh. (*Trondheim*) **34** (1961), 91.
In this note the author proves that if k is a positive integer, then the equation $u^2-(k^2+1)v^2=m$ has no solutions in positive integers u, v if $|m|<2k$, $|m| \neq t^2$. This strengthens the inequality of the title, which is an unpublished result of Davenport.
R. J. Crittenden (Providence, R.I.)

Cohen, Eckford 4723
Some analogues of certain arithmetical functions.
Riv. Mat. Univ. Parma (2) **4** (1963), 115–125.
For positive integers n and r define the unique factoring of n into relatively prime integers $\alpha_r(n)$ and $\beta_r(n)$ by the following rule: If p^k is the highest power of a prime dividing n, let p^k be a factor of $\alpha_r(n)$ or $\beta_r(n)$ according as $k \leqq r$ or $k > r$. Define $\phi_r(n) = \beta_r(n)\phi(\alpha_r(n))$, where ϕ is the Euler totient function. The author studies $\phi_r(n)$ and related functions. This involves the use of the analogue of the Möbius function defined as the multiplicative function such that $\mu_r(p^e) = p^r, -1$, or 0 according as $e=r+1$, $e=1$, or otherwise. The following typical results are cited. First, $\phi_r(n) = n \sum \mu_r(d)/d$, where the sum is taken over all divisors d of n. Also $\sum_{n \leqq x} \phi_r(n)/n = a_r x + O(\log^2 x)$, where $a_r = \prod (1-p^{-2}+p^{-r-2})$, the product being taken over all primes. The sum $\sum_{n \leqq x} \phi_r(n) = \frac{1}{2} a_r x^2 + O(x \log^2 x)$. These estimates are obtained by using classical properties of Dirichlet series.
I. Niven (Eugene, Ore.)

Rotkiewicz, A. 4724
Sur les nombres pseudopremiers triangulaires.
Elem. Math. **19** (1964), 82–83.
From the author's introduction: "Théorème: Il existe une infinité de nombres triangulaires qui sont pseudopremiers."

Shafaat 4725
On equivalence relations over integers defined by certain types of equations.
J. Natur. Sci. and Math. **4** (1964), 93–101.
Let S, Σ be subsets of the integers, and let $f(x,y)$ be a polynomial (during most of the paper $f(x,y)$ is xy or $x-y$). The author considers some cases where "$f(x,y)$ belongs to Σ" is an equivalence relation between elements x, y of S.
B. J. Birch (Manchester)

Spira, Robert 4726
Polynomial representations of sums of two squares.
Amer. Math. Monthly **71** (1964), 760–766.
Let $P_0, \cdots, P_k$ be primes congruent to 1 modulo 4, expressed as sums of squares $P_i = a_i^2 + b_i^2$. The author obtains polynomials in the a's and b's, which give all the expressions of $P_0^{t_0} \cdots P_k^{t_k}$ as a sum of two squares.
B. J. Birch (Manchester)

Pan, Cheng-dong [Pan, Cheng-Tung] 4727
On the representation of an even integer as the sum of a prime and an almost prime.
Acta Math. Sinica **12** (1962), 95–106 (*Chinese*); *translated as Chinese Math.* **3** (1963), 101–112.
Reproduction of an article already published in Sci. Sinica **11** (1962), 873–888 [MR **27** #1427].

Sunyer Balaguer, F. 4728
On families of infinite sets of natural numbers. (**Spanish**)
Math. Notae **19** (1964), 133–135.
Théorème 1: Il existe une famille dénombrable F d'ensembles infinis de nombres naturels, ayant deux à deux au plus v éléments communs et tels qu'il n'existe aucun ensemble qui ait avec chaque ensemble de F au moins un élément commun et au plus v éléments communs.

Théorème 2: Soit F une famille dénombrable d'ensembles infinis de nombres naturels, ensembles qui ont—deux à deux—au plus v éléments communs. Il existe alors un ensemble qui a avec chaque ensemble de F au moins un élément commun et au plus $v+1$ éléments communs.

Ces deux théorèmes généralisent quelques résultats de W. Sierpiński.
S. Marcus (Bucharest)

Birch, B. J. 4729
Diagonal equations over $\mathfrak{p}$-adic fields.
Acta Arith. **9** (1964), 291–300.
Let K be a $\mathfrak{p}$-adic field. It is well known that to each positive integer d there is a constant $\varphi(K,d)$ such that every form of degree d over K in $n \geqq \varphi(K,d)$ variables has a non-trivial zero in K. It has long been conjectured that $\varphi(K,d) = d^2+1$, and this has been verified for $d=2, 3$ and for certain other d, provided the residue class field of K is sufficiently large. Brauer [Bull. Amer. Math. Soc. **51** (1945), 749–755; MR **7**, 108] has shown that a $\varphi(K,d)$ can be determined provided we are given a constant $\psi(K,d)$ such that every diagonal form of degree d over K in $n \geqq \psi(K,d)$ variables has a non-trivial zero in K. When d is a prime the reviewer [Michigan Math. J. **4** (1957), 85–95; MR **18**, 793] and Gray [Doctoral Dissertation, Univ. of Notre Dame, Notre Dame, Ind., 1958] have shown $\psi(K,d) \leqq d(d-1)+1$. When K is the p-adic completion of

the rationals, Davenport and the reviewer [Proc. Roy. Soc. Ser. A **274** (1963), 443–460; MR **27** #3617] have shown that $\psi(K, d) \leq d^2+1$. In this paper the author determines a function $\psi(d)$, which is essentially a dth power, and shows that $\psi(K, d) \leq \psi(d)$ for all $\mathfrak{p}$-adic fields K. The proof involves a careful examination of the expansion of the series $(1+\sum_{i=1}^{\infty} \pi^i y_i)^d$, where π is a prime of K. The expansion varies greatly with the ramification index of K.
D. J. Lewis (Ann Arbor, Mich.)

Chawla, L. M. 4730
On some abelian groups of equivalence classes over integers and the related favourable Diophantine equations.
J. Natur. Sci. and Math. **4** (1964), 87–92.
The author describes a group he calls $J^{(\mathrm{mod}\ m)}$; this is the multiplicative group of non-zero rational numbers modulo the subgroup of mth powers. *B. J. Birch* (Manchester)

Davenport, H.; Lewis, D. J. 4731
Non-homogeneous cubic equations.
J. London Math. Soc. **39** (1964), 657–671.
Let $\phi(\mathbf{x})$ be a cubic polynomial in n variables with integral coefficients, and let C be the part of ϕ that is homogeneous of degree 3. Define the invariant $h(C)$ as the codimension of the greatest rational linear space contained in the hypersurface $C=0$, so that, for instance, $h(C)=n$ if $C=0$ has no rational point. The authors prove that if $h(C) \geq 17$ and if $\phi(\mathbf{x}) \equiv 0\ (m)$ is soluble for all m, then $\phi(\mathbf{x})=0$ has a solution in integers. Combining this with a theorem of Watson, it follows that $\phi=0$ is soluble in integers whenever $n \geq 20$, $h(C) \geq 4$, and the obvious necessary condition that $\phi \equiv 0\ (m)$ be soluble for all m is satisfied. The main part of the proof is essentially routine, using machinery developed by the authors [Amer. J. Math. **84** (1962), 649–665; MR **26** #2403] and by Davenport alone [Proc. Roy. Soc. Ser. A **272** (1963), 285–303; MR **27** #5734], but it should be added that their version of the machinery is exceedingly well-oiled. A new feature is the curious theorem that if a p-adic integral cubic ψ in at least 15 variables has a p-adic integral zero, then ψ has a non-singular p-adic zero; this can be false for a cubic polynomial in 14 variables. *B. J. Birch* (Manchester)

Muwafi, Amin 4732
Simultaneous quadratic and linear Diophantine equations.
J. Natur. Sci. and Math. **4** (1964), 163–165.
Theorem on a problem treated by L. E. Dickson and G. Pall [Pall, Duke Math. J. 8 (1941), 173–180; MR **2**, 251].
B. Stolt (Stockholm)

Oppenheim, A. 4733
The rational integral solution of the equation $a(x^3+y^3)=b(u^3+v^3)$ and allied Diophantine equations.
Acta Arith. **9** (1964), 221–226.
A method of solving the Diophantine equation

$$a(x^3+y^3) = b(u^3+v^3), \qquad (a, b) = 1,\ ab \neq 0,$$

in rational integers x, y, u, v, is presented. Apart from exceptional cases when $a(x+y)^3 = b(u+v)^3$, the remaining solutions are associated with those of a diagonal ternary quadratic equation

$$X^2+3Y^2-3nZ^2 = 0$$

by means of the expression

$$4g^2\{a(x^3+y^3)-b(u^3+v^3)\}-k(a\lambda^3-b\mu^3)\{X^2+3Y^2-3nZ^2\},$$

which vanishes identically under the substitutions

$$(X, Y, Z) = (gk, e, f), \qquad n = ab\lambda\mu,$$
$$x+y = k\lambda, \qquad u+v = k\mu,$$
$$g(x-y) = e\lambda+bf\mu^2, \qquad g(u-v) = e\mu+af\lambda^2.$$

It is indicated that a similar method applies to more general equations of the types:

(1) $$aL^n(x, y)Q(x, y) = bL^n(u, v)Q(u, v)$$

when n is an integer $\neq -2$, L is an integral linear form and Q is an integral quadratic form,

(2) $$p(x, u)Q(x, y) = q(x, u)Q(u, v),$$

where p and q are integers, depending on x and u.
J. H. H. Chalk (Toronto, Ont.)

Swinnerton-Dyer, H. P. F. 4734
Rational zeros of two quadratic forms.
Acta Arith. **9** (1964), 261–270.
Let f, g be quadratic forms in 11 variables defined over the rationals; and suppose that for all real λ, μ (not both 0) the form $\lambda f+\mu g$ is indefinite. The author proves that under these conditions f, g have a common non-trivial rational zero. The reality condition is obviously necessary. Previously, Mordell [Abh. Math. Sem. Univ. Hamburg **23** (1959), 126–143; MR **21** #3378] proved a related result assuming 13 variables. The idea behind the proof is to show that, in general, f can be expressed in the form $x_1x_2+\cdots+x_7x_8+h(x_9,\ x_{10},\ x_{11})$ in such a way that $g(x_1, 0, x_3, 0, x_5, 0, x_7, 0, \cdots, 0) = k(x)$ has a non-trivial rational zero. The agreement that f can be so expressed is very intricate and involves some careful byplay between the real condition and the p-adic conditions necessary for the solubility of $k(x)$. Ad hoc arguments are needed when f cannot be so expressed.

Since quadratic forms in 4 variables over the rationals may not have non-trivial rational zeros, clearly 9 variables are needed. On the other hand, Dem'janov [Izv. Akad. Nauk SSSR Ser. Mat. **20** (1956), 307–324; MR **18**, 284] (see also Birch, Lewis and Murphy [Amer. J. Math. **84** (1962), 110–115; MR **25** #52]) has shown that two forms in 9 variables over the rationals have a common non-trivial p-adic zero for each p. Thus one might hope that 9 variables would suffice. The author discusses the possibilities of proving the existence of common non-trivial rational zeros of f, g assuming 9 or 10 variables and possibly a stronger real condition.
D. J. Lewis (Ann Arbor, Mich.)

Williams, K. S. 4735
A note on the quadratic form $ax^2+2bxy+cy^2$.
Math. Gaz. **48** (1964), 290–291.
The author gives a simple geometric proof of the following well-known result. Let Λ be a 2-dimensional lattice of

determinant $d(\Lambda) \neq 0$. Suppose $a > 0$, $ac > b^2$. Then there is a point $(u, v) \neq (0, 0)$ of the lattice Λ such that

$$au^2 + 2buv + cv^2 \leqq \frac{2}{\sqrt{3}} \sqrt{((ac-b^2)d(\Lambda))}.$$

D. J. Lewis (Ann Arbor, Mich.)

Forder, H. G. 4736

A simple proof of a result on diophantine approximation.

Math. Gaz. **47** (1963), 237–238.

Let p_n/q_n denote the $(n+1)$st convergent in the simple continued fraction for an irrational number ξ, and let a_n denote the $(n+1)$st quotient, so that $q_{n+1} = a_n q_n + q_{n-1}$. A famous theorem of Borel states that at least one of any three successive convergents satisfies

$$\left| \frac{p_n}{q_n} - \xi \right| < \frac{1}{\sqrt{5}q_n^2}.$$

The author gives an elegant and simple proof along the following lines: If the inequality is false for $n = r-1, r, r+1$, the fact that the convergents are successively closer to ξ implies that both ratios q_{r-1}/q_r and q_{r+1}/q_r satisfy the quadratic inequality $t^2 - \sqrt{5}t + 1 \leqq 0$ and therefore lie in the open interval joining $\frac{1}{2}(\sqrt{5} \pm 1)$. This interval has length 1, so $a_r = q_{r+1}/q_r - q_{r-1}/q_r < 1$, a contradiction.

T. M. Apostol (Pasadena, Calif.)

Wright, E. M. 4737

Approximation of irrationals by rationals.

Math. Gaz. **48** (1964), 288–289.

Refer to the foregoing review [#4736]. If k is a positive integer, let $\alpha = \frac{1}{2}(k^2+4)^{1/2}$. Using Forder's simple method, the author proves that if

$$\left| \frac{p_n}{q_n} - \xi \right| \geqq \frac{1}{2\alpha q_n^2}$$

for $n = r-1,\ r,\ r+1$, then $a_r < k$. He then uses this to prove that for any ξ either (i) there is an infinity of rational approximations p/q to ξ such that $q|q\xi - p| < (k^2+4)^{-1/2}$, or (ii) $a_r \leqq k-1$ for all $r \geqq$ some r_0.

T. M. Apostol (Pasadena, Calif.)

Krätzel, E. 4738

Ein Gitterpunktsproblem.

Acta Arith. **10** (1964/65), 215–223.

Denote by $R'_{2k,2}(x)$ the number of integer lattice points (a, b) in the region $a^{2k} + b^{2k} \leqq x$, adjusted if necessary by counting any lattice points on the boundary with multiplicity $\frac{1}{2}$; let V be the volume of the region. Write $F * G$ for the convolution $\int F(\tau)G(t-\tau)\,d\tau$, and $S(x)$ for $\pi^{-1} \sum (\sin 2\pi nx)/n$. The author shows that

$$R'_{2k,2}(x) = \frac{d}{dx}[S(x^{1/2k}) * S(x^{1/2k})],$$

generalising the classical Hardy identity for $R'_{2,2}$; many details are omitted. He attempts to estimate $|R'_{2k,2} - V|$, but makes an error at the head of page 221. The result he is trying to prove was apparently obtained by D. Cauer [Univ. Göttingen, Göttingen, 1914]; far better results may be deduced from the theorem of van der Corput (see Landau, *Vorlesungen über Zahlentheorie*, Hirzel, Leipzig, 1927, esp. Teil 8, Kap. 11–12).

B. J. Birch (Manchester)

Shimura, Goro 4739

On the field of definition for a field of automorphic functions.

Ann. of Math. (2) **80** (1964), 160–189.

As shown in detail by the author in a series of earlier publications [see, for example, same Ann. (2) **79** (1964), 369–409; MR **28** #2104], the study of families of polarised abelian varieties with prescribed algebras of endomorphisms is directly related to certain discontinuous groups and the fields of automorphic functions belonging to these groups. This study is continued here, and leads to the determination of the arithmetical field of definition for the field of moduli of such families in several important cases, viz., the following: (i) The given algebra of endomorphisms L is a totally real algebraic number-field F; (ii) L is a totally indefinite quaternion algebra over F; (iii) L is a totally imaginary quadratic extension of F. In each one of these cases, the family of abelian varieties can be defined by data consisting of a vector-space V over L, an anti-hermitian form T on V, and a lattice M in V; these data determine a semisimple Lie group G and an "arithmetical" discrete subgroup Γ of G; if H is the Riemannian symmetric space belonging to G (with its natural complex structure), the field of moduli of the family in question is in many cases the field of automorphic functions for $\Gamma\backslash G$, or else it is a suitably determined subfield of that field. The purpose of the paper is to show, under suitable restrictions (sometimes of rather delicate arithmetical nature) on the lattice M, that the arithmetical field of definition for the field of moduli is the rational number-field in cases (i) and (ii) and in a certain subcase of (iii), and that otherwise, in case (iii), it is an unramified class-field over a totally imaginary quadratic extension of a totally real field, whose explicit description is given in terms of class-field theory. The proof relies heavily on previous work by the same author, and particularly upon his earlier results on the arithmetical theory of the groups in question. Case (iii), involving, as it does, a generalization of the main result in the theory of complex multiplication for abelian varieties [cf. the author and Y. Taniyama, *Complex multiplication of abelian varieties and its applications to number theory*, Chapter IV, § 15, Math. Soc. Japan, Tokyo, 1961; MR **23** #A2419] and of the concept of "dual CM-types" [ibid., Chapter II, § 8], is by far the most difficult one; a decisive step in the proof consists in showing that a "generic" abelian variety in the family under consideration can be specialized, over an arbitrarily large number-field, to an abelian variety with complex multiplication [as defined, loc. cit., Chapter II, § 6], to which the theory of complex multiplication can then be applied.

A. Weil (Princeton, N.J.)

Ankeny, N. C.; Onishi, H. 4740

The general sieve.

Acta Arith. **10** (1964/65), 31–62.

The authors derive both upper and lower bounds for a large class of sieve problems, on the basis of Selberg's method. Let T be an infinite set of primes, and let T_Y denote the set of all $p \in T$ with $p \leqq Y$. For each N let

S_N be a given set of positive integers. With a fixed $\lambda > 0$, it is required to obtain upper and lower bounds for $M(S_N, T_{N^\lambda})$, i.e., the number of integers $m \in S_N$ which are not divisible by any $p \in T_{N^\lambda}$.

In the following, $\alpha, \beta, \delta, C_1, C_2$ are fixed positive numbers. It is assumed that for each N there exists a positive multiplicative function f_N such that, for every square-free d, the number of m with $m \in S_N$, $m \equiv 0 \pmod{d}$, equals $N(f_N(d))^{-1} + R_d(N)$, where $(f_N(p))^{-1} < 1 - \delta$ for all $p \in T$, and

$$\sum_{d \leq N^\beta} (6\delta^{-2})^{\nu(d)} |R_d| = O(N(\log N)^{-\alpha-2})$$

($\nu(d)$ represents the number of prime divisors of d). Moreover, it is assumed that

$$\sum_{p<X} p(f_N(p))^{-1} < C_1 X (\log X)^{-1} \quad \text{for } X < \log N,$$

$$\sum_{p<X} (p(f_N(p))^{-1} - \alpha) < C_2 X (\log X)^{-2} \text{ for } \log N < X < N^\lambda.$$

With the abbreviation

$$B_\alpha(N) = \Gamma(\alpha) \prod (1 - (f(p))^{-1})(1 - p^{-1})^{-\alpha},$$

where p runs through T_{N^λ}, the main result is as follows.

The expression

$$M(S_N, T_{N^\lambda})(B_\alpha(N))^{-1} N^{-1} (\log N)^\alpha$$

lies between positive bounds depending on α, β, λ only. The bounds which are actually obtained contain $J_\alpha(\beta/2\lambda)$ and $G_\alpha(\beta/2\lambda)$, where J_α and G_α are related to special solutions of the differential-difference equation

$$\tau'(u) = -u^{-1}\{\alpha\tau(u-1) - (\alpha-1)\tau(u)\}.$$

Some applications are given. Let $d_1, \cdots, d_a$ be distinct integers which do not form a complete set of residues for any prime, and $K(\chi) = \prod_{i=1}^a (\chi + d_i)$. Then if a is sufficiently large, there exist infinitely many n for which $\nu(K(n)) < a(\log a + 2)$. Another application is that if we assume the extended Riemann hypothesis, then there exist infinitely many primes q such that $q+2$ has at most three prime factors.

{The reviewer was unable to check as many details as he wished, because of many minor errors and omissions. For example, the very crucial Lemma 3.1 is obscure in several respects, and instead of a proof, there is a reference that seems to be irrelevant.} *N. G. de Bruijn* (Eindhoven)

Fluch, Wolfgang 4741

Zur Abschätzung von $L(1, \chi)$.

Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II **1964**, 101–102.

The author shows that there exists an absolute constant c with the following property: For every integer $k \geq 2$ and for every real non-principal character mod k, we have either $L'(1, \chi) > 1$ or $L(1, \chi) \geq c/(\log k)$ (or both).

N. G. de Bruijn (Eindhoven)

Grosswald, E. 4742

A proof of the prime number theorem.

Amer. Math. Monthly **71** (1964), 736–743.

The proof uses the Riemann-Lebesgue theorem for an infinite interval. It is similar in principle to the one outlined (in small print) on pp. 34–35 of the reviewer's Cambridge Tract [*The distribution of prime numbers*, Cambridge Univ. Press, London, 1932], but works with $\Pi(x)$ and $\log \zeta(s)$ (in the notation of the Tract) instead of $\psi(x)$ and $-\zeta'(s)/\zeta(s)$. It is claimed that the proof, though not as short as some or as elementary as others, "seems to have the advantage of great clarity". Such judgments are highly personal; the reviewer merely records his inability to share this view of the proof as presented. Some observations of a more objective kind may, however, be offered. (1) It seems to be assumed (statement and proof of Lemma 1) that, if $f(t)$ is differentiable in a finite interval $[\varepsilon, T]$, then $|f'(t)|$ is integrable in $[\varepsilon, T]$. (2) On p. 740 the wording suggests (in two places) that uniform convergence is sufficient to justify term-by-term integration, even when the range of integration is infinite. (3) The second reference to the Euler-Maclaurin sum formula on p. 737 might well be amplified; it is not clear (to the reviewer) how this formula facilitates the passage from $\zeta(1+it) \neq 0$ to an explicit lower estimate of $|\zeta(1+it)|$ or a uniform upper estimate of $|\log \zeta(\sigma + it)|$ for $1 \leq \sigma \leq c$ (a need not explicitly recognized in the application of Cauchy's theorem at the bottom of p. 740). (4) The argument under (c) on pp. 741–742 is basically sound (though needlessly complicated), but contains statements that are incorrect or inadequate as they stand.

A. E. Ingham (Cambridge, England)

Wohlfahrt, Klaus 4743

Über die Nullstellen einiger Eisensteinreihen.

Math. Nachr. **26** (1963/64), 381–383.

The author computes the values of the absolute elliptic invariant $J(\tau_0)$ for the zeros of the normalized Eisenstein series (constant term 1) E_k for $k = 12, 16, 18, 20, 22, 26$. These values are rational numbers, but possibly $1728J(\tau_0) \notin Z$. Furthermore, the explicit quadratic equation is given for $J(\tau)$, where τ stands for the two zeros of E_{24} in a fundamental domain.

O. F. G. Schilling (Lafayette, Ind.)

Maier, Wilhelm 4744

Aus der analytischen Zahlentheorie.

S.-B. Sächs. Akad. Wiss. Leipzig Math.-Natur. Kl. **105**, *no.* 4, 15 *pp.* (1963).

An expository article on the distribution of primes, and the analytic functions which have played an important role in research in that subject.

W. J. LeVeque (Ann Arbor, Mich.)

Shanks, Daniel 4745

On maximal gaps between successive primes.

Math. Comp. **18** (1964), 646–651.

Let $p(n)$ be the first prime that follows a run of n or more consecutive composite integers. The asymptotic behaviour of $p(n)$ for large n is not known. The author gives a heuristic probability argument which suggests the relation $\log p(n) \sim n^{1/2}$. By referring to tables on the distribution of primes he shows that $1.166 \leq n^{-1/2} \log p(n) \leq 1.729$ for $n \leq 219$.

T. M. Apostol (Pasadena, Calif.)

Sierpinski, W. [Sierpiński, Wacław] 4746

Les binômes x^2+n et les nombres premiers.

Bull. Soc. Roy. Sci. Liège **33** (1964), 259–260.

This note contains a brief, elementary proof of the theorem that for each positive integer m there exists a positive integer n such that x^2+n is a prime for more than m distinct values of x. The proof is based on the inequality $\pi(n)>n/(12\log n)$, $n\geq 2$. *R. D. James* (Vancouver, B.C.)

Cohen, Eckford 4747

Errata: "A corollary of the Goldbach conjecture".

Duke Math. J. **30** (1963), 683.

The author deletes the Introduction of an earlier paper [same J. **29** (1962), 625–629; MR **26** #88; errata, MR **26**, p. 1544] and replaces it by another.

Moser, W. O. J. 4748

A generalization of some results in additive number theory.

Math. Z. **83** (1964), 304–313.

The author proves the following two results. (I) Let $0\leq f(x)\leq 1$, $0\leq g(x)\leq 1$, $f(x)+g(x)=1$ for $0\leq x\leq 1$, $f(x)=g(x)=0$ otherwise. Let $\alpha=\inf_{0\leq x\leq 1}(1/x)\int_0^x f(y)dy$, $\alpha_0=\int_0^1 f(y)\,dy$. For $0\leq t\leq 1$ put $D(t)=\int_0^{1-t}f(x)g(x+t)dx$, $C=\alpha_0+\sup D(t)$. The author proves

$$C>\alpha+\frac{1+\sqrt{\alpha}+\alpha}{(1+\sqrt{\alpha})^2}\alpha(1-\alpha).$$

(II) Let $f(x)$, $g(x)$ and α_0 be defined as in (I). Let $D(t)=\int_0^1 f(x)g(x+t)dx$. Then if $\alpha_0=\frac{1}{2}$, we have $\max D(t)>0.17675n$. (I) implies a similar result for sequences of integers, and the author states that it implies Kasch's inequality [F. Kasch, same Z. **62** (1955), 368–387; MR **17**, 712]. (II) implies a result by L. Moser [Acta Arith. **5** (1959), 117–119; MR **21** #5594].

In (II) the definitions of $f(x)$ and $g(x)$ in this review are more general than those that appear in the paper. The reviewer assumes that the author intended them to be as stated. *H. B. Mann* (Madison, Wis.)

Rotkiewicz, André 4749

Sur les nombres pseudopremiers pentagonaux.

Bull. Soc. Roy. Sci. Liège **33** (1964), 261–263.

A positive integer n is pseudoprime if n is composite and n divides 2^n-2. The author has shown [see, e.g., #4724 above] that there are infinitely many triangular pseudoprimes and only two square pseudoprimes $<10^{12}$. Here he shows that there are infinitely many pentagonal pseudoprimes. (A pentagonal number is one of the form $\frac{1}{2}k(3k-1)$, where k is a positive integer.) *J. B. Kelly* (Tempe, Ariz.)

Davenport, H.; Erdős, P. 4750

A theorem on uniform distribution. (Russian summary)

Magyar Tud. Akad. Mat. Kutató Int. Közl. **8** (1963), 3–11.

Let $z_1<z_2<\cdots$ be a sequence of positive real numbers such that $z_n\to\infty$ as $n\to\infty$, and let α be a positive irrational number. For given λ, $0<\lambda<1$, let $F(N)$ be the number of positive integers $k\leq N$ for which $k\alpha$ falls in one of the intervals $(z_j, z_j+\lambda(z_{j+1}-z_j))$. If $F(N)/N\to\lambda$ as $N\to\infty$, for each λ, then the sequence $\{k\alpha\}$ is said to be uniformly distributed relative to the sequence $\{z_j\}$. We suppose that $z_{j+1}\sim z_j$ as $j\to\infty$, since this is easily seen to be necessary for uniform distribution. It follows from work of the reviewer [Pacific J. Math. **3** (1953), 757–771; MR **15**, 511] and Davenport and the reviewer [Michigan Math. J. **10** (1963), 315–319; MR **27** #3619] that $\{k\alpha\}$ is uniformly distributed relative to $\{z_j\}$ for almost all $\alpha>0$ if $z_{j+1}-z_j$ is monotonic. In the present paper this monotonicity condition is considerably weakened. Namely, it is a consequence of the slightly more general theorem actually proved, that if the number of z_j not exceeding N is $O(N^{2-\delta})$, for some fixed $\delta>0$, then $\{N\alpha\}$ is uniformly distributed relative to $\{z_j\}$ for almost all $\alpha>0$.

W. J. LeVeque (Ann Arbor, Mich.)

Mendès France, Michel 4751

Représentation des nombres réels.

C. R. Acad. Sci. Paris **258** (1964), 4643–4645.

A toute suite réelle positive $f(n)$ on associe l'application $\alpha\to\sum_{n=1}^{\infty}2^{-n}\exp i\pi[\alpha f(n)]$ où $[x]$ représente la partie entière de x. Soit $E(f)$ l'image de $[0,1]$ par cette application (c'est $[0,1]$ si $f(n)=2^n$). Théorème 1 (cas particulier): Si $f(n)=o(2^n)$, $E(f)$ est de mesure nulle; la dimension de Hausdorff de $E(f)$ est majorée par $\limsup\log^+ f(n)/(n\log 2)$. Théorème 2 (sans démonstration): S'il existe deux suites d'entiers p_n et q_n tendant vers ∞ telles que $f(p_n+q_n)=f(q_n)$, $E(f)$ est un ensemble d'unicité pour les séries trigonométriques. *J.-P. Kahane* (Orsay)

Ikeda, Masatoshi 4752

Zum Existenzsatz von Grunwald.

J. Reine Angew. Math. **216** (1964), 12–24.

The Grunwald theorem, which was corrected by S. Wang [Ann. of Math. (2) **51** (1950), 471–484; MR **11**, 489] and generalized by H. Hasse [J. Reine Angew. Math. **188** (1950), 40–64; MR **12**, 677], deals with the existence of a Galois extension of a given algebraic number field, which possesses a given Galois group and which satisfies certain local conditions. In the present paper the author generalizes the Grunwald theorem by solving the following problem: Given a Galois extension K/k of an algebraic number field k and a group extension E of a finite abelian group A by the Galois group of K/k, does there exist an abelian extension Ω/K which is a solution of the imbedding problem corresponding to E and which satisfies certain local conditions? The author gives a solution of this problem in the form of some necessary and sufficient conditions, assuming that A is a cyclic l-group of odd order and that E is a split extension. *Rimhak Ree* (Vancouver, B.C.)

Vaitkjavičus, I. 4753

On the distribution of prime numbers of an imaginary quadratic field in sectors. (Russian. Lithuanian and German summaries)

Litovsk. Mat. Sb. **3** (1963), no. 2, 17–52.

The author's aim is an estimate for the minimal norm of a prime ideal in a given class and a given sector in any imaginary quadratic field. In the proof of the principal lemma he does not follow the intricate method of Linnik and Rodosskiĭ [cf. K. Prachar, *Primzahlverteilung*, pp. 331–348, Springer, Berlin, 1957; MR **19** 393], but

reaches his goal by some incorrect arguments. {The principal error is on p. 32, where the exponent

$$\frac{20\omega-14\omega^2+1}{4\omega^2-\omega+1}\,\omega$$

of DP should be without the factor ω.} *E. Fogels* (Riga)

Deuring, M. 4754

★Die Klassenkörper der komplexen Multiplikation.

Enzyklopädie der mathematischen Wissenschaften: Mit Einschluss ihrer Anwendungen, Band I 2, Heft 10, Teil II (Article I 2, 23).

B. G. Teubner Verlagsgesellschaft, Stuttgart, 1958. *60 pp. DM* 15.00.

Table of contents: (A) Funktionentheoretische Grundlagen; (B) Zahlentheoretische Grundlagen; (C) Der erste Hauptsatz; (D) Der zweite Hauptsatz; (E) Bemerkungen und Literatur.

Davenport, H.; Lewis, D. J. 4755

Character sums and primitive roots in finite fields.

Rend. Circ. Mat. Palermo (2) **12** (1963), 129–136.

Let p be a prime, and denote by $[p^n]$ the finite field with p^n elements. Let χ be a nonprincipal character of the multiplicative group of nonzero elements of $[p^n]$. Each element ξ of $[p^n]$ has a unique representation in the form

$$\xi = x_1\omega_1+\cdots+x_n\omega_n,$$

where $\omega_1, \cdots, \omega_n$ form a basis for $[p^n]$ over $[p]$. Let B be the set of elements ξ for which $N_j<x_j\leq N_j+H_j$ $(j=1, \cdots, n)$, where the N_j and H_j are integers satisfying $0\leq N_j<N_j+H_j<p$ $(j=1, \cdots, n)$. It is shown that then

$$\left|\sum_{\xi\in B}\chi(\xi)\right| < \{p^{1/2}(\log p+1)\}^n,$$

and that when $H_1=\cdots=H_n=H$ and $\delta>0$, there exist $\delta_1(\delta)>0$ and $p_1(\delta)$ such that if $H>p^{n/2(n+1)+\delta}$ and $p>p_1(\delta)$, then

$$\left|\sum_{\xi\in B}\chi(\xi)\right| < (p^{-\delta_1}H)^n.$$

The first of these inequalities generalizes one due to G. Pólya [Nachr. Ges. Wiss. Göttingen Math.-Phys. Kl. **1918**, 21–29] and J. Vinogradoff [Ž. Fiz.-Mat. Obšč. Permsk. Gos. Univ. No. 1 (1918), 94–98] relative to the rational integers, while the second is analogous to a result of D. A. Burgess [Proc. London Math. Soc. (3) **12** (1962), 179–192; MR **24** #A2569]. For $n=1$, the second estimate is valid for $H>p^{1/4+\delta}$, as is the estimate of Burgess.

W. J. LeVeque (Ann Arbor, Mich.)

FIELDS AND POLYNOMIALS

See also 4752, 4775, 5150, 5151, 5155, 5156.

Alanen, J. D.; Knuth, D. E. 4756

A table of minimum functions for generating Galois fields of GF(p^n).

Sankhyā Ser. A **23** (1961), 128.

Minimum polynomials of degree 3, 4, 5 are given over finite fields of order 11, 13, 17 for statistical users.

A. T. James (New Haven, Conn.)

Narkiewicz, W. 4757

On transformations by polynomials in two variables.

Colloq. Math. **12** (1964), 53–58.

Let K be a field and X a subset of K. The author has proved [Acta. Arith. 7 (1961/62), 241–249; MR **26** #110; ibid. 8 (1962/63), 11–19; MR **26** #4987] that for a wide class of fields (including all finitely generated extensions of the rationals) if $P(t)$ is a non-linear polynomial with coefficients in K such that $P(X)=X$, then X must be finite. He now considers the analogous problem for transformations in K^n defined by n polynomials in n variables. Now, there are non-linear polynomials with infinite invariant sets. For $n=2$ the following theorem is proved. Let K be a finite algebraic extension of the rationals and let $F_1(x, y)$, $F_2(x, y)$ be homogeneous polynomials with coefficients in K of degrees m_1, m_2, respectively. Suppose that $F_1(x, y)$ and $F_2(x, y)$ have no non-trivial common factor and that m_1, m_2 are greater than 2. Then the transformation T defined in K^2 by $(x, y)\to(F_1(x, y), F_2(x, y))$ has no infinite invariant sets. The theorem holds also in the case $m_1=m_2=2$. The proof depends on the work of the first of the papers referred to. *J. V. Armitage* (Durham)

Aguiló Fuster, Rafael 4758

Application of Dedekind's method to a non-Archimedean ordered field. (Spanish)

Collect. Math. **15** (1963), 77–90.

Let K be a non-archimedean ordered field. The author shows that the Dedekind cut extension Φ of K is a hemiring, that is, an object $\langle R, \cdot, +, 0\rangle$ such that $\langle R, \cdot\rangle$ is a commutative semi-group, $\langle R, +, 0\rangle$ is a commutative monoid, and the set of elements of R with additive inverse forms a subring of R. In a natural way he finds the maximal subfield Σ of Φ and shows (1) $\Sigma\neq\Phi$, and (2) the ordered field Σ is isomorphic to the completion of K by Cauchy sequences. *B. Brainerd* (Toronto, Ont.)

Thaler, Alvin I. 4759

On the Newton polytope.

Proc. Amer. Math. Soc. **15** (1964), 944–950.

In the sequel $K=\{a, a_{ij}, \cdots\}$ denotes a complete field with respect to a non-archimedean rank one valuation $a\to\text{ord }a$ with value group $\mathfrak{G}=\{r, s, \rho, \cdots\}$ which is dense in the additive group of real numbers $R=\{\mu, \nu, b, \cdots\}$. Let $\mathfrak{K}=\{\xi, \eta, \cdots\}$ denote the algebraic closure of K with the corresponding unique prolongation of the valuation. Define Γ_b as $\{\xi\in K:\text{ord }\xi=b\}$. Next set $v(f;\mu,\nu)=\min_{i,j}(\text{ord }a_{ij}+i\mu+j\nu)$ for $f=f(x, y)=\sum a_{ij}x^iy^j\in K[x, y]$. Then the graph of $\Pi_f(X, Y)=\sup_{\mu,\nu\in R}[v(f;\mu,\nu)-\mu X-\nu Y]$ in an (X, Y, Z)-space is called the Newton polytope of f. Furthermore, let $V(f)=\{(\xi,\eta)\in K\times K:f(\xi,\eta)=0\}$, and call $r\in R$ x-distinguished on $V(f)$ if there exist infinitely many $s\in\mathfrak{G}$ such that $V(f)\cap(\Gamma_r\times\Gamma_s)\neq\varnothing$. The author proves the following noteworthy results on the zeros of polynomials f. (1) Let $r, s\in\mathfrak{G}$, and let P_{rs} be the lower plane of support of the Newton polytope of f with $\partial Z/\partial X=-r$, $\partial Z/\partial Y=-s$. Then f has a zero (ξ,η) with ord $\xi=-r$, ord $\eta=-s$ if and only if P_{rs} contains an edge of the polytope. (2) Suppose $f(0,0)\neq0$. Then $\rho\in\mathfrak{G}$ is x-distinguished on $V(f)$ if and only if there is an edge of the Newton polytope of f with direction numbers $(1, 0, -\rho)$. The proofs of these theorems involve skilful geometric considerations in reductions to the one variable case.

O. F. G. Schilling (Lafayette, Ind.)

Bialynicki-Birula, A. 4760
[Bialynicki-Birula, Andrzej]
On the inverse problem of Galois theory of differential fields.
Proc. Amer. Math. Soc. **15** (1964), 960–964.

If $\mathscr{F}$ is a differential field of characteristic 0 with algebraically closed field of constants $\mathscr{C}$, then the Galois group of a strongly normal extension of $\mathscr{F}$ is (isomorphic with) an algebraic group defined over $\mathscr{C}$. Given $\mathscr{F}$, it is natural to ask what algebraic groups occur in this way. The author proves, in the case in which the differential field is ordinary, that if the transcendence degree of $\mathscr{F}$ over $\mathscr{C}$ is finite and nonzero, then every connected nilpotent affine algebraic group G defined over $\mathscr{C}$ occurs. The idea of the proof is to consider the field $\mathscr{F}(G)$ of rational functions on G which are defined over $\mathscr{F}$, and to extend the given derivation on $\mathscr{F}$ to a derivation on $\mathscr{F}(G)$ which is left-invariant under every element of G which is rational over $\mathscr{C}$; then $\mathscr{F}(G)$, together with this derivation, is a strongly normal extension of $\mathscr{F}$ on the Galois group G.

E. R. Kolchin (New York)

ABSTRACT ALGEBRAIC GEOMETRY

See also 4739, 4806, 5257, 5258.

Harder, Günter 4761
★Über die Galois-Kohomologie der Tori.
Dissertation zur Erlangung des Doktorgrades der Mathematisch-Naturwissenschaftlichen Fakultät der Universität Hamburg, Hamburg, 1963. iii + 71 *pp.*

Let k be a global field, i.e., either a finite algebraic number field or an algebraic function field of one variable over a finite constant field. Let T be an algebraic torus defined over k, i.e., an algebraic group defined over k which is isomorphic over the algebraic closure of k with the product $G_m \times \cdots \times G_m$, G_m being the multiplicative group of the universal domain. Viewing G_m as a trivial torus defined over k, it is natural to ask to what extent the arithmetical properties of G_m can be generalized for an arbitrary torus T defined over k. In this paper the author considers the Hasse principles. A torus T defined over k is said to be split by a finite Galois extension K of k if there exists an isomorphism $T = G_m \times \cdots \times G_m$ defined over K. The author says that the Hasse principle holds for T if the map φ: $H^1(K/k, T) \to H^1(K/k, T_A)$ is injective; here $T_A = T_{A_k}$ means the adelization. The main results are the following: the kernel of φ is always finite (§ 7, Satz 3), and the Hasse principle holds if every Sylow subgroup of the Galois group of K/k is cyclic (§ 6, Satz 8). This paper begins with careful preparations (§ 1–§ 5) which make it self-contained. [Cf. the reviewer's works, Ann. of Math. (2) 74 (1961), 101–139; MR 23 #A1640; ibid. (2) 78 (1963), 47–73; MR 28 #94.]

T. Ono (Philadelphia, Pa.)

Tjurin, A. N. 4762
On the classification of two-dimensional fibre bundles over an algebraic curve of arbitrary genus. (Russian)
Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 21–52.

Atiyah and Grothendieck gave the classification of vector bundles over an elliptic and over a rational curve [Atiyah, Proc. London Math. Soc. (3) 7 (1957), 414–452; MR 24 #A1274; Grothendieck, Amer. J. Math. 79 (1957), 121–138; MR 19, 315]. The author considers this problem for curves of arbitrary genus. But in this case there is no universal family of bundles, i.e., the functor which to any algebraic variety E associates the set of all families of two-dimensional bundles over the algebraic curve X, parametrized by E, is not representable. Then he introduces the notion of exceptional one-dimensional sub-bundle of a two-dimensional bundle. He proves that any two-dimensional bundle over the curve X of genus g has at most $2g$ exceptional sub-bundles. The new object consisting of a two-dimensional bundle and a fixed exceptional sub-bundle of minimal height is called a quasi-bundle. The author proves that for quasi-bundles there is a universal family whose base B is the union of a finite number of algebraic varieties.

E. Lluis (Mexico City)

Voskresenskiĭ, V. E. 4763
The behaviour of semi-simple algebraic groups under extension of the ground field. (Russian)
Dokl. Akad. Nauk SSSR **158** (1964), 767–769.

Suppose G is a connected semi-simple algebraic group defined over an algebraic number field k. The author asks which G satisfy: (A) G splits (i.e., has a splitting maximal torus) over almost all $\mathfrak{p}$-adic extensions $k_\mathfrak{p}$ of k. The question is answered completely by cohomology. G corresponds to an element h of $H^1(k, A_{\bar{k}})$, where $A_{\bar{k}}$ is the automorphism group (over the algebraic closure $\bar{k}$ of k) of the Lie algebra of G. Let $\beta(h)$ be the natural image of h in $H^1(k, W)$, where W is the quotient of $A_{\bar{k}}$ by its maximal connected subgroup. Then G satisfies (A) if and only if $\beta(h) = 0$.

E. C. Dade (Pasadena, Calif.)

LINEAR ALGEBRA

See also 4705, 4818, 5108, 5144, 5371, 5376, 5377, 5639.

Marcus, Marvin; Gordon, William R. 4764
Inequalities for mappings on spaces of skew-symmetric tensors.
Duke Math. J. **31** (1964), 691–696.

Let a positive definite Hermitian matrix A of order mk be partitioned into m^2 submatrices A_{ij} $(i, j = 1, 2, \cdots, m)$ each of order k; i.e., A_{ij} is the submatrix of A lying in rows $k(i-1)+1, \cdots, ki$ and columns $k(j-1)+1, \cdots, kj$. For $t = 1, \cdots, m$ let B_t denote the matrix of order t whose (i, j)th entry is $\det A_{ij}$, and let A_t be the principal submatrix of A formed by the first tk rows and columns. By determining the eigenvalues and eigenvectors of a certain Hermitian operator on an appropriate Grassmann space, the authors prove the inequality $(\det B_{t-1})/(\det A_{t-1}) \leq (\det B_t)/(\det A_t)$ for $t = 2, \cdots, m$. Equality holds if and only if $A_{it} = A_{ti} = 0$ $(i = 1, 2, \cdots, t-1)$. As $\det B_1 = \det A_1$, this result implies the inequality $\det A_t \leq \det B_t$ obtained by R. C. Thompson [Canad. Math. Bull. 4 (1961), 57–62; MR 26 #133].

Ky Fan (Evanston, Ill.)

Yeung, Yik-Hoi Au 4765
Another proof of the theorems on the eigenvalues of a square quaternion matrix.
Proc. Glasgow Math. Assoc. **6**, 191–195 (1964).

From the author's introduction: "The nature of the eigenvalues of a square quaternion matrix had been considered by Lee [Proc. Roy. Irish Acad. Sect. A 52 (1949), 253–260; MR 12, 153] and Brenner [Pacific J. Math. 1 (1951), 329–335; MR 13, 312]. In this paper the author gives another elementary proof of the theorems on the eigenvalues of a square quaternion matrix by considering the equation $Gy=\mu\bar{y}$, where G is an $n\times n$ complex matrix, y is a nonzero vector in C^n, μ is a complex number, and $\bar{y}$ is the conjugate of y."

Westwick, R. 4766

Linear transformations on Grassmann spaces.

Pacific J. Math. 14 (1964), 1123–1127.

Let U be an n-dimensional vector space over an algebraically closed field, and let G_{nr} be the Grassmann variety (lying in the space $\Lambda^r U$) representing the r-dimensional subspaces of U. Any non-singular linear mapping of U onto itself induces a linear transformation of $\Lambda^r U$ in which G_{nr} is invariant. In this paper it is proved, conversely, that if $n\neq 2r$, any linear transformation of $\Lambda^r U$ which leaves G_{nr} invariant is induced by a linear mapping of U. When $n=2r$, such a linear transformation is either of the type just described or is the composite of such a transformation with that induced by a correlation of the r-dimensional subspaces of U.

J. A. Todd (Cambridge, England)

Samuel, Isaac 4767

Formes réduites des déterminants caractéristiques des matrices d'ordre pair.

C. R. Acad. Sci. Paris 258 (1964), 420–421.

Let a square matrix of even order $2n$ be partitioned in four square matrices of order n,

$$\mathscr{A}=\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}.$$

Provided A_{21}^{-1} exists, the characteristic polynomial of $\mathscr{A}$ can be obtained by suitable elimination in the form

$$\det[\lambda^2 I-\lambda(A_{22}+A_{21}A_{11}A_{21}^{-1})+(A_{21}A_{11}A_{21}^{-1}A_{22}-A_{21}A_{12})].$$

F. L. Bauer (Munich)

Samuel, Isaac 4768

Formes réduites des déterminants caractéristiques des matrices.

C. R. Acad. Sci. Paris 259 (1964), 1615–1617.

Generalization of a previous result [#4767] to matrices of arbitrary order and a variety of partitions.

F. L. Bauer (Munich)

Erdős, P.; Rényi, A. 4769

On random matrices. (Russian summary)

Magyar Tud. Akad. Mat. Kutató Int. Közl. 8 (1963), 455–461 (1964).

The permanent of an n-square matrix $A=(a_{ij})$ is defined by $\sum_{\sigma\in S_n}\prod_{i=1}^n a_{i\sigma(i)}$. Let $\mathscr{A}(n,N)$ denote the set of all n-square (0, 1) matrices with N elements equal to 1 and n^2-N elements equal to 0. Let $P(n,N)$ be the probability that a matrix M chosen at random from $\mathscr{A}(n,N)$ has a positive permanent. The authors prove that if $N(n)=n\log n+cn+o(n)$, where c is any real constant, then

$$\lim_{n\to\infty}P(n,N(n))=e^{-2e^{-c}}.$$

This result can be interpreted as follows. If, under the above conditions, the permanent of M is 0, then this is probably due to M having a zero row or a zero column.

H. Minc (Santa Barbara, Calif.)

ASSOCIATIVE RINGS AND ALGEBRAS

See also 4657, 4778, 5250.

Bergman, G. M. 4770

A ring primitive on the right but not on the left.

Proc. Amer. Math. Soc. 15 (1964), 473–475.

A ring B is right [left] primitive in case it possesses a faithful simple right [left] B-module M. (In this case, B is isomorphic to a dense ring of linear transformations in the right [left] vector space M over K, where K is the field $\mathrm{Hom}_B(M,M)$.) The author presents the first known example of a ring B which is right but not left primitive. The example is, of course, ingenious, and is too complicated to completely describe in this space. The starting point is the field $D=Q(X)$ of rational functions over the rational number field Q, and the ring $A=D[Y,\alpha]$ of noncommuting polynomials $\sum_{i\geq 0}d_iY^i$ defined by the rule $Yd=\alpha(d)Y\ \forall\, d\in D$, where α is the map $d(x)\mapsto d(X^2)$. The first result states that any subring B of A containing both X and Y is right primitive. The author notes that D is a right A-module under the rule $r\cdot Y=r^*\ \forall\, r\in D$, where r^* is the unique element of D satisfying $\frac{1}{2}[r(x)+r(-X)]=r^*(X^2)$, and that the right B-submodule of D generated by X is faithful and simple. By extending the "cyclotomic" valuations (for odd n) of D to A, he obtains a subring B of A, which is the intersection of the valuation subrings corresponding to the extended valuations, which contains both X and Y, and which is not left primitive. It is remarkable that the counterexample B is an integral domain which is embedded in a field (the left quotient field of A).

Carl Faith (Princeton, N.J.)

Wall, C. T. C. 4771

Graded Brauer groups.

J. Reine Angew. Math. 213 (1963/64), 187–199.

Let A be a finite-dimensional associative algebra with 1 over a field k of characteristic $\neq 2$. A is called graded if $A=A_0\oplus A_1$, A_i are nonzero subspaces and $A_iA_j\subseteq A_{i+j}$ (i,j are taken mod 2). A subspace B of A is called graded if $B=B\cap A_0\oplus B\cap A_1$. A graded algebra A is called central if (center of A) $\cap A_0$ consists only of multiples of 1, and is called simple if it has no proper ideals. A typical example of a graded central simple algebra is the Clifford algebra $C(q)$, q being a quadratic form over k, and the motivation of this work was the observation of the fact that when q is the orthogonal sum of q_1,q_2, $C(q)\neq C(q_1)\otimes C(q_2)$ in the usual sense of (ungraded) algebras. The author first determines the structure of a graded central simple algebra $A=A_0\oplus A_1$. Either A or A_0, but not

both, is a central simple (ungraded) algebra over k (Lemma 4). The case where A is central simple is denoted by $(+)$ and where A_0 is so by $(-)$. Then, by the usual structure theory of central simple algebras, it is shown that A is determined by the invariants (ε, n, D, a), where $\varepsilon = \pm$, n a positive integer, D a central division algebra, and $a \in k^*/k^{*2}$ (Theorem 1). Next, he shows that if A, B are graded central simple, then so is $A \otimes B$, where $\otimes$ means the graded tensor product (Theorem 2). He then defines the type of A as the triple (ε, a, D) and proves that the product of types is well-defined and that the types form a group, the graded Brauer group (Theorem 3). (Thus, in the sense of graded algebras, one gets $C(q) = C(q_1) \otimes C(q_2)$ (Theorem 4). It is remarked that, for $C(q)$, ε is the parity of the number of variables of q, a is the discriminant of q, and D is essentially the Hasse-Witt invariant.) The paper concludes with some modifications in characteristic 2. *T. Ono* (Philadelphia, Pa.)

Yuan, Shuen 4772

Differentiably simple rings of prime characteristic.

Duke Math. J. **31** (1964), 623–630.

Let D be a set of derivations on a commutative ring R with identity 1, and assume that 0 and R are the only ideals I of R satisfying $dI \subseteq I$ for all $d \in D$. L. R. Harper, Jr. [Trans. Amer. Math. Soc. **100** (1961), 63–72; MR **24** #A116] showed that if, furthermore, R is a finite-dimensional algebra over an algebraically closed field C, then R is of the form $C[t_1, \cdots, t_n]/(t_1^p, \cdots, t_n^p)$.

In this paper, the author first remarks that R is the direct sum of a field F and the radical N of R, and obtains Harper's conclusion (where C is replaced by F) under the milder assumption that N/N^2 or D^*/ND^* is of finite dimension over F, where D^* is the restricted derivation Lie algebra on R generated by D.

Rimhak Ree (Vancouver, B.C.)

Faith, Carl 4773

Noetherian simple rings.

Bull. Amer. Math. Soc. **70** (1964), 730–731.

The author outlines a proof of the following theorem: If R is a simple ring with identity which contains a minimal complement (= closed = uniform) right ideal, then R is the endomorphism ring of a unital torsion-free module over an integral domain. Consequently, a right noetherian simple ring with identity is isomorphic to the endomorphism ring of a unital torsion-free module of finite rank over an integral domain. Since a right artinian ring with identity is right noetherian, this theorem generalizes the Wedderburn-Artin theorem, which states that a right artinian simple ring with identity is isomorphic to the ring of endomorphisms of a unital module over a division ring.

A. J. Douglas (Sheffield)

Hayes, Allan 4774

A characterisation of f-rings without non-zero nilpotents.

J. London Math. Soc. **39** (1964), 706–707.

An f-ring is a lattice-ordered ring satisfying the condition $a \cap b = 0$, $c \geq 0$, implies $ca \cap b = ac \cap b = 0$. The following non-lattice-theoretic characterization of f-rings without non-zero nilpotents is obtained: A partially ordered ring R without non-zero nilpotent elements is an f-ring if and only if for each $a \in R$ there exist elements a_1 and a_2 in R with $a_1 \geq 0$ and $a_2 \geq 0$ such that $a = a_1 - a_2$ and $a_1 a_2 = a_2 a_1 = 0$. *R. S. Pierce* (Seattle, Wash.)

Weinert, H. J. 4775

Ein Struktursatz für idempotente Halbkörper.

Acta Math. Acad. Sci. Hungar. **15** (1964), 289–295.

Let H be a semifield with non-commutative, idempotent addition, not containing a zero element. The author considers the equivalence on H defined by $x+y+x=x$ and $y+x+y=y$. This is compatible with both addition and multiplication on H: $C_a + C_b = C_{a+b}$, $C_a C_b = C_{ab}$. The class C_e containing the unit element of H forms a subsemifield. The quotient system G is an additively commutative semifield which is universal among all additively commutative homomorphic images of H. $G^\times$ is then the factor group $H^\times/C_e^\times$. Thus, the author discusses also the above decomposition in connection with Schreier's theory of group extensions. *Y. Utumi* (Leeds)

NON-ASSOCIATIVE ALGEBRA

See also 4809, 5116.

Block, Richard E.; Zassenhaus, Hans 4776

The Lie algebras with a nondegenerate trace form.

Illinois J. Math. **8** (1964), 543–549.

A Lie algebra L is said to have trace form (x, y) if there is a representation Δ of L such that $(x, y) = \mathrm{tr}(\Delta x \Delta y)$ for all $x, y \in L$. Over fields of characteristic zero, the structure of Lie algebras with nondegenerate trace forms has been determined by Borel and Mostow [Ann. of Math. (2) **61** (1955), 389–405; MR **16**, 897]; they are direct sums of the center and a semisimple ideal. Over algebraically closed fields F of characteristic $p > 3$, the Lie algebras with nondegenerate trace form and no abelian ideals are determined by results of the first author [Proc. Amer. Math. Soc. **11** (1960), 377–379; MR **22** #5697; Canad. J. Math. **14** (1962), 553–564; MR **25** #3973] and by results of Mills and the reviewer [the reviewer, Mem. Amer. Math. Soc. No. **19** (1956); MR **17**, 1108; Mills and the reviewer, J. Math. Mech. **6** (1957), 519–548; MR **19**, 631; Mills, ibid. **6** (1957), 559–566; MR **19**, 632]; they are direct sums of simple Lie algebras which are analogues of simple complex Lie algebras. In the present paper, the authors drop the assumption that L has no abelian ideals. If F is algebraically closed of characteristic $p > 3$, then L is the direct sum of ideals, orthogonal with respect to the trace form, and each ideal is either (a) one-dimensional; (b) simple with nondegenerate form; or (c) isomorphic to the Lie algebra of all n-by-n matrices with $p \mid n$. When F is not assumed algebraically closed, it is proved that L is the orthogonal direct sum of indecomposable ideals L_i, each L_i being either (a) one-dimensional; (b) simple; or (c) such that its center Z_i is contained in the derived algebra L_i', $L_i'' = L_i'$, $Z_i \cong L_i/L_i'$, and L_i'/Z_i is simple.

G. B. Seligman (New Haven, Conn.)

Cohn, P. M. 4777

Subalgebras of free associative algebras.

Proc. London Math. Soc. (3) **14** (1964), 618–632.

In an earlier paper [Proc. Cambridge Philos. Soc. 57 (1961), 18–30; MR 22 #9514] the author characterized free associative algebras (f.a.a.) in terms of a degree function which satisfies a generalized Euclidean algorithm. Since $F[x^2, x^3]$ is a non-free subalgebra of the f.a.a. $F[x]$, it is natural to ask how to pick out those subalgebras of an f.a.a. which are free.

Let A be an f.a.a. over a field F with a degree function d, i.e., d is a discrete valuation with rank 1 of A, and let B be a subalgebra of A with 1. A family $\{u_i\}_{i\in I}$ of elements of A is called B-independent if, for every family $\{b_i\}_{i\in I}$ of elements of B, almost all zero,

$$d(\sum u_i b_i) = \max\{d(u_i) + d(b_i)\}.$$

Theorem: Let A be an f.a.a. over a field F with the degree function defined by a free generating set. If B is any subalgebra of A such that A is B-free as a right B-module, and has a B-independent basis, then B is an f.a.a. over F.

This theorem, together with that of Poincaré, Birkhoff and Witt, yields a nice proof of the Širšov-Witt theorem which states that every subalgebra of a free Lie algebra is free. Next follow some results on the problem of deciding whether or not a subset of an f.a.a. A generates a free subalgebra and some results on the structure of the automorphism group of A. These results are used to study the automorphism group of a free Lie algebra L and to prove that every Lie automorphism of L is a succession of elementary (Lie) transformations.

G. Leger (Medford, Mass.)

Roy, Amit 4778

On a characterisation of Clifford algebras.

Math. Z. **85** (1964), 241–244.

Author's introduction: "Sridharan [Trans. Amer. Math. Soc. **100** (1961), 530–550; MR **24** #A754] considered filtered algebras over a commutative ring K whose associated graded algebras are isomorphic to the symmetric algebra of a free K-module M. He showed that such algebras are suitable generalisations of enveloping algebras of Lie algebra structures on M. In this note we ask an analogous question: namely, that of determining filtered algebras whose associated graded algebras are isomorphic to the exterior algebra on a free K-module M. If Q is a quadratic form on M, it is easily verified (Proposition 1) that the Clifford algebra of Q is an algebra of this type. We prove here that if 2 is invertible in K, any filtered algebra whose associated graded algebra is isomorphic to the exterior algebra on M is in fact a Clifford algebra."

G. Leger (Medford, Mass.)

Smith, David A. 4779

On fixed points of automorphisms of classical Lie algebras.

Pacific J. Math. **14** (1964), 1079–1089.

Let A be the automorphism group of a semi-simple Lie algebra over an algebraically closed field of characteristic zero. Let $n(A_i)$ denote the minimal multiplicity of 1 as characteristic root for elements of a connected (algebraic) component A_i of A, and let $m(A_i)$ denote the minimal dimension of fixed-point spaces for elements of A_i. Jacobson [same J. **12** (1962), 303–315; MR **26** #6222] showed that $n(A_i) = m(A_i)$, and determined these numbers. In this paper the above result is extended to the case when the ground field is infinite. For finite ground fields, only the Lie algebras of the types A, B, C, and D are considered, and the same result is obtained for $A_i = G$, the automorphism group of the Lie algebra.

Rimhak Ree (Vancouver, B.C.)

Yamaguti, Kiyosi 4780

On the theory of Malcev algebras.

Kumamoto J. Sci. Ser. A **6**, 9–45 (1963).

This is a general study, along familiar lines, of the structure and representation of a class of algebras defined by identities. Let A be alternative and let A_L be the algebra whose product operation is commutation in A. The alternative identity gives rise to identities in A_L which are then taken to define the class of Malcev algebras. These algebras are closely related to the author's general Lie triple systems and consequently to Lie algebras. For example, Malcev algebras are precisely those algebras M which can be obtained in the following way. Let L be a Lie algebra and D a subalgebra with a complementary D-invariant subspace M in L. Make M an algebra by letting xy be the D component of $[xy]$ for x and y in M.

A standard way to construct a Lie algebra L related as above to a given Malcev algebra M is to form the semi-direct sum of M with its derivation algebra D. It turns out that if M is derived from the Cayley algebra over an algebraically closed field, then L is B_3.

Solvability, the radical, and semi-simplicity are defined for Malcev algebras, and many of the standard general results for Lie algebras go through. Several notions of representation are discussed; in particular, the class of mappings induced in M by representations of its "holomorph" L, and the representations which are derived from an identity satisfied by multiplication operators and which thereby guarantee a "regular" representation. The main results here relate the solvability of M to the solvability of the Lie algebra generated by the representing transformations.

Finally, a cohomology notion based on a specific cochain construction and a class of representations is developed along the lines of the Lie theory with very similar results. If the notion of representation is appropriately chosen, the low-dimensional cohomology groups have the standard interpretation in terms of derivations and extensions.

W. G. Lister (Stony Brook, N.Y.)

HOMOLOGICAL ALGEBRA

See also 4657, 4780, 4798.

Ehresmann, Charles 4781

Catégories ordonnées, holonomie et cohomologie.

Ann. Inst. Fourier (*Grenoble*) **14** (1964), *fasc.* 1, 205–268.

From the author's introduction: "Le but de cet article était de développer une théorie de la cohomologie d'une catégorie ordonnée à valeur dans une catégorie et de l'appliquer au cas d'un espace feuilleté. Mais comme l'exposé complet de cette théorie s'est révélé trop long, il a fallu remanier le plan, de sorte que seules les cohomologies d'ordre 0 et 1 sont considérées ici.

"Deux possibilités s'offrent pour définir la cohomologie dans le cas considéré: (1) Généraliser les méthodes utilisées

dans le cas de la cohomologie pour un faisceau, en définissant les classes de cohomologie par un procédé de récurrence à partir des classes d'ordre 0. . . . (2) Généraliser la construction de la cohomologie d'un groupe, à l'aide des foncteurs Ext^n.

"Comme ces deux méthodes, pour les cohomologies d'ordre supérieur, ne s'appliquent que si l'on part d'une catégorie quasi-inductive, le premier problème est de plonger une catégorie ordonnée dans une catégorie quasi-inductive. Nous montrons que la catégorie des atlas réguliers résout cette question pour les groupoïdes ordonnés réguliers. Dans le cas général, il faut utiliser des catégories quotient de la catégorie des fusées et ceci se trouvera dans la suite du présent article."

H.-J. Hoehnke (Berlin)

Kelly, G. M. 4782a

Complete functors in homology. I. Chain maps and endomorphisms.

Proc. Cambridge Philos. Soc. **60** (1964), 721–735.

Kelly, G. M. 4782b

Complete functors in homology. II. The exact homology sequence.

Proc. Cambridge Philos. Soc. **60** (1964), 737–749.

One of the author's objectives in this series of papers is to analyse closely what one might fairly mean by a "complete system of invariants" for the objects of any given category. The second objective is to construct complete functors on several categories of importance in homology theory, and thus to find a complete set of invariants of a chain map, a chain endomorphism, or a pair consisting of a complex and a subcomplex.

In the first paper, some general results on complete functors are given, and then invariants of chain maps and chain endomorphisms are studied. The second paper uses the results of the first, together with a detailed and functorial treatment of the mapping-cylinder construction, to show that the exact homology sequence of a pair of complexes is a complete system of invariants of the pair.

S.-T. Hu (Los Angeles, Calif.)

Mitchell, Barry 4783

The full imbedding theorem.

Amer. J. Math. **86** (1964), 619–637.

Theorem: A small abelian category has a full exact imbedding into a category of modules over some ring. This is the result which Freyd adopts as thematic for his book [*Abelian categories*, Harper & Row, New York, 1964; MR **29** #3517].

The idea is to imbed the category $\mathscr{A}$ in the dual of the category $\mathscr{L}(\mathscr{A}, \mathscr{G})$ of kernel-preserving functors from $\mathscr{A}$ to abelian groups by the representation functor $A \to \mathrm{Hom}(A, -) = H^A$. This imbedding is full and exact by the Yoneda theorem. But Freyd [loc. cit.] and Gabriel [Bull. Soc. Math. France **90** (1962), 323–448] showed that $\mathscr{L}(\mathscr{A}, \mathscr{G})$ is abelian. It is easy to see that it has exact direct limits and generator $\sum_{A \in \mathscr{A}} H^A$; it follows from results of Grothendieck [Tôhoku Math. J. (2) **9** (1957), 119–221; MR **21** #1328] that it has an injective cogenerator, and thus its dual is a projective generator U'. Taking $U = \sum_{A \in \mathscr{A}} U'$, the ring is $\mathrm{Hom}(U, U)$ and the imbedding $A \to \mathrm{Hom}(U, H^A)$.

A. Heller (Urbana, Ill.)

GROUP THEORY AND GENERALIZATIONS

See also 4657, 4704, 4730, 5152, 5154, 5236, 5323, 5415, 5592, 5594, 5598.

Baumgartner, Ludwig 4784

★Gruppentheorie.

Vierte, erweiterte Auflage. Sammlung Göschen, Band 837/837a.

Walter de Gruyter & Co., Berlin, 1964. 190 *pp.* (3 *inserts*) *DM* 5.80.

The third edition (1958) of this little book was described in MR 21 #1997. This new edition consists essentially of the previous edition plus three additional chapters, which are described in the Table of Contents as follows.

(IX) Abelian groups. 45. Representation of an element of a cyclic group. 46. The cyclic group as direct product of subgroups. 47. Basis of a group. 48. Direct product of given cyclic groups. 49. Isomorphy of Abelian groups with factor groups of U_n (the direct product of n infinite cyclic groups). 50. The subgroups of U_n. 51. Fundamental theorem on finitely-generated Abelian groups. 52. Finite Abelian groups.

(X) Groups with operators. 53. Operator domain of a group; admissible subgroups. 54. Examples of admissible subgroups. 55. Extension of previous concepts and results to groups with operators. 56. A more general view of operator domains. 57. Common operator domain of several groups. 58. Operator isomorphy and homomorphy. 59. Change in the group concept through introduction of operators. 60. The homomorphy and isomorphy theorems for groups with operators. 61. Operator automorphy and endomorphy.

(XI) p-Groups and p-Sylow groups. 62. Results essential to investigation of p-groups. 63. Double coset decomposition of a group and consequences. 64. p-Groups. 65. On the center of a p-group. 66. p-Sylow groups. 67. Theorems on p-Sylow groups. 68. Sylow's theorem.

A third table (on the alternating group A_4 and the dihedral group D_4) has been added, and there are also 57 more exercises (for a total of 151) with solutions indicated.

Although this enlarged version contains a surprising amount of group theory, considering its small size and cost, the book remains something of a curiosity.

W. E. Deskins (E. Lansing, Mich.)

Scott, W. R. 4785

★Group theory.

Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964. xi+479 *pp.* $13.25.

The first sentence of the preface ("This book contains most of the standard basic theorems in group theory, as well as some topics which have not heretofore appeared in book form.") provides a reasonably accurate description. The complete table of contents is as follows.

(1) Introduction. 1.1 Preliminaries. 1.2 Definitions and first properties. 1.3 Permutation groups. 1.4 Examples of groups. 1.5 Operations with subsets. 1.6 Subgroups. 1.7 Cosets and index. 1.8 Partially ordered sets.

(2) Isomorphism theorems. 2.1 Homomorphisms. 2.2 Factor groups. 2.3 Isomorphism theorems. 2.4 Cyclic groups. 2.5 Composition series. 2.6 Solvable groups. 2.7 Operator groups. Homomorphisms. 2.8 Operator groups. Factor groups. 2.9 Operator groups. Isomorphism

theorems. 2.10 Operator groups. Composition series. 2.11 Operator groups. Examples.

(3) Transformations and subgroups. 3.1 Transformations. 3.2 Normalizer, centralizer, center. 3.3 Conjugate classes. 3.4 The transfer.

(4) Direct sums. 4.1 Direct sum of two subgroups. 4.2 Direct sum and product of a set of groups. 4.3 Subdirect products. 4.4 Direct sum of simple groups. 4.5 Endomorphism algebra. 4.6 Remak-Krull-Schmidt theorem. 4.7 Generalizations.

(5) Abelian groups. 5.1 Direct decompositions. 5.2 Divisible groups. 5.3 Free Abelian groups. 5.4 Finitely generated Abelian groups. 5.5 Direct sums of cyclic groups. 5.6 Vector spaces. 5.7 Automorphism groups of cyclic groups. 5.8 Hom(A, B).

(6) p-Groups and p-subgroups. 6.1 Sylow theorems. 6.2 Normalizers of Sylow subgroups. 6.3 p-Groups. 6.4 Nilpotent groups. 6.5 Applications.

(7) Supersolvable groups. 7.1 M-groups. 7.2 Supersolvable groups. 7.3 Frattini subgroup. 7.4 Fitting subgroup. 7.5 Regular p-groups.

(8) Free groups and free products. 8.1 Definitions and existence. 8.2 Presentations. 8.3 Some examples. 8.4 Subgroups of free groups.

(9) Extensions. 9.1 Definitions. 9.2 Semi-direct products. 9.3 Hall subgroups. 9.4 Extensions: General case. 9.5 Split extensions. 9.6 Extensions of Abelian groups. 9.7 Cyclic extensions.

(10) Permutation groups. 10.1 Intransitive groups. 10.2 Transitive groups and representations. 10.3 Regular permutation groups. 10.4 Multiply transitive groups. 10.5 Primitive and imprimitive groups. 10.6 Some multiply transitive groups. 10.7 Numerical applications. 10.8 Some simple groups.

(11) Symmetric and alternating groups. 11.1 Conjugate classes. 11.2 Infinite groups of finite permutations. 11.3 Normal subgroups of symmetric and alternating groups. 11.4 Automorphism groups. 11.5 Imbedding and splitting theorems.

(12) Representations. 12.1 Linear groups. 12.2 Representations and characters. 12.3 The p^iq^j theorem. 12.4 Representations of direct sums. 12.5 Induced representations. 12.6 Frobenius groups. 12.7 Representations of transitive groups. 12.8 Monomial representations. 12.9 Transitive groups of prime degree.

(13) Products of subgroups. 13.1 Factorizable groups. 13.2 Nilpotent times nilpotent. 13.3 Special cases of nilpotent times nilpotent. 13.4 Nilpotent maximal subgroups. 13.5 Transfer theorems. 13.6 Generalized dihedral times odd-order nilpotent. 13.7 S-rings. 13.8 Primitive S-rings over groups of small order. 13.9 S-rings over Abelian groups. 13.10 Nilpotent times dihedral or dicyclic. 13.11 Primitive S-rings over dihedral and dicyclic groups. 13.12 Dihedral or dicyclic times dihedral or dicyclic.

(14) The multiplicative group of a division ring. 14.1 Wedderburn and Cartan-Brauer-Hua theorems. 14.2 Conjugates. 14.3 Subnormal subgroups. 14.4 Subgroups of division rings.

(15) Topics in infinite groups. 15.1 FC groups. 15.2 Composition subgroups and subnormal subgroups. 15.3 Complete groups. 15.4 Existence of infinite Abelian subgroups. 15.5. Miscellaneous exercises. 15.6 A final word.

At the end comes a bibliography of about 160 items, a 4-page index of notation, and a 5-page index.

This large and attractively-printed book has a number of noteworthy features of which the following are only a few. Definitions are presented carefully and precisely so that there is little chance of misinterpretation. A generous number of nontrivial examples is given, and each section closes with a selection of exercises which serve to amplify and illustrate the material covered. Chapter 10 provides a good introduction to permutation groups and includes a development of the Mathieu groups, as well as a discussion of sharply (exactly) doubly and triply transitive groups. The material on products of subgroups (Chapter 13) includes some very interesting and elegant results (such as the Wielandt-Kegel theorems on products of nilpotent subgroups and the Iwasawa characterization of finite supersolvable groups), but the title of the chapter is hardly descriptive. And in Chapters 14 and 15 the author has gathered together additional results from recent papers (some as yet unpublished) which undoubtedly reflect his personal interests.

Of course, the reviewer had some adverse reactions to the book. In spite of the extensive index of notation the author has failed to use the available alphabets and type faces to the reader's advantage. For example, the use of Greek letters for permuted objects and bold-face letters to denote mappings would make Chapter 10 much easier to read. Although the book is advertised as a reference on group theory, the bibliography is not impressive in either size or completeness, and some discussions suffer from curious omissions. In Section 10.6, for example, near-fields are introduced for the purpose of classifying the exactly 2-transitive finite groups, yet no nontrivial example of a near-field is given, and Section 13.4 could easily have included the discussion from J. Thompson's paper (referred to in the book) which is more complete than the author's. On the other hand, some items are included unnecessarily, such as a section (5.6) on vector spaces (no one unfamiliar with vector spaces is apt to be reading page 114 of this book) and elementary material on finite fields (on pages 118, 119, and 282). Presumably, such items are put in to make the book a text which is "as self-contained as possible", but most users will probably wish that the space had been filled in other ways.

The presentation is generally adequate and straightforward, and although a few "Hom's", exact sequences, and commutative diagrams appear (in Chapter 9), the language of homological algebra is not featured. The text seems rather free of misprints.

Certainly the good features of this book far overshadow the others, and any person interested in algebra should find it a useful and informative addition to his library.

W. E. Deskins (E. Lansing, Mich.)

Tschernikow, S. N. [Černikov, S. N.]; 4786
Schmidt, O. J. [Šmidt, O. Ju.];
Nowikow, P. S. [Novikov, P. S.]
★Endlichkeitsbedingungen in der Gruppentheorie.
Mathematische Forschungsberichte, XX.
VEB Deutscher Verlag der Wissenschaften, Berlin, 1963. 88 *pp.* *DM* 14.50.

The present volume consists of translations of three papers: S. N. Černikov [Uspehi Mat. Nauk 14 (1959), no. 5 (89), 45–96; MR 22 #5679] on finiteness conditions in the general theory of groups, O. Ju. Šmidt [*Selected works* (Russian), pp. 298–300, Izdat. Akad. Nauk SSSR, Moscow, 1959; MR 21 #1259] on the local finiteness of a class of infinite

periodic groups, and P. S. Novikov [Dokl. Akad. Nauk SSSR **127** (1959), 749–752; MR **21** #5680] on periodic groups. The text is augmented by comments and additional references by the first author.

Levin, Frank 4787

One variable equations over groups.

Arch. Math. **15** (1964), 179–188.

The problem of adjoining roots of polynomial equations to fields has interesting analogues in the theory of groups. If $g_1, g_2, \cdots$ are elements of a group G, $x_1, x_2, \cdots, x_n$ are unknown elements of G or of an overgroup of G, and $f(x_1, \cdots, x_n, g_1, g_2, \cdots)$ is a monomial expression in these, a set of m equations

$$f_r(x_1, \cdots, x_n, g_1, g_2, \cdots) = 1$$

is said to form a system of m equations in n variables over G. The equations may or may not have a solution in any overgroup. Thus $x^{-1}gx = 1$, $g \neq 1$, has no solution in any overgroup.

The author proves that if the above set of equations have a solution $x_i = a_i$ in the group $G' = \mathrm{gp}(G, a_1, \cdots, a_n)$, then there exists an overgroup G'' of G' and an element a such that $G'' = \mathrm{gp}(G, a)$.

A group G is (m, n) algebraically closed if any system of m equations in n variables which has a solution in an overgroup of G has a solution in G itself.

The author proves that if G is $(m, 1)$ algebraically closed, it is (m, n) algebraically closed. An application is to prove a theorem originally proved by G. Higman, B. H. Neumann and Hanna Neumann [J. London Math. Soc. **24** (1949), 247–254; MR **11**, 322], namely, that a countable group can be embedded in a two-generator group. In particular, one of the two generators may be chosen of arbitrary order. *D. E. Littlewood* (Bangor)

Rapaport, Elvira Strasser 4788

Groups of order 1.

Proc. Amer. Math. Soc. **15** (1964), 828–833.

If the group with presentation $G = (x_1, x_2, \cdots, x_n; R_1, \cdots, R_n)$ is trivial, then $x_1, \cdots, x_n$ are words in suitable conjugates of $R_1, \cdots, R_n$. The author shows, as an application of Grushko's theorem, that these words form a simple set (a simple set of words in the symbols $y_1, \cdots, y_t$ is a set of elements of the free group F on $y_1, \cdots, y_t$ which can be extended to a set of free generators of F). There follow some brief remarks on various questions connected with the problem of deciding whether or not a group with given presentation is trivial. *P. J. Higgins* (London)

Gaughan, Edward D. 4789

The index problem for infinite symmetric groups.

Proc. Amer. Math. Soc. **15** (1964), 527–528.

For an infinite set M with cardinal X, let $S(X, Y) = \{\sigma: \sigma$ is a permutation on M such that $|\mathrm{spt}\,\sigma| < Y\}$, where $\mathrm{spt}\,\sigma = \{m \in M : \sigma(m) \neq m\}$. Generalizing the results of Onofri [Ann. Mat. Pura Appl. (4) 7 (1929), 103–130], Higman [Publ. Math. Debrecen **3** (1954), 221–226; MR **17**, 234] and Scott [Univ. Kansas Dept. of Math. Rep. No. 5 (1956), 1–22], the author proves the following: If $\aleph_0 < Y \leq X^*$, where X^* is the successor of X, then $S(X, Y)$ has no proper subgroups of index less than X. *H. Nagao* (Osaka)

Wagner, Ascher 4790

A theorem on doubly transitive permutation groups.

Math. Z. **85** (1964), 451–453.

The following result is proved. Let G be a doubly transitive group on a set Ω consisting of $n+1$ points, where n is odd. Further, let the subgroup G_a of G fixing the point a contain a subgroup F_a of even order which is a Frobenius group on the set $\Omega - a$. Then G_a is primitive on $\Omega - a$. The proof uses only elementary properties of permutation groups. *W. Feit* (New Haven, Conn.)

Held, Dieter 4791

Engel conditions and direct decompositions into groups of coprime order.

Illinois J. Math. **8** (1964), 582–585.

Theorem: The following properties of a finite group G and the set of primes π are equivalent. (I) G is a direct product of a π-group and a π'-group. (II) $x^{(3)} \circ y = y^{(3)} \circ x = 1$ for every primary π-element x and every primary π'-element y. (III) (a) $x^{(i)} \circ y = y^{(i)} \circ x = 1$ for every primary π-element x and every primary π'-element y and almost all i. (b) If the simple factor F of G is neither a π-group nor a π'-group, then elements of order 2 in a Sylow subgroup of F commute. The author also raises the question as to whether condition (III) (b) can be deleted in the theorem above. *W. Feit* (New Haven, Conn.)

Russakov, S. A. 4792

Non-simplicity of certain classes of finite groups. (Russian)

Dokl. Akad. Nauk BSSR **8** (1964), 429–431.

Let G be a group of order $g = mn$, where m is the largest π-divisor of g, π a non-empty set of primes. Let $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, where $p_1 < p_2 < \cdots < p_r$ and each $\alpha_i > 0$. G is defined to be π-discrete if G has a normal series whose consecutive factors have orders $p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots, p_r^{\alpha_r}, n$, respectively. The author has previously introduced the concepts of $\pi\delta$-, $\pi\Delta$-, $\pi\Delta_1$-groups [Dokl. Akad. Nauk SSSR **127** (1959), 270–271; MR **21** #5676]. In this note it is shown that if $\varphi(m)$ and n are relatively prime, where φ is the Euler function, then $\pi\Delta$-, $\pi\Delta_1$-, strongly π-solvable groups are π-discrete. Moreover, under the same assumptions, $\pi\Delta$-, $\pi\Delta_1$-groups are strongly π-solvable. *P. Fong* (Berkeley, Calif.)

Thompson, John G. 4793

Normal p-complements for finite groups.

J. Algebra **1** (1964), 43–46.

For a finite group G, let $m(G)$ be the minimum number of generators of G, let $d(G)$ be the maximum of $m(A)$ for abelian subgroups A of G, and let $J(G)$ be the subgroup of G generated by the abelian subgroups A of G with $m(A) = d(G)$. The following theorem is proved. If P is a Sylow p-subgroup of G (p odd), and both the centralizer of the center of P and the normalizer of $J(P)$ have normal p-complements, then G has a normal p-complement. The main theorem of an earlier paper by the author [Math. Z. **72** (1959/60), 332–354; MR **22** #8070] is a corollary. *Graham Higman* (Oxford)

Wong, W. J. 4794

A characterization of the Mathieu group M_{12}.

Math. Z. **84** (1964), 378–388.

Author's introduction: "A special case of a theorem of R. Brauer [Proc. Internat. Congr. Mathematicians 1954 (Amsterdam) Vol. I, pp. 209–217, Noordhoff, Groningen, 1957; MR 20 #1709] may be regarded as a characterization of the Mathieu group M_{11} of order 7920 in terms of the centralizers of its involutions. In this paper we give a similar characterization of the Mathieu group M_{12} of order 95040. Theorem: Let $C(\pi_0)$ be the centralizer in M_{12} of an involution π_0 in the centre of a 2-Sylow subgroup of M_{12}. Let G be a finite group such that (i) G contains an involution π whose centralizer $C(\pi)$ in G is isomorphic to $C(\pi_0)$; and (ii) G does not contain three mutually non-conjugate involutions. Then, either G is isomorphic with M_{12}, or G has a unique proper non-trivial normal subgroup N. In the latter case, N is elementary Abelian of order 8, and G/N is isomorphic with the simple group GL(3, 2) of order 168. We prove this theorem by means of computations with characters of G. We can also show that in the second case G is isomorphic with a certain group of non-linear transformations on a 4-dimensional vector space over GF(2)."

W. Moser (Montreal, Que.)

Abian, Alexander; Rinehart, Daryl 4795

Honest subgroups of abelian groups.

Rend. Circ. Mat. Palermo (2) **12** (1963), 353–356.

Let H be any subgroup and let T be the torsion subgroup of an abelian group A. H is called honest if $na \neq 0 \in H$ implies $a \in H$, where $a \in A$ and n is an integer. Equivalently, H is honest if and only if $gp(a) \cap H = \{0\}$ for every $a \in A - H$. If H is honest, then clearly H is pure. Theorem 1: If H is honest then either $H \subset T$ or $T \subseteq H$. Theorem 2: If $H \subset T$, then H is honest if and only if (i) T is p-primary for some prime p, (ii) H is a direct sum of cyclic subgroups of order p, and (iii) H is a direct summand of A. Theorem 3: If $T \subseteq H$, then H is honest if and only if A/H is torsion-free.

S. Lipschutz (Levittown, N.Y.)

Laffer, W. B.; Mann, H. B. 4796

Decomposition of sets of group elements.

Pacific J. Math. **14** (1964), 547–558.

By $A+B$ [$A-B$] is meant the set of (Abelian group-) sums $a+b$ [$a-b$], $a \in A$, $b \in B$. If C is expressible as a sum $A+B$ of sets each of which has more than two elements, C is said to be decomposable. Let G be cyclic of order $n(n+1)+1$, and suppose no set C of $n+1$ elements exists such that $G = C-C$. [This is true, in particular, for values of $n \leq 1600$ that are not prime powers.] Then every set of n^2 elements of G has a decomposition of the form $\{a_1, a_2\}+B$. Theorem 2: If a set C of three elements is decomposable, then $C = \{a, a+b, a+2b\} = \{a-x, a+b-x\}+\{x, x+b\}$. Theorem 3 provides a partial generalization of Theorem 2 for a set of more than three elements. Theorem 4: For every $s>1$, there exists an Abelian group containing an indecomposable set of $s+1$ elements. A second portion of the paper concerns sums of sets of ordinary integers; gap theorems are given. In particular, Theorem 12 asserts that if $A+B \sim P$, the set of primes, then neither A nor B can be finite, where "$\sim$" means "$=$ except for 0%".

J. L. Brenner (Palo Alto, Calif.)

Rangaswamy, K. M. 4797

Full subgroups of abelian groups.

Indian J. Math. **6** (1964), 21–27.

Let G be an abelian group. A subgroup F is full if F contains the socle of G. A subgroup H is absorbing if $x \in H$ whenever $nx \in H$ for some integer n. Clearly, an absorbing subgroup is full. Conversely (Theorem 2), a full subgroup F is absorbing if it is neat, that is, $F \cap pG = pF$ for every prime p. The author investigates some properties of neat subgroups and characterises the class of full subgroups which is (Theorem 5) the class of subgroups having a unique neat hull.

Ti Yen (E. Lansing, Mich.)

Walker, Carol Peercy 4798

Properties of Ext and quasi-splitting of Abelian groups.

Acta Math. Acad. Sci. Hungar. **15** (1964), 157–160.

Let (*) $0 \to A \to G \xrightarrow{t} B \to 0$ be an exact sequence of Abelian groups representing an element of $\mathrm{Ext}(B, A)$. It is of finite order in $\mathrm{Ext}(B, A)$ if and only if for some integer n the sequence $0 \to A/A[n] \to (A+nG)/A[n] \xrightarrow{t'} nB \to 0$ is splitting exact, where $t'(x+A[n]) = t(x)$. If $A[n]=0$ or if $B[n]=0$, then the condition for (*) to belong to $\mathrm{Ext}(B, A)[n]$ is the splitting exactness of $0 \to A \to A+nG \to nB \to 0$. It is also shown that an Abelian group G is quasi-isomorphic to a group that splits (over its torsion subgroup) if and only if $0 \to G_t \to G \to G/G_t \to 0$ represents an element of finite order in $\mathrm{Ext}(G/G_t, G_t)$, where G_t denotes the torsion subgroup of G. The existence of a non-splitting group which is quasi-isomorphic to a splitting group follows. [Ti Yen has shown (unpublished) that for groups G of finite torsion-free rank the quasi-isomorphy to a splitting group implies that the group G itself splits.]

L. Fuchs (Budapest)

Higman, Graham 4799

Amalgams of p-groups.

J. Algebra **1** (1964), 301–305.

The main aim is the proof of the following elegant theorem. Let $A \cup B$ be an amalgam of two finite p-groups with amalgamation $U = A \cap B$. Then $A \cup B$ is embeddable in a finite p-group if there are chief series of A and B which induce by intersection the same chief series of U. This has, among other things, the following consequences. (1) If U is cyclic, $A \cup B$ is embeddable in a finite p-group. This answers a question put to the author by G. Baumslag. (2) If U is normal in A and in B, and $\mathrm{Aut}_A(U)$ and $\mathrm{Aut}_B(U)$ together generate a p-group, $A \cup B$ is embeddable in a finite p-group. The proof of the main theorem is via wreath products, and the principal steps are as follows. Let A, H by any two groups and θ any homomorphism from A to H, with kernel X. Then a counter-map θ^* is a map from H to A such that $(a\theta \cdot h)\theta^*\theta = a\theta \cdot h\theta^*\theta$ for all a in A, all h in H. Countermaps always exist. To each a in A define an element f_a of the cartesian power X^H by

$$f_a(h) = [(a\theta \cdot h)\theta^*]^{-1} a \cdot h\theta^*.$$

It is readily seen that $a \to (a\theta, f_a)$ is an embedding of A in the wreath product $X \wr H$, which the author terms a standard embedding. There are, of course, very many different standard embeddings. Lemma 1: Let U be a subgroup of A and θ a homomorphism from A to H with kernel X. Given any standard embedding $u \to (u\theta, f_u)$ of U in $X \wr H$, we can find a standard embedding $a \to (a\theta, g_a)$ of A in $X \wr H$ such that $f_u(h)$ and $g_u(h)$ are conjugate in A for all u in U, h in H. This has the important corollary: If $U \cap X$ is in the centre of A, then any standard embedding of U

in $X \wr H$ can be extended to an embedding of A in $X \wr H$. Lemma 2: Let $X \cup Y$ be a normal subamalgam of $A \cup B$ (that is, $X \lhd A$, $Y \lhd B$, and $X \cap U = Y \cap U$), and suppose that $Z = X \cap U$ is central in A and in B. If $X \cup Y$ can be embedded in T and $(A/X) \cup (B/Y)$—with the obvious amalgamation—in H, then $A \cup B$ can be embedded in $T \wr H$. The main theorem is then proved by induction on the product $|A|\,|B|$ and looking at the subgroups of order p in the designated chief series of A and B.

J. Wiegold (Cardiff)

Alperin, J. L. 4800

Centralizers of abelian normal subgroups of p-groups.

J. Algebra **1** (1964), 110–113.

Theorem: "If E is a subgroup of a p-group G, maximal subject to being normal abelian and of exponent p^n, then any element of order at most p^n which centralizes E lies in E, unless perhaps $p = 2$ and $n = 1$." The case $n = 1$ implies a lemma by W. Feit and J. Thompson [Pacific J. Math. **13** (1963), 775–1029; MR **29** #3538]. The dihedral group of order 16 shows that $p^n \neq 2$ is essential.

Corollary: "Let G be a 2-group which has no three-generator abelian normal subgroups. If H is a normal subgroup of G and $H \cong Z_4 \times Z_4$, then any element of order at most four which centralizes H lies in H. Furthermore, the centralizer $C(H)$ of H is a metacyclic group." The second assertion in the corollary is based upon results of N. Blackburn [Proc. London Math. Soc. (3) **11** (1961), 1–22; MR **23** #A208]. *S. Lipschutz* (Levittown, N.Y.)

Cárdenas, Humberto; Lluis, Emilio 4801

On subgroups of a Sylow p-group of the symmetric group S_{p^2}. (Spanish)

An. Inst. Mat. Univ. Nac. Autónoma México **1** (1961), 19–25.

A detailed discussion of the Sylow subgroups mentioned in the title, as well as of related subgroups of the symmetric group on p^2 elements.

H. W. Brinkmann (Swarthmore, Pa.)

Cárdenas, Humberto; Lluis, Emilio 4802

The normalizer of the Sylow p-group of the symmetric group S_{p^2}. (Spanish)

An. Inst. Mat. Univ. Nac. Autónoma México **2** (1962), 1–7.

A continuation of a paper by the same authors [#4801] to the study of the normalizers of the Sylow subgroups of the symmetric group on p^2 elements.

H. W. Brinkmann (Swarthmore, Pa.)

Kovács, L. G.; Neumann, B. H.; de Vries, H. 4803

Some Sylow subgroups.

Proc. Roy. Soc. Ser. A **260** (1961), 304–316.

If Π denotes a set of prime numbers, a Π-group is a periodic group whose elements have orders divisible by primes in Π only. If G is a group, H is a Π-subgroup of G, then H is contained in a maximal or Sylow Π-subgroup of G.

E. V. Schenkman has proved (oral communication) that if a group G is countable, periodic and locally soluble, then to every pair Π, Π' of complementary sets of primes there exists a Sylow Π-subgroup S and a Sylow Π'-subgroup T such that $ST = G$, $S \cap T = \{1\}$. Conditions for periodicity and local solubility cannot be omitted.

The author shows that, under the conditions of Schenkman's theorem: (i) there may be Sylow Π-subgroups that are complemented, but not by Sylow Π'-subgroups; (ii) there may be Sylow Π-subgroups that are not complemented at all; (iii) there may be continuously many non-isomorphic Sylow Π-subgroups for fixed Π.

For Schenkman's groups and, more generally, for countable locally finite groups all Sylow Π-subgroups for fixed Π have the same order. The author shows that for uncountable groups even this weak proposition may fail to hold. *D. E. Littlewood* (Bangor)

Černobyl'skaja, É. I.; Cikunov, I. K. 4804

Sylow p-subgroups of the orthogonal and symplectic groups of a hyperbolic space. (Russian. English summary)

Ukrain. Mat. Ž. **15** (1963), 290–298.

A nondegenerate hyperbolic space V_{2n} is one with the metric $(X, Y) = \sum_1^n (x_{2i-1}y_{2i} - x_{2i}y_{2i-1})$ [symplectic case] or the metric $(X, Y) = 2\sum_1^n x_{2i-1}y_{2i}$ [orthogonal case]. The characteristic of the underlying field is p, $p \neq 2$; its order is $q = p^k$. The paper is concerned with the Sylow p-subgroup of the group of isometries $G(V_{2n})$ of V_{2n}. These Sylow subgroups have orders $q^{n(n-\eta)}$, where $\eta = 0$ [orthogonal case] or $\eta = 1$ [symplectic case].

Theorem: S is a Sylow subgroup of $G(V_{2n})$ if and only if (a) there is a maximal invariant isotropic subspace $U_n \subset V_{2n}$, and (b) S induces on U_n a Sylow p-subgroup of the group of isometries of U_n. Let S and a maximal invariant isotropic subspace U_n be given; $SU_n \subset U_n$. The subgroup H of S that fixes U_n elementwise is a normal divisor of S; $|H| = q^{(n+1-2\eta)n/2}$. If W_n is a complementary isotropic subspace, then every transformation in S can be written hd, where $h \in H$ and $d \in D_W$, the subgroup of S that transforms W into itself. Moreover, $S = T \cdot H$, where $T \cap H = 1$, $T = S \cap G(U_n)$. In the symplectic [orthogonal] case, H is isomorphic to the additive group of $n \times n$ [skew-] symmetric matrices. This isomorphism can be extended to S in such a way that if $d \in D_W$, then dhd^{-1} corresponds to a conjunctive transformation $\Gamma A \Gamma'$ on this additive group. *J. L. Brenner* (Menlo Park, Calif.)

Wohlfahrt, Klaus 4805

An extension of F. Klein's level concept.

Illinois J. Math. **8** (1964), 529–535.

Let Γ be the modular group, the group of all 2×2 matrices with rational integral elements and determinant 1; let $\Gamma(n)$ be the principal congruence subgroup of level n, the group of all matrices in Γ that are congruent to ± 1 modulo n. F. Klein defined a subgroup G of Γ to be of level ("Stufe") n provided n is the smallest positive integer for which $\Gamma(n) \subset G \subset \Gamma$. Then G is called a congruence subgroup.

The author extends the level concept as follows. Every parabolic matrix P in Γ is conjugate over Γ to a translation by an integer m; $|m|$ is called the amplitude of P. Let $G \subset \Gamma$; then the level of G is defined to be the least common multiple of the amplitudes of all parabolic elements in G. When G is a congruence subgroup, the

Klein level agrees with the level thus defined. The new definition has already demonstrated its importance in (as yet unpublished) research.

Using the new definition, the author proves a number of theorems. Let $\Delta(m)$ be the normal closure in $\Gamma(m)$ of $\begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix}$. It is known that $\Delta(m)=\Gamma(m)$ if $1\le m\le 5$; otherwise $\Delta(m)$ is of infinite index in $\Gamma(m)$ [cf. M. I. Knopp, Proc. Amer. Math. Soc. **14** (1963), 95–97; MR **26** #230]. The author reproves this theorem. He then constructs an infinite set of normal subgroups of Γ of level $m\ge 6$ which, however, are not congruence subgroups in the sense of Klein and so contain no principal congruence subgroup. These are different from the groups of I. Reiner [Illinois J. Math. **2** (1958), 142–144; MR **20** #2379].

Next, by means of a theorem of Fricke, the author proves an arithmetic characterization of Γ^6. Finally, he shows that every subgroup of Γ of index ≤ 6 is a congruence subgroup. *J. Lehner* (College Park, Md.)

Platonov, V. P. 4806

Splittable linear groups. (Russian)

Dokl. Akad. Nauk SSSR **151** (1963), 286–289.

Let $L_n(P)$ be the full linear group of $n\times n$ matrices over the perfect field P. A subgroup G of $L_n(P)$ is splittable if it contains the semisimple and unipotent components g_s and g_u of every element g of G. An algebraic subgroup G of $L_n(P)$ is splittable and the author enunciates with brief indications of proofs a number of theorems to show that the known properties of algebraic G extend to all splittable G. For some of them he has to assume that P is of characteristic 0. At the end there is a brief discussion of the Lie group case.

Typical results are the following. Theorem 1: If G is a splittable locally nilpotent group, then its sets G_s, G_u of semisimple [unipotent] elements are invariant subgroups and $G=G_s\cdot G_u$. Theorem 6: Suppose Char $P=0$, and let $G\subset L_n(P)$ be a soluble group and c^∞ (i.e., $c^\infty\bar{G}\subset G$, where $c^\infty\bar{G}=\bigcap c^i\bar{G}$, $c^0\bar{G}=\bar{G}$, $c^i\bar{G}=\overline{[c^{i-1}\bar{G},\bar{G}]}$ and the bar denotes the Zariski closure). Then G is splittable if and only if it is an S-group ($G=DG_u$, where D is any maximal subgroup consisting only of elements of G_s).

J. W. S. Cassels (Cambridge, England)

Steinberg, Robert 4807

Differential equations invariant under finite reflection groups.

Trans. Amer. Math. Soc. **112** (1964), 392–400.

From the author's introduction: "In this paper we study the characteristic functions of those differential operators with constant coefficients that are invariant under finite linear groups, especially under finite reflection groups. Our main result is a manifold of characterizations of finite reflection groups in terms of the above-mentioned characteristic functions."

J. A. Todd (Cambridge, England)

Suprunenko, D. A. 4808

On maximal commutative matrix algebras and maximal commutative matrix groups. (Russian)

Dokl. Akad. Nauk BSSR **8** (1964), 425–428.

Let Δ be a commutative field with more than two elements. The author shows that there is a one-to-one correspondence between L, the maximal commutative subalgebras of the algebra Δ_n of $n\times n$ matrices over Δ on the one hand, and K, the maximal abelian subgroups of the general linear group $\mathrm{GL}(n,\Delta)$ on the other. The correspondence is established by assigning to any algebra $A\in L$ the group A^* of its invertible elements, and to any group $\Gamma\in K$ its linear hull $[\Gamma]$. Conjugacy is preserved both ways. In the exceptional case $\Delta=\mathrm{GF}(2)$ with $n>1$, L has more elements than K. *K. A. Hirsch* (London)

Veldkamp, F. D. 4809

Isomorphisms of little and middle projective groups of octave planes.

Nederl. Akad. Wetensch. Proc. Ser. A **67** = *Indag. Math.* **26** (1964), 280–289.

If C is a Cayley division algebra over a commutative field K of characteristic $\ne 2, 3$, then there is a (by now standard) way to construct a projective plane $\mathbf{P}(C)$ in the Jordan algebra A of hermitian 3×3 matrices over C. On A there is defined a cubic form det. Let S be the group of linear transformations of A leaving det invariant, and let G be the group of linear transformations t of A such that $\det tx=\nu(t)x$ with $\nu(t)\in K$. The factor groups PS, PG of S, G by their centers are the little and the middle projective groups of $\mathbf{P}(C)$. Now the following result is proved: If φ is any isomorphism of PS [PG] onto PS' [PG'], where PS' and PG' are the little and the middle projective groups of some other Cayley plane $\mathbf{P}(C')$, then there is a projective transformation or a correlation τ of the projective plane $\mathbf{P}(C)$ onto $\mathbf{P}(C')$ such that $\varphi(\eta)=\tau^{-1}\eta\tau$ for all $\eta\in PS$ [PG], and this also forces the underlying fields to be isomorphic. For the little projective group of a Cayley plane this was proved by Tae-Il Suh [same Proc. **65** (1962), 320–330; MR **25** #2510a; ibid. **65** (1962), 331–339; MR **25** #2510b]. *O. H. Kegel* (Frankfurt a.M.)

Gudivok, P. M.; Drobotenko, V. S.; Lihtman, A. I. 4810

On representations of finite groups over the ring of residue classes modulo m. (Russian)

Ukrain. Mat. Ž. **16** (1964), 82–89.

Let G have order s; let T be the ring of residues mod m. G has indecomposable representations of arbitrarily high degree over T if and only if for some prime p, either (1) $p|s$, $p^2|m$, or (2) $p|m$ and the p-Sylow subgroups of G are not cyclic. If $p^2|m$, $s=p^e$, then G has an irreducible representation of arbitrarily high degree.

J. L. Brenner (Palo Alto, Calif.)

Isaacs, I. M.; Passman, D. S. 4811

Groups with representations of bounded degree.

Canad. J. Math. **16** (1964), 299–309.

The finite group G is said to have representation bound n (r.b. n) if every irreducible (complex, linear) representation of G is of degree $\le n$. The main result of this paper is that there exists a natural number $f(n)$ such that every finite group with r.b. n has an abelian subgroup of index at most $f(n)$. An upper estimate of $f(n)$ is given. After a suitable generalization of the notion of irreducible representation and its degree to arbitrary groups, the above

theorem is extended to arbitrary groups such that all irreducible representations are of finite degree bounded by n. *O. H. Kegel* (Frankfurt a.M.)

Matuljauskas, A.; Matuljauskene, M. 4812

On integral representations of a group of type (3, 3). (Russian. Lithuanian and German summaries)

Litovsk. Mat. Sb. 4 (1964), 229–233.

Authors' summary: "Jede ganzzählige Darstellung einer Gruppe von Typus (3, 3) zerfällt in eine endliche Anzahl s unzerfällbarer inäquivalenten Bestandteile ($s \leq 9, 12$), wenn es in dieser Darstellung eine Matrix gibt, die der Gleichung $x^2+x+1=0$ genügt. Die Grade unzerfällbarer Bestandteile sind durch die Zahl 6 beschränkt."

Tamaschke, Olaf 4813

S-rings and the irreducible representations of finite groups.

J. Algebra 1 (1964), 215–232.

An S-ring T of a finite group $\mathfrak{G}$ of order g over the complex number field $\mathbf{C}$ is a subring of the group algebra Γ of $\mathfrak{G}$ over $\mathbf{C}$, having as $\mathbf{C}$-basis the t sums τ_i of elements of certain trivially intersecting sets $\mathfrak{T}_i$ of t_i elements of $\mathfrak{G}$ (called T-classes of $\mathfrak{G}$), such that the inverses of the elements of $\mathfrak{T}_i$ form a T-class $\mathfrak{T}_j=\mathfrak{T}_i^*$, and the union of all $\mathfrak{T}_i$ is $\mathfrak{G}$. The T-class $\mathfrak{T}_1$ containing the identity E of $\mathfrak{G}$ is a subgroup $\mathfrak{L}$ of $\mathfrak{G}$, and T is called unitary if $\mathfrak{T}_1=E$. Commutative S-rings are subrings of the center of Γ. Other important S-rings have as T-classes the double cosets $\mathfrak{L}G\mathfrak{L}$ of $\mathfrak{L}$. Complex functions on $\mathfrak{G}$ that are constant on each $\mathfrak{T}_i$ form an algebra $T^\#$ of T-class functions. For each τ_i, assemble in lexicographic order the $t=\sum y_\nu^2$ coefficients $f_{\alpha\beta}^{(\nu)}(\tau_i)=F_{hi}$ in a complete set of r inequivalent representations $\tau\to F_\nu(\tau)$ of T of degrees y_ν, and thus form the $t\times t$ matrix F. Also, taking any complete set of irreducible $\mathbf{C}$-matrix representations $G\to(d_{\kappa\lambda}^{(\rho)}(G))$ of $\mathfrak{G}$ of degrees x_ρ, let $z_h=\sum x_\rho$, and $\Phi_h(G)=\sum x_\rho d_{\kappa\lambda}^{(\rho)}(G)$, summed for all ρ, κ, λ such that $d_{\kappa\lambda}^{(\rho)}(\tau_i)=F_{hi}$ for all i. Then $\Phi_h(G)=\Phi_{hi}$ for all $G\in\mathfrak{T}_i$. These functions Φ_h form a $\mathbf{C}$-basis of $T^\#$, and $\eta_h=\sum_i \Phi_{hi}\tau_i^*/g$ are matrix units for the representations F_ν. Let F^* and Φ^* replace F and Φ when $F_\nu'(\tau^*)$ replaces $F_\nu(\tau)$ and $d_{\lambda\kappa}^{(\rho)}(G^{-1})$ replaces $d_{\kappa\lambda}^{(\rho)}(G)$. Define the $t\times t$ diagonal matrices $T=\operatorname{diag}\{t_i\}$, $Z=\operatorname{diag}\{z_h\}$. Important results in the paper include: $ZF=\Phi T$, $\Phi'F^*=\Phi^*F'=gE_t$, and hence $F'ZF^*=gT$, $\Phi^*T\Phi'=gZ$. Also

$$(\det F/gt_1^{t-1})^2 = \pm(g/t_1)^{t-2}\prod(t_i/t_1)/\prod z_\nu^{y_\nu^2}$$

is a rational integer, which is a square if $\det F$ is rational. Furthermore, when $\det\Phi$ is an algebraic integer, then $g^{t-2}\prod z_\nu^{y_\nu^2}/\prod t_i$ is an integer.

J. S. Frame (E. Lansing, Mich.)

Puttaswamaiah, B. M.; Robinson, G. de B. 4814

Induced representations and alternating groups.

Canad. J. Math. 16 (1964), 587–601.

In § 1 an induced representation of a group is considered. In the rest of the paper the authors discuss the representation theory of the alternating group. In § 2 the matrix representing $(1, 2)(r, r+1)$ in an irreducible representation of A_n is given explicitly, in § 3 the minimal left ideals of the group ring of A_n are determined, in § 4 it is shown how the result in § 1 can be applied to determine the irreducible components in an induced representation of A_n, and in § 5 the modular representations and the block structure of A_n are considered. The authors conclude the discussion by giving the decomposition matrices (mod 2, 3) of A_n for $n=3, 4, 5, 6, 7, 8$. {Remark: Since $G=FH$ under the assumption of Theorem 1.4, (1.7) is a special case of Mackey's subgroup theorem and Corollary 1.9 says nothing. In Corollary 1.11 the assumptions of the irreducibilities of θ and ϕ are not necessary.}

H. Nagao (Osaka)

Enochs, Edgar 4815

Extending isomorphisms between basic subgroups.

Arch. Math. 15 (1964), 175–178.

Let G be an Abelian p-group. Under what condition can every isomorphism between two basic subgroups of G be extended to an automorphism of the whole group G? In this paper, the author proves that when G contains no elements of infinite height, G has the above property if and only if G is a closed p-group. (The "if" part was proved by L. Ja. Kulikov [Mat. Sb. (N.S.) 16 (58) (1945), 129–162; MR 8, 252].) *K. Honda* (Tokyo)

Hosszú, M. 4816

On the explicit form of n-group operations.

Publ. Math. Debrecen 10 (1963), 88–92.

It is known [E. L. Post, Trans. Amer. Math. Soc. 48 (1940), 208–350; MR 2, 128] that an n-group can be embedded in a group in which its operation is a continued product. This paper exhibits the operation of an n-group in terms of a group operation defined on the same set, and an automorphism of this group.

H. A. Thurston (Vancouver, B.C.)

Ljapin, E. S. 4817

★Semigroups.

Translations of Mathematical Monographs, Vol. 3.

American Mathematical Society, Providence, R.I., 1963. vii+447 pp. \$21.70.

The Russian edition of this book has already been reviewed [Fizmatgiz, Moscow, 1960; MR 22 #11054]. There is an extensive bibliography of papers on semigroups published before 1959; the list includes many entries not in the original text. *W. D. Munn* (Glasgow)

Ballantine, C. S. 4818

A semigroup generated by definite matrices.

J. Analyse Math. 12 (1964), 257–275.

In this paper the author is concerned with the semigroup (under matrix multiplication) $\mathscr{S}$ which is generated by the cone of non-positive definite $N\times N$ matrices as infinitesimal maps, that is, $\mathscr{S}$ is defined as the smallest closed (under the usual topology) semigroup whose cone $\underline{\mathscr{S}}$ of infinitesimal maps contains all non-positive definite $N\times N$ matrices. He shows that $\mathscr{S}$ has no other infinitesimal maps, that is, $\underline{\mathscr{S}}$ is just the cone of non-positive definite $N\times N$ matrices (Theorem 3.1). Except for this theorem, the results on $\mathscr{S}$ are confined to the case $N=2$. The main result is to characterize $\mathscr{S}$ by means of an inequality on the elements of those matrices S which belong to $\mathscr{S}$

(Theorem 2.1) and by finding the extremal matrices of $\mathscr{S}$ (Theorem 2.2). He has some results on the possibility of factoring a 2×2 rotation matrix into a product of positive definite matrices (Theorems 2.3 and 2.4). He also shows that all non-trivial (closed) subsemigroups of $\mathscr{S}$ (in the 2×2 case) which (like $\mathscr{S}$) are orthogonally invariant and generated by their respective cones of infinitesimal maps constitute a one-parameter family $\{\mathscr{S}(m): 0\leq m\leq 1\}$ and $\mathscr{S}(m)$ decreases as m increases, with $\mathscr{S}(0)=\mathscr{S}$, $\mathscr{S}(1)$ the obvious Abelian subsemigroup, and the remaining $\mathscr{S}(m)$ each isomorphic to $\mathscr{S}$ (Theorem 4.1). The isomorphism is explicitly given. *T.-S. Wu* (Stanford, Calif.)

Čupona, G'orgi 4819

On completely simple semigroups. (Serbo-Croatian summary)

Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hrvatske Ser. II **18** (1963), 159–164.

Author's note: "Most of the results of this paper are contained in the book by A. H. Clifford and G. B. Preston [*The algebraic theory of semigroups*, Vol. I (in Exercises 14 and 15, p. 84, Section 2.7, and Exercise 2(b), p. 97, Section 3.2), Amer. Math. Soc., Providence, R.I., 1961; MR **24** #A2627], which was not available to the author at the time he submitted the paper for publication." *W. E. Deskins* (E. Lansing, Mich.)

Čupona, G'orgi 4820

On semigroups S in which each proper subset Sx is a group. (Serbo-Croatian summary)

Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hrvatske Ser. II **18** (1963), 165–168.

The author sketches here a description of the semigroups listed in the title. If S is such a semigroup and $P=\{x: Sx=S\}$, $Q=\{x: x\subset Sx\neq S\}$, and $B=\{x: x\notin Sx\neq S\}$, then $S=Q\cup B\cup P$ with $Q\cap B=Q\cap P=B\cap P=\varnothing$. Q is nonempty and there exists a group G and a set A such that $Q\cong G\times A$, where multiplication in $G\times A$ is defined by $(x,\alpha)(y,\beta)=(xy,\beta)$ for $x,y\in G$ and $\alpha,\beta\in A$. Furthermore, P is a left simple semigroup (if nonempty) and B can be mapped into Q. Then S is characterized in terms of a lengthy list of relations between Q, B, and P. These semigroups form a subclass of the types studied by Št. Schwarz [Acta Sci. Math. (Szeged) **21** (1960), 125–134; MR **24** #A776] and R. Hrmová [Mat.-Fyz. Časopis Sloven. Akad. Vied **11** (1961), 75–80]. *W. E. Deskins* (E. Lansing, Mich.)

Drbohlav, Karel 4821

On finitely generated commutative semigroups.

Comment. Math. Univ. Carolinae **4** (1963), 87–92.

Let S_1 be the additive semigroup of all non-negative integers. S_n is defined by $S_n=S_1\oplus S_{n-1}$ recursively, namely, the direct sum of S_1 and S_n. This paper gives a theorem that the maximum condition for congruence relations holds in S_n ($n=1,2,\cdots$). The proof is given by means of induction on n. For this purpose, several lemmas and a few concepts are prepared. For example, one lemma says that the maximum condition for ideals holds in S_n. The following ideal plays an important rôle. For a congruence C on S_n and $a_1,a_2\in S_1$, an ideal $J(C,a_1,a_2)$ of S_{n-1} is defined to be the set of all $b_1\in S_{n-1}$ such that $a_1\oplus b_1 C a_2\oplus b_2$ for some $b_2\in S_{n-1}$. As the author remarks, the theorem is still true for any finitely generated commutative semigroup. The reviewer adds that all congruence relations on S_n are determined in terms of ideals of S_n and subgroups of a certain group associated with S_n. Then the theorem of this paper can be obtained more simply as one of the consequences. *T. Tamura* (Davis, Calif.)

Kazim, M. A.; Husain, F. 4822

On the postulates defining a subtractive group.

Math. Student **31** (1963), 95–107 (1964).

The authors consider a set S with two binary operations (written ab and $a\underline{b}$, respectively) satisfying: $a,b\in S$ implies ab, $a\underline{b}\in S$, $(a\underline{b})\underline{c}=\underline{a}(\underline{b}c)$, and $a\underline{b}=c$ if and only if $\underline{c}a=b$. They show that S can be turned into a group in such a way that $a\underline{b}=ab^{-1}$ and $\underline{a}b=a^{-1}b$. Thus, the three postulates constitute an alternate way of defining a group. The group is commutative if and only if the following additional postulate is satisfied: $a\underline{b}=\underline{a}b$. These four postulates are shown to be independent, and some consequences of them are derived (e.g., $(a\underline{b})(\underline{a}c)=c\underline{b}$). *E. J. Tully, Jr.* (Davis, Calif.)

Losey, Gerald 4823

On the structure of π-regular semigroups.

Proc. Amer. Math. Soc. **15** (1964), 955–959.

A semigroup S is called π-regular if for every $a\in S$ the descending chain condition on $\{Sa^n;\ n=0,1,\cdots\}$ and $\{a^nS;\ n=0,1,\cdots\}$ holds; equivalently, $xa^{n+1}=a^n=a^{n+1}y$ for some $x,y\in S$ and for some integer $n>0$ [see Azumaya, J. Fac. Sci. Hokkaido Univ. Ser. I **13** (1954), 34–39; MR **16**, 788; Drazin, Amer. Math. Monthly **65** (1958), 506–514; MR **20** #5217; the author and Schneider, Pacific J. Math. **11** (1961), 1089–1098; MR **25** #137]. The author discusses the structure of π-regular semigroups. The author calls a semigroup P a proto-group if P contains a unique maximal group G and if for each $a\in P$ there is an integer $n>0$ such that $a^n\in G$. A proto-group whose unique idempotent is zero is called a nil semigroup. Clearly, proto-groups are π-regular. The most important theorems in this paper are the following. A semigroup S is π-regular if and only if S is the union of proto-groups. If S is commutative or if all the idempotents of S are primitive [see the definition of "primitive" in Clifford and Preston, *Algebraic theory of semigroups*, Vol. I, Amer. Math. Soc., Providence, R.I., 1961; MR **24** #A2627], then S is π-regular if and only if S is a disjoint union of proto-groups. The remaining part of this paper is devoted to the study of the structure of proto-groups. Every proto-group P is obtained as a split extension of a nil semigroup N by a group G. (If $N\subseteq P$, if P is homomorphic onto G and if a subsemigroup of P is isomorphic onto G under the homomorphism of P to G, then P is called a split extension of N by G.) Also a protogroup P is an ideal extension of a group G by a nil semigroup N [see Clifford and Preston, loc. cit.]. The reviewer adds that P is uniquely determined in terms of G, N and a certain mapping of $N-\{0\}$ into G, and that the proto-groups could be treated as a special case of unipotent inversible semigroups due to the reviewer [Kôdai Math. Sem. Rep. **1954**, 93–95; MR **16**, 443]. *T. Tamura* (Davis, Calif.)

Petrich, Mario 4824

The maximal semilattice decomposition of a semigroup.
Math. Z. **85** (1964), 68–82.

First, the author gives proofs for his theorems published earlier in an abstract [see Bull. Amer. Math. Soc. **69** (1963), 342–344; MR **26** #5084]. Next he discusses the N-classes and the semilattice Y of all N-classes: several necessary and sufficient conditions are given for every N-class to be simple, left simple, completely simple, or a group, and for the semilattice Y to be linearly ordered. Moreover, some necessary and sufficient conditions are also given for Y to be linearly ordered and at the same time for every N-class to be simple, left simple, completely simple, or a group. Finally, explicit expressions for $N(x)$ are given for some special classes of semigroups.

G. Szász (Nyiregyháza)

Sankaran, N. 4825

Group embedding and duality in semigroups.
Publ. Math. Debrecen **10** (1963), 63–68.

Let S be a commutative cancellative semigroup with identity. A semigroup $(S, \mathfrak{A})$ is called a uniform semigroup if it has a uniform structure $\mathfrak{A}=\{U_\alpha\}$ such that $(x, y)\in U_\alpha$, $U_\alpha\in\mathfrak{A}$ if and only if $(x+z, y+z)\in U_\alpha$ for each $z\in S$. Let C^+ be the set of all non-negative real numbers and C the set of all real numbers. A continuous additive homomorphism of a uniform semigroup S into C^+ (or into C) is called a character. The set D of all characters of S relative to C^+ is a semigroup, but the set G of all characters of S relative to C is a group. The author proves that D can be embedded into a group if the smallest group H containing an isomorphic image of D (called a group embedding of D) is algebraically isomorphic to G. In the case where G and H have certain uniform structures, G is unimorphic to H (i.e., G is uniformly continuous to H and vice versa). Combining the above theorem with Krishnan's result [J. Indian Math. Soc. (N.S.) **24** (1960), 283–318], the author obtains theorems on the group embedding of the direct sum of dual semigroups with certain uniform structures.

T. Tamura (Davis, Calif.)

Warne, R. J. 4826

Homomorphisms of d-simple inverse semigroups with identity.
Pacific J. Math. **14** (1964), 1111–1122.

Let S be a d-simple inverse semigroup with identity, namely, a d-simple semigroup with identity such that any two idempotents of S commute. The structure of S is determined by the structure of its right unit semigroup P which satisfies: (1) it is right cancellative, (2) P has an identity, (3) the intersection of two principal left ideals of P is a principal left ideal of P. {With respect to the terminology, "d-simple (same as D-simple) inverse semigroup", see Clifford and Preston [*The algebraic theory of semigroups*, Vol. I, Amer. Math. Soc., Providence, R.I., 1961; MR **24** #A2627]; with respect to its structure, see Clifford [Amer. J. Math. **75** (1953), 547–556; MR **15**, 98].} S is isomorphic with the semigroup consisting of all ordered pairs (a, b) of elements of P in which equality is defined by $(a, b)=(a', b')$ if and only if $a'=\rho a$, $b'=\rho b$, where ρ is a unit in P. The author determines all homomorphisms of a d-simple inverse semigroup S with identity. Let S and S^* be two such semigroups and let P and P^* be their right unit subsemigroups. The first main result is that the homomorphism M of S into S^* is determined in terms of a certain homomorphism N (called an sl-homomorphism) and an element k of P^* in the following way: $(a, b)M=[(aN)k, (bN)k]$, where the square brackets indicate an element of S^*. In particular, S is isomorphic with S^* if and only if P is isomorphic with P^*. If M is an isomorphism of S with S^*, M is determined by an isomorphism N of P with P^* in a one-to-one fashion: $N\leftrightarrow M$. The second main result is concerned with the determination of homomorphisms of P into P^*. Let U be the group of units of P. Then U is a right normal divisor of P [Rédei, Acta Sci. Math. (Szeged) **14** (1952), 252–273; MR **14**, 614]. If L denotes the congruence on P modulo U, then P/L satisfies (1), (2), (3); thus P is a Schreier extension of U by P/L. For P^*, we define U^* and L^* in the same way. A homomorphism M of P into P^* is determined in terms of a homomorphism f of U into U^*, a homomorphism g of P/L into P^*/L^*, and a function h of P/L into U^* such that $(A, a)M=[(Af)(ah), ag]$, where $(A, a)\in P$, $A\in U$, $a\in P/L$. To illustrate the theory, a few examples are given.

T. Tamura (Davis, Calif.)

Yusuf, S. M. 4827

A structure theorem for inverse semigroups.
J. Natur. Sci. and Math. **4** (1964), 103–108.

The author establishes a structure theorem for inverse semigroups which is too involved to reproduce here.

J. Kist (University Park, Pa.)

Yusuf, S. M. 4828

Admissible ideals of a semigroup with operators.
J. Natur. Sci. and Math. **4** (1964), 109–124.

The author discusses admissible ideals of a semigroup with operators. He defines admissibly minimal ideals and admissibly 0-minimal ideals and gives examples to show that an admissibly minimal [admissibly 0-minimal] ideal is not necessarily a minimal [0-minimal] ideal. He defines admissibly simple [left admissibly simple, right admissibly simple], and admissibly 0-simple [left and right admissibly 0-simple] semigroups, and shows by examples that admissibly simple [admissibly 0-simple] semigroups need not be simple [0-simple]. He defines the admissible kernel of a semigroup S with operators as the intersection, when it is nonempty, of all the admissible two-sided ideals of S. He proves that an admissible kernel is an admissibly simple subsemigroup of S, and gives a necessary and sufficient condition for a semigroup with operators to be admissibly 0-simple. He also proves that an admissibly 0-minimal two-sided ideal of a semigroup with operators is either a zero semigroup or an admissibly 0-simple subsemigroup of S. He gives a condition which is sufficient, but not necessary, for an admissibly 0-minimal right [left, two-sided] ideal of a semigroup with operators to be 0-minimal. In the final section, he discusses admissible Rees factor semigroups and compares them with admissible factor inverse semigroups.

J. Kist (University Park, Pa.)

Majumdar, Subrata 4829

Homomorphisms of an inverse semigroup.
J. Natur. Sci. and Math. **4** (1964), 125–132.

The author considers certain aspects of the normaliser

and the centralizer of a subset of an inverse semigroup. In addition, he attempts to generalize Chawla's results [same J. 2 (1962), 110–112; MR 26 #3792] to the case of an inverse semigroup. *J. Kist* (University Park, Pa.)

Jakubík, Jan [Jakubík, Ján] 4830

Lexicographic products of partially ordered groupoids. (Russian. German summary)

Czechoslovak Math. J. **14** (89) (1964), 281–305.

$(G, \cdot, \leq)$ is called a u_1-groupoid if it is a partially ordered groupoid with a neutral element e and such that for any $x, y, z \in G$, $x < y \Rightarrow (xz < yz \;\&\; zx < zy)$ and the same holds with "incomparable" instead of "$<$". Let A, B be u_1-subgroupoids of G such that (i) $AB = G$, (ii) $a_1b_1 = a_2b_2 \Rightarrow a_1 = a_2 \;\&\; b_1 = b_2$, (iii) $g_1 = a_1b_1 \;\&\; g_2 = a_2b_2 \Rightarrow g_1g_2 = (a_1a_2)(b_1b_2) \;\&\; [g_1 < g_2 \Leftrightarrow a_1 < a_2 \vee (a_1 = a_2 \;\&\; b_1 < b_2)]$; then G is called a lexicographic product of A, B: $G = A \circ B$. The operation $\circ$ is associative.

If for any pair A, B of lexicographic factors of a u_1-groupoid G we have $\{a | e \leq a \in A\} \subset \{b | e \leq b \in B\} \Rightarrow A \subset B$, and the same holds with "$\geq$" instead of "$\leq$", G is called a u-groupoid. For any pair $G = A_1 \circ \cdots \circ A_n$, $G = B_1 \circ \cdots \circ B_m$ of lexicographic decompositions of a u-groupoid G there exist isomorphic (but not necessarily identical) refinements. Similar results hold for infinite lexicographic products.

Let $\mathcal{G}$ be the set of all lexicographic decompositions (without trivial factors) of a u-groupoid G. $\mathcal{G}$ is ordered by putting $\alpha \leq \beta$ if α is a refinement of β. In $\mathcal{G}$ an equivalence relation $\sim$ is introduced and in $\mathcal{G}/\sim$ order is defined in a natural way; then this quotient structure is a lattice.

V. Devidé (Zagreb)

Sade, A. 4831

Isotopies d'un groupoïde avec son conjoint.

Rend. Circ. Mat. Palermo (2) **12** (1963), 357–381.

If E is the set of elements and $\times$ the operation of a groupoid, its conjoint is the groupoid with elements E and operation $*$ for which $x * y = z$ if and only if $y \times x = z$.

If the groupoid is a quasigroup, its reciprocal is the groupoid with elements E and operation $\ominus$ for which $z \ominus y = x$ if and only if $x \times y = z$, and its adjoint is the conjoint of the reciprocal of its conjoint.

An isotopy is a trio ξ, η, ζ of permutations of E for which $x\xi \cdot y\eta = z\zeta$ if and only if $x \times y = z$ for some operation $\cdot$ on E.

The author investigates isotopies between a groupoid and its conjoints, adjoints, and reciprocals.

H. A. Thurston (Vancouver, B.C.)

Sade, A. 4832

Le groupe d'anti-autotopie et l'équation

$$Q(X, X^{-1}, 1) = QP^{12} = Q.$$

J. Reine Angew. Math. **216** (1964), 199–217.

A groupoid $E(\cdot)$ is a set closed under the operation $(\cdot)$. The dual of $E(\cdot)$ is the groupoid $E(\circ)$ defined by the condition $\forall x, y \in E$, $x \cdot y = z \Leftrightarrow y \circ x = z$. The dual of $G = E(\cdot)$ is designated by GP^{12}, where P^{12} stands for the permutation interchanging the first two of the three letters in the equation $x \cdot y = z$. An isotopy mapping a groupoid on its dual is an anti-autotopy. Sometimes the autotopies and anti-autotopies of a groupoid form a group, which then is called its anti-autotopy group. (X, Y, Z) is the notation for the isotopy $E(\cdot) \to E(\circ)$ with $x \cdot y = z \Leftrightarrow xX \circ yY = zZ$.

The author studies the conditions under which a groupoid has an anti-autotopy group. He also discusses the properties of such a group, the isotopies preserving it, and the group generated by the autotopies and the anti-autotopies in the case where no anti-autotopy group exists. Special attention is given to commutative groupoids and their isotopies and, in particular, to those having an autotopy of the type $(X, X^{-1}, 1)$. This leads to the construction of a commutative quasigroup Q which provides the solution to the equation $Q(X, X^{-1}, 1) = Q$.

R. Artzy (Princeton, N.J.)

Querré, Julien 4833

Ordres maximaux.

C. R. Acad. Sci. Paris **259** (1964), 1467–1470.

Author's summary: "Dans cette note, sont étudiées d'une part, les ordres maximaux d'un monoïde résidué, puis d'une algèbre, d'autre part la détermination des sous-monoïdes résidués clos d'un monoïde résidué; les deux problèmes étant d'ailleurs liés."

P. F. Conrad (Canberra)

TOPOLOGICAL GROUPS AND LIE THEORY

See also 4657, 4779, 4862, 4868, 5227, 5234, 5592, 5594.

Hofmann, Karl Heinrich 4834

Tensorprodukte lokal kompakter abelscher Gruppen.

J. Reine Angew. Math. **216** (1964), 134–149.

Let $\mathfrak{A}$ be the category of all locally compact Abelian groups and $\mathfrak{B}$ a subcategory. The map $b: G \times H \to A$ $(G, H, A \in \mathfrak{A})$ is a bimorphism if $x \to b(x, h)$ and $y \to b(g, y)$ are morphisms whenever $g \in G$ and $h \in H$ are kept fixed, and if b is continuous at $(0, 0)$. The pair $(T, \otimes)$ with T a topological group and $(x, y) \to x \otimes y$ a bimorphism of $G \times H$ into T is called a $\mathfrak{B}$-tensor product of G and H if (1) the subgroup generated by all $x \otimes y$ is dense in T and (2) to any bimorphism $b: G \times H \to A$, with $A \in \mathfrak{B}$, there is a morphism $b': T \to A$ such that $b(x, y) = b'(x \otimes y)$. If $(T_i, \otimes_i)$ are two $\mathfrak{B}$-tensor products of G and H, and if T_i are locally compact or if one of the T_i is compact, then the $\mathfrak{B}$-tensor products are isomorphic. Several statements can be established showing that $\mathfrak{B}$- and $\mathfrak{C}$-tensor products coincide, where $\mathfrak{B}$ is some smaller, and $\mathfrak{C}$ is some larger, subcategory of $\mathfrak{A}$. Here there is no general construction for the tensor product as in the pure algebraic case. In case we take for $\mathfrak{B}$ the category $\mathfrak{K}$ of all compact groups, then the $\mathfrak{K}$-tensor product of G and H exists and T is the character group of the discrete group of all bimorphisms of G, H into the group of reals mod 1. In several cases the $\mathfrak{A}$-tensor product can be determined more explicitly. Application of tensor products to the theory of Schur multiplicators is also given [cf. the author's paper, Math. Z. **79** (1962), 389–421; MR 27 #3739]. *L. Fuchs* (Budapest)

Platonov, V. P. 4835

Some classes of topological groups. (Russian)

Dokl. Akad. Nauk SSSR **158** (1964), 784–787.

For this review, G will denote a locally compact topological group and G_0 will denote the component of the identity. Nine theorems and related corollaries are announced without proof. Theorem 2: The group G satisfies the descending chain condition for closed Abelian subgroups if and only if G_0 is a compact Lie group and G/G_0 is a torsion group satisfying the descending chain condition for Abelian subgroups. The author completely characterizes (Theorems 4 and 5) the groups G containing no nontrivial compact subgroups; all such groups are Lie groups. A group G is said to be complete (or divisible) in the sense of Chernikov (CSC) if $\{x^n : x \in G\}$ generates G for all n. Theorem 7: G is CSC if and only if G/G_0 is CSC. Results on locally projective nilpotent and locally projective solvable groups are given. {The author's Theorem 6 also appears as Corollary 2 in Mycielski [Colloq. Math. **5** (1958), 162–166; MR **20** #6479].}

K. A. Ross (New Haven, Conn.)

Bredon, Glen E. 4836

A new treatment of the Haar integral.

Michigan Math. J. **10** (1963), 365–373.

The author presents a new and interesting proof of the existence and uniqueness of a right invariant integral on a locally compact group G. Let L be the set of all continuous real-valued functions defined on G with compact support, and L^+ the subset of L consisting of non-negative functions. For any function f defined on G and any $y \in G$ let $R_y(f)(x) = f(xy)$. For $f, g \in L^+$, we shall say that g dominates f (notation $g \gtrsim f$) if there exist functions $g_1, \cdots, g_n \in L^+$ and elements $x_1, \cdots, x_n \in G$ such that $g = \sum g_i$ and $g'(x) = \sum R_{x_i}(g_i)(x) \geq f(x)$ for all $x \in G$.

Let $g \in L^+$, $g \not\equiv 0$. For any $f \in L^+$ it is shown that the closures of the sets of real numbers $\{s | sg \gtrsim f\}$ and $\{s | sg \lesssim f\}$ intersect in exactly one point, which is defined to be the integral I_g of f. It then follows that one obtains an integral that is uniquely defined up to a multiplicative constant on L^+ which is right invariant. This can then be extended in a unique way to L.

A. Hajian (Boston, Mass.)

Coleman, Sidney 4837

The Clebsch-Gordan series for SU(3).

J. Mathematical Phys. **5** (1964), 1343–1344.

The author gives a simple derivation of the Clebsch-Gordan formula for SU(3) by first decomposing the tensor product of two irreducible representations into a sum of a particular kind of reducible representations, and then reducing these. *A. Kleppner* (College Park, Md.)

Ernest, John 4838

A new group algebra for locally compact groups.

Amer. J. Math. **86** (1964), 467–492.

Let G be a separable locally compact group. In this paper the author discusses what he calls the "big group algebra" $A(G)$ of G. This turns out to be the second dual of the group C^*-algebra $C^*(G)$ of G and is thus a von Neumann algebra whose normal $*$-representations are in natural one-to-one correspondence with the unitary representations of G. The author's method of defining $A(G)$ is somewhat novel. Fix an infinite-dimensional separable Hilbert space H, and let G^c be the collection of all (concrete) unitary representations of G on H. By an option we mean a function J on G^c whose values are operators on H such that (i) $\sup\{\|J(L)\| : L \in G^c\} < \infty$, and (ii) if $M, N \in G^c$, and U is an isometry of $H \oplus H$ onto H, then $J(U(M \oplus N)U^{-1}) = U(J(M) \oplus J(N))U^{-1}$. The options form a $*$-algebra, in fact, a von Neumann algebra, under the natural pointwise operations; this is the algebra $A(G)$. It contains the other well-known group algebras ($L_1(G), C^*(G)$ and the measure algebra $M(G)$) as weakly dense subalgebras.

{The author has pointed out to the reviewer an error in the last part of the paper. The extension described in Theorem 8.5 (mislabelled 8.3) is not unique. The word "uniquely" in Theorem 8.5 should be deleted. Theorem 9.2 must then be revised to state simply that every invertible element in $A(G)$ which preserves tensor powers is a unitary element. The very interesting question now arises: How much bigger than G itself is the group G_0 of those unitary elements of $A(G)$ which, in addition to satisfying (i) and (ii), preserve tensor products. (For compact groups, of course, $G = G_0$; this is Tannaka's duality theorem.)} *J. M. G. Fell* (Seattle, Wash.)

Kostant, Bertram 4839

Eigenvalues of the Laplacian and commutative Lie subalgebras.

Topology **3** (1964), *suppl.* 2, 147–159.

Soient $\mathfrak{k}$ une algèbre de Lie semi-simple compacte et $\mathfrak{g}$ l'algèbre de Lie complexifiée de $\mathfrak{k}$. Soit θ la représentation linéaire de $\mathfrak{g}$ dans l'algèbre extérieure $\Lambda(\mathfrak{g})$ qui prolonge la représentation adjointe et pour laquelle $\theta(x)$ est une dérivation de $\Lambda(\mathfrak{g})$ quel que soit $x \in \mathfrak{g}$. Pour tout entier $p > 0$, soit S_p la réunion de la famille $\Lambda^p(\mathfrak{a}) \subset \Lambda^p(\mathfrak{g})$ où $\mathfrak{a}$ parcourt l'ensemble des sous-algèbres commutatives de dimension p dans $\mathfrak{g}$ et soit A_p le sous-espace engendré par S_p dans $\Lambda^p(\mathfrak{g})$. L'auteur démontre les théorèmes suivants: (1) tout élément décomposable de A_p appartient à S_p; (2) si $\mathfrak{b}$ est une sous-algèbre résoluble maximale de $\mathfrak{g}$ et si $\mathfrak{a}$ est un idéal commutatif de dimension p de $\mathfrak{b}$, $\Lambda^p(\mathfrak{a})$ engendre dans $\Lambda^p(\mathfrak{g})$, considéré comme module sur $\mathfrak{k}$ par restriction de θ, un sous-module simple $V_{\mathfrak{a}}$ contenu dans A_p; (3) A_p est somme directe de la famille de sous-modules $V_{\mathfrak{a}}$ où $\mathfrak{a}$ parcourt l'ensemble des idéaux commutatifs de dimension p de $\mathfrak{b}$; (4) si $\mathfrak{a}$ et $\mathfrak{a}'$ sont deux idéaux commutatifs distincts dans $\mathfrak{b}$, les modules $V_{\mathfrak{a}}$ et $V_{\mathfrak{a}'}$ ne sont pas isomorphes.

Ces résultats sont démontrés en mettant le sous-espace A_p en rapport avec les valeurs propres du Laplacien $L = d\partial + \partial d$, où ∂ est l'opérateur bord de $\Lambda(\mathfrak{g})$ et où d est l'adjoint de ∂ pour la forme hermitienne que définissent sur $\Lambda(\mathfrak{g})$ la forme de Killing de $\mathfrak{g}$ et la conjugaison par rapport à $\mathfrak{k}$. L'auteur montre que, sur $\Lambda^p(\mathfrak{g})$, les valeurs propres de L sont $\leqq \frac{1}{2}p$ et que A_p est l'ensemble des $u \in \Lambda^p(\mathfrak{g})$ tels que $L(u) = \frac{1}{2}pu$. Cette caractérisation de A_p permet d'obtenir des renseignements détaillés sur sa structure de module sur $\mathfrak{k}$. *J. L. Koszul* (Grenoble)

Neumark, M. A. [Naimark, M. A.] 4840

★**Lineare Darstellungen der Lorentzgruppe.**

Hochschulbücher für Mathematik, Band 57.

VEB Deutscher Verlag der Wissenschaften, Berlin, 1963. ii + 391 *pp.* *DM* 49.50.

A translation into German, with a special foreword by the author, of the classical book *Linear representations of the Lorentz group* (Russian) [Fizmatgiz, Moscow, 1958; MR **21** #4995]. Translations into English and French have already appeared [Engl. transl., Macmillan, New York, 1964; French transl., Dunod, Paris, 1962; MR **25** #2112].

Auslander, L.; Green, L.; Hahn, F. 4841

★Flows on homogeneous spaces.

With the assistance of L. Markus and W. Massey, and an appendix by L. Greenberg. Annals of Mathematics Studies, No. 53.

Princeton University Press, Princeton, N.J., 1963. vii + 107 *pp.* $2.75.

This book is concerned primarily with recent work of the authors in which they study dynamical systems based on manifolds, especially solvmanifolds and nilmanifolds. Some of the results have been announced earlier (L. Auslander, F. Hahn and L. Marcus [Bull. Amer. Math. Soc. **67** (1961), 298–299; MR **22** #A636]; L. W. Green [ibid. **67** (1961), 414–415; MR **22** #A3800]), and the book repeats, in outline, results of A. Mal'cev [Izv. Akad. Nauk SSSR Ser. Mat. **13** (1949), 9–32; MR **10**, 507], G. D. Mostow [Ann. of Math. (2) **60** (1954), 1–27; MR **15**, 853], L. Auslander [Amer. J. Math. **82** (1960), 689–697; MR **22** #A242], I. M. Gel'fand and S. V. Fomin [Uspehi Mat. Nauk (N.S.) 7, no. 1 (47), 118–137 (1952); MR **14**, 660], F. Mautner [Ann. of Math. (2) **65** (1957), 416–431; MR **18**, 929] (and others) needed to make the study self-contained. If G is a solvable connected Lie group and H is a closed subgroup, then G/H is called a solvmanifold. If G is a nilpotent connected Lie group and H is a closed subgroup, G/H is called a nilmanifold. The authors discover that flows on nilmanifolds provide a rich source of examples of minimal sets. For example they provide an infinite number of minimal flows which are distal but not equicontinuous to answer a question of R. Ellis [Pacific J. Math. 8 (1958), 401–405; MR **21** #96]. A solution is given to the problems of determining all connected, simply connected, non-compact, three-dimensional Lie groups G which have a discrete subgroup D such that G/D is compact; then, for each such G, of determining all of the discrete subgroups for which G/D is compact; finally, giving a description of the dynamical behavior of the flows induced on the compact spaces G/D by the one-parameter subgroups of G. The results provide a wide variety of behavior. An application of nilflows is given to obtain a generalization of the Kronecker approximation theorem [J. Koksma, *Diophantische Approximationen*, J. Springer, Berlin, 1936]. Permanent regional transitivity is established for geodesic flows on frames based on three-dimensional compact manifolds of constant negative curvature. *W. R. Utz* (Zbl **106**, 368)

FUNCTIONS OF REAL VARIABLES

See also 4861, 5012, 5057, 5215.

Lipiński, J. S. 4842

On periodic extensions of functions.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 373–375.

The authors prove the existence of infinite sequences $\{I_n\}_{n=1}^{\infty}$ of bounded intervals such that: (1) Every real-valued function on $\bigcup I_n$ which is bounded and uniformly continuous can be extended to a continuous periodic function on the whole real line, and (2) The set of periods of all such extensions has the power of the continuum. Part (1), with "almost-periodic" instead of "periodic", was proved by S. Hartman and C. Ryll-Nardzewski [same Bull. **11** (1963), 427–429; MR **29** #6253].

R. R. Goldberg (Evanston, Ill.)

Sindalovskiĭ, G. H. 4843

The continuity of functions of several variables with respect to congruent sets. (Russian)

Uspehi Mat. Nauk **19** (1964), *no.* 4 (118), 201–207.

Soit $f(P)$, $P=(x_1, x_2, \cdots, x_n)$, une fonction réelle, définie et mesurable dans le domaine $G\subset R^{(n)}$. La fonction $f(P)$ est, par définition, Q-continue sur l'ensemble $E\subset G$, si pour tout point $P_0\in E$, on a

$$\lim_{P\to P_0, P\in Q(P_0)} f(P)=f(P_0);$$

ici on suppose que l'ensemble $Q(P_0)$ est, pour tout $P_0\in E$, congruent à un ensemble fixe, mesurable, Q, ayant l'origine O comme point d'accumulation. On dira que l'ensemble Q appartient à la classe (A) si mes $Q>0$ dans chaque voisinage de O. Théorème 1: Si la fonction $f(P)$, mesurable dans G, est Q-continue sur l'ensemble $E\subset G$ et si l'ensemble Q appartient à la classe (A), alors $f(P)$ est continue, au sens habituel, presque partout dans E.

Pour chaque ensemble Q n'appartenant pas à la classe (A) on peut construire une fonction $f(P)$ mesurable et partout discontinue, mais Q-continue presque partout.

Théorème 2: Si la fonction $f(P)$, mesurable dans G, est Q-continue dans G et si Q est un ensemble de deuxième catégorie dans un certain ensemble ouvert $Q_1\supset Q$, alors $f(P)$ est continue, au sens habituel, presque partout dans G. *S. Marcus* (Bucharest)

Marcus, Solomon 4844

Sur les dérivées dont les zéros forment un ensemble frontière partout dense.

Rend. Circ. Mat. Palermo (2) **12** (1963), 5–40.

The author calls G a Pompeiu function on $[a, b]$ it if has a bounded derivative on $[a, b]$ and if $Z=\{x: G'(x)=0\}$ is a set without interior points, everywhere dense on $[a, b]$. The paper is devoted to a study of such functions and their modifications, e.g., G is of type P_2 if, in the above definition, "bounded derivative" is replaced by "finite derivative"; and G is a Pompeiu function in a generalized sense if, in the above definition, "has a bounded derivative" is replaced by "is continuous and has at each point a derivative, finite or infinite". Using classical results (the bibliography contains 72 titles), the author obtains properties of the various sorts of functions and considers questions such as the measure of Z, the integrability of the derivatives of the functions, etc. In addition, there are sections with the following titles: Improvement of a theorem of Pompeiu; Topological characterizations of the derivative of a Pompeiu function; Operations with Pompeiu functions; Pompeiu functions and derivatives in approximate sense; Pompeiu graph of a derivative and its topological properties; A "discontinuous" analogue of functions of type P_1 in a generalized sense; Open problems; Other applications of Pompeiu functions.

R. A. Rosenbaum (Middletown, Conn.)

Rinehart, R. F.; Wilson, Jack C. 4845

Higher and iterated Hausdorff derivatives.

Rend. Circ. Mat. Palermo (2) 11 (1962), 258–261.

Let $\mathscr{A}$ be a finite-dimensional linear associative algebra with identity over F, the real or complex field. Let f be a function with domain and range in $\mathscr{A}$. Let $\varepsilon_1, \cdots, \varepsilon_n$ be a basis for $\mathscr{A}$ with ε_1 the identity of $\mathscr{A}$. Let

$$\xi = \sum_{i=1}^{n} x_i\varepsilon_i, \qquad x_i \in F, \qquad f(\xi) = \sum_{i=1}^{n} f_i(x_1, \cdots, x_n)\varepsilon_i.$$

Let the component functions $f_i(x_1, \cdots, x_n)$ possess all partial derivatives of mth order in an open set $\mathscr{N}$ of $\mathscr{A}$. Then the mth differential of $f(\xi)$ at a fixed $\xi \in \mathscr{N}$ is defined to be the multilinear function of m variables,

$$d^m(f(\xi); \delta', \cdots, \delta^{(m)}) =$$

$$\sum_{r=1}^{n} \sum_{s_1, s_2, \cdots, s_m = 1}^{m} \frac{\partial^m f_r}{\partial x_{s_m} \partial x_{s_{m-1}} \cdots \partial x_{s_1}} d_{s_1}' d_{s_2}'' \cdots d_{s_m}^{(m)} \varepsilon_r,$$

where $\delta^{(j)} = \sum_{i=1}^{n} d_i^{(j)}\varepsilon_i$, $d_i^{(j)} \in F$, and the partial derivatives are evaluated at ξ. If the mth differential is expressible in the form

$$d^m(f(\xi); \delta', \cdots, \delta^{(m)}) =$$

$$\sum_{i_1, \cdots, i_{m+1} = 1}^{n} g_{i_1 \cdots i_{m+1}} \varepsilon_{i_1} \delta' \varepsilon_{i_2} \delta'' \cdots \delta^{(m)} \varepsilon_{i_{m+1}}$$

(i.e., as a multilinear function of $\delta', \cdots, \delta^{(m)}$, with coefficients in $\mathscr{A}$), where $g_{i_1 \cdots i_{m+1}}$ depend only on ξ, then f is said to be H-differentiable of order m at ξ.

If $f(\xi)$ is H-differentiable of order m in an open set $\mathscr{N}$ of $\mathscr{A}$, then the mth order Hausdorff derivative $d^m f(\xi)/d\xi^m$ of $f(\xi)$ at $\xi \in \mathscr{N}$ is defined to be the "detached coefficient" of the $\delta', \cdots, \delta^{(m)}$ in the mth differential,

$$\frac{d^m f(\xi)}{d\xi^m} = \sum_{i_1, \cdots, i_{m+1} = 1}^{n} g_{i_1 \cdots i_{m+1}} \varepsilon_{i_1} \cdots \varepsilon_{i_{m+1}}.$$

Another type of higher derivative is the iterated first order Hausdorff derivative, $f^{(m)}(\xi)$, defined inductively by $f^{(m)}(\xi) = [f^{(m-1)}]'(\xi)$, $f'(\xi) = df(\xi)/d\xi$, $\xi \in \mathscr{N}$, with an associated concept of iterative H-differentiability of order m.

These two concepts of higher differentiability are not equivalent. However, if $f(\xi)$ is H-differentiable of orders 1 and m in $\mathscr{N}$, then it is shown that $f^{(m)}(\xi)$ exists in $\mathscr{N}$ and that $f^{(m)}(\xi) = d^m f(\xi)/d\xi^m$. *S. Marcus* (Bucharest)

Matuszewska, W. 4846

On a generalization of regularly increasing functions.

Studia Math. 24 (1964), 271–279.

Let φ be continuous and non-decreasing on $[0, \infty)$, vanishing only when $t = 0$, and $\varphi(t) \to \infty$ as $t \to \infty$. If

$$\underline{k}_\varphi(\lambda) = \liminf_{t \to \infty} \frac{\varphi(t)}{\varphi(\lambda t)},$$

$$\bar{k}_\varphi(\lambda) = \limsup_{t \to \infty} \frac{\varphi(t)}{\varphi(\lambda t)},$$

set

$$S_\varphi = \lim_{\lambda \to 0^+} \frac{\log \underline{k}_\varphi(\lambda)}{-\log(\lambda)} = \sup_{0<\lambda<1} \frac{\log \underline{k}_\varphi(\lambda)}{-\log(\lambda)},$$

$$S_\varphi^1 = \lim_{\lambda \to 1^-} \frac{\log \underline{k}_\varphi(\lambda)}{-\log(\lambda)} = \inf_{0<\lambda<1} \frac{\log \underline{k}_\varphi(\lambda)}{-\log(\lambda)}.$$

Similarly, define σ_φ and σ_φ^1 using $\bar{k}_\varphi$. If $S_\varphi^1 = \sigma_\varphi^1 = r_\varphi$ and $r_\varphi < \infty$ (the quantities are all non-negative), then $\bar{k}_\varphi(\lambda) = \lambda^{-r_\varphi} = \underline{k}_\varphi(\lambda)$, so that φ is regularly increasing (and conversely).

Denote by K_C the subclass of those φ such that $\lim_{t\to\infty} \varphi(t\alpha(t))/\varphi(t) = 1$ for all positive functions α on $[0, \infty)$ for which $\alpha(t) \to 1$ as $t \to \infty$.

The class K_C is characterized to be those φ such that $c(\alpha)\varphi(t) \leq \varphi(\alpha t) \leq d(\alpha)\varphi(t)$ for all $t \geq t_0$ and every $\alpha > 1$, where $1 < d(\alpha) < \infty$, $1 \leq c(\alpha)$ and $c(\alpha)$, $d(\alpha) \to 1$ as $\alpha \to 1$. This class contains the slowly varying and all regularly increasing functions. An example indicates a means of constructing others.

Denote by K_C^* those $\varphi \in K_C$ for which the constant $c(\alpha)$ above can be chosen greater than 1 for each $\alpha > 1$. Such functions include convex, as well as integrals of, functions in K_C. Moreover, K_C and K_C^* are shown to be closed under multiplication and composition of functions. Suppose now that φ satisfies the further conditions $\lim_{t\to 0} \varphi(t)/t = 0$, $\lim_{t\to\infty} \varphi(t)/t = \infty$. Set $\varphi^*(s) = \sup_{t \geq 0}(st - \varphi(t))$, the complementary function to φ (there is a misprint in the definition of φ^*). Then $\sigma_\varphi^1 < \infty$ implies $\sigma_\varphi^1 \geq 1$ and $1/S_{\varphi^*}^1 + 1/\sigma_\varphi^1 \leq 1$, while $1 < S_\varphi^1 < \infty$ implies $1/S_\varphi^1 + 1/\sigma_{\varphi^*}^1 \geq 1$. Some further relations between φ and φ^* are also derived, subject to various conditions on φ.

J. V. Ryff (Cambridge, Mass.)

Bhakta, P. C. 4847

On bounded variations.

Ganita 14 (1963), 59–65.

A number of simple and easily obtained results, in the nature of exercises in advanced calculus, on sequences of functions of bounded variation, employing in essence the semi-continuity of the total variation and similar facts.

John W. Green (Los Angeles, Calif.)

Ciganok, I. I. [Cyganok, I. I.] 4848

On the superpositions of N-functions. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 993–997.

Une fonction positive et convexe sur $(0, \infty)$ telle que $\lim_{u\to 0} M(u)/u = 0$, $\lim_{u\to\infty} M(u)/u = \infty$, est dite de la classe N. Supposons que $M(u)$ admette une dérivée à droite $p(u) \geq 0$. La fonction complémentaire à M est définie par $N(v) = \int_0^{|v|} q(s)\,ds$ avec $q(s) = \sup_{p(t) \leq s} t$. Évidemment $N(v) \in$ N [voir M. A. Krasnosel'skiĭ et Ja. B. Rutickiĭ, *Convex functions and Orlicz spaces* (Russian), Fizmatgiz, Moscow, 1958; MR **21** #5144]. Dans cette note l'auteur donne des relations de la forme $N_1[N_2(v)] \sim N_1(v)N_2(v)/|v|$, où $\sim$ désigne que $N_1(k_1 u) \leq N_2(u) \leq N_1(k_2 u)$, $u \geq u_0$, $k_1, k_2 > 0$ fixe, sous les conditions que (*) $|u| M_i(u) \leq M_i(ku)$, $i = 1, 2$, et $M_1(u) \leq M_2(ku)$, $k > 0$. *M. Tomić* (Belgrade)

Ciganok, I. I. [Cyganok, I. I.] 4849

On a class of N-functions. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 1284–1287.

La condition (*) dans la note précédente [#4848] est remplacée par (**) $\Phi[M(u)] \leq M(ku)$, $\Phi(u) \in$ N. On donne des propriétés des fonctions $N \in$ N satisfaisant à (**). Par exemple: $k_1 v M^{-1}(k_1 v) \leq N(v) \leq k_2 v M^{-1}(k_2 v)$, $k_1 \leq k_2$. M^{-1} désigne la fonction inverse de M. *M. Tomić* (Belgrade)

MEASURE AND INTEGRATION

See also 4694, 4713, 4826, 4841, 4906, 5053, 5112, 5135, 5263, 5267, 5268.

Saks, Stanisław 4850

★**Theory of the integral.**
Second revised edition. English translation by L. C. Young. With two additional notes by Stefan Banach. *Dover Publications, Inc., New York*, 1964. xv + 343 *pp.* $2.25.

An unabridged and corrected republication of the second revised edition, published by Stechert, New York, in 1937 as Volume VII in the "Monografie Matematyczne" series.

Fiorenza, Renato 4851

Sul prolungamento finitamente o numerabilmente additivo di una funzione modulare in un reticolo.
Ricerche Mat. **12** (1963), 270–301.

Problems on extensions of measures have been investigated by a number of authors, some of which are listed in the bibliography of eight items. Here the problem is to extend measures from one lattice to a larger one by the method of extension by continuity of the measure function. In order to give a sense to continuity it is necessary to introduce suitable topologies. The author gives sufficient conditions for the uniqueness of such an extension which appear general enough to include some well-known conditions. Moreover, it is proved that these conditions are necessary for unicity if only certain types of topologies are to be accepted. It is apparent here that it is just this kind of mixing of topology and measure theory that seems to conflict with the views of A. Denjoy [see, e.g., C. R. Acad. Sci. Paris **257** (1963), 2776–2781; MR **27** #5884].

E. Baiada (Modena)

Neubrunn, T. 4852

Metric spaces associated with a measure space. (Slovak. Russian and English summaries)
Acta Fac. Natur. Univ. Comenian. 7, 663–673 (1963).

Let $(X, \mathfrak{M}, \mu)$ be a measure space such that $\mu(E) > 0$ for every non-empty set E in $\mathfrak{M}$, and let $\{E_t\}_{t\in T}$ be a family of disjoint measurable sets such that $\bigcup E_t = X$ and $\sup\{\mu(E_t) : t \in T\} < \infty$. The author defines a metric on $\mathfrak{M}$ by the formula $\rho(E, F) = A(a)$, where a denotes the bounded function $\mu[(E \dot{-} F) \cap E_t]$, $t \in T$, and A is a functional on the space $l_\infty(T)$ of bounded real-valued functions on T satisfying the following conditions: (1) $A(a) = 0$ if and only if $a = 0$, (2) if $a \leq b$, then $A(a) \leq A(b)$, (3) $A(a+b) \leq A(a) + A(b)$; $E \dot{-} F$ denotes the symmetric difference. {Reviewer's comments: Apparently the author has meant condition (2) to be in the form: (2′) if $0 \leq a \leq b$, then $A(a) \leq A(b)$, namely, he claims that the norm $\|a\| = \sup|a(t)|$ satisfies these conditions, and he uses $A(a)$ only for non-negative a's. In Theorem 3.1, the author assumes that A is additive, which is impossible unless T consists of only one point.}

An example is given to show that the topology of this metric may depend essentially on the choice of the family $\{E_t\}$. Theorem 2.2: If there exists an integer k such that each set E_t is contained in at most k sets of another family $\{F_u\}_{u\in U}$ of disjoint measurable sets such that $\bigcup F_u = X$ and $\sup \mu(F_u) < \infty$, then the convergence determined by $\{F_u\}$ implies that determined by $\{E_t\}$.

The author also defines another pseudo-metric in the case when the measure space is σ-finite and $X = \bigcup E_n$, $E_n \subset E_{n+1}$, $\mu(E_n) < \infty$, using the formula

$$\bar{\rho}(E, F) = \operatorname{Lim} \frac{\mu[(E \dot{-} F) \cap E_n]}{\mu[(E \cup F) \cap E_n]},$$

where Lim denotes a Banach translation-invariant limit (the fractions are meant to be 0 when the denominators vanish). If F and E are of finite measure, then $\bar{\rho}(E, F)$ coincides with a metric introduced by E. Marczewski and H. Steinhaus [Colloq. Math. **6** (1958), 319–327; MR **21** #2721]; if they are of infinite measure, $\bar{\rho}(E, F)$ may be equal to 0 even though $E \dot{-} F$ is not a null set.

Z. Semadeni (Poznań)

Santagati, Giuseppe 4853

Ulteriori osservazioni sulla limitatezza di una funzione numerica definita in un reticolo d'insiemi.
Ricerche Mat. **12** (1963), 262–269.

In a previous paper [same Ricerche **12** (1963), 110–120; MR **27** #3762], conditions were given on the set R of subsets of a given set and a real-valued function φ defined on R to be relatively bounded (for notations and definitions refer to the review cited above).

Here, refinements of the results of the preceding paper are obtained; the hypothesis of conditioned $\cup$-completeness is replaced by its consequence (see the review cited): For any pair $X \supseteq Y$ of R and any real positive σ there exists $I_\sigma \in R$ such that $X \supseteq I_\sigma \supseteq X - Y$ and $\sup|\varphi(J)| < \sigma$ for J in $R_{I_\sigma \cap Y}$; continuity from inside of φ is replaced by the condition $\lim \varphi(X_n) = 0$ for any sequence of disjoint sets $X_n \subseteq X \in R$. It is pointed out that if the whole set of preceding conditions is admitted, this last condition is sufficient and necessary for boundedness of φ. These results are used to derive known theorems. *E. Baiada* (Modena)

Santagati, Giuseppe 4854

Sopra un teorema della teoria del prolungamento di misure.
Matematiche (Catania) **18** (1963), 73–82.

Two dual theorems on extension of measures due to U. Barbuti [Ricerche Mat. 8 (1959), 145–162; MR **23** #A3223] were proved again by G. Letta [ibid. 8 (1959), 300–319; MR **23** #A3224] by making use of a rather deep lemma of Carathéodory. The proofs exhibited here do not use this lemma. *E. Baiada* (Modena)

Leslie, Joshua 4855

Sur l'intégration dans les groupes de Whitehead gradués.
C. R. Acad. Sci. Paris **259** (1964), 34–37.

Let $X = \bigcup X_n$ be the strict inductive limit of an increasing sequence of compact metric spaces X_n, each of finite dimension. If $\dim(X_i) \leq n_i$, there exists a topological structure $\mathfrak{M}$ on X less fine than the initial structure, and a distance ρ compatible with $\mathfrak{M}$, such that the mth Hausdorff measure associated with ρ vanishes on X_i for every real number $m > n_i$. If X is also a topological group, and if $X_p \cdot X_q \subset X_{p+q}$ for all integers p and q, and $X_n^{-1} \subset X_n$, then X is called a graded group of Whitehead. Let $G = \bigcup G_n$ be such a group, with $\dim(G_i) = n_i$. Then there exists a sequence of left-invariant Borel measures M_i such that

(i) $M_j(G_k)=0$ for $n_k<j$; (ii) $M_j(G_k)>0$ for $n_k\geqq j$; and nontrivial in the sense that each is positive and finite for some Borel set. *J. C. Oxtoby* (Bryn Mawr, Pa.)

Gadžiev, A. D. 4856

On the speed of convergence of a class of singular integrals. (Russian. Azerbaijani summary)

Izv. Akad. Nauk Azerbaĭdžan. SSR Ser. Fiz.-Mat. Tehn. Nauk **1963**, *no.* 6, 27–31.

On dira que $\phi_\lambda(t,x)$ remplit la condition (A) si: (1) elle est négative et, pour λ et x fixes, est croissante sur $[a,x]$ et décroissante sur $[x,b]$; et si l'on a

$$\int_a^b \phi_\lambda(t,x)\,dt = 1,$$

(2) pour chaque $y\neq x$ et pour $\lambda\to\infty$, on a $\phi_\lambda(y;x)=O(\Delta_\lambda)$, où

$$\Delta_\lambda = \int_{x-\delta}^{x+\delta} |t-x|^{\alpha+1}\phi_\lambda(t,x)\,dt \qquad (\alpha>0)$$

et $\Delta_\lambda\to 0$ pour $\lambda\to\infty$. Posons

$$\gamma(t) = \frac{f(t)-f(x)}{t-x}-f'(x) \quad \text{pour } t\neq x,$$
$$= 0 \qquad\qquad \text{pour } t = x.$$

Théorème 1: Considérons l'intégrale singulière

$$(1) \qquad W_\lambda(x) = \int_a^b K_\lambda(t,x)\,df(t)$$

et supposons que $K_\lambda(t,x)$ satisfait la condition (A), tandis que f est une fonction bornée, telle que l'intégrale (1) existe pour une certaine valeur de x. Si pour cette x et pour $t\to x$ on a

$$\gamma(t) = o(|t-x|^{\alpha+1}),$$

où $\alpha>0$ est arbitraire, alors pour $\lambda\to\infty$ on a

$$|W_\lambda(x)-f'(x)| = o(\Delta_\lambda).$$

Théorème 3: Soit f une fonction bornée, assurant, pour une certaine valeur de x, l'existence de l'intégrale (1). Si, pour cette x, la dérivée à droite $f_+'(x)$ et la dérivée à gauche $f_-'(x)$ existent et sont finies, alors $\lim_{\lambda\to\infty} W_\lambda(x)=(f_+'(x)+f_-'(x))/2$. *S. Marcus* (Bucharest)

Mikusiński, J. 4857

Sur une définition de l'intégrale de Lebesgue.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 203–204.

It is well known that a function f on R^q is Lebesgue integrable if and only if there exist characteristic functions $\{f_n\}$ and numbers $\{\lambda_n\}$ such that (1) $f(x)=\sum\lambda_n f_n(x)$ for almost all x, and $\sum|\lambda_n|\int f_n<\infty$. The author proves that the sequences $\{f_n\}$, $\{\lambda_n\}$ can be so chosen that equality (1) holds for at least all x for which the series converges absolutely. *M. Esser* (Dayton, Ohio)

Flett, T. M. 4858

On transformations in R^n and a theorem of Sard.

Amer. Math. Monthly 71 (1964), 623–629.

An elementary proof is given of the theorem that if D is open in R^n, f a continuously differentiable mapping of D into R^n, and $J(x)$ the Jacobian of f at x, then the image $f(E)$ of any measurable subset E of D is measurable and of measure at most the integral of $|J(x)|$ over E. The proof is based on a simple geometric inequality independent of whether $J(x)$ is zero on a subset of D or not. The formula for change of variables in a multiple integral follows then easily from this theorem. Another elementary proof of the same theorem is due to J. Mařík [Czechoslovak Math. J. **6** (81) (1956), 212–216; MR **18**, 880]. *A. C. Zaanen* (Leiden)

Kellerer, Hans G. 4859

Schnittmass-Funktionen in mehrfachen Produkträumen.

Math. Ann. **155** (1964), 369–391.

From the author's introduction: "Let (M,K,μ) be the product of the σ-finite measure spaces $(M_1,K_1,\mu_1),\cdots,(M_n,K_n,\mu_n)$. We can formulate the following "bisection problem": Can every set A in K be decomposed into two disjoint subsets A^1, A^2 in K such that, for each $i=1,\cdots,n$, $\mu_i(A_x^1)=\mu_i(A_x^2)$ for $(\prod_{k\neq i}\mu_k)$-almost all x in $\prod_{k\neq i}M_i$? (Here $A_x^j=\{y\in M_i|(x_1,\cdots,x_{i-1},y,x_{i+1},\cdots,x_n)\in A^j\}$.) Obviously the μ_i must be assumed to be atomless. In that case the question was answered affirmatively for $n=2$ in an earlier paper of the author [same Ann. **146** (1962), 103–128; MR **26** #5117]. Indeed, a more general result was proved there for $n=2$, namely: Given K-measurable functions $0\leq f\leq g$, there exists a set A in K such that, for each $i=1,\cdots,n$,

$$\int_{M_i} f\,d\mu_i = \int_{M_i}\chi_A g\,d\mu_i$$

$(\prod_{k\neq i}\mu_k)$-almost everywhere.

"In § 1 of this paper the preceding result is proved for arbitrary n. § 2 investigates what can be said when one drops the assumption that the μ_i are atomless. This enables one to study the totality of the 'cross-sectional measure functions' (Schnittmass-Funktionen) of sets A in K. If $T_1,\cdots,T_l$ are arbitrary subsets of $\{1,\cdots,n\}$, a criterion is given in § 3 for given functions $f_{T_1},\cdots,f_{T_l}$ to be cross-sectional measure functions for some subset A in K; indeed, one can even impose an auxiliary condition $A\subset A_0$ $(A_0\in K)$. In § 4 the author looks briefly at the problem of generalizing these results to infinite products. Finally, in § 5, the results are applied to the following 'marginal problem': What conditions must be satisfied by the bounded general measures $\nu_{T_k}|K_{T_k}$ $(1\leq k\leq l)$, and measures $\underline{\nu}\leq\bar{\nu}$ on K, so that there should exist a general measure $\nu|K$ with the following properties:

$$\nu^{T_k} = \nu_{T_k} \quad \text{for } k=1,\cdots,l, \qquad \underline{\nu}\leq\nu\leq\bar{\nu}.$$

(Here ν^{T_k} is the cross-sectional measure corresponding to the measure ν and the set T_k of components of the measure product.)" *J. M. G. Fell* (Seattle, Wash.)

Villegas, C. 4860

On qualitative probability σ-algebras.

Ann. Math. Statist. **35** (1964), 1787–1796.

According to the author, the first clear and precise statement of the axioms of qualitative probability was given by de Finetti. He further states that a more detailed treatment was given later by Koopman and that de Finetti and Koopman derived a probability measure from a qualitative probability under the assumption that, for any integer n, there are n mutually exclusive, equally probable events. L. J. Savage has shown, again according to the author,

that this strong assumption is unnecessary, proving that if a qualitative probability is only fine and tight, then there is one and only one probability measure compatible with it.

The author feels that since no property equivalent to countable additivity has been used as yet in the development of qualitative probability theory, and that since the concept of countable additivity is of such fundamental importance in measure theory, it is to be expected that an equivalent property would be of interest in qualitative probability theory, and that, in particular, it would simplify the proof of the existence of compatible probability measures.

Such a property is introduced in this paper, under the name of monotone continuity. It is shown that if a qualitative probability is atomless and monotonely continuous, then there is one and only one probability measure compatible with it, and this probability measure is countably additive. It is also proved that any fine and tight qualitative probability can be extended to a monotonely continuous qualitative probability, and therefore, contrary to what might be expected, there is no loss in generality if we consider only qualitative probabilities which are monotonely continuous.

The author states that at the present time there is still a controversy over the interpretation which should be given to the word probability in the scientific and technical literature. Although the author subscribes to the opinion that this interpretation may be different in different contexts, in this paper he does not enter into this controversy. *R. V. Chacon* (Columbus, Ohio)

Ébanoidze, T. A. 4861

Functions with a countable number of arguments. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **35** (1964), 265–270.

Certain properties are established for functions with a denumerable number of arguments. The results have applications in the study of infinite systems of multidimensional, non-linear, regular, and singular integral equations. Two earlier papers of the author deal with such systems [Trudy Vyčisl. Centra Akad. Nauk Gruzin. SSR **1** (1960) 51–55; MR **26** #2840; ibid. **1** (1960), 57–61; MR **26** #1718].

H. P. Thielman (Alexandria, Va.)

Itô, Takashi; Amemiya, Ichiro 4862

A simple proof of the theorem of P. J. Cohen.

Bull. Amer. Math. Soc. **70** (1964), 774–776.

Let G be a locally compact abelian group. Eliminating all connections with the Littlewood conjecture, the authors give an elegant and simple proof of P. J. Cohen's result [Amer. J. Math. **82** (1960), 191–212; MR **24** #A3231] identifying the idempotent measures on G. The proof utilizes Cohen's main idea, but uses only one or two of his lemmas (the main estimate being one).

I. Glicksberg (Seattle, Wash.)

Ito, Yuji 4863

Finite invariant measures for temporally homogeneous Markov processes.

Ergodic Theory (Proc. Internat. Sympos., Tulane Univ., New Orleans, La., 1961), *pp.* 151–172. *Academic Press, New York,* 1963.

Let $P(x, B)$ be the transition probability of a temporally homogeneous Markov process defined on a measure space $(X, \mathscr{B}, m)$. A measure ν is said to be invariant under $P(x, B)$ if $\int_X P(x, B)\nu(dx) = \nu(B)$ for every $B \in \mathscr{B}$.

The author discusses necessary and sufficient conditions for the existence of a finite measure ν which is invariant under $P(x, B)$ and is equivalent to m. This paper contains the main results and an outline of the proofs. A detailed exposition with complete proofs appears in Trans. Amer. Math. Soc. **110** (1964), 152–184 [MR **28** #1276].

A. Hajian (Boston, Mass.)

R.-Salinas, Baltasar 4864

[Rodríguez-Salinas, Baltasar]

The extension problem. (Spanish. English summary)

Ann. Mat. Pura Appl. (4) **64** (1964), 133–189.

Let G be a transformation group operating on a set E, and let $\mathscr{S}_0$ be a ring of subsets S of E. The article deals with a finite measure μ_0 (author's Spanish: "extensión") defined on $\mathscr{S}_0$. The measure $\bar{\mu}_0$ is an ultracompletion of μ_0 if $\bar{\mu}_0$ is an extension ("prolongación") of μ_0 to the class $\mathscr{S}$ of all subsets of every S in $\mathscr{S}_0$. If $\bar{\mu}_0 = \mu_0$, μ_0 is called an ultracomplete measure. A measure μ is said to be k-compatible with G if its domain is invariant under G and there exists a real-valued function k, defined on G, such that $\mu(\alpha S) = k(\alpha)\mu(S)$ for every $\alpha \in G$ and S in the domain of μ (if $k \equiv 1$, μ is invariant under G). G is a measurable group if there exists a measure μ^* defined on the algebra of all subsets of G, such that $\mu^*(G) = 1$ and μ^* is invariant under the group of left translations of G.

The author investigates the following oscillation functions: (1) $\omega(\mu, G) = \sup\{\mu(\alpha S) - \mu(S) \mid \alpha \in G, S \subset E\}$, where μ is an ultracomplete measure with $\mu(E) = 1$; (2) $\omega(E, G) = \inf\{\omega(\mu, G) \mid \mu \in M\}$, where M is the set of all ultracomplete measures μ with $\mu(E) = 1$; (3) $\omega_0(\bar{\mu}_0, G) = \sup\{\bar{\mu}_0(\alpha S)/\mu_e(\alpha S) - \bar{\mu}_0(S)/\mu_e(S) \mid \alpha \in G, S \subset E\}$, where μ_e is the outer measure on $\mathscr{S}$ formed in the usual way from a given measure μ_0 k-compatible with G and not identically zero on $\mathscr{S}_0$, $\bar{\mu}_0$ is an ultracompletion of μ_0, and $\mu_e(S) \neq 0$; (4) $\omega_0(G) = \inf\{\omega_0(\bar{\mu}_0, G) \mid \bar{\mu}_0 \in M_0\}$, where M_0 is the set of all ultracompletions of μ_0. The main results are the following. (1) Necessary and sufficient condition for the existence of an ultracomplete invariant (with respect to G) measure μ on $\mathscr{S}$ with $\mu(E) = 1$: $\omega(E, G) = 0$. (2) Necessary and sufficient condition for the existence of a k-compatible ultracompletion of a given k-compatible measure μ_0 defined on $\mathscr{S}_0$: $\omega_0(G) = 0$. Specifically, this ultracompletion exists if G is a measurable group.

Instances are established in which these conditions are or are not verified, and the results are applied to transformations of Euclidean, elliptic, hyperbolic, and affine spaces, leading, in the first case, to conclusions of Banach and Hausdorff on the existence of invariant ultracompletions related to Lebesgue measure in E_1 and E_2 and their non-existence in E_n if $n \geq 3$.

J. F. Smith (Syracuse, N.Y.)

Zitarosa, Antonio 4865

Sull'esistenza di una misura invariante.

Ricerche Mat. **13** (1964), 106–117.

Let S be a set, $\mathscr{A}$ a relatively $\cup$-normal lattice of subsets of S, τ a one-to-one mapping of S onto S such that τ and

τ^{-1} are $\mathscr{A}$-measurable, $\mathscr{B}(\mathscr{A})$ the least relatively complemented and countably $\cap$-complete lattice containing $\mathscr{A}$. Main result: There exists an invariant (with respect to τ) non-null measure on $\mathscr{B}(\mathscr{A})$ if and only if there exists an $\mathscr{A}$-compact set $C \in \mathscr{B}(\mathscr{A})$ and a positive, finite, finitely additive set function β defined on $\mathscr{B}(\mathscr{A})$, such that $\lim'' n^{-1} \sum_{k=1}^{n} \beta(\tau^{-k}(C)) > 0$. (A set $C \subset T$ is $\mathscr{A}$-compact if every countable cover of C by sets of $\mathscr{A}$ contains a finite cover of C.) This is an extension of a result of U. Barbuti [Ann. Scuola Norm. Sup. Pisa (3) **15** (1961), 105–114; MR **23** #A3823]. Then there are derived some generalisations of a theorem of Krylov and Bogoljubov [Ann. of Math. (2) **38** (1937), 65–113] and a theorem of Oxtoby and Ulam [ibid. (2) **40** (1939), 560–566; MR **1**, 18].

N. Dinculeanu (Bucharest)

Zitarosa, Antonio 4866

Sulle misure invarianti, equivalenti ad una misura assegnata.

Ricerche Mat. **13** (1964), 135–144.

Let S be a set, $\mathscr{B}$ a σ-algebra of subsets of S, τ a one-to-one mapping of S onto S such that τ and τ^{-1} are $\mathscr{B}$-measurable, and m a measure on $\mathscr{B}$. Main result: There exists an invariant (with respect to τ) measure on $\mathscr{B}$ equivalent to m if and only if every invariant set $A \in \mathscr{B}$ with $m(A) > 0$ contains an invariant set $I \in \mathscr{B}$ with $m(I) > 0$ such that there exists an invariant measure on $\mathscr{B}_I$ absolutely continuous with respect to the restriction of m to $\mathscr{B}_I$.

There are then given some generalisations of a theorem of P. Halmos [Ann. of Math. (2) **48** (1947), 735–754; MR **9**, 137] and a theorem of E. Hopf [Trans. Amer. Math. Soc. **34** (1932), 373–393]. *N. Dinculeanu* (Bucharest)

Satō, Tokui 4867

Sur l'analyse générale. IV. Entropie dans la cybernétique.

Ann. Mat. Pura Appl. (4) **61** (1963), 287–296.

Author's summary: "Donner un exemple d'applications d'analyse générale comme suite des articles [mêmes Ann. (4) **47** (1959), 253–317; MR **25** #5021; ibid. (4) **52** (1960), 363–383; MR **25** #5022; ibid. (4) **59** (1962), 55–64]."

J. Chover (Madison, Wis.)

Tomatsu, Sizuo 4868

Entropy of ergodic automorphisms on toroidal groups. (Japanese summary)

Sci. Rep. Fac. Lib. Arts Ed. Gifu Univ. Natur. Sci. **3** (1962/63), 168–172.

The author computes the entropy of an ergodic automorphism of an n-dimensional torus, using the dual group and Fourier expansions, and he proves a result stated by Ja. Sinaĭ [Dokl. Akad. Nauk SSSR **124** (1959), 768–771; MR **21** #2036a; errata, MR **21**, p. 1599].

J. Chover (Madison, Wis.)

Dževaršeĭšvili, A. G. 4869

On the existence of a singular integral. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **34** (1964), 529–534.

Let Γ be a simple closed curve, and let $S(t_1, t_2)$ denote the shortest arc length on Γ connecting points with parameter values t_1 and t_2. If the curve Γ satisfies the condition $S(t_1, t_2) \leq K|t_1 - t_2|$ and one other condition too lengthy to state here, then the author shows that the singular integral

$$S(f, t_0) = \lim_{\varepsilon \to 0} \frac{1}{\pi i} \int_{\Gamma_\varepsilon} \frac{f(t)\,dt}{t - t_0}$$

exists for almost all $t_0 \in \Gamma$ if $f \in L_1(\Gamma)$. Here Γ_ε denotes Γ with a suitably small section around t_0 cut out.

H. S. Wilf (Philadelphia, Pa.)

Perevalov, G. E. 4870

An analytic expression for the linear measure of a planar connected set. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 626–638.

The author extends to an arbitrary plane continuum K of finite Carathéodory length the classical formula expressing the length of a non-parametric Lipschitzian curve as the integral of $(1+y'^2)^{1/2}$. Here K need not be a countable sum of simple arcs; nevertheless, by making $y(x)$ many-valued, with K as its graph, and by replacing the integrand by a corresponding sum involving derivatives of various "branches", he obtains the desired formula, provided the axes are first rotated, if necessary. (It is a question of avoiding at most countably many such rotations; otherwise, the formula must be replaced by an inequality.) The formula applies equally to any plane connected set of finite length, but the length is then unaltered by passing to the closure, which is again a plane continuum.

L. C. Young (Madison, Wis.)

Mookhopadhyaya, A. K. 4871

On transformation of measurable sets.

Indian J. Mech. Math. **2** (1964), 21–23.

Let $P \subset E^n$ be a set of strictly positive Lebesgue measure in an n-dimensional Euclidean space and let $T_1, \cdots, T_t$ be a set of linear transformations, viz.,

$$T_k: x_i' = \sum_{j=1}^{n} \alpha_k a_{ij}^k x_j \qquad (i = 1, \cdots, n;\ k = 1, \cdots, t),$$

where $0 < |\alpha_k| \leq 1$ and (a_{ij}^k) is orthogonal. If x_0 is a point of density of the set P, then there are two spheres $K(x_0, r')$ and $K(x, r)$ such that (i) $K(x_0, r') \subseteq K(x, r)$, and (ii) for every $x \in K(x_0, r')$ there are vectors $h(x)$ such that to each $h(x)$ there correspond vectors $a_k(x) \in P \cap K(x_0, r)$ satisfying $a_k(x) = x + T_k h(x)$ $(k = 1, \cdots, t)$.

The above theorem was proved by the reviewer [J. Math. Soc. Japan **13** (1961), 13–19; MR **24** #A1988] in the case in which all (a_{ij}^k) are equal to the unit matrix. The proof of the present author is different from that of the reviewer and seems to be shorter. It is not difficult to see that the conclusion of the above theorem holds if T_k are any regular linear operators, finite in number. Besides this result, there are two more theorems, one of which is an application of the above theorem. *S. Kurepa* (Zagreb)

Young, L. C. 4872

Some extremal questions for simplicial complexes. V. The relative area of a Klein bottle.

Rend. Circ. Mat. Palermo (2) **12** (1963), 257–274.

Part IV, by W. H. Fleming and the author, appeared in same Rend. (2) **12** (1963), 200–210 [MR **29** #5987]. For

a given oriented $(k-1)$-dimensional polytopic cycle C in E^n, $1<k<n-1$, and for each positive integer m, let a_m denote the infimum of $|V|$ for all oriented polytopes V having mC as boundary. Here, $|V|$ means the area of V. The author produces an example of a 1-dimensional cycle C for which $a_2 < 2a_1$, the significant point being that strict inequality occurs. The relevance of this example lies in the fact that a variational algorithm of the author [Riv. Mat. Univ. Parma 5 (1954), 255–268; MR 18, 402] cannot solve the least area problem for integral currents as formulated by Federer and Fleming [Ann. of Math. (2) 72 (1960), 458–520; MR 23 #A588]. A similar example has been produced by Almgren [Topology 1 (1962), 257–299; MR 26 #4355].

W. P. Ziemer (Bloomington, Ind.)

Young, L. C. 4873

Generalized varieties as limits.

J. Math. Mech. 13 (1964), 673–692.

To treat higher-dimensional variational problems, the author has enlarged the class of parametric varieties to obtain a class of generalized varieties with satisfactory compactness properties, in a way similar to his generalized surfaces [see the author, Mem. Amer. Math. Soc. No. 17 (1955); MR 19, 559; J. Math. Mech. 11 (1962), 615–646; MR 25 #5162]. A similar enlargement is in the basis of the theory of normal and integral currents [see H. Federer and W. H. Fleming, Ann. of Math. (2) 72 (1960), 458–520; MR 23 #A588]. Those enlargements correspond, in a certain sense, to generalizations of the notion of polytope with positive real or integer coefficients, respectively. In this paper, two classes of generalized varieties, occurring in applications of the theories of L. C. Young and of H. Federer and W. H. Fleming, are characterized as similar closures of convenient classes of simplicial varieties (see Theorems A and B below).

A generalized variety L is a non-negative linear functional in the class of k-integrands, i.e., real-valued continuous functions $f(x, J)$, $x \in R^n$, where J is a k-vector of R^n with the homogeneity condition $f(x, \lambda J) = \lambda f(x, J)$, $\lambda \geq 0$. Track, boundary or pattern of L stands, respectively, for the restriction of L to integrands that are linear, exact and of the form $f(J)$. A generalized variety has a track (or boundary) A or B when its track (or boundary) coincides with that of a mixed σ-variety or σ-variety, respectively (i.e., varieties expressible as a finite, or countable, sum of Lipschitzian varieties with positive real or integer coefficients, respectively). A micro-variety at x is a generalized variety whose support is the single point x. The micro-pattern of L at x is the pattern of its microvariety at x. A generalized variety B is a generalized variety with track B. A pattern P has a resultant (B, m) if $P = \lim P_n$, where P_n are patterns of polytopes, each with the same boundary as the mth multiple of a simplex. A generalized variety B has at x a micro-resultant B if it has a micro-pattern with a resultant (B, m), where m is the local multiplicity $m(x)$.

A generalized variety is called admissible A if it has a boundary A, and admissible B if it is a generalized variety B and possesses at almost every x a micro-resultant B. Then, the main theorems of this paper are: Theorem A [B]: In order that L be a generalized variety admissible A [B], it is necessary and sufficient that there exist, for $n = 1, 2, \cdots$, a polytope L_n and a σ-polytope S_n both with positive real [integer] coefficients such that: (i) $\lim L_n = L$, (ii) $\lim S_n = 0$, (iii) $L_n + S_n$ has the same boundary as L.

The proof of Theorem A is based on techniques developed by W. H. Fleming and the author [Trans. Amer. Math. Soc. 76 (1954), 457–484; MR 16, 721], and in the proof of Theorem B conveniently modified results of the above-cited paper of H. Federer and W. H. Fleming are used.

U. D'Ambrosio (Providence, R.I.)

FUNCTIONS OF A COMPLEX VARIABLE

See also 4650, 4869, 4909, 5015, 5024, 5025, 5028, 5129, 5150, 5151, 5167, 5370.

Davis, Philip J. 4874

Triangle formulas in the complex plane.

Math. Comp. 18 (1964), 569–577.

The simplest triangle formula in the complex plane is as follows. Let T denote a triangle of vertices z_1, z_2, z_3, and let A be its area. If $f(z)$ is regular in the closure of T, then

$$\frac{1}{2A}\iint_T f''(z)\,dxdy = f(z_1, z_2, z_3),$$

where the right side denotes the second divided difference of $f(z)$. After discovering this relation, the author learned that it was found by T. Motzkin and the reviewer in 1952 [Proc. Amer. Math. Soc. 3 (1952), 517–526; MR 14, 157]. "The object of the present paper is to publicize this interesting formula, to give an alternate proof of it, and to derive a number of consequences and results related to the alternate approach." We cite one of the consequences: Let $z_1, \cdots, z_n$ designate the vertices in counter-clockwise order of a convex polygon P. Write $z_0 = z_n$, $z_1 = z_{n+1}$. Then, for all $f(z)$ regular in the closure of P

$$\iint_P f''(z)\,dz = 2\sum_{j=1}^{n} \frac{A(z_{j-1}, z_j, z_{j+1})}{(z_j - z_{j-1})(z_j - z_{j+1})} f(z_j),$$

where A is the area of the triangle having the vertices indicated.

I. J. Schoenberg (Madison, Wis.)

Ferentinou-Nikolakopoulou, Iōannas 4875

[Ferentinou-Nicolacopoulou, Jeanne]

★Two new methods for the localization of zeros of polynomials in one variable. (Greek)

Thesis, Athens University, Athens, 1964. viii + 134 pp.

Both methods are based on the following theorem, which, although a simple corollary of well-known results, proves to be quite efficient. Let $f(x) = x^n + \sum_{j=1}^k a_j x^{n-n_j}$ with a_j, x complex, $n \geq n_j > 0$. Choose any real θ_j and non-negative real m_j; then any positive ξ such that $\sum_{j=1}^k m_j \xi^{\theta_j - n_j} \leq 1$ and $\xi^{\theta_j} m_j \geq |a_j|$ is an upper bound of the moduli of the roots of $f(x)$. In this theorem the exponents are not necessarily integers, in which case x^s is taken to be $\exp s[\log r + i(\varphi + 2K\pi)]$, where $x = re^{i\varphi}$, $0 \leq \varphi < 2\pi$, and K is some fixed integer; for positive x, however, we take $K = 0$ unless x^s appears in the polynomial whose roots are investigated.

The first part of this work consists of a systematic use of this theorem to compute various upper bounds. The freedom of choice of the θ_j and m_j leads to a huge collection of bounds, among which are included several obtained previously by others (e.g., Carmichael, Landau, Karamata, Kojima, Cowling and Thron) in the same, or sometimes in a weaker, form.

In the second part another method (a little too complicated to reproduce here) is developed, which consists essentially in reducing the problem to another involving a polynomial of fewer terms, so that by successive applications only two terms are left.

The volume ends with a collection of 117 formulas established in the text and an extensive bibliography.

P. C. Deliyannis (Chicago, Ill.)

Walsh, J. L.; Shisha, O. 4876

Extremal polynomials and the zeros of the derivative of a rational function.

Proc. Amer. Math. Soc. **15** (1964), 753–758.

Let m_j and n_j be positive integers and $N=\sum_1^m m_j-\sum_1^n n_k$. Let $R(z)=\prod_{j=1}^m (z-\alpha_j)^{m_j}\prod_{k=1}^n (z-\beta_k)^{-n_k}$ and $\omega(z)=\prod_{j=1}^m (z-\alpha_j)\prod_{k=1}^n (z-\beta_k)$. Let P be the class of polynomials $P(z)=Nz^{m+n-1}+a_1z^{m+n-2}+\cdots+a_{m+n-1}$, where the a_j are arbitrary, with $P(\alpha_j)=m_j\omega'(\alpha_j)$ for $j=1, 2, \cdots, m$. The authors establish as their principal theorem that the finite zeros of the derivative of R are the zeros of the polynomial of class P for which the weighted Tchebycheff norm, $\max|P(\beta_j)/n_j\omega'(\beta_j)|$, $j=1, 2, \cdots, n$, is a minimum. This theorem is analogous to the result due to Fekete and von Neumann when $R(z)$ is a polynomial. Together with the authors' previous results on restricted infrapolynomials, this theorem may be used to prove some well-known results of Bôcher and others on the location of the finite zeros of the derivative of a rational function.

{Reviewer's note: The results stated in the authors' remarks 1°, 2° and 3° are covered by the reviewer [*The geometry of the zeros of a polynomial in a complex variable*, p. 71, Amer. Math. Soc., New York, 1949; MR **11**, 101; ibid, p. 19; see also Trans. Amer. Math. Soc. **32** (1930), 81–109, p. 108 and Bull. Amer. Math. Soc. **42** (1936), 400–405, p. 401].}

M. Marden (Milwaukee, Wis.)

Rubinstein, Z.; Walsh, J. L. 4877

On the location of the zeros of a polynomial whose center of gravity is given.

J. Analyse Math. **12** (1964), 129–142.

In recent papers [J. Math. Pures Appl. (to appear); Proc. Amer. Math. Soc. **15** (1964), 354–360; MR **28** #4092; also #4876 above] Walsh and others established some new coincidence theorems regarding points in a unit circle. For example, if $m_k>0$ and $|\alpha_k|\leqq r$ for all k, and if $|z|>r$ and $\sum_{j=1}^n m_j\alpha_j{}^k=0$ for $k=1, 2, \cdots, p$, then the point α defined by the equation $\sum_{k=1}^n m_k(z-\alpha_k)^{-1}=\sum_{k=1}^n m_k(z-\alpha)^{-1}$ satisfies the inequality $|\alpha|\leqq r^{p+1}|z|^{-p}$. These lemmas are in the present paper applied further to yield theorems like the following. (I) Let the zeros $\alpha_1, \alpha_2, \cdots, \alpha_n$, the zeros of the nth degree polynomial f, satisfy the conditions $|\alpha_j|\leq 1$ and $\sum_{i=1}^n \alpha_i{}^k=0$ for $j=1, 2, \cdots, n$; $k=1, 2, \cdots, p$. If g_1 and g_2 are arbitrary functions and if z with $|z|>1$ is a zero of $h(z)=g_1(z)f(z)-g_2(z)f'(z)$, then $|z^p(z-ng_2/g_1)|\leq 1$. (II) Let $f(z)=a_mz^m+a_{m-p}z^{m-p}+\cdots+a_sz^s+a_0$, $p\geqq 1$, $s\geqq 1$; $g(z)=\sum_0^n b_kz^k$; $h(z)=\sum_0^m a_kg(k)z^k$. If all the zeros of f lie in the ring $0\leqq r_1\leqq |z|\leqq r_2\leqq\infty$ and those of g in the region $0\leq \rho_1\leq |z/(z-m)|\leq \rho_2\leqq\infty$, then all the zeros of h lie in the ring

$$r_1\min(1, \rho_1{}^{n/s}) \leqq |z| \leqq r_2\max(1, \rho_2{}^{n/p}).$$

In the case $p=0$, (I) reduces to a result implied by the reviewer [Bull. Amer. Math. Soc. **42** (1936), 400–406], and in the case $p=s=1$, (II) reduces to a result of the reviewer [*The geometry of the zeros of a polynomial in a complex variable*, Amer. Math. Soc., New York, 1949; MR **11**, 101].

M. Marden (Milwaukee, Wis.)

Tonkov, T. T. 4878

A bound for the moduli of zeros of polynomials. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 748–749.

It follows from a theorem of J. L. Walsh [Trans. Amer. Math. Soc. **24** (1922), 163–180] that the moduli of the zeros of the polynomial $f(z)=z^n+a_1z^{n-1}+\cdots+a_n$ are bounded above by

$$|a_n|^{1/n}+|a_{n-1}|^{1/(n-1)}+\cdots+|a_2|^{1/2}+|a_1|.$$

The author gives a simple proof of this fact and also finds a lower bound for any z such that $f(z)=0$, namely,

$$|a_n|^{1/n}-|a_{n-1}|^{1/(n-1)}-\cdots-|a_2|^{1/2}-|a_1| \leq z.$$

A. J. Lohwater (Providence, R.I.)

Viten'ko, I. V.; Kostovskiĭ, A. N. 4879

Determination of principal indices of Laurent series. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 732–734.

If $f(z)$ is given by a convergent Laurent series in the annulus $r<|z|<R$, if $r<x<R$, and if $f(z)\neq 0$ for $|z|=x$, then the principal index of the given series is

$$I(x) = \frac{1}{2\pi i}\int_{|z|=x}\frac{f'(z)}{f(z)}\,dz.$$

The authors establish certain conditions for the existence of $I(x)$ for a given series. The conditions involve the size of the coefficients of the series for $\prod_\omega f(\omega z^{1/k})$, where ω runs through the kth roots of unity.

H. S. Wilf (Philadelphia, Pa.)

Dickson, D. G. 4880

Analytic mean periodic functions.

Trans. Amer. Math. Soc. **110** (1964), 361–374.

Soient μ une mesure à support compact dans le plan complexe, $f(z)=\int\exp(zw)d\mu(w)$, P le diagramme conjugué de f (=l'intersection des supports convexes des mesures μ' telle que $f(z)=\int\exp(zw)d\mu'(w)$), R un ouvert du plan complexe. Une fonction φ, analytique dans $R+P$ et solution dans R de l'équation (1) $\int\varphi(z+w)d\mu(w)=0$, est dite moyenne-périodique dans R par rapport à f; on note $\varphi(z)\in K$, et on munit l'espace vectoriel K de la topologie de la convergence compacte. Le problème principal est de donner des conditions garantissant que les exponentielles monômes (2) $z^h\exp(\zeta z)\in K$ forment un système total dans K; des réponses partielles ont été données par L. Schwartz [Ann. of Math. (2) **48** (1947), 857–929; MR **9**, 428], A. Leont'ev [Uspehi Mat. Nauk **11** (1956), no. 5 (71), 26–37; MR **18**, 728] et le rapporteur [*Lectures on mean periodic functions*, Tata Inst. Fundamental Res., Bombay, 1959]. Etant donné $\tau>0$ et β complexe, l'auteur dit qu'une suite de contours emboités Γ_p ($p=1, 2, \cdots$) est (τ, β) associée à f si (I) Γ_p est contenu dans $r_p\leq |z|\leq Cr_p$ (C constante, $r_p\to\infty$) et $\log\lambda_p=o(r_p)$, λ_p étant la longueur de Γ_p, (II) pour tout $\varepsilon>0$, $|f(z)\exp(-\beta z)|>\exp(-(\tau+\varepsilon)r_p)$ quand $z\in\Gamma_p$, p assez grand. Comme conséquences de théorèmes

de représentation de $\varphi \in K$ par des séries d'exponentielles polynômes (2), sous diverses conditions, l'auteur montre que K est engendré par les exponentielles polynômes dès que, pour tout $\beta \in P$, il existe des contours $(0, \beta)$ associés à f. Des conditions pour qu'il en soit ainsi sont ensuite données, ainsi que des théorèmes d'unicité affirmant que le développement de φ en exponentielles polynômes (2) ne dépend pas de f. *J.-P. Kahane* (Orsay)

Besicovitch, A. S. 4881

On analytic functions.

Math. Gaz. **48** (1964), 270.

Let f and g be two general (multiple-valued) analytic functions. Suppose that for each z in a domain D the set of values of f is the same as the set of values of g. Then f and g are the same analytic function, i.e., are obtainable by analytic continuation from some common initial element. *R. P. Boas, Jr.* (Evanston, Ill.)

Distler, Raymond J. 4882

The domain of univalence of certain classes of meromorphic functions.

Proc. Amer. Math. Soc. **15** (1964), 923–928.

Let K be a non-empty closed subset of the complex plane, and let $U(K)$ denote the set of all points P in the plane such that K "subtends an angle less than $\pi/2$". Let $F(K)$ denote the family of functions of the form

$$f(z) = \sum_{k=1}^{n} \frac{A_k}{(z-a_k)}, \qquad A_k > 0,$$

where the a_k are points of K. Then the author's main result is the following. If $U(K)$ is not empty, then each member of $F(K)$ is univalent in $U(K)$; moreover, if D is a different domain such that $U(K) \subset D$, then there is a function in $F(K)$ that is not univalent in D. This result generalizes one due to Čakalov [C. R. Acad. Sci. Paris **242** (1956), 437–439; MR **17**, 724], and it is related to results due to the reviewer [Michigan Math. J. **5** (1958), 91–94; MR **20** #2453] and Cowling and Royster [J. Math. Soc. Japan **13** (1961), 104–108; MR **23** #A3838]. *M. Reade* (Ann Arbor, Mich.)

Pommerenke, Ch. 4883

Lacunary power series and univalent functions.

Michigan Math. J. **11** (1964), 219–223.

The author proves the following theorem. If

$$f(z) = a_0 + \sum_{k=1}^{\infty} (a_{n_k-p} z^{n_k-p} + \cdots + a_{n_k} z^{n_k})$$

is analytic and different from zero in $|z| < 1$, where p is a fixed non-negative integer and $n_{k+1}/n_k > \lambda > 1$ $(k \geq 1)$, then for $0 \leq q \leq p$, $a_{n_k-q} \to 0$ $(k \to \infty)$.

It is shown, using a result essentially due to Fejér and Fekete, that if $f(z) \neq 0$ $(|z| < 1)$, then there is an associated sequence of polynomials which do not vanish in $|z| < 1$. Hence it is proved that the assumption that the theorem is false leads to a contradiction. The proof is not difficult, but it is very ingenious.

The paper concludes with two applications. One is to functions analytic in $|z| < 1$ with non-vanishing first derivative, and the other is to functions analytic in $|z| > 1$ with a simple pole at $z = \infty$ and non-vanishing first derivative. In both cases the functions are assumed to have associated lacunary series expansions. *J. Clunie* (London)

Aksent'ev, L. A. 4884

Indices of functions on Riemann surfaces and their applications. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, no. 4 (41), 3–8.

Extension of known index theorems for meromorphic functions on a plane region to corresponding situations for Riemann surfaces. *M. H. Heins* (Urbana, Ill.)

Th'em, Le Van [Thiem, Le-Van] 4885

On the type of Riemann surfaces defined by the linear substitution group. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 853–857.

Definition: A covering surface over the extended plane is said to admit a modular end provided that there exists a separating cross-cut so that on one of the components of the complement of the cross-cut the projection map is equivalent to the projection map of a conformal universal covering of the extended plane punctured in q (≥ 3) distinct points restricted to a component of the domain less a cross-cut joining two distinct logarithmic singularities. Theorem: A covering surface over the extended plane admitting a modular end is hyperbolic. *M. H. Heins* (Urbana, Ill.)

Kuramochi, Zenjiro 4886

Potentials on Riemann surfaces.

J. Fac. Sci. Hokkaido Univ. Ser. I **16**, 5–79 (1962).

The author begins with the statement: "The purpose of the present paper is to simplify and to extend almost all theorems contained in previous papers." He refers to Osaka Math. J. **8** (1956), 119–137 [MR **18**, 120]; ibid. **8** (1956), 145–186 [MR **19**, 23]; ibid. **10** (1958), 119–136 [MR **20** #3271]; Proc. Japan Acad. **36** (1960), 118–122 [MR **22** #11121]. Actually, there are many improvements, both in proofs and results. However, there remain some non-trivial ambiguities in his theory. The main criticism lies in the lack of a rigorous statement and a proof of the Dirichlet principle.

Let R be an open Riemann surface and let an open "disk" R_0 be excluded from R. First, various kinds of harmonic functions are introduced on subdomains of $R - \bar{R}_0$. One of them is the function $N(p, q)$ which is a harmonic function of p in $R - \bar{R}_0$ outside of a positive logarithmic pole at a point q, vanishes on the boundary ∂R_0, and has a minimum Dirichlet integral in a neighborhood of the ideal boundary of R. This function plays a rôle similar to that of $K(p, q) = G(p, q)/G(p, q_0)$ in the theory of the Martin boundary, where G is a Green's function. Thus, the so-called Kuramochi boundary B is defined. Next, a positive superharmonic function U in $R - \bar{R}_0$ is called $\bar{\text{s}}$uperharmonic if ${}_DU \leq U$ in $R - \bar{R}_0 - \bar{D}$ for every relatively compact subdomain D of $R - \bar{R}_0$, where ${}_DU$ is, roughly speaking, the harmonic function in $R - \bar{R}_0 - \bar{D}$ which coincides with U on ∂D, which vanishes on ∂R_0, and which has a minimum Dirichlet integral in a neighborhood of the ideal boundary of R. Some properties of $\bar{\text{s}}$uperharmonic functions are proved and every $\bar{\text{s}}$uperharmonic function vanishing on ∂R_0 is represented by the potential with kernel $N(p, q)$ of a measure on $(R - \bar{R}_0) \cup B$. A function ${}_AU$, similar to a reduced function in the theory of the Martin boundary, is defined for any closed subset A of B. The set of points $p \in B$ with ${}_{(p)}(N(z, p)) = 0$ is denoted by B_0, and B_1 is defined by

$B - B_0$. It is proved that every potential $\int_B N d\mu$ can be written as $\int_{B_1} N d\mu_1$, and the latter measure μ_1 is called a canonical mass distribution. However, its uniqueness is not established. A harmonic superharmonic function U vanishing on ∂R_0 is called minimal if the $\overline{\text{s}}$uperharmonicity of V and $U - V$ implies that V is a constant multiple of U. The author proves that U is minimal if and only if it is a constant multiple of some $N(z, p)$ with $p \in B_1$. Up to this point, the discussions are parallel to those in the theory of the Martin boundary.

The author proceeds to determine at every point of B the value of any $\overline{\text{s}}$uperharmonic function U, vanishing on ∂R_0, as the limit of a certain kind of mean value of U. As a function on $(R - \bar{R}_0) \cup B$, U is seen to be lower semicontinuous. The local superharmonicity on B is defined and it is shown that U is locally superharmonic at each point of B_1. Finally, a maximum principle and the existence of an equilibrium potential are discussed.

The book of Constantinescu and Cornea [*Ideale Ränder Riemannscher Flächen*, Springer, Berlin, 1963; MR **28** #3151] gives proofs, often by different methods, to many results of the present paper. The part parallel to the theory of the Martin boundary will be made elementary, somewhat along the original line of Kuramochi, in a forthcoming paper of the reviewer. *M. Ohtsuka* (Hiroshima)

Ozawa, Mitsuru 4887

On the growth of analytic functions.

Kōdai Math. Sem. Rep. **16** (1964), 98–100.

The author continues his study of the distribution of values of meromorphic functions on Riemann surfaces [same Rep. **15** (1963), 245–256; MR **28** #5175; ibid. **16** (1964), 40–43; MR **29** #3630]. In contrast to the general theorems of L. Sario [Acta Math. **109** (1963), 1–10; MR **26** #2602], the emphasis here is placed on actual existence questions for functions with prescribed growth. Slow-growing functions are not subject to the main theorems of Sario.

The present paper presents two examples of surfaces tolerating no meromorphic functions of prescribed slow growth. Specifically, consider a plane E with slits S_i clustering only at ∞. By "sewing" two such planes together, a parabolic surface W is obtained, which, outside of a compact subset, has a natural holomorphic projection on the z-plane. In terms of it a characteristic function $T(|z|, f)$ can be defined for an f meromorphic on W. If $n(a, E)$ denotes the number of slits of E in $\{|z| < a\}$, then, for all nonconstant f,

$$\limsup T(r, f)/\log r \geq \limsup \log n(r, E)/\log r \quad \text{as } r \to \infty.$$

The second example is a hyperbolic surface with a similar property. *B. Rodin* (La Jolla, Calif.)

Ozawa, Mitsuru 4888

Remarks on unramified abelian covering surfaces of a closed Riemann surface.

Kōdai Math. Sem. Rep. **16** (1964), 101–104.

O_{MD} is the class of Riemann surfaces which cannot be realized over the Riemann sphere with finite spherical area; $O_{AB}{}^0$ consists of surfaces containing no subregion on which a nonconstant bounded analytic function exists whose real part vanishes continuously on the relative boundary. Let W be an unramified, unbounded, abelian covering surface of a closed surface R. A condition on the group Γ of cover transformations is given which implies that $W \in O_{AB}{}^0 \cap O_{MD}$. It is based on the results of T. Kuroda [Nagoya Math. J. **10** (1956), 27–50; MR **18**, 290] and A. Mori [J. Math. Soc. Japan **6** (1954), 162–176; MR **16**, 581]. If Γ is free abelian of rank r, then a characteristic function is defined on W in terms of which there is a meromorphic function of order r. *B. Rodin* (La Jolla, Calif.)

Ahlfors, Lars V. 4889

Extension of quasiconformal mappings from two to three dimensions.

Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 768–771.

The reviewer has shown that if f is a quasiconformal mapping of the 3-dimensional half-space $x_3 > 0$ onto itself, then f induces a quasiconformal mapping φ of the boundary plane $x_3 = 0$ onto itself [Trans. Amer. Math. Soc. **103** (1962), 353–393; MR **25** #3166]. In the present important paper, the author shows conversely that if φ is a quasiconformal mapping of the plane $x_3 = 0$ onto itself, then there exists a quasiconformal mapping f of the half-space $x_3 > 0$ onto itself which has φ as its induced boundary correspondence. Thus one obtains a characterization for the boundary correspondences, induced by quasiconformal mappings of a 3-dimensional half-space onto itself, similar to that obtained in two dimensions by Beurling and Ahlfors [Acta Math. **96** (1956), 125–142; MR **19**, 258]. To prove his extension theorem, the author first observes that, since an arbitrary plane quasiconformal mapping can be expressed as the composition of mappings with dilatation arbitrarily close to 1, it is sufficient to show that mappings φ with small dilatation can be extended. Next, given a φ with sufficiently small dilatation, he constructs for each integer N a piecewise linear homeomorphism f_N of $x_3 \geq 0$ onto itself such that $f_N(x) = \varphi(x)$ for $x = (m/N, n/N, 0)$, where m and n are integers, and such that the dilatation of f_N does not exceed a fixed bound. Then a subsequence of $\{f_N\}$ converges to a homeomorphism f of $x_3 \geq 0$ onto itself which is quasiconformal in $x_3 > 0$ and agrees with φ in $x_3 = 0$. *F. W. Gehring* (Cambridge, Mass.)

Ahlfors, Lars V. 4890

Finitely generated Kleinian groups.

Amer. J. Math. **86** (1964), 413–429.

Let Γ denote a group of Möbius transformations A acting on the extended plane. By Σ is meant the set of the accumulation points of the orbits of Γ. The set Σ is closed. When it contains at least three points but is not the extended plane, Γ is called a Kleinian group. The complement of Γ, denoted by $\Omega(\Gamma)$, is called the set of discontinuity. The quotient $S = \Omega/\Gamma$ has a natural complex structure which renders the projection map π taking a point of Ω into its orbit holomorphic. The components S_i of S are Riemann surfaces.

By $Q(\Gamma)$ is meant the space of holomorphic functions φ on Ω satisfying: $\varphi[A(z)][A'(z)]^2 = \varphi(z)$, $A \in \Gamma$, $\int_{\Omega/\Gamma} |\varphi|\, dx\,dy < +\infty$. Fixing $\varphi_0 \in Q(\Gamma)$, not identically zero, one introduces the Beltrami differentials $\nu = \varphi_0{}^{-1}|\varphi_0|\beta$, where β is bounded and measurable on Ω and automorphic with respect to Γ. By $N(\Gamma)$ is meant the set of ν satisfying $\int_{\Omega/\Gamma} \varphi\nu\, dx\,dy = 0$ for all $\varphi \in Q(\Gamma)$. Given a Beltrami differential ν, one introduces

$$f(\zeta) = -\pi^{-1} \int_\Omega \nu(z) \left[\frac{1}{z - \zeta} - \frac{\zeta}{z - 1} - \frac{1 - \zeta}{z}\right] dx\,dy.$$

It is shown that if Γ is finitely generated, then $Q(\Gamma)$ is finite-dimensional. The basic lemmas are: (1) $\nu \in N(\Gamma)$ if and only if $f=0$ on Σ; (2) $\nu \in N(\Gamma)$ if and only if $f[A(z)] = f(z)A'(z)$, $A \in \Gamma$.

Several unsolved problems are stated. One is: If Γ is finitely generated, does Σ have zero area?

The author points out that the proof of the principal theorem of the paper asserting the finitary character of S when Γ is finitely generated is incorrect and that the validity of the assertion remains unsettled. Theorem 6 remains valid in spite of a recurrence of the error of the proof just referred to, thanks to a result of R. Accola. On page 418, line 13, replace restriction by restrictions and on line 23 replace O by o.

M. H. Heins (Urbana, Ill.)

Ahlfors, Lars V. 4891

Eine Bemerkung über Fuchssche Gruppen.

Math. Z. **84** (1964), 244–245.

Let Γ denote a properly discontinuous group of Möbius transformations on the unit disk. Let $\lambda^2 = \sum_{A\in\Gamma} |A'|^2$, and let $\rho(z) = (1-|z|^2)^{-1}$. It is shown that $\sup_{|z|<1} \lambda/\rho < +\infty$. The proof depends upon a study of $\sum_{A\in\Gamma_0} (1-|A|^2)^2$, where Γ_0 is a finite subset of Γ.

M. H. Heins (Urbana, Ill.)

Čibrikova, L. I. 4892

Solution of the Riemann boundary-value problem for automorphic functions for groups characterised by two invariants. (Russian)

Dokl. Akad. Nauk SSSR **141** (1961), 47–50.

This paper contains announcements of results submitted and published at the same time as the author's longer paper [Izv. Vysš. Učebn. Zaved. Matematika **1961**, no. 6 (25), 121–131; MR **28** #227].

{The μ's of the present paper are replaced by λ's in the longer paper cited above.}

Čibrikova, L. I. 4893

On a particular case of the Riemann problem for automorphic functions. (Russian)

Kazan. Gos. Univ. Učen. Zap. **122** (1962), *kn.* 3, 81–94.

Sei Γ eine abzählbare Funktionalgruppe von linear-gebrochenen Substitutionen, die einen Fundamentalbereich mit nur endlich vielen Seiten besitzt, und sei S Existenzgebiet einer einfachen Γ-automorphen Funktion sowie L eine stückweise glatte Kurve in S. Unter dem Riemannschen Problem versteht Verfasser, zu vorgegebenen Funktionen $G \neq 0$ und g auf L, die die Hölderbedingung erfüllen, eine "stückweise holomorphe", Γ-automorphe Funktion Φ zu finden, die sich auf L stetig fortsetzen läßt und auf L die Randbedingung $\Phi^+ = G\Phi^- + g$ erfüllt. Diese Aufgabe ist vom Verfasser in früheren Arbeiten unter speziellen Annahmen über L gelöst worden. Hier wird (mit denselben Methoden) der Fall behandelt, daß L aus Γ-kongruenten Bögen besteht.

J. Spilker (Freiburg)

Kamthan, Pawan Kumar 4894

Proximate order (R) of entire functions represented by Dirichlet series.

Collect. Math. **14** (1962), 275–278.

Let $f(s) = \sum_1^\infty a_n \exp(s\lambda_n)$, $0<\lambda_1<\cdots<\lambda_{n-1}<\lambda_n \to \infty$, $\liminf_{n\to\infty} (\lambda_n - \lambda_{n-1}) > 0$, be an entire function of order ρ $(0<\rho<\infty)$ and let $\rho(\sigma)$ such that $\lim_{\sigma\to\infty} \rho(\sigma) = \rho$ and $\lim_{\sigma\to\infty} \sigma\rho'(\sigma) = 0$ be a Ritt proximate order of $f(s)$ as defined by Sunyer Balaguer [Proc. Amer. Math. Soc. **4** (1953), 310–322; MR **14**, 738].

Then

$$\limsup_{\sigma\to\infty} \left(\log\left\{\sup_{-\infty<t<\infty} |f(\sigma+it)|\right\}\Big/\exp[\sigma\rho(\sigma)]\right) = A$$

if and only if $\limsup_{n\to\infty} \varphi(\lambda_n)|a_n|^{1/\lambda_n} = (A\rho e)^{1/\rho}$, where $\varphi(t)$ is defined as the solution of $t = \exp[\rho(\log\varphi)\log\varphi]$, $t>t_0$.

A. G. Azpeitia (Amherst, Mass.)

Wittich, H. 4895

Über eine Borelsche Identität.

Math. Z. **84** (1964), 233–243.

Let $a_0, a_1, \cdots, a_n; g_1, g_2, \cdots, g_n$ be entire functions satisfying the following conditions:

(i) $m(r, a_k) = o\left(\sum_{j=1}^n m(r, \exp(g_j))\right)$ $(k = 0, 1, \cdots, n)$,

where $m(r,f)$ denotes the mean of Nevanlinna of the function f; (ii) no g is constant. Then, an identity such as

$$\sum_{k=1}^n a_k(z)\exp(g_k(z)) \equiv a_0(z)$$

implies

$$\sum_{k=1}^n c_k a_k(z)\exp(g_k(z)) \equiv 0,$$

where the c's are constants, not all zero. The author proves this result and remarks that it is a sharpened form (using the notions of Nevanlinna's theory) of older results of E. Borel and P. Csillag. The paper contains also a study of the distribution of values of the entire transcendental solutions of a linear differential equation $L(w) = \sum_{j=0}^n a_j(z)w^{(n-j)} = 0$. The author shows that if w is an entire solution such that $m(r, a_j) = o(m(r, w))$ $(j = 0, 1, \cdots, n)$, if ϕ is an entire function such that $m(r,\phi) = o(m(r, w))$ and $L(\phi) \not\equiv 0$, then the zeros of $w - \phi$ are not deficient in the sense of Nevanlinna. He indicates a sharper form of this result in the special case of polynomial a's and considers also an extension of a result of Saxer [Math. Z. **17** (1923), 206–227].

A. Edrei (Syracuse, N.Y.)

Ferlan, Nives Maria 4896

Sul minimo modulo delle funzioni analitiche.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **35** (1963), 463–465.

Announcement of inequalities of the Milloux-Schmidt type for functions analytic in a disk. Here the role of the minimum modulus $m(r)$ is played by $\bar{m}(r) = \sup_{0\le t<r} m(t)$. Cognate inequalities involving integrals of $\log \bar{m}(r)$ and $\log M(r)$ are stated. Applications to entire functions.

M. H. Heins (Urbana, Ill.)

Petrenko, V. P. 4897

On the deficiencies of a meromorphic function. (Russian)

Dokl. Akad. Nauk SSSR **158** (1964), 1030–1033.

The author proves that for a meromorphic function $f(z)$ of finite lower order $\lambda > \frac{1}{2}$

$$\sum_a (\delta(a, f))^{1/2} < K\lambda^{1/2},$$

where K is an absolute constant. The same estimate is also valid if $\lambda < \frac{1}{2}$ and $f(z)$ has at least two deficient values. The example $f(z) = \int_0^z e^{-t^\lambda}\, dt$ ($\lambda = 1, 2, \cdots$) shows that the result is best possible as far as the dependence on λ is concerned. *W. H. J. Fuchs* (Ithaca, N.Y.)

Suvorov, G. D. 4898

A fundamental theorem on boundary correspondence for a sequence of topological mappings of class $\widetilde{BL}_k$ of plane regions. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 1152–1162.

A real-valued, continuous function f in a plane domain D of the finite xy-plane belongs to the class BL_k provided it has first-order generalized partial derivatives (in the sense of S. L. Sobolev) and

$$\iint_D [f_x^2 + f_y^2]\, dx dy \leqq k < \infty.$$

A mapping $T(z) = f_1(x, y) + if_2(x, y)$ of D belongs to the class $\widetilde{\mathrm{BL}}_k$ if on every open set D_m in D on which $|T(z)| < m$ the functions f_1 and f_2 belong to some class BL_k, and if, moreover,

$$\iint_{D_m} \frac{\sum_{j=1,2} \operatorname{grad}^2 f_j}{\left(1 + \sum_{j=1,2} f_j^2\right)^2}\, dx dy \leqq k.$$

The author's main result concerns a sequence of one-to-one homeomorphisms T_n of simply connected plane domains A_n onto plane domains B_n. Suppose the sequences $\{A_n\}$ and $\{B_n\}$ converge (in the sense of Carathéodory) to nondegenerate kernels A_0 and B_0, respectively; suppose all the T_n and T_n^{-1} belong to some class $\widetilde{\mathrm{BL}}_k$ (k independent of n) on their domains; finally, suppose that $\{T_n\}$ and $\{T_n^{-1}\}$ converge almost uniformly to mappings T_0 and T_0^{-1} between A_0 and B_0, and that $T_0, T_0^{-1} \in \widetilde{\mathrm{BL}}_k$. Then the sequence $\{T_n\}$ determines a one-to-one correspondence between prime ends of the sequences $\{A_n\}$ and $\{B_n\}$. *G. Piranian* (Ann Arbor, Mich.)

Simonenko, I. B. 4899

The Riemann boundary-value problem for n pairs of functions with measurable coefficients and its application to the study of singular integrals in weighted L_p spaces. (Russian)

Dokl. Akad. Nauk SSSR **141** (1961), 36–39.

Announcement of results appearing in Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 277–306 [MR **29** #253].

Simonenko, I. B. 4900

A maximal boundary property of functions with integral representations of a certain type. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 1301–1302.

Let $\varphi(z)$ be an H_p function holomorphic in the unit circle C. At each point $t \in C$ as vertex is attached a sector S_t lying completely in C. Inspired by the work of Calderón and Zygmund [Acta Math. **88** (1952), 85–139; MR **14**, 637] the author extends a theorem of M. Riesz [Math. Z. **27** (1927), 218–244] concerning the norm $\|M\|_{L_p}$ of the function $M(t) = \sup_{z \in S_t} |\varphi(z)|$. The result is given without proof. *D. Mitrović* (Zagreb)

Zverovič, È. I. 4901

A boundary-value problem of Carleman type for multiply connected domains. (Russian)

Mat. Sb. (*N.S.*) **64** (**106**) (1964), 618–627.

Let D be a finite $(m+1)$-connected domain with Liapunov contour L, consisting of simple, closed curves $L_0, L_1, \cdots, L_m$, where L_0 encloses $L_1, \cdots, L_m$. Denote the complement of $D + L$ by D^-. Let $\alpha(t) = \sum_{k=0}^m \omega(t, L_k)\alpha_k(t)$, $t \in L$, where $\alpha_k(t)$ is an orientation-preserving homeomorphism of L_k into itself and $\omega(z, L_k)$ is the harmonic measure of L_k with respect to D. For $\alpha_k(t)$ it is assumed that $\alpha_k[\alpha_k(t)] \equiv t$ and that $\alpha_k'(t)$ is Hölder-continuous. For $G(t) \neq 0$ and Hölder-continuous on L the author considers the problem of determining a function $\Phi(z)$, single-valued and analytic in D, Hölder-continuous in $\bar{D}$, such that the following boundary relation holds: (*) $\Phi^+[\alpha(t)] = G(t)\overline{\Phi^+(t)}$. Denoting the number of linearly independent solutions of (*) by l and $\sum_{k=0}^m \kappa_k$ by κ, where $\kappa_k = \operatorname{ind} G(t)|_{L_k}$, the author shows that when $\kappa < 0$, $l = 0$; when $\kappa > 2m-2$, $l = \kappa - m + 1$; and when $0 \leqq \kappa \leqq 2m-2$, $\max\{0, \kappa - m + 1\} \leqq l \leqq 1 + [\kappa/2]$. *J. F. Heyda* (King of Prussia, Pa.)

Kaz'min, Ju. A. 4902

On a completeness criterion. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 549–556.

Let G be a simply connected domain lying in the finite part of the complex plane and bounded by a closed rectifiable curve Γ. Denote the space of functions regular in G with the topology of uniform convergence on compact sets by $A^-(G)$, and let $A^+(G)$ be its conjugate space (identified with a certain space of functions regular on Γ and outside G and vanishing at infinity). Let $\{h_n\}$ be a sequence of functions which is closed in $A^+(g)$, and let $f \in A^-(G)$. Define

$$f_n(z) = \frac{1}{2\pi i} \int_{C_n} \frac{f(t)h_n(t)}{t - z}\, dt, \qquad n = 0, 1, 2, \cdots,$$

where C_n is a Jordan curve lying in G such that h_n is regular on C_n and the domain exterior to it, and z is in the interior domain. The author shows that $\{f_n\}$ is complete in $A^-(G)$ if and only if f is not a rational function. He also applies this criterion to some specific sequences of functions. *H. H. Wicke* (Albuquerque, N.M.)

POTENTIAL THEORY

See also 4978.

Calderón, A. P.; Zygmund, A. 4903

On higher gradients of harmonic functions.

Studia Math. **24** (1964), 211–226.

Let $U(x) = U(x_1, \cdots, x_n)$ be a real-valued harmonic function defined on a domain of n-dimensional Euclidean space E_n ($n \geq 2$). The authors prove that the pth power of

the norm of the gradient of order m of U, viz., $|\mathrm{grad}_m U| = [\sum_{|\alpha|=m} (D^\alpha U)^2(\alpha!)^{-1}]^{1/2}$, where $\mathrm{grad}_m U = (D^\alpha U)_{|\alpha|=m}$, $D^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, is subharmonic for $p \geq (n-2)/(m+n-2)$. This generalizes a result of Stein and Weiss for $|\mathrm{grad}\, U|$ [Acta Math. **103** (1960), 25–62; MR **22** #12315]. A sufficient condition is given that a set of functions $(U_\alpha)_{|\alpha|=m}$, C^∞ in a sphere $|x| < R$, be the gradient of order m of a harmonic function. This result is applied to the set of functions $f_\beta(x, t) = R^\alpha P_t f$, f a real-valued function in $L^p(E_n)$, $p \geq 1$, $P_t f$ the Poisson integral of f and $R^\alpha = \prod R_j^{\alpha_j}$, R_j a Riesz transform and $\beta = (\alpha_1, \cdots, \alpha_n, k)$, $|\beta| = m$. The $f_\beta(x, t)$ are harmonic and $[\sum f_\beta^2(x, t)(\beta!)^{-1}]^{l/2}$ is subharmonic for $l \geq (n-1)/(m+n-1)$.

J. Mitchell (Madison, Wis.)

Položil, G. M. [Položil, G. N.] 4904

On the representation of p-harmonic functions by harmonic functions and their derivatives. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 1277–1280.

Suppose that $u(x, y)$ is a p-harmonic function with characteristic function $p(x, y)$ [see the author, Dokl. Akad. Nauk SSSR **58** (1947), 1275–1278; MR **9**, 507; ibid. **60** (1948), 769–772; MR **10**, 698]. The problem is to represent $u(x, y)$ in the form of a linear combination of a harmonic function $U(x, y)$ and its partial derivatives of nth order. A necessary and sufficient condition is given, which involves the characteristic function $p(x, y)$; this condition can be given explicitly whenever $n = 1, 2, 3$.

A. J. Lohwater (Providence, R.I.)

Ciesielski, Z. 4905

Brownian motion, capacitory potentials and semi-classical sets. I.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 265–270.

Let D be a domain in euclidean n-space $\mathbf{R}^n$ ($n \geq 2$) with a Green's function G and let K be a compact subset of D. We assume that G is normalized in the probabilistic sense, i.e., G is the potential-theoretic Green's function divided by half of the area of the unit sphere in $\mathbf{R}^n$. Let $W_K = G_{\mu_K}$ be the capacitory potential (of the capacitory distribution μ_K) of K with respect to this kernel G. Denote by $T_K(y)$ the total time spent in K by a Brownian path starting in $y \in D$ before reaching the boundary of D; denote by $U_K(y) = P[T_K(y) > 0]$ the probability that $T_K(y)$ is strictly positive. The set K is said to be semi-classical if $U_K = W_K$ on D (or, what amounts to the same, if $W_K(y) = U_K(y)$ for all $y \in D \setminus K$). In the case where the n-dimensional Lebesgue measure of K is > 0, different characterizations of a semi-classical set K are given. They are based on a modification of a method used by M. Kac [Proc. Second Berkeley Sympos. Math. Statist. and Prob. 1950, pp. 189–215, Univ. of California Press, Berkeley, Calif., 1951; MR **13**, 568] which consists in calculating the distribution of $T_K(y)$ by means of the eigenvalues λ_j and the eigenfunctions φ_j of the integral equation $\int_K G(x, y)\varphi(x)\,dx = \lambda\varphi(y)$ ($y \in K$). In particular, the following condition characterizes a semi-classical K:

$$W_K(y) = \lim_{t \to 0+} \sum_{j=1}^{\infty} e^{-t/\lambda_j} \frac{1}{\lambda_j} \int_K \varphi_j(x)\,dx \int_K G(x, y)\varphi_j(x)\,dx$$

for all $y \in D$. This generalizes a result of M. Kac ["Potential theory in three dimensions", Mimeographed notes, Rockefeller Institute, New York] proved for $n = 2, 3$ and for K with additional properties.

H. Bauer (Hamburg)

Nozaki, Yasuo 4906

On Riemann-Liouville integral of ultra-hyperbolic type.

Kōdai Math. Sem. Rep. **16** (1964), 69–87.

In the space considered, the points have the form (x, y), where x and y are, respectively, k- and m-dimensional vectors, and where the square of the norm is $x^2 - y^2$. The Laplacian operator and the Riemann-Liouville integral of order α are defined correspondingly, and are shown to verify the formal extensions of formulae used, for $k = 1$, by M. Riesz [Acta Math. **81** (1949), 1–223; MR **10**, 713].

L. C. Young (Madison, Wis.)

SEVERAL COMPLEX VARIABLES

See also 4739, 5259, 5260.

Šafeev, M. N. 4907

On functions of two complex variables, analytic in a hypersphere. (Russian)

Ukrain. Mat. Ž. **12** (1960), 101–106.

Aĭzenberg, L. A. 4908

Integral representations of functions holomorphic in n-circular domains ("Continuation" of Szegő kernels). (Russian)

Mat. Sb. (N.S.) **65 (107)** (1964), 104–143.

The investigations in this paper are carried out only in the case $n = 2$. The author considers a full Reinhardt region D with boundary ∂D, and D^+ and ∂D^+ denote the traces of D and ∂D in the positive quadrant. For non-negative integers m, n the term $d_{mn}(D) = \sup_D |w^m z^n|$ is introduced. For non-negative real numbers (r, ρ) the set $\Delta(r, \rho) = \{(re^{i\theta_1}, \rho^{i\theta_2}) \mid 0 \leq \theta_\nu < 2\pi\}$ is introduced. Let μ denote a measure on ∂D^+ such that every open set is measurable.

The principal result is that a holomorphic function $h(\alpha, \beta) = \sum a_{mn}\alpha^m\beta^n$ can be used as the kernel in an integral formula

$$f(w, z) = \frac{1}{(2\pi i)^2} \int_{\partial D^+} d\mu$$

$$\times \int_{\Delta(|\zeta|, |\eta|)} f(\zeta, \eta) h\left(w\zeta \frac{\overline{\varphi_1(|\zeta|, |\eta|)}}{|\zeta|^2}, z\bar{\eta} \frac{\overline{\varphi_2(|\zeta|, |\eta|)}}{|\eta|^2}\right) \frac{d\zeta}{\zeta} \wedge \frac{d\eta}{\eta},$$

with bounded μ-measurable functions φ_1, φ_2 different from zero μ-almost everywhere on ∂D^+ if and only if φ_1 and φ_2 can be chosen as solutions to the momentum problem

$$a_{mn}^{-1} = \int_{\partial D^+} \varphi_1^m(|\zeta|, |\eta|)\varphi_2^n(|\zeta|, |\eta|)\,d\mu$$

subject to the condition

$$\limsup_{m+n \to \infty} \sqrt[m+n]{\left(|a_{mn}| d_{mn}(D) \sup_{\partial D^+} |\zeta|^{-m}|\eta|^{-n} \left|\varphi_1^m(|\zeta|, |\eta|)\varphi_2^n(|\zeta|, |\eta|)\right|\right)} \leq 1,$$

where sup is the effective supremum with respect to μ.

If the measure μ is given, it is possible to establish an integral formula

$$f(w, z) = \frac{1}{(2\pi i)^2} \int_{\partial D^+} d\mu \int_{\Delta(|\zeta|, |\eta|)} f(\zeta, \eta) h(w\bar{\zeta}, z\bar{\eta}) \frac{d\zeta}{\zeta} \wedge \frac{d\eta}{\eta},$$

where h is analytic, if and only if μ satisfies the condition that the effective supremum on ∂D^+ of $|\zeta|^m|\eta|^n$ with respect to μ is equal to the maximum on ∂D^+ of $|\zeta|^m|\eta|^n$ for every pair of non-negative integers m, n.

Integral formulae by Szegö, Bergman, Temljakov and others are special cases of the formulae derived in the present paper. *H. Tornehave* (Copenhagen)

Gross, Fred 4909

Entire functions all of whose derivatives are integral at the origin.

Duke Math. J. **31** (1964), 617–622.

An entire function f of two variables is strongly transcendental if no partial derivative of f is identically zero; it is a Hurwitz function if f and all its derivatives are integers at $(0, 0)$. Let $\varphi(r) = \max_n r^n/n!$, and let $\psi(r)$ be any increasing function that increases more rapidly than any power of r. The author extends results of Sato and Strauss [J. Math. Soc. Japan (to appear)] for the one-variable case. For example, a Hurwitz function whose maximum modulus satisfies (for large r_1, r_2) $M(r_1, r_2) < \varphi(r_1) + \varphi(r_2) + r_1^n + r_2^n$ must be a polynomial, but $r_1^n + r_2^n$ cannot be replaced by $\psi(r_1) + \psi(r_2)$. A Hurwitz function with $M(r_1, r_2)^2 < \varphi(r_1)^2\varphi(r_2)^2 + r_1^n r_2^n$ cannot be strongly transcendental, but it can be if $M(r_1, r_2) < \varphi(r_1)\varphi(r_2) + \psi(r_1)\psi(r_2)$. If f is a Hurwitz function with $M(r_1, r_2) < \varphi(r_1)\varphi(r_2) + r_1^n r_2^n$ and for each N there exist i, j, k, l with $k \geqq i > n$, $l \geqq j > N$, and $i < k$ or $j < l$, such that $\partial^{i+j}f/\partial z_1^i \partial z_2^j$ and $\partial^{k+l}f/\partial z_1^k \partial z_2^l$ are not zero at $(0, 0)$, then either $l/j > 4$ or $k/i > 4$ for large l and k.

R. P. Boas, Jr. (Evanston, Ill.)

Rényi, C. 4910

On some questions concerning lacunary power series of two variables.

Colloq. Math. **11** (1963/64), 165–171.

Let $f(z, w)$ be a transcendental entire function. If the power series of f from a point (a, b) arranged as a sum of homogeneous polynomials has the property that non-vanishing terms occur with frequency 0, the power series is called D-lacunary. The author proves that the power series of f from (a_1, b_1) and (a_2, b_2), where $a_1 \neq a_2$ and $b_1 \neq b_2$, will not be D-lacunary simultaneously except if f can be written in the form $\sum (w - \alpha z - \beta)P_n(z)$, where the factors P_n are polynomials. If the power series of f from (a, b) arranged as a power series in $w - b$ has the property that non-vanishing terms occur with frequency 0, the power series is called C-lacunary. If no partial derivative of f vanishes identically, the power series of f from (a_1, b_1) and (a_2, b_2), where $a_1 \neq a_2$, will not be C-lacunary simultaneously. If the planar frequency of non-vanishing terms of the power series measured with respect to rectangles [squares] is 0, the power series is called A-[A*-] lacunary. The author shows by examples that only much weaker results can hold in these cases.

H. Tornehave (Copenhagen)

Ronkin, L. I. 4911

On the conjugate orders and types of entire functions of several variables. **(Russian)**

Ukrain. Mat. Ž. **16** (1964), 408–413.

Let f be an entire function of n complex variables. Let B denote the set of points $(a_1, \cdots, a_n)$ of the positive cone of the n-dimensional real space for which $\log|f|$ is majorized asymptotically by $|z_1|^{a_1} + \cdots + |z_n|^{a_n}$. The boundary S of B is called the order surface of f. The author proves that the order surfaces of entire functions are characterized by the following two properties: (i) If $(a_1, \cdots, a_n) \in B$, then every point $(b_1, \cdots, b_n)$ where $b_j \geqq a_j$, $j = 1, \cdots, n$, belongs to B; (ii) The mapping $z_j \to z_j^{-1}$, $j = 1, \cdots, n$, maps B onto a convex region. For every point $(\rho_1, \cdots, \rho_n) \in S$ the set of points $(a_1, \cdots, a_n)$ of the positive cone of the n-dimensional real space for which $\log|f|$ is majorized by $a_1|z_1|^{\rho_1} + \cdots + a_n|z_n|^{\rho_n}$ is denoted B_ρ and its boundary is called the type surface of f corresponding to $(\rho_1, \cdots, \rho_n)$. The type surfaces are characterized by the properties that the complementary set of B_ρ is logarithmically convex, and that it, together with a point $(a_1, \cdots, a_n)$, contains every point $(b_1, \cdots, b_n)$, where $b_j \leqq a_j$, $j = 1, \cdots, n$. *H. Tornehave* (Copenhagen)

Forster, Otto; Ramspott, Karl Josef 4912

Über die Darstellung analytischer Mengen.

Bayer. Akad. Wiss. Math.-Natur. Kl. S.-B. **1963**, *Abt. II*, 89–99 (1964).

The following theorems are proved. (i) Let X be a holomorphically complete, reduced, n-dimensional complex space. Then, for every closed ideal a in the ring $\mathfrak{H}(X)$ of holomorphic functions on X (equipped with the topology of compact convergence), there exist $f_1, \cdots, f_n \in a$ such that the varieties of $(f_1, \cdots, f_n)$ and of a coincide. A counterexample shows that the proof of this theorem fails in the algebraic case. (ii) Let X be a holomorphically complete, n-dimensional complex manifold. Then, for the holomorphic vector bundle W of rank d over X, the $\mathfrak{H}(X)$-module $H^0(X, \mathcal{O}(W))$ is generated by $d + [n/2]$ elements. In particular, every purely $(n-1)$-dimensional analytic subset of a holomorphically complete, n-dimensional complex manifold can be written as the set of common zeros of $1 + [n/2]$ holomorphic functions.

H. Röhrl (La Jolla, Calif.)

Spallek, Karlheinz 4913

Tensorielle Beschränkungen analytischer Garben.

Math. Z. **84** (1964), 448–463.

This paper contains a generalization of the classical Hartogs-Osgood theorem to sections in analytically coherent sheaves. Let B be an open subset in $\mathbf{C}^n$, $\mathcal{O}$ the sheaf of germs of holomorphic functions on B, and $\mathcal{H}$ a coherent subsheaf of the coherent sheaf $\mathcal{G}$ over B, $j: \mathcal{H} \to \mathcal{G}$ being the injection. Furthermore, let $\mathcal{J}$ be a coherent sheaf of ideals on B. Then the image of $\mathcal{H}$ under the composite homomorphism

$$\mathcal{H} \xrightarrow{\cong} \mathcal{H} \otimes_{\mathcal{O}} \mathcal{O} \to \mathcal{H} \otimes_{\mathcal{O}} \mathcal{O}/\mathcal{J} \xrightarrow{j \otimes \mathrm{id}} \mathcal{G} \otimes_{\mathcal{O}} \mathcal{O}/\mathcal{J}$$

is called the tensorial restriction of $\mathcal{H}$ to $\mathcal{J}$ with respect to $\mathcal{G}$ and is denoted by $\mathcal{H}(\mathcal{G})|\mathcal{J}$. In a straightforward fashion one defines now the tensorial restriction

of a homomorphism $h: \mathscr{G} \to \mathscr{G}'$ satisfying $h(\mathscr{K}) \subset \mathscr{K}'$. A similar construction can be carried out for germs. Next, given the integers l and m and the family $\mathfrak{A} = \{A_z^l\}$ of germs of l-dimensional analytic planes in points $z \in B$, one defines $\mathscr{K}[\mathfrak{A}, m]$ to be the subsheaf of $\mathscr{G}$ consisting of all those germs g_z that are in $\mathscr{K}_z(\mathscr{G}_z)|\mathscr{I}^m(A_z^l)$ for all $A_z^l \in \mathfrak{A}$. For sufficiently large families $\mathfrak{A}$, the existence of an analytic set $A \subset B$ of codimension $\geq l+1$ can be established such that $\mathscr{K}[\mathfrak{A}, 1] \subset \mathscr{K}[A]$ holds, the latter being the coherent subsheaf of $\mathscr{G}$ consisting of all those germs presentable by local sections g satisfying $g|\, U - A \in \Gamma(U - A, \mathscr{K})$. If, in addition, for every $z \in B$ the primary decomposition of $\mathscr{K}_z$ in $\mathscr{G}_z$ has only primary components of dimension $\geq n-l$, then $\mathscr{K}[\mathfrak{A}, 1] = \mathscr{K}$. The same conclusion can be drawn from the hypothesis $\operatorname{codim}_z(\mathscr{G}_z/\mathscr{K}_z) \geq n-l$. Several more statements of this type are proved.

H. Röhrl (La Jolla, Calif.)

SPECIAL FUNCTIONS

See also 5075, 5514.

Abramowitz, Milton; Stegun, Irene A. (Editors) 4914

★**Handbook of mathematical functions with formulas, graphs, and mathematical tables.**

National Bureau of Standards Applied Mathematics Series, 55.

For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964. xiv + 1046 *pp*. $6.50.

This volume is the culmination of seven years of labor by 26 authors, mostly associates of the National Bureau of Standards, and the two editors, who also served as authors. In each of its 29 chapters are given definitions, formulas, short tables, graphs, polynomial approximations, and bibliographies of the pertinent functions.

The titles of the chapters are as follows. Mathematical Constants. Physical Constants and Conversion Factors. Elementary Analytical Methods. Elementary Transcendental Functions. Exponential Integral and Related Functions. Gamma Function and Related Functions. Error Function and Fresnel Integrals. Legendre Functions. Bessel Functions of Integer Order. Bessel Functions of Fractional Order. Integrals of Bessel Functions. Struve Functions and Related Functions. Confluent Hypergeometric Functions. Coulomb Wave Functions. Hypergeometric Functions. Jacobian Elliptic Functions and Theta Functions. Elliptic Integrals. Weierstrass Elliptic and Related Functions. Parabolic Cylinder Functions. Mathieu Functions. Spheroidal Wave Functions. Orthogonal Polynomials. Bernoulli and Euler Polynomials. Riemann Zeta Function. Combinatorial Analysis. Numerical Interpolation, Differentiation and Integration. Probability Functions. Miscellaneous Functions. Scales of Notation. Laplace Transforms.

Many of the tables are new. Each column of a table for which interpolation is feasible is supplied with two indicators: one gives the maximum error in case linear interpolation is used, the other indicates how many points of Lagrangian interpolation are needed to exploit the full accuracy of the local table. There are many illustrative examples. The difficult format problems have been solved successfully.

The widespread availability of automatic computers has not eliminated the need for mathematical tables, but rather has modified the kind of information required. The needs now are small basic tables for reconnaissance handwork and for possible inclusion in special subroutines, together with copious lists of formulas for the generation, transformation and approximation of functions.

This magnificent volume is destined to become the new standard reference for workers in the field of computational mathematics.

D. H. Lehmer (Berkeley, Calif.)

Curtis, A. R. 4915

★**Coulomb wave functions.**

Prepared under the direction of The Coulomb Wave Functions Panel of the Mathematical Tables Committee. Royal Society Mathematical Tables, Vol. 11.

Published for the Royal Society at the Cambridge University Press, New York, 1964. xxxv + 209 *pp*. $15.00.

These functions are solutions of the differential equation

$$y'' + (a + 2x^{-1} - L(L+1)x^{-2})y = 0,$$

where $x > 0$, a is real and L is zero or a positive integer (in the tables $L = 0, 1$ or 2) which arises by separating off the radial component in the Schrödinger equation for an electron in a Coulomb field. There is some difficulty in deciding on the standard solutions to be tabulated, having regard to the convenience of the user (e.g., with regard to interpolation in energy parameter a) and taking account of the necessity of having the solutions "numerically" as well as analytically independent. The actual solutions chosen are too complicated to be given here and are different in the ranges $a \geq 0$, $a < 0$. Suffice it to say that 6D tables of the solutions and their first x-derivatives are given (or readily obtainable) for all x and for $a = -2(.2)2$. [For large x, the independent variables x^{-1} or $x^{-1/2}$ are used.] The values are said to be good within .6 units in the last place.

To aid interpolation in the x-direction by Everett's formula with throw-back, second and fourth differences (modified when necessary) are given. Auxiliary functions for interpolation in the irregular solution near $x = 0$ are given.

There is an elaborate introduction which discusses the choice of solution, recurrence relations, asymptotic and other representations. Interpolation is illustrated by worked examples, and the preparation and checking of the table is described carefully. Much of the material was obtained by integrating the differential equation by the Runge-Kutta method, sometimes using a double precision program. The values of some of the functions for large arguments were computed automatically from a Chebyshev expansion (in terms of the reciprocal variable), the coefficients of which were obtained by desk calculations. Special tablemaking subroutines were employed, e.g., to produce the modified differences from the tabular values, and for editing purposes.

The tables were reproduced photographically from copy prepared by a card-controlled typewriter. A few imperfections are clarified in an inset.

There is a substantial bibliography. For some further references see the chapter by M. Abramowitz on Coulomb Wave Functions in #4914 reviewed above.

John Todd (Pasadena, Calif.)

Curtis, A. R. 4916

★**Tables of Jacobian elliptic functions whose arguments are rational fractions of the quarter period.**

National Physical Laboratory Mathematical Tables, Vol. 7. Department of Scientific and Industrial Research.

Her Majesty's Stationery Office, London, 1964. iii+81 *pp*. 15*s*.; $3.00.

This table gives values to 20D of $\mathrm{sn}(u, k)$, $\mathrm{cn}(u, k)$, $\mathrm{dn}(u, k)$, where $u = mK/n$ for $m = 1(1)(n-1)$, $n = 2(1)15$. The values of k are those which correspond to $q = 0(.005).35$ and lie in the range 0 to .99933. For each value of q, the corresponding values of k, k', $\theta = \sin^{-1} k$, K, K', K/K' are given to 20D.

The calculations were performed, at triple-precision, on the DEUCE, summing the Fourier series for the theta function for angular argument π/n. From these theta values the elliptic functions for argument K/n were formed and from these the addition formulae gave the values for argument mK/n. Various functional relations were satisfied to within a unit in the 25th decimal. It is stated that the tabular values are in error by less than half a unit in the 20th decimal.

An alternative computation based on the Gauss arithmetic-geometric mean is described; the use of this is indicated for the case when tabulations are required with k as the independent variable.

{(1) These tables were prepared to facilitate computations occurring in a method of filter design proposed by S. Darlington [J. Math. Phys. Mass. Inst. Tech. **18** (1939), 257–353; MR **1**, 275]. We note that a similar scheme was developed by W. Cauer [see, e.g., Math. Z. **38** (1933), 1–44; *Synthesis of linear communication networks*, Vols. 1, 2, second edition, McGraw-Hill, New York, 1958; MR **20** #4416], and that tables to assist in its use were prepared by W. Glowatzki [Abh. Bayer. Akad. Wiss. Math.-Natur. Kl. (N.F.) No. 67 (1955); MR **16**, 1153]. See also R.A.-R. Amer and H. R. Schwarz [Mitt. Inst. Angew. Math. Zürich No. 9 (1964); MR **29** #4639]. (2) These tables have an interesting application in the theory of alternating implicit direction methods. W. B. Jordan [see E. L. Wachspress, J. Soc. Indust. Appl. Math. **11** (1963), 994–1016; MR **29** #6623] obtained the optimal parameters for the solution of the "model" problem, and these are essentially among the values of the elliptic functions tabulated, the k of the elliptic functions being related to the estimates of the eigenvalues in the discrete version of the "model" problem.} *John Todd* (Pasadena, Calif.)

Karpov, K. A. 4917

★**Tables of the functions $F(z) = \int_0^z e^{x^2}\,dx$ in the complex domain.**

Translated by D. E. Brown. Mathematical Tables Series, Vol. 23. A Pergamon Press Book.

The Macmillan Co., New York, 1964. xxv+497 *pp*. (3 *inserts*) $20.00.

This is an English version of the Russian book [Izdat. Akad. Nauk SSSR, Moscow, 1958] noted in MR **24** #B1297, a companion to the author's earlier volume [*Tables of the function $w(z) = e^{-z^2}\int_0^z e^{x^2}\,dx$ in a complex region* (Russian), Izdat. Akad. Nauk SSSR, Moscow, 1954; MR **16**, 749]. Together, these volumes make readily accessible values, to 5D or to 5S, of the error function in the whole complex plane, with a polar argument; for a cartesian argument see the tables of V. N. Faddeeva and N. M. Terent'ev [*Tables of values of the function $e^{-z^2}(1 + 2i\pi^{-1/2}\int_0^z e^{t^2}\,dt)$ for a complex argument* (Russian), GITTL, Moscow, 1954; MR **16**, 960; English transl., M. D. Friedman, Newtonville, Mass., 1956; MR **18**, 155].

The actual contents of this table are: (1) the real and imaginary parts of $F(\rho e^{i\theta}) = u + iv$ for $0 \leq \rho \leq \rho_0 = \rho_0(\theta)$, $\frac{1}{4}\pi \leq \theta \leq \frac{1}{2}\pi$, where ρ_0 ranges from 5 to 3; the intervals in ρ, θ vary and are indicated on an insert. (2) The values of $F(\rho)$ for $\rho = 0(.001)10$. Second differences are given in the range [3, 10] and an insert table of $\frac{1}{4}t(1-t)$ and a nomogram to calculate the second difference correction in Bessel interpolation is provided.

In the introduction various integral representations, series expansions and differential equations for u and v are given, together with graphs and reliefs. In particular, asymptotic series appropriate for calculating u and v beyond the range of tabulation are given; auxiliary tables to facilitate such calculations are provided.

In the main table linear interpolation along radii is adequate but on the circles $\rho = \mathrm{const}$, higher-order Lagrangian interpolation is recommended. [A diagram indicates which order is required; generally 4-point is sufficient.] Worked examples illustrate interpolation.

The tables were computed using a piecewise approximation to the solution of simultaneous second-order differential equations for u and v by polynomials of degree 14 in ρ; the evaluation of the polynomials was carried out by tabulators [K. A. Karpov, Dokl. Akad. Nauk SSSR **62** (1948), 741–744; MR **10**, 405].

The printing is clear, but in the review copy pages 216–217 were blank. The translation of the introduction is adequate. *John Todd* (Pasadena, Calif.)

Schrutka v. Rechtenstamm, Guntram 4918

Tabelle der (Relativ)-Klassenzahlen der Kreiskörper, deren φ-Funktion des Wurzelexponenten (Grad) nicht grösser als 256 ist.

Abh. Deutsch. Akad. Wiss. Berlin Kl. Math. Phys. Tech. **1964**. *no.* 2, 64 *pp*.

Eastham, M. S. P. 4919

On polylogarithms.

Proc. Glasgow Math. Assoc. **6**, 169–171 (1964).

The nth-order polylogarithm, $\mathrm{Li}_n(z)$, is defined by $\mathrm{Li}_n(z) = \sum z^r r^{-n}$ $(n = 2, 3, \cdots)$. Extend the sequence by $z\,\mathrm{Li}_n'(z) = \mathrm{Li}_{n-1}(z)$. Then it is shown that no recurrence formula of the form $\sum_{j=0}^{m} A_j(z)\,\mathrm{Li}_{n-j}(z) = 0$ holds, where $A_j(z)$ are algebraic functions of z, $A_0(z)$ is not identically zero, and $m \geq 1$. Also, a generating function for $\mathrm{Ai}_n(z)$, $n = 2, \cdots$, is found and it is used to sum some series involving $\mathrm{Li}_n(z)$. *R. A. Askey* (Madison, Wis.)

Tricomi, Francesco 4920

Sul comportamento asintotico della funzione gamma incompleta $\Gamma(\alpha, x)$ al simultaneo divergere di α e x.

Univ. Nac. Tucumán Rev. Ser. A **14** (1962), 333–339.

In an earlier paper [Math. Z. **53** (1950), 136–148; MR **12**, 553] the author investigated the asymptotic behaviour of $\Gamma(\alpha + 1, x)$ as $\alpha \to \infty$ while $y = (x - \alpha)/\sqrt{(2\alpha)}$ is bounded. He now gives a new method for carrying out the same investigation. The earlier method produced a series in

descending powers of α for a certain multiple of $\Gamma(\alpha+1, x)$. The new method yields a more complicated series, but it is an easy matter to extract the leading terms of the expansion in descending powers of α.

A. Erdélyi (Edinburgh)

Glasser, M. L. 4921

Note on the evaluation of some Fermi integrals.

J. Mathematical Phys. 5 (1964), 1150–1152.

The author evaluates integrals of the type

$$\int_0^\infty f(E)g(E, \gamma, \zeta)\, dE,$$

either explicitly or asymptotically (ζ, γ large), where $f(E)$ is an analytic function that possibly contains algebraic or logarithmic branch-point singularities and $g(E, \gamma, \zeta) = \frac{1}{4}\operatorname{sech}^2[\frac{1}{2}\gamma(E-\zeta)]$, where $\gamma = 1/kT$ and ζ is the Fermi energy parameter. Several examples of frequent occurrence in theoretical physics are discussed.

C. J. Bouwkamp (Eindhoven)

Bhowmick, K. N. 4922

Some recurrence relations for a generalized Struve transform.

Ganita 14 (1963), 89–97.

The author obtains various recurrence relations for the generalized Struve transform $F_{\mu,\nu}^{\lambda}[g]$ which is defined by means of

$$F_{\mu,\nu}^{\lambda}[g] = \int_0^\infty (xy)^\mu H_\nu^\lambda(xy) g(y)\, dy,$$

where

$$H_\nu^\lambda(x) = \sum_{r=0}^\infty \frac{(-1)^r(\frac{1}{2}x)^{\nu+2r+1}}{\Gamma(r+\frac{3}{2})\Gamma(\nu+\lambda r+\frac{3}{2})} \quad (\lambda > 0).$$

We cite as a sample the relation

$$\lambda x^{1-\nu}\frac{d}{dx}[x^\nu F_{\mu,\nu}^\lambda[g(x)]] =$$

$$\{(\mu+\nu+\eta+1)-2\nu-1\}F_{\mu,\nu-1}^\lambda[g(x)] + F_{\mu+1,\nu-1}^\lambda[g(x)].$$

W. A. Al-Salam (Lubbock, Tex.)

Braaksma, B. L. J. 4923

Asymptotic expansions and analytic continuations for a class of Barnes-integrals.

Compositio Math. 15, 239–341 (1964).

The H-function is defined by

$$H(z) = \frac{1}{2\pi i}\int_C \frac{\prod_1^n \Gamma(1-a_j+\alpha_j s)\prod_1^m \Gamma(b_j-\beta_j s)}{\prod_{m+1}^q \Gamma(1-b_j+\beta_j s)\prod_{n+1}^p \Gamma(a_j-\alpha_j s)} z^s\, ds,$$

where $0 \le n \le p$, $1 \le m \le q$, α_j, β_j are positive numbers and a_j, b_j may be complex. It is also assumed that the poles of the numerator are simple and that the contour C can be drawn so as to divide the complex s-plane into two parts so that the poles of $\Gamma(1-a_j+\alpha_j s)$, $j = 1, 2, \cdots, n$, all lie on the left of C and those of $\Gamma(b_j-\beta_j s)$ on the right.

The H-function contains a vast number of well-known analytic functions as special cases and also an important class of symmetrical Fourier kernels of a very general nature.

If $\mu = \sum_1^q \beta_j - \sum_1^p \alpha_j$ and $\beta = \prod_1^p \alpha_j^{\alpha_j}\prod_1^q \beta_j^{-\beta_j}$, then (Theorem 1) the author proves that the H-function exists for $\mu > 0$ and $z \ne 0$, or $\mu = 0$ and $0 < |z| < \beta^{-1}$. Theorem 2 gives conditions for the analytic continuation of $H(z)$ into the domain $|z| > \beta^{-1}$.

In Theorems 3 to 9, inclusive, the author discusses the asymptotic expansion of $H(z)$ as $|z| \to \infty$, when $\mu > 0$. Such asymptotic expansions are sometimes exponentially large, sometimes of algebraic order and sometimes exponentially small, mainly according to the value of $\arg z$. The discussion is complete, but it is not possible to give details in a brief review.

The H-function reduces to the G-function when $\alpha_j = 1$, $j = 1, 2, \cdots, p$, and $\beta_j = 1$, $j = 1, 2, \cdots, q$. In Theorems 10 to 17, inclusive, the author discusses the asymptotic behaviour of the G-functions as $|z| \to \infty$. In the case when $n = 0$ both the H- and the G-functions have exponentially small asymptotic expansions.

In the rest of the paper the author discusses the asymptotic expansions of hypergeometric series which are closely connected with the H-function. They are

$$\chi(z) = \sum_{\nu=0}^\infty \frac{\prod_1^n \Gamma(\alpha_j\nu+a_j)z^\nu}{\nu!\prod_1^q \Gamma(\beta_j\nu+b_j)\prod_{n+1}^p \Gamma(1-a_j-\alpha_j\nu)},$$

where, as before, α_j ($j = 1, 2, \cdots, p$) and β_j ($j = 1, 2, \cdots, q$) are positive numbers, while a_j and b_j can be complex. The following particular cases of $\chi(z)$ are also discussed in detail:

$${}_p\psi_q(z) = \sum_{\nu=0}^\infty \frac{\prod_1^p \Gamma(\alpha_j\nu+a_j)z^\nu}{\nu!\prod_1^q \Gamma(\beta_j\nu+b_j)},$$

$$\phi(z) = \sum_0^\infty \frac{z^\nu}{\nu!\Gamma(b-\sigma\nu)},$$

where $0 < \sigma < 1$ and b is real.

The author gives a comprehensive discussion of the asymptotic behaviour, as $|z| \to \infty$, of all these hypergeometric functions.

The author fails to give, at the end of the paper, an index of symbols with references to the place in the paper where they are defined. Such an index should be given for all papers of this size.

C. Fox (Montreal, Que.)

Verma, A. 4924

Integration of bilateral hypergeometric series with respect to their parameters.

J. London Math. Soc. 39 (1964), 673–684.

A bilateral series is a hypergeometric series which is infinite in both directions. In this paper, the author evaluates certain general contour integrals of Barnes's type, where the integrand involves one or two of these bilateral series. He also gives the corresponding results involving basic bilateral series.

L. J. Slater (Cambridge, England)

Verma, Arun 4925

Series involving products of two E-functions.

Proc. Glasgow Math. Assoc. 6, 172–176 (1964).

The author evalutes six bilateral series involving products of two E-functions. The series are too complicated to reproduce here. *W. A. Al-Salam* (Lubbock, Tex.)

Al-Salam, W. A.; Carlitz, L. 4926

Bateman's expansion for the product of two Bessel functions.

Ann. Mat. Pura Appl. (4) **65** (1964), 97–111.

It is shown that the general solution of the functional equation

$$\text{(F)}\quad g_\mu[(1-x)(1-y)z]g_\nu(xyz) = \sum_{n=0}^{\infty}\frac{(-1)^n n!A_n(x)A_n(y)z^n g_{\mu+\nu+2n+1}(z)}{(\mu+1)_n(\nu+1)_n(\mu+\nu+n+1)_n}$$

($A_n(x)$ a polynomial in x of degree n; $\mu, \nu > -1$) of the form

$$g_\nu(z) = \sum_{n=0}^{\infty}(-1)^n c_n{}^\nu z^n,$$

$c_1{}^\nu \neq 0$ and continuous in ν, occurs when

$$c_n{}^\nu = \frac{a^{2n}}{n!(\nu+1)_n},\quad a \text{ an arbitrary constant.}$$

Bateman showed that $\Gamma(\nu+1)z^{-\nu/2}J_\nu(2\sqrt{z})$ satisfies (F), where $J_\nu(z) = \sum_{n=0}^{\infty}(-1)^n(\frac{1}{2}z)^{\nu+2n}/n!\Gamma(\nu+n+1)$ is the Bessel function of order ν.

A generalization of (F) is also considered.

A. E. Danese (Buffalo, N.Y.)

Epstein, Leo F. 4927

Some infinite sums involving zeros of $J_0(x)$.

J. Res. Nat. Bur. Standards Sect. B **68B** (1964), 17–26.

Let β_n denote the nth positive zero of the Bessel function $J_0(x)$. Sums of the form

$$F_r(\phi) = \sum_{n=1}^{\infty}(1/\beta_n)^{2r}\exp(-\beta_n{}^2\phi)$$

and

$$G_r(\phi) = \sum_{n=1}^{\infty}\frac{\exp(-\beta_n{}^2\phi)}{\beta_n{}^{2r-1}J_1(\beta_n)},$$

where r is an integer and $\phi > 0$, occur in a study of diffusion processes under conditions of axial symmetry. For $r = 0(1)3$, the author derives approximations in powers of $\phi^{1/2}$ as $\phi \to 0$ in the case of F_r. Analogous expansions are obtained for G_r and a few more complicated functions involving error functions. These expansions are useful in the numerical evaluation of the several functions as ϕ tends to zero from above. *C. J. Bouwkamp* (Eindhoven)

Heading, J. 4928

Transition point values.

J. London Math. Soc. **39** (1964), 466–480.

The value of the parabolic cylinder function $D_{-1/2+ia^2/2}(z)$ at the transition point $z = 2^{1/2}e^{\pi i/4}a$ is determined asymptotically for large complex values of a by the method of steepest descents, and also by application of the asymptotic theory of ordinary differential equations. The results obtained agree with those previously given by the reviewer [J. Res. Nat. Bur. Standards Sect. B **63B** (1959), 131–169; MR **22** #781].

F. W. J. Olver (Washington, D.C.)

Jorna, S. 4929

Derivation of Green-type, transitional, and uniform asymptotic expansions from differential equations. I. General theory, and application to modified Bessel functions of large order.

Proc. Roy. Soc. Ser. A **281** (1964), 99–110.

Consider the differential equation

$$d^2y/dx^2 = \chi(a^2, x)y,$$

in which a is a large parameter and $\{\chi(a^2, x)\}^{-1/4}$ is a bounded, slowly-varying function of x in the region of integration. It is well known that uniform asymptotic solutions exist of the form

$$\text{(1)}\quad y = \chi^{-1/4}\exp\left\{\pm\int^x \chi^{1/2}\,dx\right\}\sum_{s=0}^{\infty}(-1)^s v_s(a^2, x),$$

where $v_0(a^2, x) = 1$, and the higher terms $v_s(a^2, x)$ are given by recurrence relations. Furthermore, in one case in which these conditions are violated and the region of integration contains a simple zero of $\chi(a^2, x)$—a "transition point" of the differential equation—uniform asymptotic expansions for the solutions can be found in terms of Airy functions.

The novelty of the present paper is the way in which these two kinds of expansions, particularly the second, are derived. The $v_s(a^2, x)$ in (1) are obtained by solving an integro-differential equation by successive approximation. The same integro-differential equation is used in the case of a transition point; the expansions of the solutions in Airy functions are obtained with the aid of the Mellin transform and the Nörlund summation operator.

The methods are illustrated by application to Bessel functions of large order, and the results compared with those of the reviewer [Philos. Trans. Roy. Soc. London Ser. A **247** (1954), 328–368; MR **16**, 696]. Much of the analysis is purely formal, with no attempt at determining the regions of validity in the complex plane.

F. W. J. Olver (Washington, D.C.)

Jorna, S. 4930

Derivation of Green-type, transitional, and uniform asymptotic expansions from differential equations. II. Whittaker functions $W_{k,m}$ for large k, and for large $|k^2 - m^2|$.

Proc. Roy. Soc. Ser. A **281** (1964), 111–129.

The methods described in the preceding paper [#4929] are applied to the Whittaker form of the confluent hypergeometric equation

$$\frac{d^2W}{dz^2} = \left(\frac{1}{4} - \frac{k}{z} + \frac{m^2 - \frac{1}{4}}{z^2}\right)W,$$

in the cases (i) k large and m fixed, (ii) $|k^2 - m^2|$ large. Asymptotic expansions in terms of elementary functions are obtained, valid in regions excluding the transition points $z = 4k$ (case (i)), $z = 2k \pm 2(k^2 - m^2)^{1/2}$ (case (ii)). Uniform expansions in terms of Airy functions which are valid in regions containing the transition points are also obtained. The results are compared with those of earlier writers. The special cases of Weber parabolic cylinder functions, and especially Poiseuille functions, are examined in detail. *F. W. J. Olver* (Washington, D.C.)

Lindsey, William C. 4931

Infinite integrals containing Bessel function products.

J. Soc. Indust. Appl. Math. **12** (1964), 456–464.

The author evaluates the integral

$$\int_0^\infty x^m \exp(-kx^2) I_{m-1}(bx) I_n(\lambda x^2)\, dx,$$

where $k>\lambda$, n and m are integers and $I_l(z)$ is the modified Bessel function of imaginary argument and order l, and gives the result in a very complicated form. The result is suitable for making numerical computations.

Brij Mohan (Varanasi)

Srivastava, K. N. 4932

On analogies between some series containing Bessel functions and certain integrals.

Ricerca (Napoli) (2) **14** (1963), *maggio-agosto*, 22–26.

The author shows that

$$2a^{-2}\sum_{s=1}^{\infty}\frac{\alpha_s{}^{\nu}J_\mu[\sqrt{(\alpha_s{}^2+\beta^2)}]}{(\alpha_s{}^2+\beta^2)^{\mu/2}J_{\nu+1}(\alpha_s a)}J_\nu(\alpha_s\gamma) =$$

$$\int_0^\infty x^{\nu+1}(x^2+\beta^2)^{-\mu/2}J_\mu[\sqrt{(x^2+\beta^2)}]J_\nu(x\gamma)\, dx,$$

where $\beta>0$, $\mu>\nu>-1$ {the paper has erroneously >1}, $a>1$, $0<\gamma<1$ or $1<\gamma<a$, and the α_s are the positive roots of $J_\nu(\alpha a)=0$. The proof is given by representing

$$f(\gamma) = \beta^{\nu-\mu+1}\gamma^\nu(1-\gamma^2)^{(\mu-\nu-1)/2}J_{\mu-\nu-1}[\beta\sqrt{(1-\gamma^2)}] \qquad \text{if } 0<\gamma<1,$$

$$= 0 \qquad \text{if } \gamma>1$$

{the paper has erroneously $J_{\mu-\nu-1}[\beta(1-\gamma^2)]$} on the one hand as a Fourier-Bessel series for $0<\gamma<a$, and on the other hand as a Hankel transform.

There are five other similar formulae. {If $F(\gamma)$ vanishes for $\gamma\geq a$, then there is an obvious relation between the Fourier-Bessel coefficients for the interval $(0,a)$, and the Hankel transform, of F. The author's results are trivial examples of that relation.}

A. Erdélyi (Edinburgh)

Srivastava, K. N. 4933

On some integrals involving Gegenbauer polynomial and Chebychev polynomial of first kind.

Ricerca (Napoli) (2) **14** (1963), *gennaio-aprile*, 10–16.

The following result is obtained:

$$\int_0^1 \frac{C_n{}^\lambda[(xz+1)^{1/2}]C_{n-1}^{-\lambda}\left[\left(\frac{xz+1}{z+1}\right)^{1/2}\right]}{x^{1/2-\lambda}(1-x)^{1/2+\lambda}(xz+1)^{1/2}}\, dx =$$

$$\frac{2\pi}{\cos\pi\lambda}(z+1)^{(n-2\lambda-1)/2}{}_2F_1[n,-\lambda;1;-z],$$

$-\frac{1}{2}<\lambda<\frac{1}{2}$, $C_n{}^\lambda$ the Gegenbauer or ultraspherical polynomial of degree n and order λ, by using a certain explicit expression for $C_n{}^\lambda$. The case $\lambda=0$, $C_n{}^0=T_n$, the Chebyshev polynomial of the first kind, is of interest.

A. E. Danese (Buffalo, N.Y.)

Mukherjee, S. N. 4934

On some integrals involving Gegenbauer polynomials and associated Legendre polynomials.

J. Sci. Res. Banaras Hindu Univ. **14** (1963/64), no. 1, 145–150.

Integrals such as

$$\int_0^\infty x^\sigma(1-x^2)^{-\mu/2}P_\nu{}^\mu(x)J_\lambda(2\alpha tx)J_\delta(2\beta tx)\, dx,$$

$$\int_0^\infty x^\sigma(1-x^2)^{\delta-1/2}C_\nu{}^\delta(x)J_\lambda(2\alpha tx)J_\delta(2\beta tx)\, dx,$$

$$\int_0^1 x^\sigma(1-x^2)^{-\mu/2}P_\nu{}^\mu(x)\,{}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_p\end{matrix};\pm x^2t^2\right]J_\mu(2xt)\, dx,$$

where $P_\nu{}^\mu$ are the associated Legendre polynomials and $C_\nu{}^\delta$ are the Gegenbauer polynomials, are evaluated by expanding the Bessel functions in series and integrating term by term. The results are too complicated to be reproduced here.

A. E. Danese (Buffalo, N.Y.)

Scott, E. J. 4935

A formula for the derivatives of Tchebychef polynomials of the second kind.

Amer. Math. Monthly **71** (1964), 524–525.

By using properties of Gegenbauer polynomials, the author derives a formula for the derivatives of Tchebychef polynomials U_n analogous to one for Legendre polynomials conjectured by D. Đoković [Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 65–69 (1961), 5–8; MR **25** #2245] and proved by the reviewer [Amer. Math. Monthly **70** (1963), 643–644; MR **28** #263]. The reviewer has recently learned that the following formula, which includes both the Legendre and Tchebychef formulas as special cases, was proved by L. Gegenbauer [Denkschr. Akad. Wiss. Math.-Natur. Cl. Wien. **48** (1884), 293–316]:

$$[C^\nu_{n+\alpha p}(x)]^{(\alpha p)} = \frac{2^{\alpha p}\Gamma(\nu+\alpha p)}{\Gamma(\nu)}\sum C^\nu_{n_1}C^\nu_{n_2}\cdots C^\nu_{n_{\alpha q+1}},$$

where the C's are the Gegenbauer polynomials, $\nu=p/q$, α is an integer, and the sum is over all subscripts whose sum is n.

Mary L. Boas (Chicago, Ill.)

Haradze, A. K. 4936

On the representation of ultraspherical polynomials in the form of a differential operator containing the generating function of these polynomials. (Russian)

Dokl. Akad. Nauk SSSR **158** (1964), 59–61.

The following representations of ultraspherical polynomials are established

$$y^{n/2\lambda}P_n{}^{(\lambda)}(x) = P_n{}^{(\lambda)}(z)+\binom{\omega}{1}ty^{1/2\lambda}P^{(\lambda)}_{n-1}(z)+\cdots$$

$$+\binom{\omega}{n}t^ny^{n/2\lambda}P_0{}^{(\lambda)}(z),$$

$$y^{(n+\lambda)/\lambda}P_n{}^{(\lambda)}(x) = D_n{}^{(\lambda)}[y] \equiv$$

$$\frac{y^{(n)}}{n!}+\binom{\omega}{1}ty^{1/\lambda}\frac{y^{(n-1)}}{(n-1)!}+\cdots+\binom{\omega}{n}t^ny^{n/\lambda+1},$$

where $z=y^{1/2\lambda}(x-t)$, $\omega=2\lambda+n-1$, $y=(1-2xt+t^2)^{-\lambda}$ is the generating function of these polynomials. It is obvious that the function $(1-2x_it+t^2)^{-\lambda}$ is the solution of the equation $D_n{}^{(\lambda)}[y]=0$, where x_i are the roots of the polynomials $P_n{}^{(\lambda)}(x)$.

B. S. Popov (Skopje)

Haradze, A. K. 4937

On sequences of polynomials of Appel type satisfying a recurrence relation. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 963–967.

Applying the technique of Carlitz's previous paper [Portugal. Math. **20** (1961), 43–46; MR **24** #A266], the author proves the following result. Let $\{f_n(x)\}$ be a sequence of the class of Appel polynomials satisfying

$$(*)\quad f_{n+1}(x)-2(x+b_n)f_n(x)+(x+b_n)^2f_{n-1}(x)+c_nf_{n-2}(x)=0,\qquad c_n>0,\ n\geq 2,$$

where b_n, c_n are independent of x. Then the solution of (*) is

$$f_n(x)=\alpha^n\left[H_{3;n}\left(\frac{x+\beta}{\alpha}\right)+Ch_{3;n-1}\left(\frac{x+\beta}{\alpha}\right)\right],$$

where α,β are constants and C an arbitrary constant. $H_{k;n}$ and $h_{k;n}$ are generalised Hermite polynomials [Chatterjee, Bull. Calcutta Math. Soc. **47** (1955), 27–41; MR **17**, 967; the author, ibid. **52** (1960), 25–34; MR **24** #A264]. *B. S. Popov* (Skopje)

ORDINARY DIFFERENTIAL EQUATIONS

See also 4650, 4807, 4895, 4929, 4930, 5128, 5129, 5131, 5386, 5393, 5529, 5670, 5671.

★**Differential equations and their applications.** 4938

Proceedings of a Conference held in Prague, September, 1962. Scientific Editor: Ivo Babuška; reviewer: Miloš Zlámal.

Publishing House of the Czechoslovak Academy of Sciences, Prague; Academic Press, New York-London, 1963. 247 *pp.* $12.00.

The proceedings of the above Prague conference contain 21 papers which will be reviewed individually.

Dou, Alberto 4939

Linear ordinary differential systems with constant coefficients. (Spanish. English summary)

Collect. Math. **14** (1962), 261–268.

The author defines a complex-valued scalar function $u(t)\in C^\infty$, on a real interval, to have finite differential dimension if the vector space generated by u and its derivatives has finite dimension. He shows that such a u is a linear combination of products of polynomials and exponentials, and obtains solutions in finite form of the system $x'=Ax+f(t)$, where A is an $n\times n$ constant matrix and f is a vector with entries having finite differential dimension. *W. J. Coles* (Salt Lake City, Utah)

Bellman, Richard 4940

On the nonnegativity of Green's functions.

Boll. Un. Mat. Ital. (3) **18** (1963), 219–221.

The non-negativity of the solutions of certain boundary-value problems of the form $Lu=v$ is established by studying the behavior as $t\to\infty$ of the related diffusion equation $\partial u/\partial t=Lu-v$. *I. Stakgold* (Evanston, Ill.)

Levin, A. Ju. 4941

A bound for a function with monotonely distributed zeros of successive derivatives. (Russian)

Mat. Sb. (*N.S.*) **64** (**106**) (1964), 396–409.

The main result [announced without proof in Dokl. Akad. Nauk SSSR **138** (1961), 37–38; MR **23** #A3310] of the paper is the following. If a real-valued function $x(t)$ of class C^n on (a,b) satisfies the conditions $x(a_1)=x'(a_2)=\cdots=x^{(n-1)}(a_n)=0$ $(a\leq a_1\leq a_2\leq\cdots\leq a_n\leq b)$ and $|x^{(n)}(t)|\leq\mu$ $(a\leq t\leq b)$, then

$$|x(t)|\leq\mu\frac{(b-a)^n}{n\left[\frac{n-1}{2}\right]!\left[\frac{n}{2}\right]!}\qquad(a\leq t\leq b),$$

and this inequality is the best possible. The proof uses the known theorem of M. G. Kreĭn and D. Milman on extreme points of regular convex sets [Studia Math. **9** (1940), 133–138; MR **3**, 90].

The author compares his result with an inequality obtained by S. N. Bernšteĭn [Izv. Akad. Nauk SSSR Ser. Mat. **14** (1950), 381–404; MR **12**, 322] without the hypothesis that the sequence $a_1,\cdots,a_n$ is non-decreasing.

In the second part of the paper the author proves that if

$$\sum_{k=1}^{n}L_k\frac{(b-a)^k}{2^kk\left[\frac{k-1}{2}\right]!\left[\frac{k}{2}\right]!}\leq 1,$$

then every non-trivial solution of the linear differential equation $x^{(n)}+p_1(t)x^{(n-1)}+\cdots+p_n(t)x=0$ $(a\leq t\leq b)$, where $p_i(t)$ is continuous on $[a,b]$ and $|p_i(t)|\leq L_i$ $(i=1,\cdots,n)$, has at most $n-1$ zeros on $[a,b]$. Thus he obtains a sharpening of an analogous estimate of Ch. de La Vallée Poussin [J. Math. Pures Appl. (9) 8 (1929), 125–144]. *Z. Opial* (Kraków)

Bačelis, R. D.; Melamed, V. G. 4942

Solution of a limiting boundary-value problem to which the generalized Stefan problem can be reduced. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 738–745.

The following free boundary diffusion problem is shown to have a solution of the form $u=\Phi(x/\sqrt{t})$, $\xi=\alpha\sqrt{t}$:

$$c(u)\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left[\lambda(u)\frac{\partial u}{\partial x}\right],$$

with boundary conditions $u(0,t)=A\neq 0$, $u(x,0)=u(\infty,t)=B$, $AB\leq 0$, and $u=0$ at the free boundary $x=\xi(t)$. In addition, at $x=\xi$ there is a discontinuity in the flux λu_x proportional to the slope of the boundary curve. It follows that the function $\Phi(y)$, $y=x/\sqrt{t}$, satisfies a nonlinear ordinary differential equation if $y\neq\alpha$ with boundary conditions $\Phi(0)=A$, $\Phi(\infty)=B$. At $y=\alpha$, Φ' has a jump discontinuity proportional to α.

The authors prove that there is a unique choice of $\Phi'(0)$ (and therefore of α) such that the conditions of the problem are satisfied. They also give a method for constructing approximate solutions. *W. T. Kyner* (Los Angeles, Calif.)

Mil'man, V. D. 4943

A transformation operator for Sturm-Liouville differential equations in the non-selfadjoint case. (Russian)

Mat. Sb. (*N.S.*) **59** (**101**) (1962), *suppl.*, 145–164.

Consider the differential equations $u_j'' - g_j(x)u_j + \lambda^2 u_j = 0$ $(x \geq 0)$ with the initial-value problems $u_j'(0) = h_j$, $u_j(0) = 1$ $(j = 1, 2)$. Denote the solution of the problem by $v_j(x, \lambda)$ $(j = 1, 2)$. Let V be a linear operator in a function space which maps the function $v_1(\cdot, \lambda)$ into the function $v_2(\cdot, \lambda)$ for all complex λ. It is proved in the paper that under certain regularity conditions on the functions g_j, the operator admits an integral representation, and properties of the operator in certain function spaces are studied.

W. Bogdanowicz (Washington, D.C.)

Lees, Milton 4944

A boundary value problem for nonlinear ordinary differential equations.

J. Math. Mech. **10** (1961), 423–430.

From the author's introduction: "The purpose of this paper is to prove the following theorem. Let $f(x, z, p)$ be a continuous function defined in the region $0 \leq x \leq 1$, $-\infty < z, p < +\infty$. Assume that

$$(1.1) \qquad f(x, z_1, p) \geq f(x, z_2, p) \quad \text{for } z_1 \geq z_2$$

and

$$(1.2) \qquad |f(x, z, p_1) - f(x, z, p_2)| \leq K|p_1 - p_2|.$$

Then there exists a unique twice continuously differentiable function $z(x)$, defined for $0 \leq x \leq 1$, such that

$$(1.3) \qquad z(0) - z_0 = z(1) - z_1 = 0$$

and

$$(1.4) \qquad z''(x) = f(x, z(x), z'(x)) \qquad \left(' = \frac{d}{dx}\right)."$$

Bihari, I. 4945

The asymptotic behaviour of a system of nonlinear differential equations. (Russian summary)

Magyar Tud. Akad. Mat. Kutató Int. Közl. 8 (1963), 475–488 (1964).

The author discusses the behavior as $t \to +\infty$ of the solutions of nonlinear systems of the form (1) $dx/dt = Ax + f(t, x)$, $x = (x_1, \cdots, x_n)$, $f = (f_1, \cdots, f_n)$, A an $n \times n$ constant matrix. As usual, the discussion is based on the comparison of the solutions of (1) with the solutions of the linear system $dy/dt = Ay$. Here $f(t, x)$ is defined for all $t \geq 0$, $x \in E_n$, and satisfies the relation

$$|f(t, x) - f(t, y)| \leq g(t)\omega(|x - y|)$$

for all $t \geq 0$, $x, y \in E_n$, where $g(t)$ is a given measurable bounded non-negative function with $\int_0^\infty g(t)dt < +\infty$, and $\omega(u)$, $u \geq 0$, is a continuous positive nondecreasing function of u with $\omega(0) = 0$, $\int_0^1 \omega^{-1}(u)du = \infty$, $\int_1^\infty \omega^{-1}(u)du = \infty$. If, in addition, every solution of (1) is bounded for $t \geq 0$, and $\omega^{-1}(u)$ satisfies a convenient condition of growth, then for every solution $x(t)$ of (1) there is a solution $y(t)$ of (2) such that $x(t) - y(t) \to 0$ as $t \to +\infty$. This theorem extends a previous one of N. Levinson and H. Weyl [N. Levinson, Amer. J. Math. **68** (1946), 1–6; MR **7**, 381; H. Weyl, ibid. **68** (1946), 7–12; MR **7**, 382]. Also it includes a previous remark of A. Wintner [ibid. **68** (1946), 13–19; MR **7**, 297]. *L. Cesari* (Ann Arbor, Mich.)

Kolodner, I. I.; Lehner, J.; Wing, G. M. 4946

On first order ordinary differential equations arising in diffusion problems.

J. Soc. Indust. Appl. Math. **12** (1964), 249–269.

In the scalar differential equation (*) $\dot{x} = g(t, x)$ let g satisfy the following set of hypotheses (H): $g(t, x)$ is a real-valued, continuously differentiable function for $t > 0$ and $x \geq 0$; $g_t < 0$ and $g_x < 0$; $g(t, 0) > 0$; $\lim_{t\to\infty} g(t, x) > -\infty$ always exists and $|g(t, 0)|$ is integrable in a neighborhood of $t = 0$. The authors prove theorems concerning existence, uniqueness and continuability of solutions of (*), as well as several results concerning the existence of extrema, inflection points, and asymptotic behavior of these solutions. Corresponding results, under a set of hypotheses weaker than (H), are stated without proof. The authors also study, under hypotheses similar to (H), an equation $\dot{y} = g(t, y, \beta)$, and obtain results concerning the influence of the parameter β upon the location of the extrema of the solutions. *T. F. Bridgland, Jr.* (Columbia, S.C.)

Popov, G. E. 4947

On the absence of periodic solutions taking the value $x = 0$ in a class of differential equations of the form $\ddot{x} + x = F(x, \dot{x})$. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, no. 3 (40), 120–122.

The theorem proved for the absence of solutions of the type mentioned in the title is for

$$F(x, y) = F_1(x) + F_2(x) + F_3(x, y),$$

$$F_1(x) = \sum_{k=1}^{\infty} a_{2k}x^{2k}, \qquad a_{2k} \geq 0,$$

$$F_2(x) = \sum_{k=1}^{\infty} a_{2k+1}x^{2k+1},$$

$$F_3(x, y) = \sum_{m,n=0}^{\infty} a_{2m+1,2n+1}x^{2m+1}y^{2n+1},$$

and all $a_{2m+1,2n+1}$ have the same sign, F_1, F_3 not identically zero. *J. K. Hale* (Providence, R.I.)

Walter, Wolfgang 4948

Bemerkungen zu verschiedenen Eindeutigkeitskriterien für gewöhnliche Differentialgleichungen.

Math. Z. **84** (1964), 222–227.

It is shown in the paper that the uniqueness criterion for solutions of ordinary differential equations given by Krasnosel'skiĭ and Kreĭn [Uspehi Mat. Nauk (N.S.) **11** (1956), no. 1 (67), 209–213; MR **18**, 38], as well as that obtained by Brauer [Canad. J. Math. **11** (1959), 527–533; MR **21** #7319], can be reduced to Kamke's criterion [Kamke, *Differentialgleichungen reeller Funktionen*, Akademische Verlag, Leipzig, 1930].

Also, a result due to the reviewer [Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **8** (1960), 661–666; MR **24** #A2074] is reproduced and reproved in the paper.

C. Olech (Kraków)

Miller, Richard K. 4949

On almost periodic differential equations.

Bull. Amer. Math. Soc. **70** (1964), 792–795.

Consider the system (E) $x' = P(t, x)$ with P continuous on

$R^1 \times D$, D a domain in R^n, uniformly continuous on $R^1 \times D^*$ if D^* is a compact subset of D, and almost periodic in t for each fixed x. A subset A of D is called quasi-invariant with respect to (E) if for every $z \in A$ there is a $P^* \in \overline{H(P)}$, the closed hull of P, and a solution $y(t)$ of $y' = P^*(t, y)$ with $y(0) = z$ such that $y(t)$ is in a compact subset of A for all real t. It is shown that for a rather wide class of perturbed systems of (E), the positive limit set of a solution is quasi-invariant if it is compact. As corollaries of this result, the author generalizes several known stability theorems, e.g., those of Amerio [Ann. Mat. Pura Appl. (4) **39** (1955), 97–119; MR **18**, 128], LaSalle [Proc. Sympos. Appl. Math., Vol. XIII, pp. 299–307; Amer. Math. Soc., Providence, R.I., 1962; MR **25** #303], and Yoshizawa [Contributions to Differential Equations **1** (1963), 371–387; MR **26** #6487].

J. Auslander (New Haven, Conn.)

Kohanovskaja, L. P. 4950

On the accuracy of the first approximation for nonstationary linear systems. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 4 (41), 75–78.

Estimates of the accuracy of the approximate amplitude and phase obtained by the method of averaging.

J. K. Hale (Providence, R.I.)

Glatenok, I. V. 4951

On the foundations of the method of harmonic balance. (Russian. English summary)

Analytic methods in the theory of non-linear vibrations (*Proc. Internat. Sympos. Non-linear Vibrations, Vol. I*, 1961), *pp.* 181–188. *Izdat. Akad. Nauk Ukrain. SSR, Kiev*, 1963.

Author's summary: "Sufficient conditions are given for the existence of a stable periodic solution of the differential equation $y = f(y, \dot y)$ in some estimated neighbourhood of an approximate solution $y = a \sin \omega t$, as defined by the method of harmonic balance, in the following cases: (1) the function $f(y, \dot y)$ is twice continuously differentiable; (2) the function $f(y, \dot y)$ is continuously differentiable; (3) the function $f(y, \dot y)$ is continuous. These conditions present some restrictions on $f(y, \dot y)$ and are not related to its piecewise linear approximation. For example, if the function $f(y, \dot y)$ is twice continuously differentiable, they will mean essentially that the Fourier coefficients of $f(a \sin u,\ a\omega \cos u)$, $f_y'(a \sin u,\ a\omega \cos u)$ and $f_{\dot y}'(a \sin u,\ a\omega \cos u)$ must be sufficiently small. A differential equation describing processes in an automatic control system of the second order is considered when, for example, the dry friction in a sensitive link is taken account of. By means of the results obtained for continuous functions $f(y, \dot y)$ one finds for this equation a neighbourhood of the approximate solution defined by the method of harmonic balance in which the exact periodic solution is contained."

Jakubovič, V. A. 4952

Absolute stability of non-linear control systems in critical cases. III. (Russian. English summary)

Avtomat. i Telemeh. **25** (1964), 601–612.

In two recent papers [Avtomat. i Telemeh. **24** (1963), 293–303; MR **28** #311a; ibid. **24** (1963), 717–731; MR **28** #311b] the author made an extensive study of nonlinear control systems of the standard form

$$\text{(1)} \qquad \dot x = Ax + a\varphi(\sigma), \qquad \dot\sigma = b'x - \rho\varphi(\sigma),$$

under the assumption that φ is continuous, $\varphi(0) = 0$, $\sigma\varphi(\sigma) > 0$. In the present paper, many of the results are extended to the direct control case ($\rho = 0$) with φ possessing some finite jumps. The general additional restriction $0 < \sigma\varphi(\sigma) < \mu_0\sigma^2$, $\sigma \neq 0$, $\mu_0 \in (0, \infty)$, is made. The required new properties of dynamical systems are explicitly proved.

The notations are as follows: x, a, b are real n-vectors, φ and σ are scalars, A is a constant real $n \times n$ matrix. [The paper has a very extensive bibliography with 19 titles.]

S. Lefschetz (Princeton, N.J.)

Jakubovič, V. A. 4953

The method of matrix inequalities in the theory of stability of non-linear control systems. I. Absolute stability of forced vibrations. (Russian. English summary)

Avtomat. i Telemeh. **25** (1964), 1017–1029.

In this paper the author discusses a control system with a forced disturbance $f(t)$:

$$\text{(1)} \qquad \dot x = Ax + a\varphi(\sigma) + f(t),$$

where the notations are those of the preceding paper [#4952], and in addition, $f(t)$ is a bounded n-vector. Several theorems on the nature of the solutions $x(t)$ of (1) are given. We mention just the following comparatively simple theorem. Let λ_h be the characteristic roots of A with $\operatorname{Re} \lambda_h < -\alpha < 0$. If σ_1, σ_2 are any two places where $\varphi(\sigma)$ is continuous ($\sigma_1 < \sigma_2$), assume that

$$0 < (\varphi(\sigma_1) - \varphi(\sigma_2))/(\sigma_1 - \sigma_2) \leq \mu_0 \leqq +\infty.$$

Suppose, also, that for all $\omega \geqq 0$, we have

$$\mu_0^{-1} + \operatorname{Re} \chi(-\alpha + i\omega) > 0, \quad \text{where } \chi(\lambda) = b'(A - \lambda I)^{-1}a,$$

and if $\mu_0 = \infty$, let $\lim_{\omega \to \infty} \omega^2 \operatorname{Re} \chi(-\alpha + i\omega) > 0$. Then the system (1) has a unique bounded solution $x_0(t)$ on $(-\infty, +\infty)$. It is exponentially stable in the large with exponent $-\alpha$. More precisely, there exist $\beta, \varepsilon > 0$ such that for $t > t_0$ any solution $x(t)$ satisfies

$$|x(t) - x_0(t)| \leq \beta \exp(-(\alpha + \varepsilon)(t - t_0))|x(t_0) - x_0(t_0)|.$$

The numbers ε, β are independent of $\varphi(\sigma)$. If $f(t)$ is periodic or almost periodic, so is $x_0(t)$. The paper concludes with 33 references. *S. Lefschetz* (Princeton, N.J.)

Karinskiĭ, S. Ju.; Sataev, A. T. 4954

Discontinuous solutions of a slope problem. (Russian. Kazak summary)

Izv. Akad. Nauk Kazah. SSR Ser. Fiz.-Mat. Nauk **1963**, *no.* 3, 71–78.

This paper is concerned with the stability form of a slope. Assuming the tension in advance, the author considers the problem of determination of this stability in particular cases and discusses the discontinuous solutions.

B. S. Popov (Skopje)

Pavljuk, I. A. 4955

On the stability of solutions of second-order differential equations (linear and non-linear). (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 315–318.

The system $y' + Q(t)y = F(t, y)$ is considered, where y is an n-vector and $Q(t)$ is a diagonal matrix. Both cases: $F(t, y)$ is linear in y, and nonlinear but $F(t, 0) = 0$, are treated, and sufficient conditions for the stability of the trivial solution are given. These conditions are derived from the well-known lemma concerning the linear integral inequality.

C. Olech (Kraków)

Lin, C. C. 4956

Some examples of asymptotic problems in mathematical physics.

Asymptotic Solutions of Differential Equations and Their Applications (*Proc. Sympos., Math. Res. Center, U.S. Army, Univ. Wisconsin, Madison, Wis.*, 1964), *pp.* 129–143. *Wiley, New York*, 1964.

A brief historical review of asymptotic approximations to the solutions of ordinary differential equations is given. Two recent examples from the author's own work in hydrodynamics and stellar dynamics, one, a fourth-order ordinary differential equation, and the other, a system of three first-order and one second-order partial differential equations, are perfunctorily described.

I. Kay (Ann Arbor, Mich.)

Ezeilo, J. O. C. 4957

An extension of a property of the phase space trajectories of a third order differential equation.

Ann. Mat. Pura Appl. (4) **63** (1963), 387–397.

For the third-order differential equation (1) $x''' + ax'' + bx' + h(x) = p(t, x, x', x'')$, equivalent to the system (2) $x' = y$, $y' = z$, $z' = -az - by - h(x) + p(t, x, y, z)$, the following theorem is proved: Suppose that (i) a, $b > 0$, $h(x)/x > 0$ for $|x| \geq 1$, (ii) $|h(x)| \to \infty$ as $|x| \to \infty$, (iii) $h'(x)$ is continuous for all x and $h'(x) \leq \gamma < ab$ for $|x| \geq 1$, (iv) there exists a constant $\Gamma > 0$, $ab > \Gamma \geq \gamma$, such that

$$\max(0, (\Phi'(x))^2 - 4\Gamma\Phi(x)) = O(|x|^\varepsilon) \quad \text{for } |x| \to \infty$$

($\Phi(x) = xh(x)$ and $0 \leq \varepsilon < 2$), (v) the function

$$\theta(R) = \max_{t \geq 0,\ |x|,|y|,|z| \leq R} |p(t, x, y, z)|$$

satisfies the condition $\theta(R) = o(h(\pm R))$ as $R \to \infty$. Then in the space (x, y, z) there exists a bounded closed surface Σ, starlike with respect to the origin, such that all trajectories of (2) cross Σ only inwards and any trajectory of (2) ultimately enters Σ.

This improves an earlier result of the same author [J. London Math. Soc. **37** (1962), 33–41; MR **25** #283] concerning the equation (1) in which the second member does not depend on x, x', x'' and $h(x)$ is assumed to satisfy considerably more restrictive conditions.

Z. Opial (Kraków)

Greguš, Michal 4958

Über die asymptotischen Eigenschaften der Lösungen der linearen Differentialgleichung dritter Ordnung.

Ann. Mat. Pura Appl. (4) **63** (1963), 1–10.

The equation considered is $y''' + 2A(x)y' + [A'(x) + b(x)]y = 0$, with $A'(x)$ and $b(x) \geq 0$ continuous in $(-\infty, \infty)$, and with $b(x)$ not identically zero in any interval. The principal results established are the following. (1) If $A(x) \leq 0$ for $x \in (-\infty, \infty)$, there exists at least one solution $y(x)$ which does not vanish in $(-\infty, \infty)$, and which is such that y and y' are monotone, $\operatorname{sgn} y(x) = \operatorname{sgn} y''(x) \neq \operatorname{sgn} y'(x)$, and $y'(x) \to 0$ as $x \to \infty$. (2) If $A(x) \leq 0$, $A'(x) + b(x) \geq 0$ for $x \in (-\infty, \infty)$, the solution $y(x)$ described in (1) has the further properties: $y''(x)$ is monotone, and $y''(x) \to 0$ as $x \to \infty$. (3) Under the hypotheses of (2), if the equation has one oscillatory solution, all solutions are oscillatory, with the exception of one solution $y(x)$, and its constant multiples, which have the following properties: $y(x) \neq 0$ for $x \in (-\infty, \infty)$, $\operatorname{sgn} y(x) = \operatorname{sgn} y''(x) \neq \operatorname{sgn} y'(x)$, y, y', and y'' are monotone, and $y'(x) \to 0$, $y''(x) \to 0$ as $x \to \infty$. (4) Under the hypotheses of (2), let the integral $\int_{x_0}^\infty t^2(A' + b)dt$ or the integral $\int_{x_0}^\infty b\,dt$ diverge. Then there exists a solution $y(x)$ which does not vanish for $x \in (-\infty, \infty)$, and which has the following properties: y, y', and y'' are monotone, $\operatorname{sgn} y = \operatorname{sgn} y'' \neq \operatorname{sgn} y'$, and $y \to 0$, $y' \to 0$, $y'' \to 0$ as $x \to \infty$.

L. A. MacColl (New York)

Červeň, J. 4959

A sufficient condition for non-oscillation of solutions of a linear third-order differential equation. (Slovak. Russian and German summaries)

Acta Fac. Natur. Univ. Comenian. **9**, 63–70 (1964).

Author's summary: "In der Arbeit wird die lineare Differentialgleichung dritter Ordnung (A), deren Koeffizienten stetige Funktionen sind, untersucht. Die Arbeit kann man in drei Teile teilen. Im ersten Teil werden nichtoszillatorische Eigenschaften der Bündel von Lösungen bei einigen Spezialbedingungen untersucht. Im zweiten Teil wird gezeigt, daß eine gewisse Bedingung eine hinreichende Bedingung für einen nichtoszillatorischen Charakter der Lösungen der Differentialgleichung (A) ist. Im dritten Teil sind Vergleichungssätze der Lösungen von zwei linearen Differentialgleichungen dritter Ordnung angeführt."

Sibuya, Yasutaka 4960

On the problem of turning points for systems of linear ordinary differential equations of higher orders.

Asymptotic Solutions of Differential Equations and Their Applications (*Proc. Sympos., Math. Res. Center, U.S. Army, Univ. Wisconsin, Madison, Wis.*, 1964), *pp.* 145–162. *Wiley, New York*, 1964.

In the differential equation

$$(1) \qquad \varepsilon^2\{y'' + R_1(z, \varepsilon)y' + R_2(z, \varepsilon)y\} = zDy$$

let y be a two-dimensional vector function of the complex variable z and of the small complex parameter ε, where $R_k(z, \varepsilon)$, $k = 1, 2$, are two-by-two matrices holomorphic at $z = \varepsilon = 0$ and D is a constant diagonal matrix. By means of a linear transformation of y and y' the equation (1) is reduced to a simpler equation of the form

$$(2) \qquad \varepsilon^2\{u'' + C_1(\varepsilon)u' + C_2(\varepsilon)u\} = zDu.$$

The method is an adaptation of techniques originating with Langer [Trans. Amer. Math. Soc. **67** (1949), 461–490; MR **11**, 438] and developed further by the author [Funkcial. Ekvac. **4** (1962), 115–139; MR **25** #5240; Arch. Rational Mech. Anal. **13** (1963), 206–221; MR **27** #369]. The first part of the calculation consists of formal operations with series in powers of ε. The proof that these series are asymptotic expansions of functions that effect the desired transformation, even in certain closed regions that contain the turning point $z = 0$, is based on an earlier result by the author [Funkcial. Ekvac. **4** (1962), 83–113; MR **25** #5250]. The transformation so constructed is

different in different sectors of the z-plane. However, as the author proves, the various solutions of (1) obtained in this manner from one solution of (2) are asymptotically equivalent.

There remains the task of solving the simplified equation (2) in a full neighborhood of the turning point. By a skillful use of the fact that these solutions are entire functions of certain auxiliary variables, the author succeeds in determining the values at $z=0$ of a fundamental system of solutions of (2) with known asymptotic expansions for $z\neq 0$.

The last section of the paper contains a brief account of an extension of the method to systems of two third-order equations; this work was done by P. F. Hsieh in his dissertation at the Univ. Minnesota, Minneapolis, Minn., 1964. *W. Wasow* (Madison, Wis.)

Dosumov, T. B. 4961

Properties of the solutions of countable systems of differential equations and their stability. (Russian. Kazak summary)

Izv. Akad. Nauk Kazah. SSR Ser. Fiz.-Mat. Nauk **1963**, *no.* 3, 59–65.

Using a slightly weaker form of the so-called "strong Cauchy condition", the author generalizes results of K. P. Persidskiĭ [Izv. Akad. Nauk Kazah. SSR Ser. Mat. Meh. No. 7 (11) (1959), 52–71; ibid. No. 9 (13) (1961), 11–34]. *M. A. Geraghty* (Huntsville, Ala.)

Ghizzetti, Aldo 4962

Formule di maggiorazione e criteri sufficienti di stabilità per gli integrali di un'equazione differenziale lineare omogenea di ordine *n*.

Atti. Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez. I (8) 7 (1963), 15–31.

In the present paper the author extends to homogeneous linear differential equations of any order

$$(1)\qquad \sum_{k=0}^{n} p_k(t)x^{(n-k)}(t) = 0$$

the criteria of stability previously given for $n=2$ [A. Ghizzetti, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **33** (1962), 219–229; MR **27** #4996]. First, a comparison theorem is given relating the solutions of (1) with the solutions of an analogous equation with constant coefficients $\sum_{k=0}^{n} q_k y^{(n-k)}(t)=0$ and distinct (real or complex) characteristic roots $\rho_1,\cdots,\rho_n$. Then the criteria of stability of the zero solution as $t\to+\infty$ are given, or of boundedness in $[0,+\infty)$ of each solution with all its derivatives of orders $\leq n-1$. Functions $g(t)$, $t\geq 0$, are defined in terms of the coefficients $p_k(t)$, q_k and roots ρ_s, so that if $g(t)$ is bounded above, then all solutions of (1) are bounded; if $g(t)\to-\infty$ as $t\to+\infty$, then all solutions of (1) approach zero as $t\to+\infty$.

L. Cesari (Ann Arbor, Mich.)

Mitropol'skiĭ, Ju. A. 4963

An investigation of non-stationary oscillations in nonlinear systems. (Russian. English summary)

Applications of the methods of non-linear vibrations to the problems of physics and technology (Proc. Internat. Sympos. Non-linear Vibrations, Vol. III, 1961), *pp.* 241–274. *Izdat. Akad. Nauk Ukrain. SSR, Kiev,* 1963.

In this paper the author investigates nonstationary oscillatory processes in nonlinear systems with slowly varying parameters. The author takes into consideration systems, even of high order, of the perturbation type, that is, containing a small parameter ε and reducing to linear ones when $\varepsilon=0$. In these systems not only the external forces may depend on time, together with their frequencies and amplitudes, but also dependent on time are a series of other parameters of the wave system as, for example, the effective mass of the system, the characteristic frequencies and the friction coefficients. In any case, it is assumed that all these parameters vary comparatively slowly. The problem is reduced to the problems studied previously by the author and treated by the Krylov-Bogolyubov-Mitropol'skiĭ method. The problem of passing through resonance in nonlinear systems with one or more degrees of freedom is considered in detail, as well as the problem of passing through parametric resonance. A number of examples are given, taken from physics and technics (resonance phenomena in accelerators, passing through resonance in gyroscopic systems, etc.). Asymptotic expansions of the solutions for ε small are given.

L. Cesari (Ann Arbor, Mich.)

Onuchic, Nelson 4964

Nonlinear perturbation of a linear system of ordinary differential equations.

Michigan Math. J. **11** (1964), 237–242.

Let us consider the systems (1) $\dot x=A(t)x+f(t,x)$, (2) $\dot y=A(t)y$, and denote by $Y(t)$ a fundamental matrix for (2). The author discusses the following problems: (i) If $x(t)$ is a solution of (1), does there exist a constant vector b such that (3) $x(t)=Y(t)[b+O(1)]$ as $t\to+\infty$? (ii) If b is a constant vector, does there exist a solution $x(t)$ of (1) such that (3) holds? Suppose that for every $M>0$ there exists a non-negative function $h_M(t)$ with the property $\|Y^{-1}(t)f(t,Y(t)x)\|\leq h_M(t)$, for all (t,x) such that $t\geq 0$, $\|x\|\leq M$, and $\int_0^\infty h_M(t)dt<+\infty$. Theorem 1: If $x(t)$ is a solution of (1) such that $Y^{-1}(t)x(t)$ is bounded as $t\to+\infty$, then there exists a constant vector b for which (3) holds. Theorem 2: The answer to (ii) is positive. The author considers also the second-order systems $\ddot x=A(t)x+f(t,x)$ and $\ddot y=A(t)y$, where $A(t)=\mathrm{diag}(a_1(t),\cdots,a_n(t))$.

C. Corduneanu (Iaşi)

Derguzov, V. I. 4965

Sufficient conditions for the stability of Hamiltonian equations with unbounded periodic operator coefficients. (Russian)

Mat. Sb. (N.S.) **64** (**106**) (1964), 419–435.

Consider the equation (1) $J\,dx/dt=H(t)x$, where x is an element of a complex separable Hilbert space, $J=-J^*$ and J^{-1} are bounded operators, $H(t)$ is a symmetric bounded operator, $H(t+T)=H(t)$, $T>0$ a constant, $H(t)=H_0(t)+H_1(t)$, where $H_0(t)$ is positive definite and $H_1(t)$ is a perturbation term. Both $H_0(t)$ and $H_1(t)$ are subjected to some additional technical assumptions. The concept of a monodromy operator $Y(T)$ of (1) is introduced. Also, the author introduces the concept of a continuously increasing family of operators $H(t,s)$, $0\leq s\leq 1$, which implies by his Theorem 2 that the monodromy operator $Y(T,s)$ of equation (1) with $H_1(t)=H(t,s)$ is such that the spectrum of $Y(T,s)$ is an increasing function of s. The

main theorem is the following: If $H(t, s)$ is a continuously increasing family of operators such that (1) with $H_1(t) = H(t, s)$ is strongly stable for $0 \leq s \leq 1$, then (1) with $H_1(t) = \bar{H}_1(t)$ is strongly stable provided that $H(t, 0) \leq \bar{H}_1(t) \leq H(t, 1)$. This theorem is then applied to the partial differential equation

$$\partial^2 v/\partial t^2 + (EJ^1/m)\partial^4 v/\partial \xi^4 + (1/m)\partial[a(\xi, \theta t)\partial v/\partial \xi]/\partial \xi = 0$$

to obtain limitations on $a(\xi, \theta t)$ to insure stability.

J. K. Hale (Providence, R.I.)

Perov, A. I. 4966

Topological characteristics of solutions of higher-dimensional differential equations. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 791–794.

The author considers the differential equation $dy/dx = Ay$. Here y is a real n-vector in E^n, x is a real m-vector in E^m, and A is a constant linear transformation from E^n to $\mathrm{Hom}(E^m, E^n)$. Assuming certain commutation hypotheses for A, one obtains a unique solution $y(x, \eta)$ satisfying the initial conditions $y(0, \eta) = \eta$. Also $y(x, \eta)$ is continuous in $E^m \oplus E^n$ and satisfies the group property $y(x_1 + x_2, \eta) = y(x_1, y(x_2, \eta))$. Thus the solutions define an action of the vector group E^m on the space E^n. Each orbit is a homomorphic image of E^m; hence the orbits can be classified topologically.

L. Markus (Minneapolis, Minn.)

Fodčuk, V. I. 4967

The existence and properties of the integral manifold of a system of non-linear differential equations with retarded argument and with variable coefficients. (Russian)

Approximate methods of solving differential equations, pp. 129–140. *Izdat. Akad. Nauk Ukrain. SSR, Kiev*, 1963.

Consider the quasi-linear, difference-differential system

$$\begin{aligned} h'(t) &= H(t)h(t) + H_1(t)h(t-\Delta) \\ (1) \qquad &\quad + Q[t, g(t), g(t-\Delta), h(t), h(t-\Delta), \varepsilon], \\ g'(t) &= \omega(t) + P[t, g(t), h(t), \varepsilon], \end{aligned}$$

where h is a vector, g is a scalar, $H(t)$ and $H_1(t)$ are bounded matrices for all t, $\omega(t)$ is a bounded function for all t, and Q and P are small perturbation terms dependent upon a small parameter ε. For the corresponding ordinary differential system ($\Delta = 0$), Mitropol'skiĭ [Ukrain. Mat. Ž. **10** (1958), 270–279; MR **21** #176a] has given conditions for the existence of an asymptotically stable integral manifold. The present author shows that the conditions and ingenious proof, due to Bogoljubov and Mitropol'skiĭ, can be adapted to the system with a delay ($\Delta > 0$).

The author does not state his results too clearly, but apparently what he proves is the following. Assume appropriate conditions of continuity, Lipschitz continuity, smallness of P and Q when h and ε are small, and exponential asymptotic stability of the trivial solution of $h'(t) = H(t)h(t) + H_1(t)h(t-\Delta)$. Then there exists a function $f(t, g, \varepsilon)$ such that, given any (t_0, g_0), if $g(t)$ is the (unique) solution of $g'(t) = \omega(t) + P[t, g(t), f(t, g(t), \varepsilon), \varepsilon]$ with $g(t_0) = g_0$ and if $h(t) = f(t, g(t), \varepsilon)$ on $[t_0 - \Delta, t_0]$, then $g(t)$ and $h(t) = f(t, g(t), \varepsilon)$ satisfy (1) for all t. Moreover, if $g(t)$, $h(t)$ is any solution of (1) for $t > t_0$ with $h(t) = \varphi(t, \varepsilon)$ on $[t_0 - \Delta, t_0]$, where $|\varphi(t, \varepsilon)|$ is sufficiently small, then $|h(t) - f(t, g(t), \varepsilon)| \to 0$ exponentially as $t \to \infty$.

R. D. Driver (Albuquerque, N.M.)

PARTIAL DIFFERENTIAL EQUATIONS

See also 4938, 4940, 4942, 5127, 5130, 5133, 5188, 5399, 5400, 5417, 5524, 5530.

Skorobogat'ko, V. Ja. [**Скоробогатько, В. Я.**] 4968

★A study of the qualitative theory of partial differential equations [**Исследование по качественной теории дифференциальных уравнений с частными производными**].

Izdat. L'vovsk. Univ., L'vov, 1961. 125 pp. 0.30 r.

This book is concerned chiefly with the stability of solutions of a system of differential equations and with an extremal principle for certain systems of differential equations. It also contains applications to physics and mechanics.

Budak, B. M.; Samarskii, A. A. [**Samarskiĭ, A. A.**]; 4969 **Tikhonov, A. N.** [**Tihonov, A. N.**]

★A collection of problems on mathematical physics.

Translated by A. R. M. Robson; translation edited by D. M. Brink. A Pergamon Press Book.

The Macmillan Co., New York, 1964. xii+770 pp. $11.50.

The original Russian was published by GITTL, Moscow, 1956 [MR **18**, 740], and a number of problems have been omitted by the translator.

The problems are intended to accompany the book, *The equations of mathematical physics* (Russian) by A. N. Tihonov and A. A. Samarskiĭ [GITTL, Moscow, 1951; MR **15**, 430]. The problems are arranged in the following categories: Classification and reduction to canonical form of second-order partial differential equations (Chapter I); Equations of hyperbolic type (Chapter II); Equations of parabolic type (Chapter III); Equations of elliptic type (Chapter IV); Equations of parabolic type (Chapter V); Equations of hyperbolic type (Chapter VI); Equations of elliptic type $\Delta u + cu = -f$ (Chapter VII). In addition, there is a supplement devoted to the special coordinate systems of mathematical physics, together with a section devoted to special functions.

Mangeron, Demetrio 4970

Su alcune formole di media per le soluzioni analitiche di certe classi di equazioni differenziali a derivate parziali.

Boll. Un. Mat. Ital. (3) **17** (1962), 380–384.

The author establishes two mean-value formulae for analytic solutions of certain second-order partial differential equations with constant coefficients, thus extending results of M. Roşculet [Com. Acad. R. P. Romine **11** (1961), 909–914; MR **24** #A2148].

A. Dou (Madrid)

Kapilevič, M. B. 4971

Singular Goursat problems in a neighborhood of a singular characteristic at zero and at infinity. (Russian)

Dokl. Akad. Nauk SSSR **137** (1961), 1287–1290.

The author considers the equation

$$(1) \qquad L(z, B, C) = xz_{xy} + A(x)z_x + B(x)z_y + C(x)z = 0$$

in the domain D $(0 \leq x \leq x_0,\ 0 \leq y \leq y_0)$ where $A(x) > 0$ and $B(x)$, $C(x)$ are C^2 functions in X $(0 \leq x \leq x_0)$. Let $z(x, y, B, C)$ denote a twice continuously differentiable solution of (1)

in D, for which (2) $z(0,y)=f(y)$, $z(x,0)=0$, $f(0)=0$. As in an earlier paper [same Dokl. **130** (1960), 487–490; MR **22** #11208], the author assumes that $f(y)\in C_y^q$ on Y $(0\le y\le y_0)$ on the y-axis.

A typical result announced by the author is the following. Let $z(x,y,b+n+1)$ be a solution of the problem $z(0,y)=(-a)^{-n-1}(b)_{n+1}f^{(n+1)}(y)$, $z(x,0)=0$ for the equation $L(z,b+n+1)=0$, where $f(y)\in C_{n+3}^{n+1}$. Then

$$z(b)=\sum_{k=0}^{n}\frac{(b)_k}{k!}\left(-\frac{x}{a}\right)^k f^{(k)}(y)+\frac{1}{n!}\int_0^x (x-\xi)^n z(\xi,y,b+n+1)\,d\xi.$$

A. J. Lohwater (Providence, R.I.)

Beals, Richard 4972

Nonlocal elliptic boundary value problems.

Bull. Amer. Math. Soc. **70** (1964), 693–696.

Let A be a properly elliptic operator of order $2p$ in a region $G\subseteq E^n$, and let B_j be a differential operator of order j defined on the boundary ∂G of G, $0\le j<2p$. The boundary-value problems considered are of the form

$$(1)\qquad Au=f \quad \text{in } G,$$

$$(2)\qquad B_k u=\sum_{j\in J} C_{kj}B_j u \quad \text{on } \partial G,\qquad k\in K,$$

where J and K are complementary subsets of $\{0,1,\cdots,2p-1\}$ and (C_{kj}) are arbitrary operators bounded in a certain sense. The operator $A(C)$ is defined as the restriction of A to those distributions u such that $Au\in L^2(G)$, such that distribution derivatives of orders $<2p$ have L^2 boundary values on ∂G, and such that (2) holds. Sufficient conditions are obtained for $A(C)$ to be closed and have its domain in $W^{2p,2}(G)$. The formal adjoint $A'(C')$ of $A(C)$ is defined, and sufficient conditions are given for them to be true adjoints. Information about the spectrum of $A(C)$ is given for second-order operators. The paper gives generalizations of some results of Bade and Freeman [Pacific J. Math. **12** (1962), 395–410; MR **29** #2421].

M. Schechter (New York)

Ladyženskaja, O. A.; Rivkind, V. Ja.; Ural'ceva, N. N. 4973

Classical solvability of diffraction problems for equations of elliptic and parabolic types. (Russian)

Dokl. Akad. Nauk SSSR **158** (1964), 513–515.

Les auteurs énoncent deux théorèmes sur la régularité de la solution du problème de transmission pour l'opérateur elliptique et parabolique du deuxième ordre sous des hypothèses assez faibles pour les coefficients, le domaine et les données. La solution en question est höldérienne jusqu'à la frontière; c'est le cas pour les premières dérivées dans les domaines de continuité des coefficients et du second membre à l'exception eventuelle des points de contact des surfaces de discontinuité des coefficients.

J. Nečas (Prague)

Tutschke, Wolfgang 4974

Über periodische Lösungen nicht notwendig selbstadjungierter elliptischer Differentialgleichungssysteme in mehrfach-zusammenhängenden Gebieten.

Math. Nachr. **26** (1963/64), 247–286.

The author discusses the system

$$\frac{\partial u^*}{\partial x_2}=a_{11}(x_1,x_2)\frac{\partial u}{\partial x_1}+a_{12}(x_1,x_2)\frac{\partial u}{\partial x_2}+\frac{\partial z}{\partial x_2}u,$$

$$-\frac{\partial u^*}{\partial x_1}=a_{21}(x_1,x_2)\frac{\partial u}{\partial x_1}+a_{22}(x_1,x_2)\frac{\partial u}{\partial x_2}-\frac{\partial z}{\partial x_1}u,$$

where z has additive periods along the boundary curves and u^* has only additive periods. In a multiply connected domain there corresponds to the differential operator

$$\sum_{i,j}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right)+\sum_k e_k\frac{\partial u}{\partial x_k}=0$$

an integrating factor μ with constant boundary values so that $\mu\cdot L\equiv\sum_{i,j}\partial(\hat a_{ij}\,\partial/\partial x_j)/\partial x_i$ can be reduced to Bianchi normal form. μ is different from zero in all of $\bar G$ and is uniquely determined up to a constant factor.

M. Coroi-Nedelcu (Bucharest)

Baiocchi, Claudio 4975

Su alcuni spazi di distribuzioni e sul problema di Dirichlet per le equazioni lineari ellittiche.

Ricerche Mat. **13** (1964), 3–29.

L'auteur généralise à des espaces de type de Sobolev, dénotés par $H^{k,r,p}(R_+{}^n)$, des résultats obtenus par Lions, Magenes et Peetre relatifs aux solutions du problème de Dirichlet pour les équations du type elliptique d'un ordre quelconque

$$A(u)=f\quad \text{sur } R_+{}^n;\qquad \frac{\partial^l u}{\partial x_n{}^l}=g_l\quad \text{pour } x_n=0,$$

où $R_+{}^n$ est le demi-espace de $x\equiv(x_1,x_2,\cdots,x_n)$ avec $x_n>0$.

M. Coroi-Nedelcu (Bucharest)

Lin', Czun-či [Lin, Chün Chi] 4976

Perturbation of solutions and perturbation of eigenvalues and eigenfunctions of second-order elliptic equations under perturbation of the boundary. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 784–787.

L'auteur considère Ω_ε, les domaines plans, à frontière Γ_ε, aux cartes polaires sous la forme $\rho=1+\varepsilon\alpha(\varphi)$, où α est une fonction analytique à periode 2π et le problème

$$(1)\qquad L_\varepsilon u_\varepsilon\equiv\Delta u_\varepsilon+a\frac{\partial u_\varepsilon}{\partial x}+b\frac{\partial u_\varepsilon}{\partial y}+cu_\varepsilon=f$$

avec a,b,c,f des fonctions analytiques dans Ω_ε et avec la condition aux limites

$$(2)\qquad u_\varepsilon|_{\Gamma_\varepsilon}=0.$$

Si le problème (1), (2) pour $\varepsilon=0$, $f\equiv0$ n'a qu'une solution triviale, la solution de (1), (2) se trouve sous la forme $u=u_0+\varepsilon u_1+\cdots+\varepsilon^n u_n+\cdots$, où $L_0u_0=f$, $u_0|_{\Gamma_0}=0$, $L_0u_n=0$, $u_n|_{\Gamma_0}=-\sum_{l=1}^n(\alpha^l/l!)\,D^{(l)}u_{n-l}$, $n=1,2,\cdots$, et $D^{(l)}$ est la dérivée selon la normale.

L'auteur considère encore le cas, où (1), (2) a une solution propre. Puis, on cherche des valeurs et fonctions propres, à savoir u_ε, λ_ε de sorte que $L_\varepsilon u_\varepsilon-\lambda_\varepsilon u_\varepsilon=0$, $u_\varepsilon|_{\Gamma_\varepsilon}=0$. Si λ_0 est simple, λ_ε, u_ε s'expriment sous la forme $\lambda_\varepsilon=\sum_{n=0}^\infty\varepsilon^n\lambda_n$, $u_\varepsilon=\sum_{n=0}^\infty u_n\varepsilon^n$ avec

$$L_0u_0-\lambda_0u_0=0,\qquad u_0|_{\Gamma_0}=0,$$

$$L_0u_n - \lambda_0 u_n = \sum_{k=1}^{n} \lambda_k u_{n-k}, \quad u_n|_{\Gamma_s} = -\sum_{l=1}^{n} (\alpha^l/l!)D^{(l)}u_{n-l},$$

$n = 1, 2, \cdots$; les fonctions $u_0, u_1, \cdots$ sont d'une certaine manière normées. *J. Nečas* (Prague)

Pulvirenti, Giuseppe 4977
Sui problemi ai limiti per l'operatore di Laplace iterato.
Matematiche (*Catania*) **17** (1962), 137–161.

Sia l'equazione

$$\Delta^2 u - \lambda u = f \quad \text{in } \Omega$$

verificante una delle seguenti coppie di condizioni

$$\text{(I)} \begin{cases} u = h_0 & \text{su } \Gamma, \\ \Delta u = g_0 & \text{su } \Gamma, \end{cases} \qquad \text{(II)} \begin{cases} u = h_0 & \text{su } \Gamma, \\ \dfrac{\partial \Delta u}{\partial \nu} = g_1 & \text{su } \Gamma, \end{cases}$$

$$\text{(III)} \begin{cases} \dfrac{\partial u}{\partial \nu} = h_1 & \text{su } \Gamma, \\ \Delta u = g_0 & \text{su } \Gamma, \end{cases} \qquad \text{(IV)} \begin{cases} \dfrac{\partial u}{\partial \nu} = h_1 & \text{su } \Gamma, \\ \dfrac{\partial \Delta u}{\partial \nu} = g_1 & \text{su } \Gamma, \end{cases}$$

ove Ω e un aperto limitato sufficientemente regolare di R^n, Γ la sua frontiera, ν la normale interna a Γ, λ una costante positiva.

L'oggetto del presente lavoro e di cercare le soluzioni negli spazi $D^{s,r}_{\Delta,\Delta^2}(\Omega)$ delle $u \in H^s(\Omega)$ tali che $\Delta u \in H^r(\Omega)$ e $\Delta^2 u \in L^2(\Omega)$ oppure negli spazi $D^{s,r}_{\Delta,\Delta^2+\lambda}(\Omega)$ delle $u \in H^s(\Omega)$ tali che $\Delta u \in H^r(\Omega)$ e $\Delta^2 u + \lambda u \in L^2(\Omega)$ già studiati da S. Agmon, A. Douglas, L. Nirenberg, J. L. Lions, e E. Magenes. In questo caso si possono fissare i dati Γ anche in coppie di spazi di distribuzioni che invece non risultano dalle tracce delle u di $H^s(\Omega)$ e di $D_{\Delta^2+\lambda}(\Omega)$ (in cui non si forma ipotesi direttamente su $\Delta(u)$.

Per s interno ≥ 0, $H^s(\Omega)$ e lo spazio delle funzioni $u \in L^2(\Omega)$ tali che $D^k u \in L^2(\Omega)$ per $|k| \leq 1$, munito del prodotto scalare $\sum (D^k u, D^k v)$, la derivazione essendo intesa nel senso delle distribuzioni.
M. Coroi-Nedelcu (Bucharest)

Pulvirenti, Giuseppe 4978
Ancora sui problemi ai limiti per l'operatore di Laplace iterato.
Matematiche (*Catania*) **18** (1963), 108–116.

A study is made of the solution of

$$\Delta^2 u + \lambda u = f \quad \text{in } \Omega,$$
$$\Delta u = g_0 \quad \text{on } \Gamma,$$
$$\gamma_1(\Delta u) = g_1 \quad \text{on } \Gamma,$$

where Ω is a bounded domain in R^n with C^∞ boundary Γ, γ_1 is the normal derivative on Γ. This boundary-value problem is not coercive, since the boundary conditions do not cover Δ^2. Therefore the usual results for such problems do not hold. It is shown that if $f \in L^2(\Omega)$, $g_0 \in H^{r-1/2}(\Gamma)$, $g_1 \in H^{r-3/2}(\Gamma)$ for $-2 \leq r < 2$, there does not always exist a solution $u \in L^2(\Omega)$ such that $\Delta u \in H^r(\Omega)$. Necessary and sufficient conditions for existence are given. On the other hand, if $f \in H^{r-2}(\Omega)$, $g_0 \in H^{r-1/2}(\Gamma)$, $g_1 \in H^{r-3/2}(\Gamma)$ for $r \geq 2$, then there exists a unique solution $u \in H^{r-2}(\Omega)$ such that $\Delta u \in H^r(\Omega)$. *M. Schechter* (New York)

Redheffer, R. M.; Straus, E. G. 4979
Degenerate elliptic equations.
Pacific J. Math. **14** (1964), 265–268.

The authors obtain the following main result (Theorem 2). Let u be differentiable in B, a region in E^n, and have continuous second derivatives except on the union of countably many smooth surfaces. (Here, a smooth surface means the set of zeros of a C^2 function.) If the conditions

(1) $\quad \sum a_{ij}(x)u_{ij}(x) \geq 0, \qquad (a_{ij}) \geq 0, \ (a_{ij}) \neq 0,$

hold on a dense subset of B, then u satisfies the maximum principle, i.e., if E is any compact subset of B, we have

$$\sup_{x \in E} u(x) \leq \sup_{x \in \partial E} u(x).$$

The whole problem here, of course, is that u might conceivably have a local maximum at a singular point, i.e., a point where either u fails to have continuous second derivatives or where (1) fails to hold. The authors show that in that case a C^2 function θ may be constructed such that $(\theta_{ij}) > 0$, and such that the function $\tilde{u} = u + \theta$ must then have a local maximum at a non-singular point, thereby violating the classical maximum principle.
Stanley Kaplan (Pisa)

Vekilov, Š. I. 4980
An application of potential theory to the solution of Dirichlet problems for an elliptic equation in a domain with corners. (Russian. Azerbaijani summary)
Izv. Akad. Nauk Azerbaĭdžan. SSR Ser. Fiz.-Mat. Tehn. Nauk **1963**, *no.* 6, 11–16.

L'auteur résout le problème de Dirichlet classique pour l'opérateur elliptique du deuxième ordre

$$\sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right)$$

à coefficients assez réguliers dans un domaine borné à trois dimensions dont la frontière Γ satisfait à la condition de Ljapunov sauf sur une courbe $L \subset \Gamma$, assez régulière, formant une arête angulaire. La solution se cherche en potentiel d'une double couche. Le problème est ainsi réduit à l'équation intégrale $\mu(x) + (1/2\pi)\int_\Gamma Q(x,\xi)\mu(\xi)d\Gamma_\xi = f(x)$. Le noyau $Q(x,\xi) = G(x,\xi) + G_1(x,\xi) + R_1(x,\xi)$ avec R_1 borné, $\int_\Gamma |G_1(x,\xi)|d\Gamma_\xi$ arbitrairement petit, $\int_\Gamma |G(x,\xi)|d\Gamma_\xi \leq \kappa < 1$. On peut alors appliquer l'alternative de Fredholm et obtenir par la méthode de T. Carleman la solution unique. On considère encore l'opérateur elliptique général du deuxième ordre. *J. Nečas* (Prague)

Mihaĭlov, V. P. 4981
The first boundary-value problem for certain semi-bounded hypo-elliptic equations. (Russian)
Mat. Sb. (*N.S.*) **64 (106)** (1964), 10–51.

Consider the differential equation

(1) $\quad P(u) = \sum_{(\alpha)=1} A^\alpha(x)D^\alpha u + \sum_{(\alpha)<1} A^\alpha(x)D^\alpha u = f(x),$

where $x = (x_0, x_1, \cdots, x_n)$ varies in a closed domain $\bar{Q}$, $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_n)$, $(\alpha) = \alpha_0/(2m_0+1) + \sum_{i=1}^n (\alpha_i/2m_i)$, and assume that $\operatorname{Re} \sum_{(\alpha)=1} A^\alpha(x)(i\xi)^\alpha \geq \theta^2(\xi_1^{2m_1} + \cdots + \xi_n^{2m_n})$ $(\theta^2 > 0)$, $A^{(2m_0+1,0,\cdots,0)}(x) \neq 0$ in $\bar{Q}$. Assume also that $2m_0 + 1 < 2m = 2\max_{1 \leq i \leq n} m_i$, that the boundary Γ of Q consists of portions Γ_i $(i = 0, \cdots, n)$ lying on $x_i =$ const and

of a complementary set Γ, and that at two points a, b of Γ the tangent (to Γ) is of the form $x_0 = \text{const}$, and one of the cylinders $|x_i - x_i^c| = C_0|x_0 - x_0^c|^p$ is contained in a Γ-neighborhood of c for $c = a$, $c = b$, where p is any number $> (2m_0+1)/2m$. Together with (1) consider the boundary conditions (2) u vanishes on $\hat{\Gamma}$ together with all its first $m-1$ derivatives, (3) $\partial^j u/\partial x_i^j = 0$ on Γ_i for $j = 0, 1, \cdots, m_i - 1$ and $i = 0, 1, \cdots, n$. For the special case $m_0 = 1$, $m_1 = \cdots = m_n$, (1) is parabolic and the existence and uniqueness of a solution of (1)–(3) were established by the author in previous works [see, for example, same Sb. (N.S.) **63** (105) (1964), 238–264; MR **29** #369]. In the present paper the author establishes analogous results for (1)–(3). He employs an extended Gårding inequality to reduce the problem (1)–(3) (or rather, a weaker version of it) to a Fredholm alternative and then uses the conditions on Γ to establish uniqueness (which implies existence). Equations other than (1) are also treated by the same method. *A. Friedman* (Evanston, Ill.)

Èidus, D. M. 4982

On the limiting amplitude principle. (Russian)

Dokl. Akad. Nauk SSSR **158** (1964), 794–797.

Let H be a Hilbert space, G a self-adjoint operator in H, $G > 0$, R_z the resolvent of G, and H_0 a subspace of H with associated orthogonal projection P. Let $w(t)$ take values in H with $d^2w/dt^2 + Gw = f\exp(-i\sqrt{(\lambda t)})$, $w|_{t=0} = 0$, and $dw/dt|_{t=0} = 0$, for $\lambda > 0$, $f \in H$, where w' is continuous on $0 \leq t < \infty$ and w'' on $0 < t < \infty$. Suppose for any $\sigma > 0$, $\lim PR_z f = v^+(\sigma)$ exists for $z \to \sigma$ when $\operatorname{Im} z > 0$ and $\lim PR_z f = v^-(\sigma)$ exists for $z \to \sigma$ when $\operatorname{Im} z < 0$, and suppose the function $\theta(\sigma) = v^+(\sigma) - v^-(\sigma)$ satisfies a Hölder condition on any interval $a \leq \sigma \leq b$, $a > 0$. Then, for example, if $\|\theta(\sigma)\| \leq c\sigma^{-\alpha}$ for $0 < \sigma \leq \delta$, $0 \leq \alpha < \frac{1}{2}$, it follows that $\exp(i\sqrt{(\lambda t)})Pw \to v^+(\lambda)$ as $t \to \infty$ (for any $\lambda > 0$). Various other limiting results as $t \to \infty$ for Pw, Pw', and Pw'' are also obtained under various hypotheses on θ.

Next the author considers the equation $\partial^2 w/\partial t^2 + gw = \exp(-i\sqrt{(\lambda t)})f(x)$ in an unbounded region $\Omega \subset \mathbf{R}^n$ under zero initial and homogeneous boundary conditions. Here $\lambda > 0$ and g is an elliptic differential operator of order $2m$. One says that the principle of limiting amplitudes holds for this equation if $\lim \exp(i\sqrt{(\lambda t)})w = v$, where v is the solution of $gv = \lambda v + f$. Various aspects of this problem are then studied under suitable hypotheses, and a number of interesting results are stated. *R. Carroll* (Urbana, Ill.)

Huet, Denise 4983

Perturbations singulières.

C. R. Acad. Sci. Paris **258** (1964), 6320–6322.

Let u_ε be the solution of a boundary-value problem of the form

$$-\varepsilon Au + u = h_\varepsilon \quad \text{in } \Omega,$$

$$B_j u = g_j \quad \text{on } \Gamma, \qquad 1 \leq j \leq m,$$

where Ω is a bounded domain in R^n with smooth boundary Γ, A is a properly elliptic operator of order $2m$, and $\{B_j\}$ is a normal set of differential boundary operators which not only covers A but satisfies an extended condition of Agmon [Comm. Pure Appl. Math. **15** (1962), 119–147; MR **26** #5288], which guarantees that the positive real axis is a ray of minimal growth of the resolvent in $L^p(\Omega)$ for the corresponding homogeneous problem. It is shown that if $h_\varepsilon \to h$ in $L^p(\Omega)$, $1 < p < \infty$, then $u_\varepsilon \to h$ in $L^p(\Omega)$, $\varepsilon u_\varepsilon \to 0$ in $W_p^{2m}(\Omega)$ as $\varepsilon \to 0$ whenever the g_j are in a suitable space. Moreover, for $r \geq 0$ ($r - 1/p$ not an integer for $p \neq 2$), if $h_\varepsilon \to h$ in $W_p^r(\Omega)$, then $u_\varepsilon \to h$ in $W_p^r(\mathfrak{A})$ and $\varepsilon u_\varepsilon \to 0$ in $W_p^{2m+r}(\mathfrak{A})$ as $\varepsilon \to 0$ for any compact subset $\mathfrak{A}$ of Ω.

M. Schechter (New York)

Jakut, L. I. 4984

Theorems of Lax for non-linear evolutionary equations. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 1304–1307.

Consider the nonlinear equation (*) $du/dt + A(t)u = \varphi(t, u)$ ($t \in \langle 0, T\rangle$) with the initial-value problem $u(0) = u_0$. Let $A(t)$, for every $t \in \langle 0, T\rangle$, be a closed linear operator from its domain $D(A) \subset E$ into a Banach space E. Assume that $D(A)$ does not depend on t, is dense in E, and, for every λ such that $\operatorname{Re}\lambda \geq 0$, the resolvent $(A(t)+\lambda I)^{-1}$ exists and $\|(A(t)+\lambda I)^{-1}\|_E \leq C/(|\lambda|+1)$ for $t \in \langle 0, T\rangle$.

Assume that there are given two Banach spaces E_1 and E_2 such that $E_2 \subset E_1 \subset E$, $D(A^\alpha(t)) \subset E_2$ and $\|v\|_{E_2} \leq C_1\|A^\alpha(t)v\|_E$ for some $\alpha > 0$ ($v \in D(A^\alpha(t))$). Let there exist a sequence of linear continuous operators $A_n(t)$ from the space E_2 into itself such that

$$\sup\{\|(A_n(t) - A(t))A^{-1}(t)v\|_{E_1} : t \in \langle 0, T\rangle\} \leq \rho_n\|v\|_{E_2}$$

for $v \in E_2$, where $\rho_n \to 0$. Assume that $L_n = \{v \in E_2 : A_n(t)v = 0\}$ does not depend on t. Assume that there exists a seminorm $\|v\|_n$ on the space E_2 such that (1) $\|v\|_n = 0$ if and only if $v \in L_n$; (2) $\|v\|_n \leq C_2\|v\|_{E_1}$ for $n = 1, 2, \cdots$ and $v \in E_2$; (3) $\|v\|_n \leq \|v\|_E + \varepsilon_n\|v\|_{E_2}$ ($\varepsilon_n \to 0$). Let S_n be the quotient space E_2/L_n with the norm $\|\bar{v}\|_{S_n} = \inf\{\|v\|_{E_1} : v \in \bar{v}\}$, where $\bar{v}$ is the coset containing v, and (4) let $\|\bar{v}\|_{S_n} \leq (1/\gamma_n)\|\bar{v}\|_n$ for $\bar{v} \in S_n$, where $\gamma_n \to 0$. Define an operator $\overline{A(t)}$ in the space S_n by the formula $\overline{A_n(t)}\bar{v} = \overline{A_n(t)v}$. Let $\varphi(t, v)$ be such that, for every fixed t, the set $\{\varphi(t, v) : v, \bar{v} \in S_n, \|v\|_{E_2} < R_2\}$ is contained in a coset of the quotient space $E_2/L_n = S_n$.

Consider the finite-difference equation corresponding to the equation $(\bar{v}_{k+1} - \bar{v}_k)/\Delta_n t + \overline{A_n(k\Delta_n t)}\bar{v}_k = \bar{\varphi}_k$, $\bar{v}_0 = \bar{u}_0$, where $\bar{\varphi}_k = \overline{\varphi(k\Delta_n t, v_k)}$ and $v_k \in \bar{v}_k$. Theorem: Let $u(t)$ be a solution of the equation (*). Assume $u_0 \in D(A^{1+\gamma_2}(0))$, $\varphi(0, u_0) \in D(A^{\gamma_2}(0))$; let $\varphi(t, u(t))$ be bounded in the norm of the space E_2 and continuously differentiable in the space E. Let $\|\varphi(t, v) - \varphi(t, w)\|_n \leq C\|v - w\|_n$ for $t \in \langle 0, T\rangle$ and $v, w \in K$, where $K = \{v \in E_1 : \|v\|_{E_1} \leq R\}$ and $K \ni u(t)$ for all $t \in \langle 0, T\rangle$. Let $A(t)A^{-1}(0)$ be strongly continuously differentiable in E and $\|I - \Delta_n t\bar{A}_n(t)\|_n \leq 1 + C\Delta_n t$ for $t \in \langle 0, T\rangle$. If $\varepsilon_n \Delta_n t^{\gamma_2 - \gamma_1} = O(\gamma_n)$, $\rho_n = O(\gamma_n)$, $\Delta_n t^{\gamma_2} = O(\gamma_n)$ ($0 < \alpha = \gamma_1 < \gamma_2 < 1$), then the solution of the finite-difference equation $v_n(t)$ converges in the norm of S_n to $u(t)$ uniformly on the interval $\langle 0, T\rangle$.

Similar results are announced for the equation of the form $du/dt + B(t, u)u = 0$. An application to quasi-elliptic equations is given. *W. Bogdanowicz* (Washington, D.C.)

Kostjučenko, A. G. 4985

The asymptotic distribution of eigenvalues of elliptic operators. (Russian)

Dokl. Akad. Nauk SSSR **158** (1964), 41–44.

The principal result announced concerns the distribution of the eigenvalues of

$$(1)\quad L=(-1)^m\sum_{k_1+\cdots+k_n=2m}A^{k_1,\cdots,k_n}(x)\frac{\partial^{2m}}{\partial x_1^{k_1}\cdots\partial x_n^{k_n}}+L_1\left(x,\frac{\partial}{\partial x}\right)+q(x)$$
$$=L_0+L_1+q(x),$$

where $x=(x_1,\cdots,x_n)$, $-\infty<x_i<\infty$. L_0 is uniformly elliptic in R_n, with uniformly Hölder-continuous coefficients; L_1 is of order $<2m$ with bounded coefficients; and as $|x|\to\infty$, $q(x)\to\infty$ restricted by a number of conditions. If $N(\lambda)$ denotes the number of eigenvalues of L less than λ, then

$$(2)\quad N(\lambda)\sim\frac{1}{(2\pi)^n\Gamma\left(\dfrac{n}{2m}+1\right)}\int_{q(x)<\lambda}\Phi(x)(\lambda-q(x))^{n/2m}\,dx,$$

where $\Phi(x)=\int_{R_n}e^{-L_0(x,s)}\,ds$.

The proof outlined starts with the parabolic equation $\partial v/\partial t=-Lv$ and proceeds to estimates on the fundamental solution $G(x,y,t)$ via the parametrix method. The identity $\int_{R_n}G(x,x,t)\,dx=\int_0^\infty e^{-\lambda t}dN(\lambda)$ and a Tauberian theorem are then used to obtain (2).

{Line 7, p. 42, should read "$(\det\|a_{ij}(x)\|)^{-1/2}$" and in line 15, p. 43, a term $L_1(y,\partial/\partial x)$ should be added inside the brackets.} *R. E. L. Turner* (Madison, Wis.)

Mangeron, D.; Krivošein, L. E. 4986

Problemi di Goursat e di Dirichlet per una classe di equazioni integro-differenziali a derivate totali. (English summary)

Rend. Accad. Sci. Fis. Mat. Napoli (4) **28** (1961), 213–224.

Authors' summary: "Some theorems are presented concerning the problems of Goursat and Dirichlet and for the class of integro-differential equations with total derivatives, where, analogously with the novel aspects of the theorems established by one of the authors [cf., e.g., D. Mangeron, Mathematica (Cluj) **14** (1938), 31–35] for the boundary-value problems $[A(x)u'+\lambda B(x)u]'+\lambda[B(x)u'+C'(x)u]=0$, $u|_{FR}=0$, $R(x_i'\leq x_i\leq x_i'')$, the symbol $'$ designates total differentiation in the sense of Picone: $u'=\partial^{2n}u/\partial x_1\partial x_2\cdots\partial x_{2n}$."

Ossicini, A. 4987

Sugli integrali doppi di espressioni lineari alle derivate parziali del 4° ordine.

Rend. Mat. e Appl. (5) **21** (1962), 100–124.

In this paper the author characterizes the operators

$$E=\sum_{s=0}^{4}\sum_{h+k=s}a_{hk}(x,y)\frac{\partial^s}{\partial x^h\partial y^k}$$

and the domains T of the plane such that the double integral

$$\Phi(u)=\iint_T E(u)\,dx\,dy$$

depends only on the values of u and its partial derivatives up to the second order at the angular points of the boundary $\mathcal{F}T$ of T. Such functionals are called point-functionals. A. Ghizzetti solved the same problem for operators of second and third order [Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **22** (1957), 276–281; MR **19**, 1061; ibid. (8) **22** (1957), 430–433; MR **19**, 1061]. The author proves first that the boundary $\mathcal{F}T$ must be composed only of arcs of characteristic curves of the operator E. Therefore the author assumes that T is bounded by characteristic curves. He considers four cases in which the four systems of characteristic curves are real, with at least two of them different; consequently, he sets $a_{40}=a_{04}=0$. Then he gives necessary and sufficient conditions (one on T and three on $\mathcal{F}T$) in order that Φ be a point-functional. If $\mathcal{F}T$ consists only of parallels to the axes, then the author proves that a linear partial differential operator of second order F exists such that

$$\Phi(u)=\int_{x_0}^{x_1}\int_{y_0}^{y_1}\frac{\partial^2F(u)}{\partial x\partial y}\,dx\,dy=$$
$$[F(u)]_{P_0}-[F(u)]_{P_1}+[F(u)]_{P_2}-[F(u)]_{P_3},$$

where $P_0(x_0,y_0)$, $P_1(x_1,y_0)$, $P_2(x_1,y_1)$, $P_3(x_0,y_1)$ are the angular points of $\mathcal{F}T$. The author considers also the subcases in which $\mathcal{F}T$ consists of arcs belonging to three or four different systems of characteristic curves. Thus the arbitrariness of the coefficients of $F(u)$ is greatly reduced. In all cases typical examples are given. The method makes use of the formulae

$$E(u)=\frac{\partial A(u)}{\partial x}+\frac{\partial B(u)}{\partial y}+R\cdot u,$$

$$\Phi(u)=\sum_{i=1}^{q}\int_{\gamma_i}\left[\sum_{j=0}^{3}\sum_{h+k=j}C_{hk}\frac{\partial^j u}{\partial s^h\partial n^k}\right]ds+\iint_T R\cdot u\,dx\,dy,$$

with A, B, R, C_{hk} conveniently defined and where s is the arc, n the normal and q the number of regular arcs γ_i of the boundary $\mathcal{F}T$. *A. Dou* (Madrid)

Šilov, G. E. 4988

Well-posed boundary-value problems in a half-space for linear partial differential equations with constant coefficients. (Russian)

Uspehi Mat. Nauk **19** (1964), no. 3 (117), 3–52.

This is primarily an expository paper. Consider the equation (1) $P(D_t,D_x)u=0$ for $x\in R^n$, $t>0$, where $D_t=\partial/\partial t$, D_x is any partial x-derivative and assume that $P(\lambda,\sigma)$ is a polynomial in λ,σ of degree m in λ, and $P(\lambda,\sigma)=\lambda^m+$lower order terms in λ. The Cauchy problem for (1) consists of finding a solution subject to initial conditions of the form $\partial^j u(x,0)/\partial t^j=\varphi_j(x)$ $(j=0,\cdots,m-1)$. This problem always has a unique solution provided the φ_j and u belong to suitable classes of functions (depending on P). The purpose of the present paper is to prove (under suitable initial conditions) existence and uniqueness of solutions in $0<t<\infty$ under the following requirements: (a) For each $t>0$, $u(x,t)$ belongs to a space $\mathcal{K}$ (independent of P), (b) $D_t^ju(x,t)$ $(j=0,\cdots,m-1)$ are bounded in $\mathcal{K}$ by $O(t^h)$ as $t\to\infty$, where h may depend on u. It should be noted that in the Cauchy problem the solution may not exist beyond some finite interval $0\leq t\leq T$. $\mathcal{K}$ is defined as the Fourier transform of H, where H is the union space

of spaces H_j $(j=1, 2, \cdots)$ and H_j consists of all the integrable functions $g(x)$ with finite norm

$$|g|_{j} = \left\{ \int_{E^n} |g(x)|^2 (1+|x|^2)^{-j} \, dx \right\}^{1/2}.$$

The main result is the following. Let $\operatorname{Re} \lambda_0(\sigma) \leq \cdots \leq \operatorname{Re} \lambda_{m-1}(\sigma)$, where $\lambda_j(\sigma)$ are the zeros of $P(\lambda, \sigma)=0$, and let A_j consist of the points σ for which $\operatorname{Re} \lambda_{j-1}(\sigma) \leq 0 < \operatorname{Re} \lambda_j(\sigma)$. Assume that on each A_j $(j=0, \cdots, m-1)$ there is given a function $\nu_j(\sigma)$ which can be extended to an element of H. Then there exists a unique solution of (1) which satisfies (a), (b) and the following conditions: The Fourier transform of $\partial^j u(x, 0)/\partial t^j$ $(j=0, \cdots, m-1)$ coincides with $\nu_j(\sigma)$ on A_j. If the $\operatorname{Re} \lambda_j(\sigma)$ do not change sign, then the solution can be represented in terms of fundamental solutions. Explicit formulas for these fundamental solutions are given in the case of homogeneous equations.

A. Friedman (Evanston, Ill.)

Adler, G. 4989

Majoration du gradient des solutions de l'équation $\Delta u - a u_t' = f$. II.

Acta Math. Acad. Sci. Hungar. **15** (1964), 259–283.

This is a continuation of a previous paper [same Acta **15** (1964), 137–152; MR **28** #4235]. Estimates of $|\operatorname{grad} u|$ are found for solutions of the equations $\Delta u - a u_t' = 0$, $\Delta u = f$, and $\Delta u - a u_t' = f$, under various sets of hypotheses concerning the domain and the data relative to initial and boundary conditions. The results involve much intricate detail, and consequently no summary of them can be given here.

L. A. MacColl (New York)

Kopaček, Irži [Kopáček, Jiří] 4990

Solution of the Cauchy problem for linear hyperbolic equations by the method of finite differences. (Russian. English summary)

Czechoslovak Math. J. **14** (89) (1964), 52–78.

Let (1) $a(x, D)u = g$ be a linear strictly hyperbolic equation of arbitrary order with variable coefficients. (Here u is supposed to be periodic in the space variables.) It has been proved [Gårding, *La théorie des équations aux dérivées partielles*, pp. 71–90, Centre Nat. Rech. Sci., Paris, 1956; MR **22** #6937] that there exists a unique solution $u \in H_0^{m+1, n-1}$ to the corresponding Cauchy problem if $g \in H^{0,n}$, and if the coefficients of $a(x, D)$ belong to Lip^n $(n \geq 1)$. The proof was based on the construction of an a priori (Friedrichs-Lewy) inequality for the given Cauchy problem and of a similar inequality in the dual norms for an adjoint problem.

The present paper closely follows Gårding's in the construction of a (discrete) F.L. inequality for the explicit difference equation (2) $a(x, \Delta)u = g$, where $a(x, \Delta)$ is obtained from $a(x, D)$ by the substitution of a centered difference operator Δ in the place of D. Then it is proved that this discrete F.L. inequality is sufficient to ensure the weak convergence, with decreasing mesh width, of the solutions of (2) to a solution $u \in H_0^{m+1, n-1}$ of (1), provided the mesh ratio is bounded. The regularity assumptions are the same as in Gårding's paper (or slightly weaker). The author also considers an implicit difference scheme of the same type. A preliminary report of these results was given in Dokl. Akad. Nauk SSSR **141** (1961), 551–554; [MR **24** #A2756].

J. Friberg (Lund)

Mel'nik, Z. O. 499

A mixed problem for certain hyperbolic systems. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 314–318.

The author considers the problem

$$\sum_{k_0+\cdots+k_n=4} a^{i}_{k_0,\cdots,k_n}(x, t)\, \partial^4 u_i / \partial t^{k_0} \partial x_1^{k_1} \cdots \partial x_n^{k_n} + \sum_{s=1}^{N} \sum_{k_0+\cdots+k_n<4} a^{i,s}_{k_0,\cdots,k_n}(x, t)\, \partial^{\Sigma k_i} u_s / \partial t^{k_0} \cdots \partial x_n^{k_n} = f_i(x, t)$$

in a region $\Omega = [0, T] \times D$, $D \subset \mathbf{R}^n$, with $\partial^m u_i / \partial t^m |_{t=0} = 0$ $(m \leq 3;\ 1 \leq i \leq N,\ x \in \bar{D})$ and $u_i|_S = \partial u_i / \partial \nu|_S = 0$ $(1 \leq i \leq N,\ 0 \leq t \leq T)$, where S is the (smooth) boundary of D, and the system is hyperbolic. Under suitable hypotheses using a method of Gårding, existence and uniqueness are proved, and an estimate $\|u\|_{W_2^3} \leq c \|f\|_{L^2}$ is obtained. Also a second-order hyperbolic system is considered.

R. Carroll (Urbana, Ill.)

Peyser, Gideon 499

The Goursat problem for hyperbolic equations.

J. Math. Mech. **10** (1961), 91–109.

The author considers the linear partial differential equation

$$(\alpha_1 \partial/\partial x + \beta_1 \partial/\partial y) \cdots (\alpha_m \partial/\partial x + \beta_m \partial/\partial y) u + \cdots = f,$$

where $x \in [0, x_0]$, $y \in [0, y_0]$, $\alpha_i = \alpha_i(x, y)$, $\beta_i = \beta_i(x, y)$, $\alpha_i^2 + \beta_i^2 = 1$, and

$$\infty > \frac{\alpha_1}{\beta_1} > \cdots > \frac{\alpha_k}{\beta_k} > 0 \geq \frac{\alpha_{k+1}}{\beta_{k+1}} > \cdots > \frac{\alpha_m}{\beta_m} \geq -\infty \qquad (k \leq m)$$

On the lines $y=0$ and $x=0$ linear boundary conditions of order $m-1$ are prescribed for $k+r$ and $m-r$, respectively, where r is given, $0 \leq r \leq m-k$; $u_{x^i y^j}(0, 0)$ are also given $(i+j \leq m-2)$. Under well-defined smoothness hypotheses on the coefficients and under sufficiently broad hypotheses on homogeneous terms in the problem (and also under certain natural hypotheses of an algebraic character on the operators in question) the author proves the existence and uniqueness of a generalized, and thus of a classical solution. We remark that under the conditions imposed on the characteristics, each of the two non-strict inequalities is assumed to be strict on the y-axis and the x-axis respectively, or else to be the identity, i.e., the author considers the non-characteristic and the characteristic problems. The work is based on an energy inequality and on one related to it which was derived in an earlier paper of the author [same J. **6** (1957), 641–653; MR **19**, 1059].

The author should also make reference to Campbell and Robinson [Proc. London Math. Soc. (3) **5** (1955), 129–147; MR **16**, 1116].

A. D. Myškis (RŽMat **1961** #10 B214)

Makai, E. 499

On the fundamental frequencies of two and three dimensional membranes. (Russian summary)

Magyar Tud. Akad. Mat. Kutató Int. Közl. **8** (1963), 109–123.

The author obtains new isoperimetric inequalities for the first eigenvalue (square of the fundamental frequency) in the three-dimensional membrane problem defined on a domain D whose outer boundary S_0 of surface area S is

convex. The domain D may have holes bounded by surfaces $S_1, S_2, \cdots$. The membrane is fixed on the outer boundary and free on the inner boundaries. The following two theorems are proved. Theorem A: Among all three-dimensional domains with given volume V and area S of the outer surface (the outer surface being convex) the spherically symmetric one bounded by two concentric spherical surfaces has the highest fundamental frequency. Theorem B: With the same assumptions as in Theorem A, $\lambda \leq (\pi S/2V)^2$. A two-dimensional analogue of (A) was given by the reviewer and Weinberger [J. Math. Anal. Appl. **2** (1961), 210–216; MR **26** #7220] while the two-dimensional analogue of (B) was an extension by Pólya [J. Indian Math. Soc. (N.S.) **24** (1960), 413–419; MR **24** #A2895] of earlier results of the author [Acta Sci. Math. (Szeged) **20** (1959), 33–35; MR **21** #3965].

The author shows that the convexity of S is necessary in (A) and (B). He also shows that the theorem of Pólya for a membrane fixed along its entire boundary cannot be extended to general multiply connected domains.

L. E. Payne (College Park, Md.)

Vichik, M. I. [Višik, M. I.] 4994

Sur les problèmes aux limites pour des équations quasi-linéaires elliptiques et paraboliques d'ordre supérieur.

Les Équations aux Dérivées Partielles (*Paris*, 1962), *pp.* 213–218. *Éditions du Centre National de la Recherche Scientifique*, *Paris*, 1963.

L'auteur trouve des solutions des équations quasi-linéaires elliptiques d'ordre $2m$ de forme de divergence, dans les espaces $W_p^{(m)}$ et les solutions des équations quasi-linéaires paraboliques dans des espaces $\mathscr{L}_p(0, T; W_p^{(m)})$. Il construit d'abord ces solutions en supposant que les paramètres du problème sont suffisamment réguliers, à l'aide de la méthode de Galerkine. Ensuite, par un théorème de densité, il démontre l'existence d'une solution sous des restrictions faibles sur les données du problème.

M. Coroi-Nedelcu (Bucharest)

Akišanov, A. 4995

Cauchy-Dirichlet problems in a multiply connected region for the heat equation. (Russian. Kazak summary)

Vestnik Akad. Nauk Kazah. SSR **1964**, *no.* 5 (230), 40–47.

On considère le système

$$(\Delta - (1/a^2)\partial/\partial t)u(x, t) = 0, \qquad t > 0,$$

$$u(x, 0) = f(x), \qquad \lim_{x \to y} u(x, t) = F(y, t), \qquad y \in \sum_{j=0}^{n-1} S^{(j)},$$

où (a) $x = (x_1, x_2, x_3)$; (b) $f(x) \in L_2(B)$, $F(x, t) \in L_2(B \times [0, T])$; (c) B est un domaine dont la frontière est formée par les surfaces de Liapunov $S^{(0)}$ et $S^{(i)}$, telles que $S^{(0)}$ est une surface fermée, qui contient dans son intérieur les surfaces $S^{(i)}$ $(i = 1, 2, \cdots, n-1)$, qui n'ont pas de points communs entre elles. La méthode d'intégration est semblable à celle utilisée par V. D. Kupradze [*Potential-theoretic methods in the theory of elasticity* (Russian), Fizmatgiz, Moscow, 1963; MR **28** #1808]. On construit d'abord un système complet de fonctions orthogonales, et puis on trouve les coefficients de Fourier du développement en série de la solution.

A. Haimovici (Iaşi)

Alihanova, R. I. 4996

On the Cauchy problem for a quasi-parabolic equation. (Russian. Azerbaijani summary)

Akad. Nauk Azerbaĭdžan. SSR Dokl. **20** (1964), *no.* 6, 9–14.

The initial-value problem

$$\frac{d}{dt} u(t, x) = -\sum_{|m| \leq 2r} \varphi^m \left[\int u^2(t, \xi)\, d\xi\right] A_m(t) D^m u,$$

where m is a vector $(m_1, \cdots, m_n)$, $|m| = |m_1| + \cdots + |m_n|$, $p^m = p_1^{m_1} \cdots p_n^{m_n}$, $D^m = D_1^{m_1} \cdots D_n^{m_n}$, $D_k = i^{-1}\partial/\partial x_k$, and $|p|^2 = |p_1|^2 + \cdots + |p_n|^2$, is solved under the hypothesis $0 \leq \sum_{|m| \leq 2r} \varphi^m A_m(t) p^m \leq C|1 + |p|^2|^r$. The solution is obtained by first taking the Fourier transform. This paper is a continuation of one written with G. N. Agaev [Akad. Nauk Azerbaĭdžan. SSR Trudy Inst. Mat. Meh. **2** (10) (1963), 129–132; MR **27** #4037].

D. C. Kleinecke (Santa Barbara, Calif.)

Cattabriga, Lamberto 4997

Equazioni paraboliche in due variabili. I.

Rend. Sem. Fac. Sci. Univ. Cagliari **31** (1961), 48–79.

The author considers the parabolic equation

$$\begin{aligned} L(x, y, D_x, D_y)u &= L'(x, y, D_x, D_y)u + L''(x, y, D_x, D_y)u \\ &= \sum_{j=0}^{n} (-1)^{p(n-j)} a_{n-j}(x, y) D_x^{2p(n-j)} D_y^{j} u \\ &\quad + \sum_{0 \leq i + 2pj < 2pn} b_{ij}(x, y) D_x^{i} D_y^{j} u = 0 \end{aligned}$$

on a region that can be a half-plane, a quarter-plane, a half-strip, or a rectangle. The initial and boundary conditions are the apparent ones. First he constructs a fundamental solution for $L'u = 0$ for the case of constant coefficients. He then proves uniqueness for $Lu = 0$ if the coefficients depend on x only for the half-strip. Uniqueness theorems for the quarter-plane and the rectangle are given for constant coefficients. He obtains the solution in closed form for the Cauchy problem and for initial-boundary problems for the other regions for the constant-coefficient case.

J. Douglas, Jr. (Houston, Tex.)

Èĭdel'man, S. D. [Эйдельман, С. Д.] 4998

★Parabolic systems [Параболические системы].

Izdat. "Nauka", Moscow, 1964. 443 *pp.* 1.14 *r.*

The author is concerned with the following topics: fundamental solutions (f.s.), differentiability of solutions, the Cauchy problem and the initial-boundary-value problem; f.s. are first constructed and then employed in a systematic manner throughout the book.

In Chapter 1, f.s. are constructed for general parabolic systems, also in the case of unbounded coefficients, and bounds on their derivatives are obtained. It is shown that the f.s. are analytic functions in the space variable if the same is true of the coefficients. The behavior of solutions near an isolated singularity is studied.

Chapter 2 deals with interior estimates of solutions, and with their smoothness and analyticity properties. Liouville-type theorems are proved.

In Chapter 3, the Cauchy problem is solved with the aid of f.s. Existence and uniqueness theorems are proved for solutions $u(x, t)$ satisfying certain growth conditions in x; these growth conditions are sharp. Next, the Cauchy

problem is considered for perturbed non-linear equations. Finally, the behavior of the solution as $t \to \infty$ is investigated.

In Chapter 4, the initial-boundary-value problems are solved, first in a half-space for equations with constant coefficients and then in arbitrary cylinders and for equations with variable coefficients. The treatment is analogous to that of Agmon [Comm. Pure Appl. Math. **10** (1957), 179–239; MR **21** #5057] for elliptic equations, and it is an original contribution of the book.

A. Friedman (Evanston, Ill.)

Fife, Paul 4999

Solutions of parabolic boundary problems existing for all time.

Arch. Rational Mech. Anal. **16** (1964), 155–186.

The author gives conditions under which there exists a unique solution of a second-order parabolic equation in the cylinder $Q\{\Omega, -\infty < t < \infty\}$ which vanishes on the lateral surface of the cylinder; here Ω is a domain in the space $(x_1, \cdots, x_n)$. The problem studied is that of the behaviour of the solutions as $t \to \infty$. The author finds conditions under which there exists a periodic solution in Q which vanishes on the lateral surface of Q. The corresponding theorems are obtained as a consequence of certain theorems for operator equations in Banach spaces. By choosing various norms (L_2, L_1, and the maximum norm), the author obtains existence theorems for the boundary-value problem in Q in the various spaces and a priori bounds for these solutions in the corresponding norms. Analogous results are also obtained for certain classes of quasi-linear parabolic equations.

O. A. Oleinik (Moscow)

Jiang, Li-shang [Chiang, Li-shang] 5000

The proper posing of free boundary problems for non-linear parabolic differential equations.

Acta Math. Sinica **12** (1962), 369–388 (*Chinese*); *translated as Chinese Math.* **3** (1963), 399–418.

The author establishes existence and uniqueness for the free boundary problems

$$(1)\quad \partial(k(x,t,u)\partial u/\partial x)/\partial x = \partial u/\partial t \qquad (0 < x < h(t),\ t > 0);$$

$$(2)\quad -k(0,t,u(0,t))\partial u(0,t)/\partial x = g(t) \qquad (t > 0,\ g \geq 0);$$

$$(3)\quad u(h(t),t) = 0 \qquad (t > 0);$$

$$(4)\quad -k(h(t),t,0)\partial u(h(t),t)/\partial x = dh(t)/dt \qquad (t > 0);$$

$$(5)\quad h(0) = 0.$$

The proof consists of two parts: (a) establishing existence and uniqueness for (1)–(3) for any given $h(t)$ satisfying $h(t) \geq 0$, $dh(t)/dt \geq 0$, $h(0) = 0$; (b) rewriting (1)–(5) in the form $h = Th$ (the definition of T employs (a)), and proving that T has a unique fixed point. The present problem was treated by the same method by Kyner [J. Math. Mech. **8** (1959), 483–498; MR **26** #1630], but the latter author did not give a proof of (a). The present proof of (a) involves derivations of a priori bounds on the derivatives of the solution (by the method of S. N. Bernstein).

A. Friedman (Evanston, Ill.)

Zagorskiĭ, T. Ja. 5001

A mixed problem for general parabolic systems in a half-space. (Russian)

Dokl. Akad. Nauk SSSR **158** (1964), 37–40.

The problem considered is to solve a $2a$-parabolic syste[m] $D_t^{n_i} u_i = A_i(D_t, D_x)u$ $(i = 1, \cdots, N)$ for $t, x_n > 0$, wit[h] Cauchy data $= 0$ for $t = 0$, and with $B(D_t, D_x)u \to f$ whe[n] $x_n \to 0$. Here B is an $aM \times N$ matrix $(M = n_1 + \cdots + n_N)$ o[f] differential operators of arbitrary order. The problem i[s] solved by the Fourier-Laplace transformation under th[e] assumption of exponential growth of f with t (and possibl[y] some of its derivatives).

J. Friberg (Lund[)]

Kikin, D. B. 500[2]

Extremal properties of the solutions of certain classes o[f] second-order partial differential equations and thei[r] applications to gas flows. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1963**, *no.* 5 (36[)], 56–58.

The following theorem is proved: If a differential operato[r] $L(u) = \Phi(x, y, u, u_x, u_y, u_{xx}, u_{yy}, u_{xy})$ is such that applyin[g] it to any arbitrary twice continuously differentiabl[e] function $f(x, y)$ at its maximum N_0 and in some neighbor[-] hood of it yields a non-positive quantity, and $\Phi_{u_{xx}}(N_0) > 0$ or $\Phi_{u_{yy}}(N_0) > 0$, then the solution $u(x, y)$ to the equatio[n] $L(u) = 0$ cannot assume a maximum value at this poin[t] or in some neighborhood of it. On the function Φ i[s] imposed the requirement that it be continuous an[d] continuously differentiable in u, u_x, u_{xx} or in u, u_y, u_{yy} i[n] its domain of definition. This theorem generalizes a resul[t] due to Chaplygin in two-dimensional gas flow.

C. K. Chu (New York[)]

Schechter, E. 500[3]

Chaplighin type methods for hyperbolic equations.

Mathematica (Cluj) **4 (27)** (1962), 383–396.

Chaplygin's method consists of constructing a solution t[o] a given non-linear problem as a limit of a monoton[e] sequence of solutions of approximate problems. Suc[h] constructions have been devised for parabolic and ellipti[c] equations. The present paper deals with the analogou[s] situation for the Cauchy and the Darboux problems fo[r] $u_{xy} = f(x, y, u, u_x, u_y)$. The assumptions under which th[e] construction can be applied are too involved to be state[d] here.

A. Friedman (Evanston, Ill.[)]

Pul'kin, S. P. 500[4]

On the solution of Tricomi's problem for an equation o[f] Chaplygin type. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1958**, *no.* 2 (3)[,] 219–226.

Teut, O. M. 500[5]

A boundary-value problem for a system of partial dif[-] ferential equations of mixed type. (Russian)

Boundary-value problems in the theory of functions of [a] complex variable, pp. 40–58. *Izdat. Kazan. Univ. Kazan*, 1962.

The traditional method of solving the Tricomi problem for equations of changing type is by the use of integra[l] equations for the unknown function value assume[d] between the domains of ellipticity and hyperbolicity an[d] the uniqueness of the solution to the main problem, t[o] determine the existence of this function [L. Bers, *Mathe[-] matical aspects of subsonic and transonic gas dynamics* p. 97, Wiley, New York, 1958; MR **20** #2960].

In this paper, the author discusses the system of equations of mixed type

$$(1)\quad \begin{aligned} u_x - v_y &= a_1(x, y)u + b_1(x, y)v + f_1(x, y), \\ u_y + (\operatorname{sign} y)v_x &= a_2(x, y)u + b_2(x, y)v + f_2(x, y) \end{aligned}$$

in a domain D bounded in the upper half-plane by a smooth curve σ joining $A = (-1, 0)$ to $B = (1, 0)$ and in the lower half-plane by the segments of the characteristics $AC: y = -x-1$, $-1 \leq x \leq 0$, and $BC: y = x-1$, $0 \leq x \leq 1$; a_i, b_i, f_i, $i = 1, 2$, are Hölder-continuous in the closure $\bar{D}$ of D for $y \geq 0$ and are continuously differentiable in $\bar{D}$ for $y \leq 0$. A pair of functions u, v is sought, satisfying (1) in D, such that

$$\text{(a)}\quad (u+v)|_\sigma = \gamma(t),$$

$$\text{(b)}\quad (u+v)|_{AC} = \psi(2x+1), \qquad -1 \leq x \leq 0,$$

for given functions γ, ψ, which are, respectively, Hölder-continuous on σ and twice continuously differentiable on $-1 \leq x \leq 1$; t is the complex coordinate on σ. Moreover, u, v are to be continuous on D, and have continuous derivatives throughout D except on the arc $-1 \leq x \leq 1$, where they experience jump discontinuities.

Under the additional conditions that along σ, $dx/ds \leq 0$, with s the variable of arc length (which is met for a semi-circle) and that $a_1 < 0$, and $(a_2 - b_1)^2 + 4a_1 b_2 < 0$, uniqueness of the solution to the problem is easily proved.

Let D_1 $[D_2]$ be the portion of D for which $y \geq 0$ $[y \leq 0]$. The author now obtains for a general function $\tau(x) = u(x, 0) + v(x, 0)$ a solution to the equations subject to condition (b) for the equations in D_2, the region of hyperbolicity, by transformation to characteristic co-ordinates. Under resulting conditions on the segment $-1 \leq x \leq 1$, and in the case when σ is the unit semi-circle, the equation (1) is rewritten in the complex form

$$w_{\bar{z}} + A(z)w + B(z)\bar{w} = F(z)$$

and solved explicitly in D_1 for $w = u + iv$ in terms of singular integrals depending on τ and on the coefficients of the equation. The author then obtains necessary differentiability theorems for w and is able thereby to derive integral equations for $u(x, 0)$, $v(x, 0)$ for which existence of a solution is equivalent to existence of a solution to the original problem, and for which the Fredholm alternative is applicable. By the uniqueness of the solution to the main problem, the proof is complete.

A. D. Solomon (New York)

Žegalov, V. I. 5006

Some problems for the equation

$$\left(\frac{\partial^2}{\partial x^2} + \operatorname{sgn} y \frac{\partial^2}{\partial y^2}\right)^n u = 0.$$

(Russian)

Boundary-value problems in the theory of functions of a complex variable, pp. 3–16. *Izdat. Kazan. Univ., Kazan*, 1962.

The author has previously considered a generalization of the Tricomi problem, here called Problem T^n, in which the differential equation to be solved is

$$(\partial^2/\partial x^2 + \operatorname{sgn} y \partial^2/\partial y^2)^n u = 0$$

[Dokl. Akad. Nauk SSSR **136** (1961), 274–276; MR **22** #8228]. It was shown that Problem T^n could be reduced to a succession of boundary-value problems of the type known as Problem T. A method was indicated for solution, but no existence proof was given. In the present paper a smoothness condition is imposed on the solutions in this succession in order to guarantee uniqueness, and the solution is constructed with the use of n-harmonic functions. The same ideas are used to treat similar generalizations of the two problems of Ju. M. Krikunov [Izv. Vysš. Učebn. Zaved. Matematika **1961**, no. 6 (25), 60–70; MR **26** #4061]. *R. N. Goss* (San Diego, Calif.)

Wiener, Klaus 5007

Über ein gemischtes Randwertproblem für die verallgemeinerte Tricomi-Gleichung.

Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe **13** (1964), 55–57.

Let $u = u(x, y)$, $0 < x < \infty$, $0 < y < \infty$, satisfy the generalized Tricomi equation

$$\text{(A)}\quad u_{yy} + y^m(u_{xx} + k^2 u) = 0, \qquad m \geq 0,\ \operatorname{Im} k > 0,$$

and the boundary conditions

$$\text{(B)}\quad u = f(x), \qquad 0 < x < 1, \quad y = 0,$$

$$\text{(C)}\quad u_y = g(x), \qquad 1 < x < \infty, \quad y = 0,$$

$$\text{(D)}\quad u_x = y^{-m}h(y), \qquad x = 0, \quad 0 < y < \infty,$$

where $f(x)$, $g(x)$, and $h(y)$ are continuous in $[0, 1]$, $[1, \infty)$, and $[0, \infty)$, respectively, and $g(x)$ and $h(y)$ are exponentially small at infinity. This mixed boundary-value problem is transformed to a Fredholm integral equation of the first kind for $u_y(x, y)$ in $x = (0, 1)$. This integral equation must be solved approximately, after which the specification of u_y on $y = 0$, $x = (0, \infty)$ permits the invocation of a known, general solution of (A). *J. W. Miles* (Canberra)

Prouse, Giovanni 5008

Soluzioni quasi-periodiche dell'equazione differenziale di Navier-Stokes in due dimensioni.

Rend. Sem. Mat. Univ. Padova **33** (1963), 186–212.

This paper treats the existence, uniqueness, and almost periodicity of bounded weak solutions $x(\xi_1, \xi_2, t)$ of the 2-dimensional Navier-Stokes system in $\Omega \times (-\infty, \infty)$, where Ω is an open set in the (ξ_1, ξ_2)-plane. Let N and N^1 be the closures with respect to the norms $\|u\|_{L^2} = (\int_\Omega |u|^2 d\Omega)^{1/2}$ and $\|u\|_{H_0^1} = (\int_\Omega |\nabla u|^2 d\Omega)^{1/2}$, respectively, of smooth divergence-free vector functions vanishing near $\partial\Omega$. Consider functions $v(t)$ from $(-\infty, \infty)$ to $L^2(\Omega)$ such that for each t, the norm

$$\|v(t)\|_{L_0^2(L^2)} = \left(\int_{-1}^0 \|u(t+\eta)\|_{L^2}^2 \, d\eta\right)^{1/2}$$

exists; then the range of v can be interpreted as lying in $L_0^2(L^2) = L^2(-1, 0; L^2(\Omega))$. One defines $v(t) \in L_0^2(H_0^1)$, $L_0^2(N^1)$, $L_0^\infty(N)$ similarly. By a weak solution of the Navier-Stokes system with homogeneous Dirichlet data, the author means a function $x(t) \in L_0^2(N^1) \cap L_0^\infty(N)$ satisfying

$$\int_{-\infty}^{\infty} \{(x(t), h(t))_{H_0^1} - (x(t), h'(t))_{L^2} + b(x(t), x(t), h(t))\}\, dt = \int_{-\infty}^{\infty} (f(t), h(t))_{L^2}\, dt$$

for every $h(t)$ of compact support which is N^1-continuous in t with $h'(t) \in L_0{}^2(N)$. (Here $b(u, v, w) = \int_\Omega \sum u_i (\partial v_j/\partial \xi_i w_j)\, d\Omega$.) The inhomogeneous term $f(t)$ is supposed to have its range in $L_0{}^1(L^2)$.

The author proves the following. (1) If $f(t)$ is L^2-bounded uniformly in t, there exists a weak solution defined for all time which is also L^2-bounded in $(-\infty, \infty)$. (2) If f is "small enough" (or, more generally, if there exists a solution that is small enough), then uniqueness holds among functions L^2-bounded in $(-\infty, \infty)$. (3) If $f(t)$ is weakly almost periodic with respect to the norm of $L_0{}^2(L^2)$ (abbreviated $L_0{}^2(L^2)$-w.a.p.) then the solution $x(t)$ is L^2-w.a.p. and $L_0{}^2(L^2)$-almost periodic ($L_0{}^2(L^2)$-a.p.). (4) If $f(t)$ is $L_0{}^2(L^2)$-a.p. and $x(t)$ is small enough, then $x(t)$ is L^2-a.p. and $L_0{}^2(H_0{}^1)$-a.p.

The results of Amerio [Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **31** (1961), 110–117; MR **25** #3389; ibid. (8) **31** (1961), 197–205; MR **26** #2879; Ricerche Mat. **9** (1960), 255–274; MR **24** #A896] on almost periodic solutions of equations in Hilbert spaces are used extensively. *P. C. Fife* (Minneapolis, Minn.)

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

See also 4699, 4709.

Weinmann, A. 5009

Asymptotic expansions of generalized Bernoulli polynomials.

Proc. Cambridge Philos. Soc. **59** (1963), 73–80.

It is known that the coefficients of all the usual central difference formulae for numerical differentiation and integration can be expressed in terms of generalized Bernoulli polynomials. The author derives asymptotic expansions for certain of these polynomials from their expression as a contour integral, thus obtaining the asymptotic form of the coefficients.

L. M. Milne-Thomson (Tucson, Ariz.)

Zeitlin, David 5010

On the sums $\sum_{k=0}^{n} k^p$ and $\sum_{k=0}^{n} (-1)^k k^p$.

Proc. Amer. Math. Soc. **15** (1964), 642–647.

Let u_k satisfy the linear difference equation of order m with real constant coefficients

$$u_{m+k} + a_1 u_{m+k-1} + \cdots + a_m u_k = 0, \qquad k = 0, 1, 2, \cdots.$$

The author derives a closed form for $\sum_{k=0}^{\infty} u_k r^k$, and thus closed forms for the sums given in the title, as well as certain other formulae.

L. M. Milne-Thomson (Tucson, Ariz.)

Koroljuk, V. S. 5011

The asymptotic behavior of solutions of difference equations with small parameters. (Russian)

Ukrain. Mat. Ž. **12** (1960), 342–346.

In analogy with a method used for differential equations by M. I. Višik and L. A. Ljusternik [Uspehi Mat. Nauk **12** (1957), no. 5 (77), 3–122; MR **20** #2539], a process is set up for obtaining an asymptotic expression (for small ε) for solutions of difference equations of the form (1) $P_\varepsilon u_\varepsilon(x) \equiv \sum_{k=-\infty}^{+\infty} p_k u_\varepsilon(x+k\varepsilon) = 0$ (x/ε an integer), with $u_\varepsilon(x)$ given for $x \leq 0$ and $x \geq 1$. Four conditions are assumed. (a) The coefficients p_k in (1) depend on ε in such a way that $\sum_{k=-\infty}^{+\infty} |k|^{N+1} |p_k| \leq C$ (independent of ε) for some positive integer N, and ε sufficiently small. (b) Let $\rho_m = (1/m!) \sum_{k=-\infty}^{+\infty} k^m p_k$ $(m = 0, \cdots, N)$. Then asymptotically $(\varepsilon \to 0)$, $\rho_m = \varepsilon^{n-m} \sum_{r=0}^{N+1} \varepsilon^r a_{mr}$ $(m = 0, \cdots, n)$, $\rho_{n+l} = \sum_{r=0}^{N+1-n-l} \varepsilon^r a_{n+l,r}$ $(l = 1, \cdots, N-n)$, with $a_{n0} \neq 0$. Here n is the order of the differential operator $L_0 = \sum_{m=0}^{n} a_{m0} d^m/dx^m$ to which P_ε of (1) passes for $\varepsilon \to 0$. (c) Let $p_0(\lambda) = \sum_{k=-\infty}^{+\infty} p_{k0} \lambda^k$, where $p_k = p_{k0} + O(\varepsilon)$, p_{k0} independent of ε. Then $n_0, n_1 > 0$ exist $(n_0 + n_1 = n)$ such that $q(\lambda) = p_0(\lambda)(\lambda - 1)^{-n_1}(1 - \lambda^{-1})^{-n_0} \neq 0$ on the circle $|\lambda| = 1$, and $\operatorname{index}_{|\lambda|=1} q(\lambda) = 0$. (d) The non-homogeneous equation $\varepsilon^{-n} P_\varepsilon u_\varepsilon = g$, with bounded exterior conditions (i.e., for $x \leq 0$, $x \geq 1$) for $\varepsilon \to 0$, is uniformly solvable in the sense that for ε sufficiently small, $\|u_\varepsilon\| \leq K\|g\|$ (K independent of ε).

The actual asymptotic expression obtained for $u_\varepsilon(x)$ involves too much detail to present here. The author notes that his method can be extended to the case in which the coefficients p_k involve x as well as ε.

I. M. Sheffer (University Park, Pa.)

Newns, W. F. 5012

On the difference and sum of a basic set of polynomials.

Pacific J. Math. **14** (1964), 639–644.

The reader is expected to be familiar with terminology and fundamental results on basic sets of polynomials, as contained in J. M. Whittaker's book, *Sur les séries de base de polynômes quelconques* [Gauthier-Villars, Paris, 1949; MR **11**, 344]. The author proves two theorems on best possible bounds for the orders of the difference set and the sum set of a basic set, as established in N. N. Mikhail and M. Nassif [Pacific J. Math. 11 (1961), 1099–1107; MR **25** #347]. By considering spaces of integral functions, he extends one of the theorems to integral functions, and he obtains related results which he believes provide a deeper insight into the problem. He notes that either approach can be used to determine the types of the sets, as well as their orders. *R. A. Rosenbaum* (Middletown, Conn.)

Choczewski, B.; Kuczma, M. 5013

Sur certaines équations fonctionnelles considérées par I. Stamate.

Mathematica (Cluj) **4** (**27**) (1962), 225–233.

The authors consider the following functional equation

$$(1) \quad \frac{xf(y) - yf(x)}{x - y} = F[u, f(u), f'(u), \cdots, f^{(n)}(u); g(u), g'(u), \cdots, g^{(m)}(u)],$$

where F and $u = u(x, y)$ are given functions and f is a function to be determined. This equation is a generalisation of some equations considered by I. Stamate [Inst. Politehn. Lucrăr. Şti. **1959**, 107–110; MR **22** #A2654]. If $u(x, y) = (x+y)/2$, then f is linear, i.e., $f(x) = a + bx$. However, if u has a more general structure, then some additional assumptions are to be made. Under suitable conditions the authors find that f has the form

$$(2) \qquad f(x) = a_0 + a_1(x - a) + a_2 \int \frac{a - xv(x)}{a(x-a)}\, dx,$$

where a_0, a_1, a_2 and a are constants, $0<a<1$, and $v(x)=-(\partial u(x,a)/\partial x)/(\partial u(x,a)/\partial y)$. In general, the function (2) need not satisfy the equation (1). The question, under which additional assumptions (2) satisfies (1), is discussed. A necessary but not sufficient condition is that $(xf(w)-wf(x))/(x-w)$ does not depend on x, where w is a function such that $u(x, w(x,z))\equiv z$ in some neighborhood.

S. Kurepa (Zagreb)

Gumowski, Igor; Clergue, Michel 5014

Sur les solutions analytiques d'une équation différentielle-fonctionnelle d'ordre 1.

C. R. Acad. Sci. Paris **253** (1961), 1522–1528.

The authors claim as their principal result the theorem that the differential-difference equation (1) $\dot u(t)=-u(t-r)$ has a unique solution $u=u^*(t,r)$, analytic with respect to t and r at $(0,0)$, with prescribed initial value $u^*(0,r)=u_0$. They propose to find u^* as the limit of the sequence of successive approximations (2) $(u_n: n=0,1,\cdots)$, where $u_0(t)=u_0$ and $u_{n+1}(t)=u_0-\int_0^t u_n(s-r)\,ds$. They assert that the requisite convergence of (2) has been shown, for $0\leq r\leq 0.3$, by a computer. {Reviewer's note: An easy estimation argument shows that (2) converges (for all t) if $r<1/e$.} The authors assert also that it can be shown by a computer that (2) diverges if $r\geq 0.4$. They cite various salient parts of the literature on (1).

W. Strodt (New York)

SEQUENCES, SERIES, SUMMABILITY

See also 5052, 5100.

Matsuyama, Noboru 5015

A note on convergence of gap series.

Sci. Rep. Kanazawa Univ. 8, 271–277 (1963).

Let $f(t)$ be an integrable function with period 1 such that the mean of f is 0 and the variance is 1. Let (n_k) be a sequence of positive integers with the Hadamard gap $n_{k+1}/n_k>c>1$ $(k=1,2,3,\cdots)$. The author proves the following three theorems. (I) If (i) $(\int_0^1 |f(t)-s_n(t;f)|^2\,dt)^{1/2}=O(1/(\log n)^\alpha)$ for $\alpha>1$, where $s_n(t:f)$ is the nth partial sum of the Fourier series of f, and if (ii) $\sum c_k^2 \log_q k<\infty$ for a q, where $\log_1 k=\log k$, $\log_{q+1} k=\log(\log_q k)$, then the series $\sum c_k f(n_k t)$ converges a.e. (II) If (i) holds for $0<\alpha<\frac12$ and $\beta>1-\alpha$ or $\frac12\leq\alpha<1$ and $\beta>\frac12$, then $\sum f(n_k t)/k^\beta$ converges a.e. (III) If (i) holds for $\alpha=1$ and $\beta>\frac12$, then $\sum f(n_k t)/\sqrt{k}(\log k)^\beta$ converges a.e.

S. Izumi (Tokyo)

Lorentz, G. G.; Zeller, K. 5016

Abschnittslimitierbarkeit und der Satz von Hardy-Bohr.

Arch. Math. **15** (1964), 208–213.

Given $\alpha\geq 0$, let $\mathbf{C}$ denote the Banach space of number sequences $\mathbf{a}=\{a_k\}_{k=0}^\infty$ such that $\sum a_k$ is C_α-summable, where C_α denotes the Cesàro mean of order α. Let $\{\sigma_m\}$ denote the C_α transform of the sequence of partial sums of $\sum a_k$, and take $\|\mathbf{a}\|=\sup|\sigma_m|$. For each non-negative integer k, let $\mathbf{e}_k$ denote the point of $\mathbf{C}$ whose nth coordinate is unity if $n=k$ and zero if $n\neq k$. Using the Hardy-Bohr characterization of C_α convergence factors, K. Zeller had previously shown that, for $\mathbf{a}\in\mathbf{C}$, the series $\sum a_k\mathbf{e}_k$ is C_α-summable (in the norm of $\mathbf{C}$) to $\mathbf{a}$.

In this paper the authors use techniques from functional analysis to obtain a more direct proof of this theorem. Apart from its functional-analytic content, the proof depends mainly on establishing the binomial coefficient inequality

$$\sum_{k=0}^{\min(m,n)}\binom{m-k+\alpha}{m-k}\binom{n-k+\alpha}{n-k}\binom{k-\alpha-2}{k}\geq 0,$$

where m and n are non-negative integers and $\alpha\geq 0$. The same method also yields a computationally simpler proof of the theorem of Hardy and Bohr.

J. D. Buckholtz (Lexington, Ky.)

Petersen, G. M.; Baker, Anne C. 5017

On a theorem of Pólya. II.

J. London Math. Soc. **39** (1964), 745–752.

This paper is the fourth in a series of papers dealing with the solution of infinite systems under certain restrictions on the coefficient matrix [see Petersen and Thompson, same J. **38** (1963), 335–340; MR **27** #3652; ibid. **39** (1964), 31–34; MR **28** #4270; Petersen and Baker, ibid. **39** (1964), 501–510; MR **29** #2280]. A matrix $\{a(i,j)\}$ which satisfies

$$\liminf_{j\to\infty}\frac{|a(1,j)|+|a(2,j)|+\cdots+|a(i-1,j)|}{|a(i,j)|}=0 \qquad (i=1,2,\cdots)$$

for all but finitely many rows, and any finite selection of rows are independent, is called a Pólya matrix. Matrices that are obtained from each other by permuting rows or by a sequence of additions of one row to another, with the restriction that no row has infinitely many others added to it, are termed equivalent. The authors prove the following. Let $A=\{a(i,j)\}$ be a matrix such that $a(1,j)\neq 0$ $(j=1,2,\cdots)$. For any selection of real numbers $\{b(i)\}$, the infinite system of equations $\sum_j a(i,j)u(j)=b(i)$ $(i=1,2,\cdots)$ has solutions $\{u(j)\}$ such that the left-hand sides converge absolutely and with any arbitrary finite subset of the $u(j)$ equal to zero if and only if A is equivalent to a Pólya matrix.

B. E. Rhoades (Berkeley, Calif.)

Slipenčuk, K. M. [Slepenčuk, K. M.] 5018

Some methods for summation of series. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 1559–1562.

$(W^{(\alpha)},\lambda)$ est un procédé régulier de sommation défini par $\lambda_{n+1}\geq\lambda_n$, $\lambda_0=0$, $W_n^{(\alpha)}=\lambda_n^{-1}\sum_{k=1}^n(\lambda_{n+1-k}-\lambda_{n+k})W_{k-1}^{(\alpha-1)}$, $W_n^0=S_n=\sum_0^k a_k$, $W_0^{(\alpha)}=0$. Si la série $\sum a_k$ est $(W^{(\alpha)},\lambda)$-sommable vers S, alors $\lambda_n^{-1}\sum_{k=1}^n(\lambda_n-\lambda_{n-k})a_k\to 0$, $n\to\infty$, est la condition nécessaire et suffisante pour la convergence de $\sum a_k$ vers S.

L'auteur donne aussi une condition telle que $|W^{(\alpha)},\lambda|$-sommabilité de $\sum a_k$ implique la convergence absolue de $\sum a_k$.

M. Tomić (Belgrade)

Slipenčuk, K. M. [Slepenčuk, K. M.] 5019

A theorem of Tauberian type for $(H^{(\alpha)},\lambda)$-methods of summation of double series. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 312–314.

Le procédé $(H^{(\alpha)}, \lambda)$ est défini par $\alpha = 1, 2, \cdots$; $\lambda_{k0} = \lambda_{0l} = 0$; $\lambda_{kl} \to \infty$, $k, l \to \infty$, $\lambda_{k,l+1} > \lambda_{kl}$, $\lambda_{k+1,l} > \lambda_{kl}$,

$$\lim_{k,l\to\infty} \frac{\lambda_{k_0 l}}{\lambda_{kl}} = \lim_{k,l\to\infty} \frac{\lambda_{kl_0}}{\lambda_{kl}} = 0 \qquad (k_0, l_0 \text{ fixes}).$$

La série double (*) $\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl}$ est $(H^{(\alpha)}, \lambda)$-sommable vers S si $H_{mn}{}^{\alpha} \to S$ quand $m, n \to \infty$, où

$$H_{mn}{}^{\alpha} = (\lambda_{m+1,n+1})^{-1} \sum_{k=0}^{m} \sum_{l=0}^{n} \Delta\lambda_{kl} H_{kl}{}^{(\alpha-1)},$$

$$\Delta\lambda_{kl} = \lambda_{k+1,l+1} - \lambda_{k,l+1} - \lambda_{k+1,l} + \lambda_{kl},\ H_{mn}{}^{(0)} = S_{mn}.$$

On établit la condition nécessaire et suffisante pour que la série double (*), $(H^{(\alpha)}, \lambda)$-sommable vers S, avec $|S_{kl}| < M$, converge vers la même somme. *M. Tomić* (Belgrade)

Kuttner, B. 5020

The high indices theorem for discontinuous Riesz means.

J. London Math. Soc. **39** (1964), 635–642.

If $0 < \lambda_n \nearrow \infty$ and $k > 0$, the series $\sum u_n$ is called (R, λ_n, k)-summable to s if $w^{-k}A_k(w) \to s$ $(w \to +\infty)$, where $A_k(w) = \sum_{\lambda_n < w} (w - \lambda_n)^k u_n$, and (R^*, λ_n, k)-summable if $\lambda_n{}^{-k}A_k(\lambda_n) \to s$ $(n \to \infty)$. It is well known that (R, λ_n, k)-summability implies convergence if the λ_n satisfy (1) $\liminf \lambda_{n+1}/\lambda_n > 1$, and since (R, λ_n, k) and (R^*, λ_n, k) are equivalent for $0 < k \leq 1$, (1) is a high-indices condition for (R^*, λ_n, k) $(0 < k \leq 1)$. The author shows that it is also a high-indices condition for $(R^*, \lambda_n, 2)$ and (R^*, c^n, k) if $c > 1$, $1 < k < 2$, whereas (1) is not a high-indices condition for (R^*, λ_n, k) $(k > 2)$, not even if $\lambda_n = c^n$ $(1 < c < c_0(k))$. However, the stronger condition

$$\liminf \lambda_{n+1}/\lambda_n > 2^{1/k}/(2^{1/k} - 1)$$

is always a high-indices condition for (R^*, λ_n, k) $(k > 0)$. The proofs concerning the case $\lambda_n = c^n$ depend on the lemma: If $k > 0$, $c > 1$, then (R^*, c^n, k) is equivalent to convergence if and only if

$$\Phi(z) = \sum_{n=0}^{\infty} (1 - c^{-n-1})^k z^n \neq 0 \qquad (|z| \leq 1, z \neq 1).$$

D. Gaier (Pasadena, Calif.)

Kuttner, B. 5021

A Tauberian theorem for discontinuous Riesz means. II.

J. London Math. Soc. **39** (1964), 643–648.

With the notations of #5020 above, the problem is under what conditions the following proposition (*) is true: (R^*, λ_n, k)-summability implies (R, λ_n, k)-summability. The author proved earlier [J. London Math. Soc. **38** (1963), 189–196; MR **26** #6649] that (*) is true if (1) $\Lambda_{n-1} = O(\Lambda_n)$, where $\Lambda_n = \lambda_{n+1}/(\lambda_{n+1} - \lambda_n)$, and (2) $u_n = o(\Lambda_n{}^k)$, provided $k > 0$ is an integer. This result is now extended to non-integer values of k if additional conditions are imposed on the λ_n; for example,

$$(3) \quad \sum_{\nu=0}^{n} \lambda_\nu{}^{q+1}(\lambda_{\nu+1} - \lambda_\nu)^{k-q-1}\Lambda_\nu{}^k = O(\lambda_{n+1}^k \Lambda_{n+1}^{q+1})$$

(some integer q)

is [together with (1) and (2)] sufficient to imply (*). Condition (3) holds if $\{\lambda_n\}$ is a logarithmico-exponential sequence. It is also proved that (R, λ_n, k) and (R^*, λ_n, k) are not equivalent for $k > 1$, a result that had been known only for $k \geq 2$. *D. Gaier* (Pasadena, Calif.)

Levinson, Norman 5022

Absolute convergence and the general high indices theorem.

Duke Math. J. **31** (1964), 241–245.

For Dirichlet series $f(x) = \sum_1^{\infty} a_n e^{-\lambda_n x}$, where the λ_n satisfy a gap condition $\lambda_{n+1}/\lambda_n \geq q^2 > 1$ $(\lambda_1 > 0)$, Zygmund proved the following relation between the l^1 norm of the coefficient sequence a and the L^1 norm of f' on $(0, \infty)$: $\|a\| \leq A_q \|f'\|$, where A_q depends only on q [Trans. Amer. Math. Soc. **55** (1944), 170–204; MR **5**, 230]. The author extends Zygmund's result to kernels $N(x)$ much more general than the power series kernel e^{-x}. Most of the conditions imposed on $N(x)$ are formulated in terms of the Fourier transform $k(w) = \int_0^{\infty} N'(x)x^{-iw}dx$. They are too numerous to be stated in a review, and are very similar to the conditions in the author's general high-indices theorem [*Gap and density theorems*, Theorem LI, Amer. Math. Soc., New York, 1940; MR **2**, 180]. The conditions on $N(x)$ are, in particular, satisfied for the Lambert series kernel $xe^{-x}/(1 - e^{-x})$. *J. Korevaar* (La Jolla, Calif.)

Pleijel, Åke 5023

On a theorem by P. Malliavin.

Israel J. Math. **1** (1963), 166–168.

The author gives a simple proof, with a slightly better remainder, of a Tauberian theorem of Malliavin [C. R. Acad. Sci. Paris **255** (1962), 2351–2352; MR **26** #1662]. Let $f(z) = \int_0^{\infty} (\lambda - z)^{-1} d\sigma(\lambda)$, with $d\sigma \geq 0$, and let L be a contour consisting of the arcs $y = \pm x^{\gamma}$, $1 \leq x < \infty$, where $0 \leq \gamma < 1$, and a connecting arc which does not meet the half-line $y = 0$, $x \geq 0$. It is assumed that $f(z) = a(-z)^{\alpha} + O(z^{\beta})$ as $z \to \infty$ along L; here $-1 < \alpha < 0$ and $\beta < \alpha$. The author denotes by $L(Z)$ the part of L from $\bar{Z}$ to $Z = X + iY$ $(X > 1)$. He observes that

$$\frac{1}{2\pi i} \int_{L(Z)} f(z)\, dz = \sigma(X) - \sigma(0) + O(X^{\alpha+\gamma}),$$

and uses this result to show that $\sigma(X) - \sigma(0) = aC(\alpha)X^{\alpha+1} + O(X^{\alpha+\gamma}) + O(X^{\beta+1}) + A$. The constant A is present only when $\beta < -1$, and the term $O(X^{\beta+1})$ has to be replaced by $O(\log X)$ when $\beta = -1$. *J. Korevaar* (La Jolla, Calif.)

Skof, Fulvia 5024

Sull'attenuazione delle condizioni tauberiane.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **35** (1963), 466–468.

Let (C) denote the class of power series $\sum a_n z^n = f(z)$ which converge for $|z| < 1$ and are such that $\lim_{r\to 1-} f(re^{i\theta}) = g(\theta)$ exists uniformly in θ $(0 \leq \theta < 2\pi)$. The author announces, without proofs, three theorems, the simplest of which is Theorem A. Let two strictly positive functions $\varepsilon(t)$ and $\psi(t)$ be given. Assume that $\varepsilon(t) \to 0$, $\psi(t)$ increases to $+\infty$, $\psi(t)/t$ decreases to 0. Then, there exists some $f(z) \in$ (C) with the following two properties. (i) $|a_n| \leq \psi(n)/n$, (ii) there are arbitrarily large n for which $|a_n| = \psi(n)/n$ and $|a_0 + a_1 \cdots + a_n| > \varepsilon(n) \log \psi(n)$.

As is pointed out by the author, this result (which sharpens an analogous theorem of Turán) is connected with Littlewood's Tauberian theorem and a result of Titchmarsh.

The two other results stated by the author have the same general character as Theorem A and are connected with a one-sided Tauberian condition studied by R. Schmidt and T. Vijayaraghavan.

A. Edrei (Syracuse, N.Y.)

Subhankulov, M. A. 5025

On a theorem of Littlewood. (Russian. Uzbek summary)

Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk **1964**, *no.* 1, 22–30.

The author proves two Tauberian remainder theorems related to the following result of Littlewood. Let $F(\sigma)$, $\sigma>0$, be the sum of an absolutely convergent Dirichlet series $\sum_{n=0}^{\infty} a_n e^{-\lambda_n \sigma}$, where $\lambda_n \uparrow \infty$. Suppose that $F(\sigma)\to l$ as $\sigma \downarrow 0$, that $a_n=O\{(\lambda_n-\lambda_{n-1})/\lambda_n\}$, and that $\lambda_{n+1}\sim\lambda_n$. Then $\sum_{n=0}^{\infty} a_n=l$. Since the conditions in the general remainder theorems are very complicated, only some of the corollaries for power series will be quoted here. Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$, where $z=re^{i\theta}$ and $a_n=O(n^{-\nu})$ with $0<\nu\leq 1$. (1) Suppose that in a region G bounded by curves of the form $|\theta|=c(1-r)^{\omega}$, $0<\omega<\nu$, and $r=e^{-c}$, one has an estimate $f(z)=l+O(|1-z|^{\alpha})$, $\alpha>0$. Then $\sum_{n\leq x} a_n=l+O(x^{-\alpha\omega})+O(x^{\omega-\nu})$. (2) If $f(z)$ is regular at the point $z=1$, then $\sum_{n\leq x} a_n=f(1)+O(x^{-\nu})$.

The author credits the second result to Fatou and the reviewer. Actually, the reviewer obtained a more general result corresponding to a one-sided Tauberian condition [Nederl. Akad. Wetensch. Proc. Ser. A **57** (1954), 46–56; MR **15**, 698]. In a third and older type of result, due to Freud [Acta Math. Acad. Sci. Hungar. **2** (1951), 299–308; MR **14**, 361, and subsequent papers] and the reviewer [Duke Math. J. **18** (1951), 723–734; MR **13**, 227, and subsequent papers], there is only a hypothesis on $f(x)-l$ for real x; in this case, the remainder estimates are usually rather weak. The first type of result of the author provides a significant intermediate form between the older second and third type of remainder theorems.

J. Korevaar (La Jolla, Calif.)

APPROXIMATIONS AND EXPANSIONS

See also 4874, 4920, 5042, 5044, 5056, 5091, 5382.

Hsu, L. C. 5026

On a kind of extended Fejér-Hermite interpolation polynomials.

Acta Math. Acad. Sci. Hungar. **15** (1964), 325–328.

Let $F_n(g(t);x)$ be the Fejér-Hermite interpolation polynomial of the function $g(x)$, $-1\leq x\leq 1$ [Natanson, *Konstruktive Funktionentheorie*, p. 397, Akademie-Verlag, Berlin, 1955; MR **16**, 1100]. If f is a real continuous function on $(-\infty,+\infty)$, for which $|f(x)|\leq M\exp_m|x|=M\exp(\cdots(\exp|x|))$ (m times), and if $\lambda_n=\log_{m+1} n$, then $F_n(f(\lambda_n t);x/\lambda_n)\to f(x)$ uniformly on compact sets. There are similar theorems for other choices of the λ_n.

G. G. Lorentz (Syracuse, N.Y.)

Guerra, S. 5027

Osservazioni su un noto teorema di Jackson.

Boll. Un. Mat. Ital. (3) **18** (1963), 57–64.

A well-known theorem of Jackson states that there exists $M>0$ such that if $f(x)$ is continuous on $[-1,1]$ with modulus of continuity $\omega(\delta)$, then for n any positive integer, there is a polynomial $P_n(x)$ of degree at most n such that,

$$|f(x)-P_n(x)| \leq M\omega(1/n).$$

Korovkin [*Linear operators and approximation theory* (Russian), Fizmatgiz, Moscow, 1959; Engl. transl., Gordon and Breach, New York, 1960; MR **27** #561] showed that $M\leq 1+\frac{1}{2}\pi^2$. Using Korovkin's methods, it is shown here that $M\leq 1$.

R. O'Neil (Houston, Tex.)

Džrbašjan, M. M.; Kitbaljan, A. A. 5028

On a generalization of the Chebyshev polynomials. (Russian. Armenian summary)

Akad. Nauk Armjan. SSR Dokl. **38** (1964), 263–270.

Avec un continuum borné K dans le plan et une suite de nombres complexes $z_0, z_1, \cdots$ situés dans le domaine contigu à K qui contient le point à l'infini, Džrbašjan avait associé dans deux travaux antérieurs [Izv. Akad. Nauk Armjan. SSR Ser. Fiz.-Mat. Nauk **10** (1957), no. 1, 21–29; MR **20** #3297; Dokl. Akad. Nauk SSSR **143** (1962), 17–20; MR **25** #207] un système $\{M_k^{(s)}(z)\}_{k=0}^{\infty}$ de fonctions rationnelles dépendant d'un paramètre s, $0\leq s\leq 1$. Dans le travail présent on considère un cas particulier, celui où K coïncide avec le segment $[-1,1]$ et $s=0$ ou 1. On démontre que la suite $\{M_k^{(0)}(z)\}_{k=0}^{\infty}$ est orthogonale sur $[-1,1]$ avec le poids $1/\sqrt{(1-x^2)}$, et la suite $\{M_k^{(1)}(z)\}_{k=0}^{\infty}$ également. mais avec le poids $\sqrt{(1-x^2)}$.

E. J. Akutowicz (Bologna)

Rusak, V. N. 5029

On approximations by rational fractions. (Russian)

Dokl. Akad. Nauk BSSR **8** (1964), 432–435.

Soit $\beta_k=r_k e^{i\vartheta_k}$ une suite donnée de nombres complexes telle que $r_k<1$ et $\sum(1-r_k)=+\infty$. La fonction

$$K_n(t,x) = \sin\left\{\int_x^t \left(\frac{1}{2}+\sum_{k=1}^{n}\frac{1-r_k^2}{1+r_k^2-2r_k\cos(u-\vartheta_k)}\right)du\right\}$$

généralise le noyau sommatoire de Dirichlet, qui en résulte lorsque $\beta_k\equiv 0$. Posons, après Fejér, pour une fonction f de période 2π:

$$(1)\qquad \Phi_n(f;t) = \frac{\int_{-\pi}^{\pi} f(x)[K_n(t,x)]^2\,dx}{\int_{-\pi}^{\pi} K_n(t,x)^2\,dx}.$$

En désignant par $\gamma_n(t)$ le dénominateur précédent, l'auteur énonce le résultat suivant. Si la fonction $f(t)$ appartient à la classe Lip α, $0<\alpha<1$, alors

$$|\Phi_n(f;t)-f(t)| \leq \left(1+\frac{2\pi^2}{1-\alpha}\right)\gamma_n(t)^{-\alpha}.$$

En remplaçant les exposants 2 dans (1) par 4, après Jackson, on obtient un noyau $D_n(f;t)$ pour lequel

$$|D_n(f;t)-f(t)| \leq \text{const}\cdot\omega(1/\gamma_n(t)),$$

avec une constante indépendante de n, t; ω désigne ici le module de continuité de la fonction $f(t)$.

E. J. Akutowicz (Bologna)

Aramă, O.; Ripianu, D. 5030

Sur la monotonie de la suite des dérivées des polynomes de Bernstein.

Mathematica (Cluj) **4** (27) (1962), 9–19.

Let $f(x)$ be real analytic on $[0, 1]$ with the additional property that all derivatives of order $\geqq 2$ are non-negative on $[0, 1]$. Let $B_n(x; f)$ denote the nth Bernstein polynomial of $f(x)$. The authors show the existence of a subinterval $[0, \lambda]$, independent of $f(x)$, such that the sequence $\{B_n(x; f)\}_{n \geqq 1}$ is non-decreasing for each x of the subinterval. The case $f(x) = x^2$ shows that λ cannot exceed $\frac{1}{2}$. The authors give reasons for believing that $\frac{1}{2}$ is an admissible value of λ, but are unable to prove that it is.

D. S. Greenstein (Evanston, Ill.)

Boehm, Barry 5031

Functions whose best rational Chebyshev approximations are polynomials.

Numer. Math. **6** (1964), 235–242.

Tschebyscheff-Approximation stetiger Funktionen auf einem reellen kompakten Intervall tritt die Frage nach dem günstigsten Verhältnis vom Zähler- zum Nennergrad der rationalen Funktion auf. In der vorliegenden Arbeit, einem Auszug aus der Dissertation des Verfassers, wird ein Teil dieser Frage geklärt: Es wird untersucht, wann die beste rationale Approximation ein Polynom darstellt. In einigen Fällen gelingt eine vollständige Charakterisierung solcher Funktionenklassen, und es ergeben sich Sätze von der folgenden Art: Die beste rationale Approximation vom Zähler- und Nennerhöchstgrad n ist für alle n genau dann ein Polynom, wenn $f(x)$ die Form $f(x) = c + dT_m(x)$ mit einem Tschebyscheffschen Polynom von geeignetem Grad m besitzt.

Bei allgemeinem Gradverhältnis wird durch spezielle Tschebyscheff-Entwicklungen die Existenz von Funktionen $f(x)$ mit der obigen Eigenschaft nachgewiesen.

G. Meinardus (Clausthal-Zellerfeld)

Briones, F. 5032

On the alternants appearing in Chebyshev's best approximation problem.

Numer. Math. **6** (1964), 211–223.

Eine auf einem reellen kompakten Intervall B reellwertige und stetige Funktion $A_N(x)$ heiße eine Alternante der Ordnung N, wenn es $N+1$ der Größe nach geordnete Punkte x_i aus B gibt, so daß

$$A_N(x_i) + A_N(x_{i+1}) = 0 \text{ und } |A_N(x_i)| = \max_{x \in B} |A_N(x)| = E_N \quad \text{für } i = 1, 2, \cdots, N$$

gilt. Derartige Alternanten treten bekanntlich als Fehlerfunktionen bei Tschebyscheffschen Approximationen auf. Ausgehend von einer komplexen Darstellung $A_N(x) = \mathrm{Re}\{E_N \cdot e^{it(x)}\}$ mit einer reellen Phasenfunktion $t(x)$ werden Alternanten der Ordnung K in der Form

$$A_K(x) = \mathrm{Re}\{(E_N)^{K/N} \cdot e^{i(K/N)t(x)}\}$$

gewonnen. Sie genügen ähnlichen Rekursionsformeln wie die Tschebyscheffschen Polynome. Mit Hilfe der so konstruierten Alternantenfamilien können Tschebyscheffsche Approximationsaufgaben angenähert gelöst werden. Dies wird an einigen Beispielen theoretischer und numerischer Art illustriert. Die gewonnene Annäherung kann durch eine iterative Methode verbessert werden. Spezielle Untersuchungen beziehen sich auf Funktionen, die durch eine Tschebyscheff-Entwicklung, allgemeiner durch eine Entwicklung nach einer orthogonalen Alternantenfamilie gegeben sind, auf Approximationen mit rationalem Gewicht und auf Approximationen mit kleinstem relativen Fehler.

G. Meinardus (Clausthal-Zellerfeld)

Brown, A. L. 5033

Best n-dimensional approximation to sets of functions.

Proc. London Math. Soc. (3) **14** (1964), 577–594.

Let E be a subset of the normed linear space B, and for a given n-dimensional subspace L_n of B define $\delta(E, L_n) = \sup\{d(x, L_n) : x \in E\}$. Let $r_n(E) = \inf\{\delta(E, L_n) : L_n \subset B\}$. If $r_n(E) = \delta(E, L_n)$ for some n-dimensional L_n, then the latter is said to be an extremal subspace for E. The author is concerned with the existence of extremal subspaces for subsets of $C(S)$, the space of real-valued continuous functions on the compact Hausdorff space S. Theorem 8 asserts that every finite subset of $C(S)$ has an extremal n-dimensional subspace. Theorem 11 shows that for each $n \geqq 1$ there is an $(n+3)$-dimensional subset of $C(S)$ which has no extremal n-dimensional subspace, provided S satisfies a certain connectedness property. (This should be contrasted with the result of Garkavi [Izv. Akad. Nauk SSSR Ser. Mat. **26** (1962), 87–106; MR **25** #429] that in any Banach space every $(n+1)$-dimensional set has an extremal n-dimensional subspace.) Theorem 12, which is a corollary to the proof of Theorem 11, shows that for each $n \geqq 1$ there exists a continuous function of the form $\sum_{i=1}^{n+3} \varphi_i(s)\psi_i(t)$ on the unit square which has no best uniform approximation by a continuous function of the form $\sum_{i=1}^{n} f_i(s)g_i(t)$. These and other related results are for the most part proved by quite geometrical methods.

R. R. Phelps (Seattle, Wash.)

Polovina, O. I. [Polovina, A. I.] 5034

On the best approximation of continuous functions on the interval $[-1, 1]$. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 722–726.

The author shows that

$$\sup_{f \in H_\omega} E_n(f) = \tfrac{1}{2}\omega\left(\frac{\pi}{n}\right) - \varepsilon_n \omega\left(\frac{\pi}{n}\right); \quad \varepsilon_n \geqq 0; \ \varepsilon_n = O\left(\frac{1}{\ln n}\right),$$

where $\omega(t)$ is an arbitrary convex modulus of continuity, and $E_n(f)$ is the best approximation of the function $f(x)$ by algebraic polynomials of degree less than or equal to $n-1$ on the interval $[-1, 1]$. H_ω is the set of continuous functions on $[-1, 1]$ with modulus of continuity bounded by $\omega(\delta)$.

Bl. Sendov (Sofia)

Wang, Xing-hua [Wang, Hsing-hua] 5035

The exact constant of approximation of continuous functions by the Jackson singular integral.

Acta Math. Sinica **14** (1964), 231–237 (*Chinese*); *translated as Chinese Math.* **5** (1964), 254–260.

Let $J_n(f)$ denote the Jackson integral of the function f; the usual notation is so changed that $J_n(f)$ becomes a trigonometric polynomial of degree n. The author shows that the smallest constant C for which $\|f - J_n(f)\| \leq$

$C\omega(f; 1/(n+1))$ holds for all $n=0, 1, \cdots$ and all continuous functions f is $C=\frac{3}{2}$. He treats similarly another singular integral.

{Reviewer's remark: A first result in this direction is due to P. C. Sikkema [Numer. Math. **3** (1961), 107–116; MR **23** #A459]. He shows that for the Bernstein polynomial $B_n(f)$, the smallest C for which $\|f-B_n(f)\| \leq C\omega(f; n^{-1})$ for all $n=1, 2, \cdots$ and all continuous f on $[0, 1]$ is $C=(4306+837\sqrt{6})/5832$. This value is attained only for $n=6$.} *G. G. Lorentz* (Syracuse, N.Y.)

Wilf, Herbert S. 5036

The stability of smoothing by least squares.

Proc. Amer. Math. Soc. **15** (1964), 933–937.

The author uses the definition of stability for a smoothing process given by I. J. Schoenberg [Bull. Amer. Math. Soc. **59** (1953), 199–230; MR **15**, 16]. This definition is not explicitly stated in this paper. Necessary and sufficient conditions for the stability of a discrete polynomial smoothing process are known. The result of this paper may be formulated as follows. Consider the continuous least squares polynomial smoothing process: $f(x)$ is given on $[-\infty, +\infty]$, $P_x(t)$ is the polynomial of degree k or less such that $\int_{x-1}^{x+1} [f(t)-P_x(t)]^2 dt$ is minimized for all such polynomials. The smoothed value of $f(x)$ is $P_x(x)$. The following theorem is established by a detailed examination of certain definite integrals of Bessel functions. There exists K_0 such that for $k \geq K_0$, the continuous least squares polynomial smoothing process is stable. It is not known whether $K_0=0$ or not.

J. R. Rice (W. Lafayette, Ind.)

Natanson, I. P. 5037

Saturation classes in the theory of singular integrals. (Russian)

Dokl. Akad. Nauk SSSR **158** (1964), 520–523.

L'auteur considère un processus d'approximation d'une fonction $f(x)$ continue et de période 2π ($f \in C_{2\pi}$) par des intégrales singulières,

$$(1) \qquad U_n(f; x) = \int_{-\pi}^{\pi} f(t)\Phi_n(t-x)\,dt,$$

où le noyau $\Phi_n(t)$ est continu, pair, positif, de période 2π et satisfait aux conditions

$$\int_{-\pi}^{\pi} \Phi_n(t)\,dt = 1, \qquad \int_0^{\pi} t\Phi_n(t)\,dt \to 0 \qquad (n \to \infty).$$

On dit, après Favard, que ce processus d'approximation est saturé avec un degré de saturation $\varphi(n)$ ($\varphi(n)>0$, $\varphi(n) \to 0$) si, pour toute fonction $f \in C_{2\pi}$ ($f \neq$ const), on a

$$\max_x |U_n(f; x)-f(x)| > a(f)\varphi(n) \quad (a(f) > 0,\ n = 1, 2, \cdots),$$

et s'il existe une fonction $f \in C_{2\pi}$ ($f \neq$ const) telle que

$$(2) \quad \max_x |U_n(f; x)-f(x)| < b(f)\varphi(n) \qquad (n = 1, 2, \cdots).$$

L'ensemble des fonctions satisfaisant à (2) est nommé la classe de saturation $(\Phi_n, \varphi(n))$.

Si l'on ajoute aux hypothèses déjà énumérées la condition que pour chaque σ, $0<\sigma<\pi$,

$$\int_\sigma^{\pi} \Phi_n(t)\,dt = o(\gamma_n), \qquad \gamma_n = \int_0^{\pi} t^2\Phi_n(t)\,dt,$$

alors on constate que le processus d'approximation (1) est saturé avec le degré de saturation γ_n et que la classe de saturation (Φ_n, γ_n) consiste des fonctions $f(x)$ de $C_{2\pi}$ telles que la dérivée $f'(x)$ appartient à la classe Lip 1.

L'auteur énonce un résultat analogue concernant la classe de saturation lorsqu'on itère l'opération U_n; ici il s'appuie sur un travail de A. Tureckiĭ [Izv. Akad. Nauk SSSR Ser. Mat. **25** (1961), 411–442; MR **23** #A1988].

E. J. Akutowicz (Bologna)

Penkov, B.; Sendov, Bl. 5038

Entropy of the set of continuous functions of several variables. (Russian)

C. R. Acad. Bulgare Sci. **17** (1964), 335–337.

Let Δ be an s-dimensional cube in R_s with edge length l, and let $C_\Delta{}^M$ be the set of all real continuous functions defined on Δ and with values in the interval $[0, M]$. If the distance between two functions is taken to be the Hausdorff distance between their graphs in R_{s+1}, one has the strong equivalences for the entropy and the capacity of the set $C_\Delta{}^M$: $H_\varepsilon(C_\Delta{}^M) \sim C_{2\varepsilon}(C_\Delta{}^M) \sim 2^{-s+1}(l/\varepsilon)^s \log_2(M/2\varepsilon)$. A similar relation holds for a certain class of closed subsets of R_{s+1}. In earlier papers [Bŭlgar. Akad. Nauk. Izv. Mat. Inst. **6** (1962), 27–50; MR **26** #590; Vestnik Moskov. Univ. Ser. I Mat. Meh. **1962**, no. 3, 15–19; MR **25** #3358a], the authors have established this in the case $s=1$. See also Clements [Canad. J. Math. **15** (1963), 422–432; MR **28** #2190]. *G. G. Lorentz* (Syracuse, N.Y.)

Sapogov, N. A. 5039

On the norms of linear polynomial operators associated with problems of approximating continuous functions. I. (Russian. English summary)

Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. **19** (1964), no. 3, 53–68.

Let ϕ_k be a normed orthogonal system of functions on $[a, b]$. For operators $U_n(f, x)$, $f \in C[a, b]$, whose values are polynomials $P_n(x)=\sum_{k=1}^n a_k\phi_k(x)$, the author proves an identity, which, for the trigonometric case, reduces to the identity of Faber, Marcinkiewicz and Zygmund. For this latter case, let $\phi_k(x)=e^{ikx}$, $U_n(\phi_k)=\sum_{|l|\leq n} \gamma_{kl}{}^{(n)}\phi_l$. If

$$\sum_{|k|\leq n} |\gamma_{kk}{}^{(n)}-1|^2 = (4\pi^{-2}\delta_n \log n)^2, \qquad 0 \leq \delta_n \leq 1,$$

and $(1-\delta_n)\log n \to \infty$ as $n \to \infty$, then $\|U_n(f)-f\| \to 0$ cannot hold for all continuous periodic functions f. This generalizes earlier results of the author [Dokl. Akad. Nauk SSSR **143** (1962), 53–55; MR **25** #381a; ibid. **143** (1962), 1286–1288; MR **25** #381b]. Operators for which $U_n(f, x)$ is an algebraic polynomial or a linear combination of characters of a compact abelian group are also studied.

G. G. Lorentz (Syracuse, N.Y.)

Schoenberg, I. J. 5040

Spline functions and the problem of graduation.

Proc. Nat. Acad. Sci. U.S.A. **52** (1964), 947–950.

Let $I=[a, b]$ and let (x_i, y_i), $i=1, 2, \cdots, n$, be given data with $a \leq x_1 < \cdots < x_n \leq b$. Define $Jf=\int_I [d^mf(x)/dx^m]^2 dx$ and $Ef=\sum_1^n [f(x_i)-y_i]^2$. A natural generalization of Whittaker's method of graduation [Proc. Edinburgh Math. Soc. **41** (1923), 63–75] is to take $m<n$, a parameter $\varepsilon>0$ and determine $f_\varepsilon^*(x)$ so that $\varepsilon Jf+Ef$ is minimized for all

functions f with square integrable mth derivative. The solution $f_\varepsilon^*(x)$ of this problem is the graduation of the data. The main results announced here are: (i) the solution $f_\varepsilon^*(x)$ is a spline function, (ii) the set $\{f_\varepsilon^*(x) | 0 \leq \varepsilon \leq \infty\}$ is the same as the set of solutions g_u^* to the problem: minimize Eg subject to the constraint $Jg = u$ for $0 \leq u \leq Js$, where $s(x)$ is the spline interpolating function $\lim_{\varepsilon\to 0} f_\varepsilon^*(x)$. Proofs and computational remarks for these (and other) results are to appear elsewhere. The periodic case is considered briefly and analogous results announced.

J. R. Rice (W. Lafayette, Ind.)

Shapiro, Harold S. 5041

Some negative theorems of approximation theory.

Michigan Math. J. **11** (1964), 211–217.

Let X be a Banach space, $C(T)$ the space of all real continuous functions on a compact (metric) space T, Λ_ω the class of all functions f on T for which $|f(t_1) - f(t_2)| \leq \omega(\rho(t_1, t_2))$, d_n a sequence with $d_n > 0$, $d_n \to 0$. The author proves that if X_n is a sequence of proper closed subspaces of X, then there is an $x \in X$ for which $\rho(x, X_n) \neq O(d_n)$. Let X_n, Y_n be two sequences of finite-dimensional subspaces of $C(T)$, let R_n be the set of all $h = f/g \in C(T)$, $f \in X_n$, $g \in Y_n$. Then for some $f \in C(T)$, $\rho(f, R_n) \neq O(d_n)$. There is also a lower estimate of the degree of approximation of the class Λ_ω by generalized rational functions.

G. G. Lorentz (Syracuse, N.Y.)

FOURIER ANALYSIS

See also 4751, 4842, 4880, 5073, 5103, 5134.

Grišin, V. B. 5042

The approximation of functions of two variables by Fourier sums. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 566–569.

Soit $\bar{H}(\omega_1, \omega_2)$ la classe des fonctions 2π-périodiques par rapport aux x et y satisfaisant aux conditions

$$|f(x_1, y) - f(x_2, y)| \leq \omega_1(|x_2 - x_1|),$$

$$|f(x, y_1) - f(x, y_2)| \leq \omega_2(|y_2 - y_1|),$$

$$|f(x_1, y_1) - f(x_1, y_2) - f(x_2, y_1) + f(x_2, y_2)| \leq \omega_1(|x_2 - x_1|)\omega_2(|y_2 - y_1|).$$

$\omega_1(u)$ et $\omega_2(v)$ sont continues, convexes et telles que $\omega_i(0) = 0$, $0 \leq \omega_i(t_2) - \omega_i(t_1) \leq \omega_i(t_2 - t_1)$, $i = 1, 2$.

On établit la formule asymptotique pour

$$\sup_{f \in \bar{H}} |f(x, y) - S_{m,n}(x, y)|,$$

où $S_{m,n}$ désigne la somme partielle de la série de Fourier de $f(x, y)$ sous la condition que $\lim[\omega_i(\delta) \log 1/|\delta|] = 0$, $\delta \to 0$, $i = 1, 2$.

M. Tomić (Belgrade)

Kokilašvili, V. M. 5043

On a function space and Fourier coefficients. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **35** (1964), 523–530.

The paper contains eight theorems involving $E_n(f)_{L_p}$, i.e., the best approximation of a 2π-periodic function in $L_p(0, 2\pi)$ by trigonometric polynomials of degree $\leq n$, $\omega_k(f, \delta)$, i.e., the modulus of continuity of degree $k \geq 1$ for f in the metric of L_p, and the Banach space $B_{p,\gamma}^{k;\varphi}$, defined as follows: $f \in B_{p,\gamma}^{k;\varphi}$ $(1 \leq \gamma < \infty)$ if

$$\int_0^1 \frac{\omega_k^\gamma(f, t)\varphi(1/t)}{t}\, dt < \infty$$

for every function $\varphi \in L_k^\infty$, where L_k^∞ is defined as the space of non-decreasing functions φ on $[0, \infty)$ with $\varphi(0) = 0$ which fulfill the following conditions: (1) There exists a number $\beta > 0$ such that $t^{-\beta}\varphi(t) = \psi(t)$ is almost increasing on $(0, \infty)$, i.e., there exists a constant A such that $\psi(t_1) \leq A\psi(t_2)$ for all $t_1 \leq t_2 \in (0, \infty)$; (2) There exists a number α, $0 < \alpha < k$, such that $\mu(t) = t^{\alpha - k}\varphi(t)$ is almost decreasing on $(0, \infty)$, i.e., $\mu(t_1) \geq B\mu(t_2)$ for all $t_1 \leq t_2 \in (0, \infty)$. Two typical results are the following. (1) If $f \in L_p(0, 2\pi)$ and $f = F + T_1$, where T_1 is the trigonometric polynomial of degree one of best approximation in L_p for f, then

$$A \int_0^1 \frac{\omega_k^\gamma(F, t)\varphi^\gamma(t^{-1})}{t}\, dt \leq \sum_{s=0}^{\infty} E_{2^s}^\gamma(f)_{L_p}\varphi^\gamma(2^s) \leq B \int_0^1 \frac{\omega_k^\gamma(f, t)\varphi^\gamma(t^{-1})}{t}\, dt.$$

(2) If $f \sim \sum_1^\infty b_n \sin nx$ and if there exists a $\tau > 0$ such that $n^{-\tau}b_n \downarrow 0$, then $f \in B_{p,p}^{k;\varphi}$ if and only if $\sum_{k=1}^\infty b_k^p k^{p-2}\varphi^p(k) < \infty$.

G. Goes (Chicago, Ill.)

Pavlovs'kii, M. M. [Pavlovskii, N. M.] 5044

On the approximation of functions satisfying a Lipschitz condition by trigonometric polynomials. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 1311–1315.

Soient KH^α la classe des fonctions 2π-périodiques satisfaisant à la condition de Lipschitz d'ordre α $(0 < \alpha \leq 1)$ et

$$T_n^N(f, x) = \frac{2}{N} \sum_{k=1}^{N} f(x_k) D_n(x - x_k),$$

$$x_k = 2k\pi/N, \ k = 1, 2, \cdots, N \leq 2n.$$

D_n est le noyau de Dirichlet. L'auteur donne la formule asymptotique pour $E_n^N(x) = \sup_{f \in KH^\alpha} |f(x) - T_n^N(f, x)|$.

M. Tomić (Belgrade)

Videnskii, V. S. 5045

On trigonometric polynomials of half-integer order. (Russian. Armenian summary)

Izv. Akad. Nauk Armjan. SSR Ser. Fiz.-Mat. Nauk **17** (1964), *no.* 3, 133–140.

L'auteur étudie certains problèmes extrémaux dans la classe $T(n + \frac{1}{2})$ des polynômes trigonométriques ayant la forme

$$s(t) = \sum_{k=0}^{n} (a_k \cos(k + \tfrac{1}{2})t + b_k \sin(k + \tfrac{1}{2})t).$$

(1) Si $s(t)$ appartient à $T(n + \frac{1}{2})$ et si $|s(t)| \leq 1$ $(-\beta \leq t \leq \beta$, $0 < \beta < \pi)$, alors

$$|s'(t)| \leq \frac{(n + \frac{1}{2}) \cos \frac{1}{2}t}{\sqrt{(\sin^2 \frac{1}{2}\beta - \sin^2 \frac{1}{2}t)}}$$

$(-\beta < t < \beta)$. La signe d'égalité a lieu seulement pour

$$\tau_{2n+1}(t) = \gamma \cos(2n + 1)\text{arc}\cos(\sin \tfrac{1}{2}t/\sin \tfrac{1}{2}\beta)$$

$(-\beta \leq t \leq \beta$, $|\gamma| = 1)$, et seulement en les $2n + 1$ zéros de

$\tau_{2n+1}(t)$ sur le segment $[-\beta, \beta]$. (2) Dans l'hypothèse de (1), on a

$$|s'(t)| \leq \tfrac{1}{2}(2n+1)^2 \operatorname{ctg} \tfrac{1}{2}\beta = \tau'_{2n+1}(\beta) \qquad (-\beta \leq t \leq \beta),$$

pourvu que n soit supérieur à $\frac{1}{2}\sqrt{(3 \operatorname{tg}^2 \frac{1}{2}\beta + 1)}$. (3) L'auteur donne également la solution de la question (1) pour une majoration $|s(t)| \leq h(t)$, $h(t)$ étant donné dans la classe $T(m+\frac{1}{2})$, $m \leq n$.

E. J. Akutowicz (Bologna)

Aljančić, S.; Tomić, M. 5046

Sur le module de continuité intégral des séries de Fourier à coefficients convexes.

C. R. Acad. Sci. Paris **259** (1964), 1609–1611.

Suppose that $\mu(x)$ $(x \geq 0)$ satisfies the conditions: (1) $0 < \mu(x) \searrow 0$ as $x \to +\infty$; (2) $\mu(x)$ is convex; and (3) $\int_1^x \mu(t)\,dt = O[x\mu(x)]$. Then one of the main theorems of the authors is as follows. Set

$$f(x) = \frac{\mu_0}{2} + \sum_{n=1}^{\infty} \mu_n \cos nx, \qquad \mu_n = \mu(n).$$

Then $f(x)$ is L-integrable in $(0, \pi)$ and

$$\sup_{|t| \leq \pi/n} \int_0^{\pi} |f(x+t) - f(x)|\, dx = O(\mu_n).$$

The integral modulus of continuity of the sine series is also given from its Fourier coefficients. From these results and the Young theorem on convolution, the authors derive the integral modulus of continuity of such series as

$$\sum \mu_n(a_n \cos nx + b_n \sin nx)$$

under appropriate conditions. This gives an alternative proof of a theorem by S. Aljančić [Math. Z. **81** (1963), 215–222; MR **27** #1774].

G. Sunouchi (Sendai)

Mitjagin, B. S. 5047

On the absolute convergence of the series of Fourier coefficients. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 1047–1050.

The main result of this paper is the following generalization of the "negative" part of the classical result of Bernstein [C. R. Acad. Sci. Paris **199** (1934), 397–400]. Let $\Psi = (\psi_n)$ be an arbitrary complete orthonormal system of functions defined on the interval $[0, 2\pi]$. Then there exists a function f in $C^{1/2}(0, 2\pi)$ such that

$$\sum_{n=1}^{\infty} \left| \int_0^{2\pi} f(t)\psi_n(t)\, dt \right| = +\infty,$$

where $C^{1/2}(0, 2\pi)$ denotes the class of all complex-valued 2π-periodic functions such that

$$\sup_{0 \leq h \leq 2\pi} \sup_{0 \leq t \leq 2\pi} |f(t+h) - f(t)| h^{-1/2} < +\infty.$$

The proof of this result is as follows. Suppose that for some Ψ the assertion of the theorem is false. Then the linear operator $I: C^{1/2} \to l_1$, with $If = (\int_0^{2\pi} f(t)\psi_n(t)\, dt)$, is well-defined. Let us consider the diagram

$$\begin{tikzcd} & C^{1/2} \arrow[r, "I"] & l_1 \arrow[dr, "J"] & \\ L_\infty \arrow[ur, "\Lambda"] \arrow[rrr, "\tilde{\Lambda}"] & & & l_2 \end{tikzcd}$$

where Λ is the operator of multiplication; $\Lambda(1) = 1$, $\Lambda(e^{int}) = |n|^{-1/2} e^{int}$ $(n = \pm 1, \pm 2, \cdots)$; J is the natural (identical) embedding of l_1 into l_2; $\tilde{\Lambda}(f) = (c_n)$, where $c_0 = \int_0^{2\pi} f(t)\, dt$ and $c_n = n^{-1/2} \int_0^{2\pi} f(t) e^{int}\, dt$ $(n = \pm 1, \pm 2)$. Then Λ is continuous [A. Zygmund, *Trigonometric series*, Vol. II, p. 136, second edition, Cambridge Univ. Press, New York, 1959; MR **21** #6498], and J maps every unconditionally convergent series into an absolutely convergent one [the reviewer and W. Szlenk, Colloq. Math. **10** (1963), 313–323; MR **29** #452]. Therefore, $\tilde{\Lambda}$ would map every unconditionally convergent series into an absolutely summable one, but this contradicts the inequality $\sum_{n=1}^{N} \|\tilde{\Lambda}\chi_n\| \geq (\log \frac{1}{2}N)^{1/2}$, where χ_n denotes the characteristic function of the interval $[2\pi(n-1)/N, 2\pi n/N]$ $(n = 1, 2, \cdots, N;\ N = 1, 2, \cdots)$. A similar result for orthonormal systems of functions of several variables is also given.

A. Pełczyński (Warsaw)

Pippert, Raymond E. 5048

On absolutely convergent trigonometric series.

Math. Z. **85** (1964), 401–406.

The well-known theorem of Fatou is as follows. Let the sequence $\{a_n\}$ satisfy $|a_{n+1}| \leq |a_n|$ for all n. If either $\sum |a_n \cos nx_0| < \infty$ for some x_0 or $\sum |a_n \sin nx_0| < \infty$ for some $x_0 \neq 0 \pmod{\pi}$, then $\sum |a_n| < \infty$. The author generalizes the condition $|a_{n+1}| \leq |a_n|$, replacing it by $|a_{n+1}| \leq K|a_n|$ for a constant K independent of n, and shows that this new condition is the best possible in a sense. Another such generalization is given also in a theorem of Salem.

However, as the author has remarked in proof, part of the result is not new. In particular, the original idea of such a generalization is due to O. Szász [Ann. of Math. (2) **47** (1946), 213–220; MR **7**, 435].

G. Sunouchi (Sendai)

Takeda, Kazuaki; Takeda, Zirô 5049

A certain series of multiplicatively orthogonal functions.

Tôhoku Math. J. (2) **16** (1964), 34–59.

An orthonormal set $\{\varphi_{n,1}(x)\}_1^{\infty}$ is defined on $[0, 1]$, each function taking the values $\sqrt{2}$, 0, $-\sqrt{2}$ on intervals of length $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{4}$. Analogous to the completion of the Rademacher functions by the Walsh functions, this set is completed by setting $\varphi_{n,2}(x) = \varphi^2_{n,1}(x) - 1$ and taking the collection of all finite products $\{\varphi_{n_1,i_1}\varphi_{n_2,i_2} \cdots \varphi_{n_k,i_k}\}$, $n_1 > n_2 > \cdots > n_k \geq 0$, $i_j = 1$ or 2, $\varphi_{0,i}(x) \equiv 1$. Let $\{\psi_n(x)\}_0^{\infty}$ denote the complete system. The following inequality is established, analogous to one proved by Paley for the Walsh functions. If $f \in L^p[0, 1]$, $1 < p < \infty$, and $f \sim \sum_0^{\infty} c_n\psi_n$, then

$$B_p \int_0^1 \left(\sum_0^{\infty} \Delta_n^2(x) \right)^{p/2} dx < \int_0^1 |f(x)|^p\, dx < B_p' \int_0^1 \left(\sum_0^{\infty} \Delta_n^2(x) \right)^{p/2} dx,$$

where B_p and B_p' are constants, and $\Delta_n(x) = \sum_{3^n}^{3^{n+1}-1} c_j\psi_j(x)$. More general orthonormal sets of step functions were studied by Ohkuma [same J. (2) **5** (1953), 166–177; MR **15**, 867].

J. J. Price (Princeton, N.J.)

Suetin, P. K. 5050

On the representation of continuous and differentiable functions by Fourier series in Legendre polynomials. (Russian)

Dokl. Akad. Nauk SSSR **158** (1964), 1275–1277.

Using A. F. Timan's refinement of Jackson's classical result on the approximations by polynomials, the author proves (Theorem I) the uniform convergence in the closed interval $[-1, +1]$ of the Legendre series of $f(x)$, provided $f(x)$ satisfies in $[-1, +1]$ a Lipschitz condition of order exceeding $\frac{1}{2}$. *E. Kogbetliantz* (New York)

Pati, T.; Ahmad, Z. U. 5051

A new proof of a theorem on the absolute summability factors of Fourier series.

Riv. Mat. Univ. Parma (2) 4 (1963), 149–158.

T. Pati and S. R. Sinha proved a theorem on the Cesàro absolute summability factors of Fourier series [Indian J. Math. 1 (1958), no. 1, 41–54; MR **22** #2837 (where the result is quoted in full)]. Here is presented a different and shorter proof of the same theorem. *L. Lorch* (Aarhus)

Sahney, B. N. 5052

On the relation between Riemann and Abel summability of trigonometrical series.

Ann. Mat. Pura Appl. (4) **63** (1963), 123–131.

Let p be an integer ≥ 1 and s_n the nth partial sum of the series $\sum a_n$. The series $\sum_{n=1}^{\infty} a_n$ is said to be summable (R', p) to the sum zero if the series $t\sum_{\nu=1}^{\infty} s_\nu(\sin \nu t/\nu t)^p$ converges for small $t>0$ and tends to zero as $t \to 0$. The author proves that if the series $\sum a_n$ is Abel summable and if $\sum_n^{2n}(|a_\nu| - a_\nu) = O(1)$, then the series is summable (R', p) to the same value for any integer $p \geq 1$.

S. Izumi (Tokyo)

Skvorcov, V. A. 5053

The theorem of du Bois-Reymond for generalized integrals and trigonometrical series which are Abel-Poisson summable. (Russian. English summary)

Vestnik Moskov. Univ. Ser. I Mat. Meh. **1964**, *no.* 4, 16–20.

Main result: There is a trigonometric series which is A-summable for each $x \in [0, 2\pi]$ to a finite and D-integrable function $f(x)$ but which is not the Denjoy-Fourier series of $f(x)$. *N. Dinculeanu* (Bucharest)

Matuš̌ů, Josef 5054

Das Dinische Integral und die Frage der Konvergenz des Fourierschen Integrals.

Publ. Math. Debrecen **10** (1963), 203–214.

Suppose that the function f is (L) integrable in $(-\infty, \infty)$ or that f is integral in any finite interval and is of bounded variation in $(-\infty, h)$ and (h, ∞), tending to zero as $t \to \pm\infty$. It is well known that if the Dini integral

$$\int_0^d \frac{(f(t+x)+f(t-x)-2f(t))}{x}\,dx$$

exists as the Lebesgue integral (absolutely convergent integral), then the Fourier integral

$$\frac{1}{\pi}\int_0^{\infty} dy \int_{-\infty}^{\infty} f(x)\cos y(x-t)\,dx$$

converges to $f(t)$.

The author gives an example of a function f such that, at a fixed point t, (i) the Dini integral exists as the conditionally convergent integral for a finite d, but (ii) the Fourier integral does not converge.

(I) is the generalization of Kac, Salem and Zygmund [Trans. Amer. Math. Soc. **64** (1948), 235–243; MR **9**, 426] and (III) generalizes the reviewer's theorem [Tôhoku Math. J. (2) **3** (1951), 89–103; MR **14**, 868].

S. Izumi (Tokyo)

Šilov, G. E. 5055

On the Fourier transform theory of infinitely differentiable, rapidly decreasing functions. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1961**, *no.* 3 (22), 174–176.

In their book on generalized functions, Gel'fand and Šilov [*Generalized functions*, Vol. 1 (Russian), Fizmatgiz, Moscow, 1958; MR **20** #4182] study the classes $S_{\alpha,A}^{\beta,B}$ of infinitely differentiable functions $\phi(x)$ $(-\infty < x < \infty)$ satisfying the conditions

$$|x^k\phi^{(q)}(x)| \leq CA_1{}^kB_1{}^qk^{k\alpha}q^{q\beta} \quad (k, q = 0, 1, 2, \cdots)$$

for any $A_1 > A$, $B_1 > B$ and C independent of A_1 and B_1. They prove that if $\alpha > 0$, $\beta > 0$ and $\alpha+\beta > 1$, then the Fourier transform takes $S_{\alpha,A}^{\beta,B}$ onto itself. The author shows here that this remains true for $\alpha \geq 0$, $\beta \geq 0$ and $\alpha+\beta > 1$. *A. Devinatz* (St. Louis, Mo.)

Bredihina, E. A. 5056

On the approximation of almost periodic functions. (Russian)

Sibirsk. Mat. Ž. 5 (1964), 768–773.

Let Q denote the set of uniformly almost periodic functions, B_σ the set of entire functions of degree $\leq \sigma$, bounded on the whole real line, and $L(f)$ the set of absolute values of Fourier coefficients of $f \in Q$. Let us write

$$A_\sigma(f) = \inf_{\varphi \in B_\sigma} \sup_x |f(x) - \varphi(x)|,$$

$$R_\sigma(f) = \sup_x |f(x) - f_\sigma(x)|,$$

where $f_\sigma \in B_\sigma$ is a uniformly almost periodic function with Fourier series $\sum_{|\lambda_k| \leq \sigma} a_k e^{i\lambda_k x}$. A positive number σ is called an isolated point of $L(f)$ if there exists $\varepsilon > 0$ such that the interval $(\sigma, \sigma+\varepsilon)$ does not contain points from $L(f)$. In the paper under review estimations of the following type $R_\sigma(f) \leq \phi(\sigma, L(f)) \cdot A_\sigma(f)$, where σ denotes the isolated point of $L(f)$, are investigated. The results are too complicated to be quoted here. *J. Albrycht* (Poznań)

Hartman, S.; Ryll-Nardzewski, C. 5057

Almost periodic extensions of functions.

Colloq. Math. **12** (1964), 23–39.

Let R be the real line. I is the class of all sets $A \subseteq R$ such that every bounded real- or complex-valued function on A which is uniformly continuous can be extended to an almost periodic (a.p.) function on R. Property I has been considered hitherto only for the case when A is a sequence of positive numbers. In this case Strzelecki [same Colloq. **11** (1963), 91–99; MR **28** #3294] has proved that $\{a_1, a_2, \cdots\} \in$ I if (*) $a_{n+1}/a_n \geq c > 1$ for $n = 1, 2, \cdots$. The authors show that $A \in$ I if there exists an $\alpha > 0$ such that $\{X \subseteq A : \inf\{|x-y| : x \neq y,\ x, y \in X\} \geq \alpha\} \subseteq$ I. Both results give, e.g., that $\bigcup_{n=1}^{\infty} I_n \in$ I whenever I_n is a sequence of

intervals of bounded length such that some sequence $a_n \in I_n$ $(n = 1, 2, \cdots)$ satisfies (*).

Similar results are obtained for almost periodicity in the sense of Stepanoff and Riemann-Stepanoff.

For any Abelian topological group G let $\mathrm{I}(G)$ denote the class of all sets $A \subseteq G$ such that every bounded real- or complex-valued function on A can be extended to a continuous a.p. function on G (thus $\mathrm{I}(R) \subset \mathrm{I}$). It is shown that $A \in \mathrm{I}(G)$ if and only if every function on A taking only the values 0 or 1 can be extended to a continuous a.p. function on G; if G is discrete, then there are $A \in \mathrm{I}(G)$ such that $\operatorname{card} A = \operatorname{card} G$; if G is compact and its topological weight is at least $2^{\aleph_0}$, then there is an infinite set $A \in \mathrm{I}(G)$. In several proofs the Bohr compactification is used. Some open problems are mentioned. Related investigations concerning extensions to continuous periodic functions are quoted. *J. Mycielski* (Wroclaw)

Kovalenko, M. K. 5058

Approximate construction of polynomials of best approximation. (Russian)

Uspehi Mat. Nauk **18** (1963), *no.* 6 (114), 201–207.

The author describes several algorithms for the construction of polynomials of best uniform approximation that can be realized on high-speed computers. Let E_m be a finite set of abscissas $u_1, u_2, \cdots, u_m$, with $u_i < u_{i+1}$, and let $f(u)$ be a real-valued function defined on E_m. The problem is to determine the polynomial P^* of degree $\leq n$, where $n < m-2$, which best approximates f on E_m. The author introduces the set G_n of all possible $(n+2)$-tuples $x = (u_{k_1}, u_{k_2}, \cdots, u_{k_{n+2}})$ of abscissas, where $1 \leq k_1 < k_2 < \cdots < k_{n+2} \leq m$. With every $x \in G_n$ there is an easily determined polynomial $P_x(u)$ of degree $\leq n$ which best approximates f on the components of x; the corresponding deviation is denoted by $L_n(f; x)$. It is well known that the polynomial P^* may be characterized as the polynomial P_x associated with an $(n+2)$-tuple x for which $L_n(f; x)$ is a maximum. The author gives a detailed definition of a number of operators Γ which take G_n into itself and have the following property. Whenever $L_n(f; x) > 0$ and $P_x \neq P^*$, then $L_n(f; \Gamma x) > L_n(f; x)$. It follows that a finite number of applications of one of the operators Γ takes x into an $(n+2)$-tuple y such that $P_y = P^*$. The methods are used to obtain approximate polynomials of best approximation on an interval. *J. Korevaar* (La Jolla, Calif.)

Taberski, R. 5059

On double integrals and Fourier series.

Ann. Polon. Math. **15** (1964), 97–115.

Let a function $f(x, y)$ be plane-measurable with respect to Lebesgue measure in the rectangle $R = [a, b; c, d]$ and let

$$\mu\{(x, y) \in R\colon |f(x, y)| > n\} = o\left(\frac{1}{n}\right) \quad \text{as } n \to \infty.$$

Write

$$[f(x, y)]_n = f(x, y), \qquad |f(x, y)| \leq n,$$
$$= 0, \qquad |f(x, y)| > n.$$

If the Lebesgue integrals

$$(\mathrm{L}) \int_a^s \int_c^t [f(x, y)]_n \, dxdy, \qquad (s, t) \in R,$$

tend uniformly to a limit $F(s, t)$ as $n \to \infty$ in R, then f said to be T-integrable on R and F is denoted by

$$(\mathrm{T}) \int_a^b \int_c^d f(x, y) \, dxdy.$$

Suppose that the functions $g_n(x, y)$ of bounded variation in $R = [a, b; c, d]$ are uniformly bounded in R for $n = 1, 2, \cdots$. Moreover, let the variations

$$\int_a^b \int_c^d |dg_n(x, y)|, \quad \int_a^b |dg_n(x, d)|, \quad \int_c^d |dg_n(b, y)|$$

remain uniformly bounded for $n = 1, 2, \cdots$, and let

$$\lim_{n \to \infty} (\mathrm{L}) \int_a^\alpha \int_c^\gamma g_n(x, y) \, dxdy = 0$$

for each $(\alpha, \gamma) \in R$. Then the author proves that

$$\lim_{n \to \infty} (\mathrm{T}) \int_a^b \int_c^d f(x, y) g_n(x, y) \, dxdy = 0$$

for any T-integrable f on R. Besides this, the author obtains a number of results on two-dimensional analogues of Riemann-Stieltjes, Lebesgue and Titchmarsh integrals. He further establishes some relations between certain classes of functions and classes of Fourier coefficients of a double Fourier series corresponding to these classes. *B. N. Prasad* (Allahabad)

Wik, Ingemar 5060

Some examples of sets with linear independence.

Ark. Mat. **5**, 207–214 (1964).

A set E is called a uniform Kronecker set if, to every continuous function f on E, of absolute value 1, and to every $\varepsilon > 0$, there corresponds a real number t such that

$$\sup_{x \in E} |f(x) - e^{itx}| < \varepsilon.$$

A set E is called a Kronecker set if

$$\inf_{\mu \in \Gamma_0} \sup_n \left| \int_E e^{inx} \, d\mu(x) \right| = 1.$$

Here Γ_0 is the class of functions μ which are constant outside E and $\int_E |d\mu| = 1$. The author proves the following results: (1) If E is a uniform K-set, then $E \cup \{0\}$ is a K-set, but the converse is not true; (2) There exists no maximal uniform K-set; (3) Let h be a continuous positive increasing function defined for $r \geq 0$ such that $h(0) = 0$ and $\lim_{r \to 0} h(r)/r = \infty$. There exists then a perfect uniform K-set of E of positive Hausdorff measure with respect to the measure function h $(M_h(E) > 0)$. *N. Dinculeanu* (Bucharest)

Korotkov, V. B.; Gil'derman, Ju. Ĭ. 5061

The Fourier transform for abstract set functions. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 844–852.

By means of a correspondence (established in their previous paper [Gil'derman and Korotkov, same Ž. **4** (1963), 1426–1430; MR **28** #2431]) between the spaces of abstract set functions $\phi_p(X, \Omega)$ and the spaces of linear operators $(L_{p'}(\Omega) \to X)$, the authors construct Fourier transforms for

abstract set functions from ϕ_p, $1 \leq p \leq 2$. Thus, fundamental theorems concerning Fourier transforms of numerical functions are generalized for Fourier transforms of abstract set functions. *J. Albrycht* (Poznań)

Beurling, Arne 5062a
Analyse spectrale de pseudomesures.
C. R. Acad. Sci. Paris **258** (1964), 406–409.

Beurling, Arne 5062b
Analyse spectrale de pseudomesures.
C. R. Acad. Sci. Paris **258** (1964), 782–785.

Beurling, Arne 5062c
Analyse spectrale de pseudomesures. Sur les mesures préhausdorffiennes dans l'analyse harmonique.
C. R. Acad. Sci. Paris **258** (1964), 1380–1382.

Beurling, Arne 5062d
Analyse harmonique de pseudomesures. Intégration par rapport aux pseudomesures.
C. R. Acad. Sci. Paris **258** (1964), 1984–1987.

Beurling, Arne 5062e
Analyse harmonique de pseudomesures. Intégration par rapport aux pseudomesures.
C. R. Acad. Sci. Paris **258** (1964), 2959–2962.

Beurling, Arne 5062f
Analyse harmonique de pseudomesures. Intégration.
C. R. Acad. Sci. Paris **258** (1964), 3423–3425.

L'auteur appelle pseudomesure toute distribution df sur la droite dont la primitive f est localement sommable. Etant donné un espace de Banach B de pseudomesures, et un compact E sur la droite, il note B_E le sous-espace de B constitué des pseudomesures dans B à support dans E, et pose les problèmes suivants: (1) Pour quels ensembles E a-t-on $B_E=\{0\}$? (2) Si $B_E \neq \{0\}$, B_E contient-il nécessairement des mesures positives? (3) ("Synthèse spectrale") Les combinaisons linéaires de mesures positives $\mu \in B_E$ sont-elles denses dans B_E?

Les notes 1 et 2 sont consacrées à un théorème fondamental (théorème 1 et corollaire). Etant donné une famille M de fonctionnelles $df \to m(df) = \int_{-\infty}^{\infty} \Phi(|f(t+\tau) - f(t)|)dt$ définies chacune par une fonction Φ strictement convexe, paire, différentiable, nulle en 0, et par un nombre $\tau \in [0, 1]$, on dit que df est une M-minorante intérieure de dg si (1°) pour toute $m \in M$, $m(df) \leq m(dg)$, (2°) df est portée par le support de dg. Alors toute pseudomesure $dg \neq 0$ admet une M-minorante intérieure df qui est une mesure positive. Plus précisément, si dg est portée par $[-a, a]$, df maximise, sur l'ensemble des M-minorantes intérieures de dg, les formes linéaires $df \to \int \varphi df$, où φ est une fonction concave positive sur $[-a-1, a+1]$ (l'intervention de la concavité de φ dans la démonstration peut causer au lecteur une certaine surprise).

La note 3 répond aux problèmes 1 et 2 lorsque $B = \hat{B}_\alpha$ $(0 \leq \alpha \leq 1)$, l'ensemble des transformées de Fourier des fonctions F localement de carré sommable telles que $\|F\| = \sup_{T \geq 1} (\frac{1}{2} T^{-\alpha} \int_{-T}^{T} |F|^2)^{1/2} < \infty$. La réponse au problème 1 fait intervenir une "mesure préhausdorffienne" de dimension $1-\alpha$, notion ad hoc, et la réponse au problème 2 est positive.

Les trois dernières notes sont consacrées à la synthèse spectrale. Etant donné une fonction $\varphi(t)$ continue $(t \geq 0)$ telle que $0 = \varphi(0) \leq \varphi(t) \leq \varphi(t+t') \leq \varphi(t) + \varphi(t')$, on désigne par $\mathscr{C}_\varphi^0$ l'espace de Banach des pseudomesures df dont les primitives satisfont

$$\lim_{t' \to t} |f(t') - f(t)|/\varphi(|t'-t|) = 0,$$

$$(\|df\| = \sup_{t,t'} |f(t') - f(t)|/\varphi(|t'-t|)).$$

Par un procédé explicite et élémentaire, l'auteur montre que la réponse au problème 3 est positive si $B = \mathscr{C}_\varphi^0$. Ce procédé lui permet de plus de définir l'intégrale $\int_0^1 g(t)\,df(t)$ sous la conditions $\int_0^1 \varphi_f(s)^{-1} d\varphi_g(s) < \infty$ (φ_f = module de continuité de f), plus large que la condition classique de Young (note 4).

Plus généralement (note 5), l'auteur définit une intégrale $\mathbf{S} g\,df$ sous la condition (*) $\int_0^\infty \|g, \tau\| \|f, \tau\| \tau^{-2} d\tau$, où $\|f, \tau\|^2 = \int |f(t+\tau) - f(t)|^2 dt$. Cette intégrale peut être définie par transformation de Fourier. Sa propriété la plus importante est une sorte de théorème de Lebesgue, affirmant, sous certaines conditions, que $\mathbf{S} g_n\,df_n$ tend vers $\mathbf{S} g\,df$.

On peut donner (note 6) des conditions sur les transformées de Fourier de g et df, garantissant (*). On en déduit une réponse positive au problème 3 pour tous les espaces B tels que (1°) B contient topologiquement $\mathscr{F}L^\infty$, (2°) toute fonctionnelle linéaire sur B s'écrit $df \to \int_{-\infty}^{\infty} G(-x)F(x)dx$, où F est la transformée de Fourier de df, et F et G satisfont

$$\int_{-\infty}^{\infty} |G(x)|^2 \lambda(x)\,dx \int_{-\infty}^{\infty} |F(x)|^2 \lambda^{-1}(x)\,dx < \infty,$$

$\lambda(x)$ étant une fonction paire, continue, positive, égale à 1 sur $[-1, 1]$ et satisfaisant $0 < \varepsilon \leq d \log \lambda(x)/d \log x \leq 2 - \varepsilon$ sur $[1, \infty[$. Cela vaut en particulier pour $B = \hat{B}_0^\alpha$ $(1 < \alpha < 2)$, l'espace des df dont les transformées de Fourier F satisfont $\int_{-T}^{T} |F(x)|^2 dx = o(T^\alpha)$ $(T \to \infty)$.

Cette série de notes est rédigée de façon assez détaillée pour que—sauf peut-être à la fin—le lecteur ait tous les éléments des démonstrations. On ne peut que souhaiter voir d'autres œuvres inédites de l'auteur paraître dans un proche avenir. *J.-P. Kahane* (Orsay)

INTEGRAL TRANSFORMS, OPERATIONAL CALCULUS

See also 5023, 5078, 5431.

Sharpe, C. B. 5063
Upper bounds for resistance and reactance integrals of analytic functions.
SIAM Rev. **6** (1964), 126–130.

The author bounds certain integrals involving the real and imaginary parts of network frequency response functions. *Stanley Katz* (New York)

Artiaga, Lucio 5064
Generalized Hilbert kernels.
Duke Math. J. **31** (1964), 471–478.

It is shown that if $f(x) \in L^2(0, \infty)$, the formula

$$G(x) = \operatorname{sgn} x f(x) - \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{\log x^2 - \log t^2}{x - t} f(t)\,dt$$

defines for all real non-zero x a function $G(x)$ of $L^2(0, \infty)$, and the reciprocal formula, in which f and G are interchanged, holds almost everywhere.

The Mellin transform of any Fourier kernel $k(x)$ satisfies $k^*(s)k^*(1-s)=1$. Taking $k^*(s)=\cot \frac{1}{2}\pi s$, we obtain the Hilbert kernel $k(x)=2/(\pi-\pi x^2)$. The above formulae arise when $k^*(s)=\cot^2 \frac{1}{2}\pi s$, and the author remarks that similar results could be obtained by taking $k^*(s)=\cot^n \frac{1}{2}\pi s$ with $n=3, 4, 5, \cdots$. *P. Heywood* (Edinburgh)

Joshi, J. M. C. 5065

On a generalized Stieltjes transform.

Pacific J. Math. **14** (1964), 969–975.

The principal theorem proved by the author is that if

$$(1) \qquad \phi(s) = \int_0^\infty e^{-sx} F(x)\, dx,$$

where

$$(2) \quad F(x) = [\Gamma(\beta+\eta+1)/\Gamma(\alpha+\beta+\eta+1)] \times \int_0^\infty (xy)^\beta {}_1F_1(\beta+\eta+1;\alpha+\beta+\eta+1;-xy) f(y)\, dy \quad (\beta \geqq 0, \eta > 0),$$

then

$$(3) \quad \phi(s) = [\Gamma(\beta+\eta+1)/\Gamma(\alpha+\beta+\eta+1)][\Gamma(\beta+1)/s] \times \int_0^\infty (y/s)^\beta F(\alpha, \beta+1; b; -y/s) f(y)\, dy,$$

which is termed a generalized Stieltjes transform.

Inversion formulas for (2), which reduce to the Laplace transform $F(x)=\int_0^\infty e^{-xy}f(y)dy$ for $\alpha=\beta=0$, as well as for (3), are also obtained. *E. J. Scott* (Urbana, Ill.)

Joshi, J. M. C. 5066

Inversion and representation theorems for a generalized Laplace transform.

Pacific J. Math. **14** (1964), 977–985.

Using operators defined by H. Kober [Quart. J. Math. Oxford Ser. **11** (1940), 193–211; MR **2**, 191], the author obtains an inversion formula and a representation theorem for the transform

$$F(x) = \frac{\Gamma(\beta+\eta+1)}{\Gamma(\alpha+\beta+\eta+1)} \times \int_0^\infty (xy)^\beta {}_1F_1(\beta+\eta+1;\alpha+\beta+\eta+1;-xy) f(y)\, dy$$

$(\beta \geq 0, \eta \geq 0)$, which reduces to the Laplace transform $F(x)=\int_0^\infty e^{-xy}f(y)dy$ when $\alpha=\beta=0$.

E. J. Scott (Urbana, Ill.)

Singh, S. P. 5067

On generalised Hankel transform and self reciprocal functions.

Gaṇita **14** (1963), 22–42.

The author defines a generalization of the Hankel transform and proves an inversion theorem similar to that for the Hankel transform. He also develops conditions that a function be its own transform.

The kernel of the new transform occupies a half page of type. *Stanley Katz* (New York)

Singh, S. P. 5068

On kernel functions.

Gaṇita **14** (1963), 105–113.

In continuation of his earlier paper [#5067], the author develops integral transforms relating functions which are, for different choices of parameters, their own generalized Hankel transforms. *Stanley Katz* (New York)

Brand, Louis 5069

A division algebra for sequences and its associated operational calculus.

Amer. Math. Monthly **71** (1964), 719–728.

Der Verfasser ersetzt in der Operatorenrechnung von J. Mikusiński die Funktionen durch Folgen und das Faltungsprodukt durch das Cauchy-Produkt. Er erhält so ein diskretes Analogon, mit dessen Hilfe sich lineare Differenzengleichungen mit konstanten Koeffizienten auflösen lassen. Diese Übertragung wurde bereits von verschiedenen Autoren vorgenommen und ist (in etwas abgeänderter Bezeichnungsweise) auch in dem Buch [*Einführung in die Operatorenrechnung*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1962; MR **25** #5347] des Referentes zu finden. *L. Berg* (Halle)

Gesztelyi, Ernő 5070

Anwendung der Operatorenrechnung auf lineare Differentialgleichungen mit Polynom-Koeffizienten.

Publ. Math. Debrecen **10** (1963), 215–243.

This paper fills a gap left in the two standard texts of Mikusiński [*Operational calculus*, Pergamon, New York, 1959; MR **21** #4333] and Erdélyi [*Operational calculus and generalized functions*, Holt, Rinehart & Winston, New York, 1962; MR **26** #557].

The algebraic derivative of an operator is defined by $D\{f(t)\}=\{-tf(t)\}$, together with the usual formula for products and quotients. The algebraic (indefinite) integral $b=\int a$ is taken to mean $Db=a$.

In Section 1, the author starts with the definition and develops operational and existence theorems. His final result is that the distribution operators, i.e., those of the form $h^{-\lambda}f/l^k$, where $h=e^{-s}$ is the shift operator $l=\{1\}$ and f is a continuous function, are algebraically integrable.

In Section 2, the author examines functions of the type $f(t)/(-t)^n$, where $f(t) \in C^\infty$. Starting with a chosen ("bezeichnete") integral of $1/s$, an iterated starred integral $\int_*^n f$ is defined, so that we may define $\{f(t)/(-t)^n\}=\int_*^n f$ to be continuous. (This treatment is related to the finite part of an expression.)

In Section 3, the algebraic logarithm is introduced in the form that if $a=Dw/w$, then $\log w=\int a$ (which is not unique). There follows a discussion of sequences of logarithms and "algebraic powers".

The final Section 4 contains some applications of the previous work to the solution of differential equations.

J. L. Griffith (Lawrence, Kans.)

Fenyő, István 5071

Sur une extension du calcul opérationnel.

Rend. Mat. e Appl. (5) **22** (1963), 382–391.

It is supposed that $f(t)$, defined on $0 \leq t \leq \infty$, is continuous except for a finite number of algebraic singularities (at

points a_j), i.e., at each point a_j there exist positive numbers r_j and ρ_j such that

$$\lim_{t\to a_j+0} (t-\tau_j)^{r_j} f(t), \qquad \lim_{t\to a_j-0} (t-\tau_j)^{\rho_j} f(t)$$

are finite. Using Hadamard's definition of Pf ("partie finie") the convolution $t^n * f(t)$ is defined by

$$\sum_{a_j<t}\left[\mathrm{Pf}\int_{a_{j-1}}^{a_j} (t-\tau)^n f(\tau)\,d\tau + \mathrm{Pf}\int_{a_j}^{a_j} (t-\tau)^n f(\tau)\,d\tau\right] + \mathrm{Pf}\int_{a_j}^{t} (t-\tau)^n f(t)\,dt.$$

Using the notation of Mikusiński, the author defines $\{f(t)\}$ by $(1/n!)s^n * t^n * f(t)$. This definition is connected with the idea of regularisation [Gel'fand and Shilov, *Generalized functions*, Vol. I, p. 45 et seq., Academic Press, New York, 1964; MR **29** #3869]. It is now possible to extend Mikusiński's operational calculus by these new operators.

The author notes that there is some connection between his work and that of Gesztelyi [#5070]. He is also able to solve a certain equation of the Volterra type.

J. L. Griffith (Lawrence, Kans.)

Vu, Tuan 5072

On a theorem of the operational calculus. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 1315–1317.

Assume that

$$(1) \qquad f(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{pt}F(p)\,dp,$$

where σ is greater than the order of growth of $f(t)$;

$$(2) \qquad \int_{\sigma-i\infty}^{\sigma+i\infty} e^{pt}F(p)\,dp$$

is differentiable for $t \geqq 0$;

$$(3) \qquad \Phi(p, w) = \int_0^\infty e^{q(t)w}t^{-1}e^{-pt}\,dt,$$

where $q(t)$ is bounded and differentiable for $t \geqq 0$ and $q(t) \equiv 0$ for $t < 0$. The author then proves that

$$f[q(t)] = \frac{1}{4\pi^2}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{pt}\left\{\int_{\sigma-i\infty}^{\sigma+i\infty} F(w)\frac{d}{dp}\Phi(p, w)\,dw\right\}dp.$$

A. H. Stroud (Lawrence, Kans.)

Aheizer, N. I. [Ahiezer, N. I.]; Krein, M. [Krein, Mark G.] 5073

★**Some questions in the theory of moments.**

Translated by W. Fleming and D. Prill. Translations of Mathematical Monographs, Vol. 2.

American Mathematical Society, Providence, R.I., 1962. v + **265** pp. $6.10.

The original Russian version was published in 1938 [Gosudarstv. Naučno-Tehn. Izdat. Ukrain., Kharkov, 1938].

Table of Contents: The L-problem of moments (Ahiezer and Krein); General theorems about positive definite functionals (Krein); On positive definite functionals in linear normed spaces (Krein); The L-problem in an abstract linear normed space (Krein); Concerning the theorem of S. Bochner, or the continuous analogue of the trigonometric moment problem (Ahiezer); Concerning a special class of entire and meromorphic functions (Krein).

INTEGRAL EQUATIONS

See also 4856, 4861, 4986, 5549.

Rybin, P. P. 5074

On a problem of construction of solutions of nonlinear integral equations in the form of Laurent series. (Russian)

Sibirsk. Mat. Ž. **2** (1961), 127–128.

The author gives an example of an equation of the form $\varphi(x) = \int \sum_{ij} \lambda^i A_{ij}(x, s)\varphi^j(s)\,ds$, where the integral is over the unit interval and the A_{ij} are all continuous and uniformly bounded in L_2 on the unit interval, such that $\lambda = 0$ is an isolated essential singularity of a solution $\varphi(x, \lambda)$.

D. C. Kleinecke (Santa Barbara, Calif.)

Ashour, Attia A. 5075

An integral equation for the associated Legendre function of the first kind.

J. Mathematical Phys. **5** (1964), 1421–1423.

The solution of the homogeneous Fredholm equation

$$\psi(\xi) = \lambda\int_0^1 M(\xi, \xi')\psi(\xi')\,d\xi',$$

where

$$M(\xi, \xi') = \int_0^\pi \frac{\cos m\phi\,d\phi}{(x^2 - 2xx'\cos\phi + x'^2)^{1/2}}$$

and $x = (1-\xi^2)^{1/2}$, is found to be the associated Legendre function $P_n^m(\xi)$, $n+m$ even, and the characteristic numbers of this kernel are obtained. The solution of the corresponding inhomogeneous equation is then found. Finally, the kernel of a homogeneous equation whose solution is $P_n^m(\xi)$, $n+m$ odd, is obtained. The integral equations considered arise in various branches of electromagnetism.

W. D. Collins (Manchester)

Bittner, Leonhard; Hirche, Joachim 5076

Über eine singuläre Integralgleichung mit Anwendung in der Theorie des schwingenden Gitters.

Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe **13** (1964), 41–53.

Die Autoren untersuchen die folgende singuläre Integralgleichung mit Cauchyschen Hauptwertintegralen

$$(1) \qquad A(t_0)\varphi(t_0) + \sum_{\nu=1}^{n}\frac{B_\nu(t_0)}{\pi i}\int_L \frac{C_\nu(t)\varphi(t)}{t-t_0}\,dt = f(t_0), \quad t_0 \in L,$$

worin L ein System von p glatten und doppelpunktfreien, getrennt liegenden Bogen L_j in der komplexen z-Ebene bezeichnen mögen. Gegeben sind die Funktionen $A(t)$, $B_\nu(t)$, $C_\nu(t)$ und $f(t)$, die auf L einer Hölderbedingung genügen sollen, während die gesuchte Funktion $\varphi(t)$ in jedem abgeschlossenen Teilbogen der L_j, der keinen der Endpunkte c enthalte, einer gleichmäßigen Hölderbedingung

$$(2) \qquad |\varphi(t_1) - \varphi(t_2)| \leq A\cdot|t_1 - t_2|^\mu, \qquad 0 < \mu \leq 1,\ A > 0,$$

genügen soll. In den Umgebungen der Endpunkte c gelte $\varphi(t) \in H^*$, d.h.

(3) $\varphi(t) = \varphi^*(t)\cdot(t-c)^{-\gamma}$ mit $\gamma = \alpha + i\beta$, $0 \leq \alpha < 1$,

und dort Hölderstetigem $\varphi^*(t)\cdot(t-c)^\gamma$ sei ein beliebiger Zweig der sich für $t \in L$ stetig ändert. Weiterhin wird vorausgesetzt:

(4) $$A^2(t) - \left[\sum_{\nu=1}^{n} B_\nu(t)\cdot C_\nu(t)\right]^2 \neq 0 \quad \text{für alle} \quad t \in L.$$

Im Spezialfall $n=1$, $B_1(t)=1$ oder $C_1(t)=1$ ist eine explizite Lösung von (1) in dem Buche von N. I. Mushelišvili [*Singular integral equations* (russisch), OGIZ, Moscow, 1946; MR 8, 586] angegeben. Wie in diesem Buche führen die beiden Autoren hier (1) auf ein Hilbertsches Randwertproblem zurück. Sie definieren dazu:

(5) $$\Phi_\nu(z) = \frac{1}{2\pi i}\int_L \frac{C_\nu(t)\varphi(t)}{t-z}\,dt \quad \text{für } \nu = 1, \cdots, n,\ z \notin L,$$

welches außerhalb L holomorphe und im Unendlichen verschwindende Funktionen sind. Die Formeln von Sokhoski-Plemelj besagen unter anderem

(6) $C_\nu(t)\varphi(t) = \Phi_\nu^+(t) - \Phi_\nu^-(t)$ für $\nu = 1, \cdots, n$, $t \in L$,

wenn $\Phi_\nu^\pm(t)$ die Randwerte von $\Phi_\nu(z)$ für $z \to t \in L$ von links bzw. rechts bei gegebener Orientierung der Bogen L_j von L bezeichnen. Nach einigen Umformungen resultieren die folgenden Randwertrelationen, die zur Integralgleichung (1) völlig äquivalent sind:

(7) $$\Phi_\mu^+(t) - \Phi_\mu^-(t) = \left[-2C_\mu(t)\cdot\sum_{\nu=1}^{n} B_\nu(t)\cdot\Phi_\nu^-(t) + C_\mu(t)\cdot f(t)\right]$$
$$\cdot\left[A(t) + \sum_{\lambda=1}^{n} B_\lambda(t)\cdot C_\lambda(t)\right]^{-1}, \qquad \mu = 1, \cdots, n,\ t \in L.$$

In Vektorschreibweise lauten die Relationen

(8) $$\Phi^+(t) = G(t)\cdot\Phi^-(t) + g(t) \quad \text{für } t \in L,$$

wobei die Elemente $G_{\mu\nu}(t)$ der Matrix $G(t)$ die spezielle Form haben

(9) $$G_{\nu\nu}(t) = 1 + \varepsilon_\nu(t)\cdot\delta_\nu(t), \qquad G_{\mu\nu}(t) = \varepsilon_\mu(t)\cdot\delta_\nu(t) \quad \text{für } \mu \neq \nu.$$

Das Hilbertsche Randwertproblem (8) für den außerhalb L holomorphen Funktionenvektor $\Phi(z)$ ist in der bekannten Weise [siehe das Buch von N. I. Mushelišvili, Kap. 18, S. 381, loc. cit.] so zu lösen, daß erst die Lösungen des homogenen Problems ($g(t)=0$) bestimmt werden und dann eine spezielle Lösung der inhomogenen Aufgabe konstruiert wird. Die Hauptschwierigkeit dabei ist das Auffinden einer homogenen Lösungsbasis, die die Autoren für die Spezialfälle (a) die $\delta_\nu(t)$ sind—abgesehen von einem gemeinsamen Faktor— Polynome, (b) die $\varepsilon_\mu(t)$ sind Polynome—bis auf einen gemeinsamen Faktor, (c) $A(t)=0$, $B_1(t)=C_2(t)$, $B_2(t)=C_1(t)$ $(n=2)$ mit $C_1\cdot C_2 \neq 0$ jedoch explizit angeben können.

Im dritten Abschnitt reduzieren die Autoren das Integralgleichungssystem für $\gamma_k(\xi)$, $k=0, \cdots, N-1$:

(10) $$q_l(x) = \frac{1}{2aN}\sum_{k=0}^{N-1}\int_{-c}^{+c}\gamma_k(\xi) \cdot \frac{e^{\pi i(l-k)/aN}\sinh(\pi(x-\xi)/aN) + \sin^2(\pi(l-k)/N)}{\sinh^2(\pi(x-\xi)/aN) + \sin^2(\pi(l-k)/N)}\,d\xi,$$

$l=0, 1, \cdots, N-1$; $-c < x < +c$; auf N Integralgleichungen vom speziellen Typus (c). Dieses Integralgleichungssytem tritt in der Aerodynamik instationärer Gitterströmungen auf und wurde von H. Söhngen und dem Referenten [Z. Angew. Math. Mech. 38 (1958), 442–465; MR 21 #1067] aufgestellt. Die explizite Lösung von (10) mit der Nebenbedingung $\gamma_k(\xi) = O(1)$ für $\xi \to c$ ist zu lang, als daß sie hier niedergeschrieben werden könnte.

E. Meister (Saarbrücken)

Čibrikova, L. I. 5077

On the solution of some complete singular integral equations. (Russian)

Kazan. Gos. Univ. Učen. Zap. **122** (1962), *kn.* 3, 95–124.

The solutions of several singular integral equations are given. Some of these equations have the Hilbert kernel $\cot\frac{1}{2}(t-x)$, and $\sin\frac{1}{2}(t-x)$. Other kernels involve Jacobi's theta function. The consideration of equations with such kernels leads to the Riemann boundary-value problem for doubly periodic functions. Equations with periodic and quasi-periodic kernels are studied. The number of solutions of Riemann's boundary-value problem corresponding to a given integral equation is determined completely by the index of the equation, and it is shown to be independent of the type of the fundamental region.

H. P. Thielman (Alexandria, Va.)

Guseinov, A. I.; Muhtarov, H. Š. 5078

Some properties of a linear singular integral operator with Hilbert kernel in a generalized Hölder class. (Russian. Azerbaijani summary)

Akad. Nauk Azerbaĭdžan. SSR Dokl. **20** (1964), *no.* 3, 3–8.

Let H be the class of periodic functions $U(x)$ of period 2π such that $|U(x+\Delta x) - U(x)| \leq \sum_{n=1}^{\infty} K_n|\Delta x|^{\delta_n}$ for some sequences $0 < \delta_n < 1$ and $0 \leq K_n$, where $\delta_n \to 0$ and $\sum_{n=1}^{\infty} K_n < \infty$. Then if $\sum_{n=1}^{\infty}(K_n/\delta_n) < \infty$, the function $V(x) = (1/2\pi)\int_{-\pi}^{\pi} U(s)\cot\frac{1}{2}(s-x)\,ds$ is also in H. If $f(x, s)$ is in H as a function of both variables and $\delta_n(s) < \delta_n(x)$, then if $\sum_{n=1}^{\infty}(K_n(s)/\delta_n(s)) < \infty$ and $\sum_{n=1}^{\infty}(K_n(x)/(\delta_n(x) - \delta_n(s))) < \infty$, the function $\varphi(x) = \int_{-\pi}^{\pi} f(x, s)\cot\frac{1}{2}(s-x)\,ds$ is in H.

D. C. Kleinecke (Santa Barbara, Calif.)

Krupnik, N. Ja. 5079

Higher-dimensional singular operators in spaces of fundamental and generalized functions. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 41–44.

The author extends some results on elliptic systems of singular integral operators on Euclidean space E_m to new classes of functions, and with fewer restrictions on the kernels than are usually made. The operators are of the form $A\varphi(x) = a(x)\varphi(x) + \lim_{\varepsilon\to 0}\int h(x, x-y)\,dy$, where $h(x, z)$ is homogeneous of degree m in z and $\int_{|z|=1} h(x, z)\,d\sigma_z = 0$, and the adjoints of these. The symbol of A is $\sigma(A)(x, \zeta) = a(x) + \lim_{\varepsilon\to 0}\int_{\varepsilon<|z|<\varepsilon^{-1}} e^{-i(z,\zeta)}h(x, z)\,dz$. $\sigma(A)$ is said to be in G_n if $\sigma(A)(x, \zeta)$ and certain of its derivatives extend continuously on $S^m \times S^{m-1}$, where S^m is the one-point compactification of the sphere and S^{m-1} is $\{|\zeta| = 1\}$; and the derivatives $D_x^{\,q}D_\zeta^{\,p}\sigma(A)(x, \zeta)$ of order $q+p$, with $q \leq n$ and $p \leq m + [m/2] + 2$, are not too bad. These operators act on the Sobolev spaces $W_p^{\,l}(E_m)$ $(1 < p < \infty,\ |l| \leq n)$, and for

$n=\infty$, on the space $\mathcal{M}$ of functions all of whose derivatives are $O(|x|^{\varepsilon-n})$ for all $\varepsilon>0$, as $|x|\to\infty$. In either case, the author states the usual theorem: If $\inf|\sigma(A)|>0$, then A has closed range of finite codimension, and a finite null space. Further, the index of A is the same on all these spaces. He does not suggest that any very new techniques are necessary for these results, or that the assumptions on $\sigma(A)$ cannot be further reduced.

He concludes with a result showing that non-zero operators of the type considered here cannot map certain types of spaces into others; these are such as to make the choice of $\mathcal{M}$ (as above) appear reasonable.

R. T. Seeley (Waltham, Mass.)

Massera, J. L. 5080

Sur une équation intégrale provenant d'un problème de mécanique des fluides. (Spanish summary)

Bol. Fac. Ingen. Agrimens. Montevideo **8** (1962), 27–43.

The author investigates the integral equation

$$\int_{x-1}^{x}[f(x)-f(t)]^{-1/2}\,dt = A,$$

where A is a constant. The integral equation is to be satisfied for $x\geqq 0$, and $f(x)$ must coincide with a given, bounded, measurable function $f_0(x)$ on the interval $-1\leqq x<0$.

A continuous function $\bar f$ is constructed to satisfy the integral equation except possibly on a set of the first category; this function $\bar f$ becomes a strict solution for all $x\geqq 0$ if f_0 satisfies the additional condition

$$f_0(y)-f_0(x) \geqq p_0(y-x) \quad \text{for some } p_0>0,$$
$$-1\leqq x\leqq y<0.$$

The author derives a number of properties of $\bar f$ and lists some related unsolved questions.

I. Stakgold (Evanston, Ill.)

Plamenevskiĭ, B. A. 5081

Singular integral equations on an infinite cylinder. (Russian)

Dokl. Akad. Nauk SSSR **158** (1964), 780–783.

Let $x\in R^k$, $y\in R^{m-k}$, $(x,y)\in R^m$ be points in Euclidean space, and let $C_{m,k}$ be the cylinder obtained from R^m by reducing the first k variables mod 1. For u in $L^1(C_{k,m})$, let $\hat u(n,y)$ (n a k-tuple of integers) be the Fourier transform of u. The author considers operators of the form

$$(1)\quad Au(\bar x,\bar y) = \sum_n\int_{R^{m-k}}\phi(\bar x,\bar y;n,y)\hat u(n,y)e^{i(2\pi n,\bar x)+i(y,\bar y)}\,dy,$$

where $\phi(\bar x,\bar y;x,y)$ is bounded, measurable, and periodic in $\bar x$ with period 1, and homogeneous of degree zero in (x,y). From the results of Marcinkiewicz and Mihlin on the multipliers of Fourier series and integrals, he concludes that if ϕ is independent of $(\bar x,\bar y)$, and if the derivatives of $\phi(\bar x,\bar y;x,y)$ with respect to x and y of order $\leqq m$ are continuous for $|x|^2+|y|^2>0$, then A is bounded on $L^p(C_{m,k})$ for $1<p<\infty$. Then, by expanding $\phi(\bar x,\bar y;x,y)=\sum a_{jl}(\bar x,\bar y)Y_{jl}(x,y)$ in spherical harmonics, he concludes that if the derivatives of $\phi(\bar x,\bar y;x,y)$ with respect to x and y of order $\leqq l$ ($l\geqq\frac12 m-1$) are square integrable on $|x|^2+|y|^2=1$, then the same result holds. Finally, if ϕ_1 and ϕ_2 have, in addition, limits as $\bar y\to\infty$, and A_j corresponds to ϕ_j as in (1), while A corresponds to the product $\phi_1\phi_2=\phi$, then $A-A_1A_2$ is compact on $L^p(C_{m,k})$ for $1<p<\infty$.

R. T. Seeley (Waltham, Mass.)

Mac Nerney, J. S. 5082

A nonlinear integral operation.

Illinois J. Math. **8** (1964), 621–638.

This paper extends the theory of solutions of a general linear integral equation developed earlier [same J. **7** (1963), 148–173; MR **26** #1726] to a nonlinear case. S is a linearly ordered set of elements x, y, z; G is a normed complete Abelian group of elements P, Q; H is the class of all functions on G to G to which $\{0,0\}$ belongs, with 1 the identity function. Then addition in H is defined in the usual way, but the product h_2h_1 is $h_2(h_1(P))$. For interval functions $U(x,y)$ to H, it is then possible to define sum integrals $\int_x{}^y$ or ${}_x\sum^y$ and product integrals ${}_x\prod^y$. Product integrals of this type were introduced by G. Birkhoff [J. Math. Phys. Mass. Inst. Tech. **16** (1937), 104–132]. As in the linear case, it is shown that there exists a correspondence between the class $\mathcal{OA}$ of additive (interval) functions $V(x,y)$ to H subject to a Lipschitz condition that there exists a non-negative additive $\alpha(x,y)$ such that for P and Q in G, we have $\|V(x,y)P-V(x,y)Q\|\leqq\alpha(x,y)\|P-Q\|$ and the class $\mathcal{OM}$ of multiplicative (interval) functions $W(x,y)$ to H satisfying the Lipschitz condition that there exists a real-valued multiplicative function $\mu(x,y)\geqq 1$ such that for all P and Q in G we have

$$\|(W(x,y)-1)P-(W(x,y)-1)Q\| \leqq (\mu(x,y)-1)\|P-Q\|.$$

This correspondence is determined by the conditions $V(x,y)P={}_x\sum^y[W-1]P$ and $W(x,y)P={}_x\prod^y[1+V]P$. If $V(x,y)$ belongs to $\mathcal{OA}$ and $u(x)$ on S to G satisfies a bounded variation condition that there exists a non-negative $\beta(x,y)$ such that $\|u(x)-u(y)\|\leqq\beta(x,y)$, then the integral $\int_x{}^y V\cdot u$ based on $V(s,t)u(t)$ exists and the solution of the integral equation $u(x)=P+\int_x{}^c V\cdot u$ is given by $u(x)=W(x,c)P$. This solution can also be obtained by iteration based on $h_n(x)=P+\int_x{}^c V\cdot h_{n-1}$, where h_0 can be assumed to belong to the extension by uniform convergence of functions of the type $u(x)$. The theory is extended to the case when S is replaced by a canonically ordered semi-group. There is an example related to the differential equation for the arctangent function.

T. H. Hildebrandt (Ann Arbor, Mich.)

Ahmedov, K. T. 5083

Particular solutions for a certain class of integro-differential equations. (Russian)

Dokl. Akad. Nauk SSSR **128** (1959), 443–446.

FUNCTIONAL ANALYSIS

See also 4838, 4852, 4902, 4965, 4985, 4987, 5033, 5038, 5039, 5055, 5061, 5073, 5079, 5188.

Douady, Adrien 5084

Espace des sous-modules d'un module de Banach.

C. R. Acad. Sci. Paris **258** (1964), 5783–5785.

The author studies the local structure of the set $\mathcal{G}_A(E)$ of

trong submodules of a Banach module E over a Banach lgebra with unit A, considered as a subspace of the nalytic Banach manifold $\mathscr{G}(E)$ of the strong subspaces, he grassmannian of E. Attention is thus focused on strong ubspaces and strong homomorphisms; each such subpace is closed and has a closed supplementary subspace, nd each such homomorphism has a strong kernel and a trong image. Theorem: Let E be a Banach A-module, $F \in \mathscr{G}_A(E)$ be 1-presentable such that $\mathrm{Ext}_A{}^1(F; E/F)$ be trong. There exists a submanifold S of $\mathscr{G}(E)$, containing F nd such that the tangent space to S at F is $\mathrm{Hom}_A(F; E/F)$, nd there is an analytic mapping ω of S into $\mathrm{Ext}_A{}^1(F; E/F)$ uch that $\omega^{-1}(0)$ and $\mathscr{G}_A(E)$ coincide around F. In addition, is tangent to 0, and its expansion limited to second order s given by $f \to f^* f_*(\alpha)$, where α is the element of $\mathrm{Ext}_A{}^1(E/F; F)$ corresponding to the exact sequence $\to F \to E \to E/F \to 0$. The proof is based on suitable implicit function and fibration arguments.

L. Nachbin (Rochester, N.Y.)

Garling, D. J. H. 5085

Weak Cauchy sequences in normed linear spaces.

Proc. Cambridge Philos. Soc. **60** (1964), 817–819.

The author proves the following propositions. (1) If a subpace of (c_0) is isomorphic to the dual of a normed linear pace, it is finite-dimensional. {From the author's arguments it follows only that if a subspace of (c_0) is isometrically isomorphic to the dual of a normed linear space, hen it is finite-dimensional. However, proposition (1) s true and follows easily from Theorem 4 of Bessaga nd Pełczyński [Studia Math. **17** (1958), 151–164; MR **22** #5872].} (2) If the unit ball of a normed linear space E is ontained in the closed absolutely convex cover of the weak Cauchy sequence of points of E, then E is finite-dimensional. (3) If $(E, \|\ \|, \|\ \|^*)$ is an infinite-dimensional -compact two-norm space, and if $(E, \|\ \|)$ is a reflexive Banach space, then there exists no γ-Cauchy sequence whose γ-closed absolutely convex cover contains the γ-ompact set $\{x : \|x\| \leq 1\}$.

A. Wiweger (Warsaw)

Fréchet, Maurice 5086

L'espace dont chaque élément est une courbe n'est qu'un semi-espace de Banach. II.

Ann. Sci. École Norm. Sup. (3) **80** (1963), 135–137.

L'auteur remplace les démonstrations figurant aux pages 65–270 d'un précédent Mémoire [Ann. Sci. École Norm. Sup. (3) **78** (1961), 241–272; MR **25** #1430] par de nouvelles émonstrations qui aboutissent aux mêmes conclusions. Il 'agit de donner un exemple dans lequel la condition de etranchement: si $\xi + \xi_1 = \xi + \xi_2$, alors $\xi_1 = \xi_2$, n'est pas oujours vérifiée.

E. R. Lorch (New York)

Edwards, R. E. 5087

A fixed point theorem with applications to convolution equations.

J. Austral. Math. Soc. **3** (1963), 385–395.

Edelstein [J. London Math. Soc. **37** (1962), 74–79; MR **24** #A2936] proved the following extension of Banach's contraction principle: If F is a self-map of a metric space M uch that $d(F(x), F(y)) < d(x, y)$ for any distinct x and y in M, and if x_0 is a point of M such that the sequence of terates $x_n = F^n(x_0)$ contains a convergent subsequence, then the limit of this subsequence is the unique fixpoint of F.

The author applies this result to a subset M of a topological vector space with a suitable choice of metric. He obtains some general theorems which are then applied to convolution equations over groups.

I. N. Baker (London)

Lindenstrauss, Joram 5088

On nonlinear projections in Banach spaces.

Michigan Math. J. **11** (1964), 263–287.

Let Y be a metric space, let X be its subspace isometric to a Banach space (i.e., if $d(\ ,\ )$ is the metric of Y and $\|\cdot\|$ is the norm of X, then $d(x_1, x_2) = \|x_1 - x_2\|$ for $x_1, x_2 \in X$). A mapping $r : Y \to X$ is said to be a uniform projection [lipschitzian projection] if r is uniformly continuous [satisfies a Lipschitz condition] and $rx = x$ for each x in X. The main subject of the paper under review is the study of such projections.

The author shows that if Y itself is a normed linear space, then the existence of such projections is closely related to the existence of bounded linear projections. In fact, the following surprising result is obtained: Let Y be a normed linear space, let X be a closed linear subspace of Y, and let there exist a uniform projection from Y onto X. Then there exists a bounded linear mapping $T : Y \to X^{**}$ such that $Tx = \kappa x$ for $x \in X$, where $\kappa : X \to X^{**}$ is the canonical embedding, and X^{**} denotes the second dual to X. Hence, if there exists a bounded linear projection from X^{**} onto X (in particular, if X is reflexive, more generally, if X is isomorphic to a conjugate space to a Banach space, or if X is an abstract L-space), then the existence of a uniform projection on X implies the existence of a bounded linear one. Dual results on uniformly continuous liftings and other interesting corollaries are given. If there is no bounded linear projection from X^{**} onto κX, then, in general, the existence of a uniform projection does not imply the existence of a bounded linear one (an example: the space m and its subspace c_0). This is a consequence of the following results generalizing results of J. R. Isbell [Pacific J. Math. **11** (1961), 609–648; MR **25** #4487]. Let $C_u(K)$ denote the space of all bounded real-valued and uniformly continuous functions (with the sup norm) on a metric space K. Then for every metric space Y which contains $C_u(K)$ isometrically there is a lipschitzian projection from Y onto $C_u(K)$. A similar result is true in the case where K is a one-point compactification of a discrete set as well as in a slightly more general situation. If K is an arbitrary compact Hausdorff space, then it is shown only that for every metric space Y containing isometrically the space $C(K) = C_u(K)$ and such that the set $Y \setminus C(K)$ is finite, there is a lipschitzian projection from Y onto $C(K)$ (with a Lipschitz constant ≤ 2).

If X is a uniformly convex Banach space with modulus of convexity $\delta(\varepsilon)$, then (1) from every metric space $Y \supset X$ with $d(y, X) \leq \alpha$ for all $y \in Y$ there is a uniform projection onto X whose modulus of continuity $\phi(\varepsilon)$ satisfies the condition $\phi(\varepsilon) \leq K\alpha\delta^{-1}(\varepsilon/\alpha)$; (2) from every metric space $Y \supset S_X(0, 1)$ ($S_X(0,1)$ denotes the unit cell of X) there is a uniform projection onto $S_X(0, 1)$ whose modulus of continuity $\phi(\varepsilon)$ satisfies the condition $\phi(\varepsilon) \leq K\delta^{-1}(\varepsilon)$ (K is a universal constant which does not depend on X and Y). It is shown that in the case of $L_p(\mu)$-spaces these moduli of continuity are in a sense the best possible. This implies in

particular that the Banach space $X=(l_{p_1}\oplus l_{p_2}\oplus\cdots)$, with $p_k\to+\infty$ is not a uniformly continuous retract of any of its uniform neighbourhoods in m. This is a solution of a problem raised by Isbell [loc. cit.].

The results given above are applied to the problem of the classification of Banach spaces with respect to the equivalent uniform structures [cf. C. Bessaga and the reviewer, General Topology and its Relations to Modern Analysis and Algebra (Proc. Sympos., Prague, 1961), pp. 87–90, Academic Press, New York, 1962; MR 26 #2855]. Banach spaces X and Y are said to be uniformly homeomorphic provided that there exists a homeomorphism T from X onto Y such that both T and T^{-1} are uniformly continuous. The author shows: (a) A space $C(K)$ is not uniformly homeomorphic to any reflexive infinite-dimensional Banach space (more generally, to any Banach space X such that there is a projection from X^{**} onto κX, but X is not a $\mathfrak{P}$-space, i.e., X does not have the extension property); (b) If $X_1=L_{p_1}(\mu_1)$, $X_2=L_{p_2}(\mu_2)$, and if $p_1>\max(2,p_2)$, then X_1 is not uniformly homeomorphic to X_2; (c) There is a reflexive space which is not uniformly homeomorphic to any uniformly convex space; (d) The space c_0 has an uncountable number of mutually non-uniformly homeomorphic closed linear subspaces.

Another application is a partial negative answer to a problem of Smirnov [cf. E. A. Gorin, Uspehi Mat. Nauk 14 (1959), no. 5 (89), 129–134; MR 22 #967] whether $C[0,1]$ is uniformly homeomorphic to a subset of l_2.

A. Pełczyński (Warsaw)

Lindenstrauss, Joram 5089

On the extension of operators with range in a $C(K)$ space.

Proc. Amer. Math. Soc. 15 (1964), 218–225.

Theorem: Let K be a compact Hausdorff space and let $X=C(K)$ be the space of all the continuous real-valued functions on K with the supremum norm. The following four statements are equivalent. (i) For every two Banach spaces $Z\supset Y$ with $\dim(Z/Y)=1$ and every operator T from Y into X with a separable range there is an extension $\hat{T}$ of T from Z into X with $\|\hat{T}\|=\|T\|$. (ii) For every two Banach spaces $Z\supset Y$ with $\dim Y=2$, $\dim Z=3$ and every operator T from Y into X there is an extension $\hat{T}$ of T from Z into X with $\|\hat{T}\|=\|T\|$. (iii) There is a $\lambda<2$ such that for every two Banach spaces $Z\supset Y$ with $\dim(Z/Y)=1$ and every operator T from Y into X with a separable range there is an extension $\hat{T}$ of T from Z into X with $\|\hat{T}\|\leq\lambda\|T\|$. (iv) K is an F-space in the terminology of the reviewer and Henriksen [Trans. Amer. Math. Soc. 82 (1956), 366–391; MR 18, 9], that is, for every $f\in C(K)$ there is a $g\in C(K)$ such that $f(k)>0$ implies $g(k)\geqq 1$ and $f(k)<0$ implies $g(k)\leqq -1$. Furthermore, if the continuum hypothesis is true, then the following statement is also equivalent to the preceding ones. (v) For every two Banach spaces $Z\supset Y$ and every operator T from Y into X with a separable range there is an extension $\hat{T}$ of T from Z into X with $\|\hat{T}\|=\|T\|$. The author remarks that the equivalence of (i) with (iv) is essentially due to Aronszajn and Panitchpakdi [Pacific J. Math. 6 (1956), 405–439; MR 18, 917].

L. Gillman (Rochester, N.Y.)

Nirenberg, R.; Panzone, R. 5090

On the spaces L^λ which are isomorphic to a B^*.

Rev. Un. Mat. Argentina 21, 119–130 (1963).

The present paper concerns the problem of Dieudonné [Arch. Math. 10 (1959), 151–152; MR 21 #3763]: "characterize abstract L-spaces which are (isometrically) isomorphic to the dual of another Banach space". The authors discuss the actual situation of this subject (an exhaustive list of references is given). Under a certain very special additional assumption (too complicated to state here) the authors show that a space $L_1(X,\Sigma,\mu)$ for σ-finite μ is not isomorphic to a dual space of a subspace of $L_\infty(X,\Sigma,\mu)$.

A. Pełczyński (Warsaw)

Onicescu, O.; Simboan, G.; Theodorescu, R. 5091

On the Weierstrass approximation theorem.

C. R. Acad. Bulgare Sci. 17 (1964), 789–792.

In this generalization of the Weierstrass theorem, the problem is to approximate a continuous, vector-valued function defined on a compact, convex subset A of a Banach space X. The approximating functions are obtained by averaging with respect to a non-negative measure on the Borel subsets of X. The usual form of the Weierstrass theorem is obtained when A is a rectangle and X is R^n.

L. de Branges (Lafayette, Ind.)

Retherford, J. R. 5092

A note on unconditional bases.

Proc. Amer. Math. Soc. 15 (1964), 899–901.

A sequence of nontrivial subspaces $\{M_i\}$ of a Banach space X is said to be a basis of subspaces for X if for every $x\in X$, x can be uniquely written $x=\sum_{i=1}^\infty x_i$, $x_i\in M_i$. The author considers necessary and sufficient conditions that the convergence be unconditional.

R. G. Douglas (Ann Arbor, Mich.)

Thorp, E. O. 5093

Some Banach spaces congruent to their conjugates.

J. London Math. Soc. 39 (1964), 703–705.

The author exhibits a large class of Banach spaces Z which are congruent (i.e., isometrically isomorphic) to their conjugates Z^*. Starting from a space $(X,|\cdot|)$ which is congruent to X^{**}, endow the direct sum $Z=X\oplus X^*$ with a norm $\|(x,x^*)\|=\|(|x|,|x^*|)\|$, derived from a norm $\|\cdot\|$ on R^2 which coincides on the non-negative quadrant with some multiple of the Euclidean one. An implementing congruence $\varphi: Z^*\to Z$ is defined by $\varphi(z^*)=(z_1^*,z_2^*)$, where $z_1^*(x)=z^*((x,0))$ and $z_2^*(x^*)=z^*((0,x^*))$. {The reviewer points out that E. R. Lorch has established a similar result, however, with the Euclidean norm in R^2 [cf. Ann. of Math. (2) 46 (1945), 468–473; MR 7, 125]; Lorch also gives a more restrictive interpretation of selfadjointness, leading to a characterization of Hilbert space.}

R. A. Hirschfeld (Nijmegen)

Yamamuro, Sadayuki 5094

On Beurling-Livingstone's theory on the Banach space with duality mapping.

Yokohama Math. J. 11 (1963), 1–4.

It is proved here that if a conditionally σ-complete Banach lattice admits a duality mapping T such that $x\cap y=0$ implies $T(x+y)=T(x)+T(y)$, then, whenever we have $x=x_1+x_2$, $y=y_1+y_2$ with $x_1\cap x_2=0$, $y_1\cap y_2=0$, $\|x_i\|=\|y_i\|$, we also have $\|x\|=\|y\|$. A discussion of some consequences is given.

P. C. Deliyannis (Chicago, Ill.)

[...]mamuro, Sadayuki 5095

Some fixed point theorems in locally convex linear spaces.

Yokohama Math. J. **11** (1963), 5–12.

[...] this paper a number of fixed-point theorems are estab[...]hed for a completely continuous map F on the closure $\bar{G}$ [...] an open set G in some locally convex topological vector [...]ace. Indicative of the nature of the results is the follow[...]g: If there exists some $a \in G$ with the property that the [...]lation $F(x) = \alpha x + (1-\alpha)a$ with x on the boundary of G [...]plies $\alpha \leq 1$, then F has a fixed point in $\bar{G}$. This contains [...]veral known theorems as special cases. The main tool in [...]is investigation is a systematic use of the degree of a [...]apping. *P. C. Deliyannis* (Chicago, Ill.)

[...]kaler, A. I. 5096

Two problems in the theory of semi-ordered spaces. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 952–954.

[...]ter einer Komponente wird das disjunkte Komplement [...]er Teilmenge eines Vektorverbandes verstanden. Der [...]ktorverband X hat die Eigenschaft (P), wenn jedes Ele[...]nt von X eine Projektion auf beliebige Komponente [...]t. Der Raum m ist ein Vektorverband mit der Eigen[...]haft (P), der Raum c_0 ist ein l-Ideal von m und der [...]otientenraum m/c_0 besitzt nicht die Eigenschaft (P) [...], c_0 haben die übliche Bedeutung). Damit ist ein Prob[...]n von Jakubík gelöst [Časopis Pěst. Mat. **84** (1959), 150–[...]1; MR **21** #7254].

Der Raum m ist ein vollständiger Vektorverband. Der [...]rfasser definiert in m eine monotone Norm so, dass die [...]llständige metrische Hülle dieses Raumes ein Vektor[...]rband mit monotoner Norm ist. Diese Hülle ist aber kein [...]llständiger Vektorverband. Damit ist ein Problem von [...] Z. Vulih gelöst. *M. Novotný* (Brno)

[...]lderón, A. P. 5097

Intermediate spaces and interpolation, the complex method.

Studia Math. **24** (1964), 113–190.

[...]is is the complete version of the author's results on [...]termediate (or interpolation) spaces, part of which were [...]nounced already several years ago at a meeting in War[...]w [see Studia Math. (Ser. Specjalna) Zeszyt 1 (1963), 31–[...]; MR **26** #5409]. Let (B^0, B^1) be a couple of Banach [...]aces, B^0 and B^1 being both continuously imbedded in [...]me topological vector space V. Let $\mathscr{F}(B^0, B^1)$ be the [...]nach space (with the natural norm) of functions with [...]lues in $B_0 + B_1$, boundedly continuous in the strip [...]$\leq s \leq 1$ of the complex $(s+it)$-plane, analytic in its [...]terior $0 < s < 1$ and such that $f(it) \in B^0$ and tends to 0 in [...] as $|t| \to \infty$, and $f(1+it) \in B^1$ and tends to 0 in B^1 as [...]$\to \infty$. The basic intermediate spaces, denoted by $B_s = [B^0, B^1]_s$, $0 \leq s \leq 1$, are now obtained as the Banach space [...]ith the natural norm) spanned by $f(s)$ when f runs [...]rough $\mathscr{F}(B^0, B^1)$. (The same definition, indeed, in a [...]mewhat more general form, was also found inde[...]ndently by Lions [C. R. Acad. Sci. Paris **251** (1960), [...]53–1855; MR **22** #9859].) It is trivial to verify the fol[...]wing interpolation theorem: Let (B^0, B^1) and (C^0, C^1) be [...]o couples and let T be a linear mapping from $B^0 + B^1$ [...]o $C^0 + C^1$ such that T (when restricted to B^0) $\in \mathscr{L}(B^0, C^0)$ [...]d (when restricted to B^1) $\in \mathscr{L}(B^1, C^1)$. Then T (when restricted to B_s) $\in \mathscr{L}(B_s, C_s)$, with a corresponding convexity inequality $M \leq M_0^{1-s} M_1^{s}$, where M_0, M_1, M are the operator norms in question. Similar results hold for multilinear mappings. For technical reasons certain other intermediate spaces, denoted by $B^s = [B^0, B^1]^s$, $0 \leq s \leq 1$, are also used, the definitions of which are analogous. It is trivial to see that $B_s \subset B^s$. If one of the spaces, say B^0, is reflexive, one can show also the converse $B^s \subset B_s$, so that $B_s = B^s$ in this case. However, a concrete counterexample shows that $B_s \neq B^s$ in general. An important result is a duality theorem claiming that $([B^0, B^1]_s)^1 = [(B^0)^1, (B^1)^1]^s$ provided $B^0 \cap B^1$ is dense in both B^0 and B^1. The theory is applied to some concrete cases, in the first place, to Banach-space-valued Banach lattices $X(B)$. It is shown that, under certain assumptions, $[X_0(B^0), X_1(B^1)]_s = X(B) = [X_0(B^0), X_1(B^1)]^s$, where $B = [B^0, B^1]_s$ and $X = X_0^{1-s} X_1^{s}$, this latter Banach lattice being obtained as follows: $\varphi \in X_0^{1-s} X_1^{s}$ if and only if there exist $\varphi_0 \in X_0$ and $\varphi_1 \in X_1$ such that, for each point x of the measure space $\mathscr{M}$ in question, $|\varphi(x)| \leq |\varphi_0(x)|^{1-s} |\varphi_1(x)|^{s}$. A similar result is given for a general class of Lipschitz-type spaces $\Lambda(X, B)$ obtained as follows: One assumes that a continuous group of operators $G(t)$, $t \in R^n$, acts in B and that X is a Banach lattice on R^n. Then $\Lambda(X, B)$ is, roughly speaking, the subspace of B defined by the relation $\|G(t)b - b\| \in X$ or a similar relation with a higher-order difference; the relation depends on the growth properties of the functions in X. The presentation is not very clear, mainly owing, in the reviewer's opinion, to the unfortunate subdivision into two parts, the first giving the definitions and main results, the second giving the detailed proofs; the reader has to spend quite a lot of time just searching for the relevant passage for the proof of each particular statement.

J. Peetre (Lund)

Gudiev, A. H. 5098

An imbedding theorem for the trace in abstract functions. (Russian)

Dokl. Akad. Nauk SSSR **147** (1962), 764–767.

L'auteur démontre de nouveaux théorèmes d'immersion pour les espaces de Sobolev $W_p^{(l)}(\Omega)$ (espace des fonctions dont les dérivées d'ordre $\leq l$ sont de puissance p sommable dans Ω, ouvert de R^n).

On désigne par $L_{(p_1, p_2)}(\Omega)$ l'espace des fonctions qui sont des restrictions à Ω de fonctions f définies dans R^n et telles que

$$\left\{\int_{R^{n-s}} \left(\int_{R^s} |f(\mathrm{x}_s, \mathrm{x}_{n-s})|^{p_1}\, d\mathrm{x}_s\right)^{p_2/p_1} d\mathrm{x}_{n-s}\right\}^{1/p_2} < +\infty$$

$(\mathrm{x}_s = (x_1, x_2, \cdots, x_s)$ décrit l'espace R^s et $\mathrm{x}_{n-s} = (x_{s+1}, \cdots, x_n)$ décrit l'espace R^{n-s}). Alors pour $n/p - l < (n-s)/p_2 + s/p_1$, $p_2 \geq p_1 \geq p > 1$, on a l'immersion de $W_p^{(l)}(\Omega)$ dans $L_{(p_1, p_2)}(\Omega)$. L'auteur démontre également des résultats plus généraux (immersion dans les espaces $L_{(p_1, p_2, \cdots, p_k)}(\Omega)$ de définition évidente). Aucune hypothèse particulière n'est faite sur Ω bien que la formule de représentation de Sobolev [*Some applications of functional analysis in mathematical physics* (Russian), Izdat. Leningrad. Gos. Univ., Leningrad, 1950; MR **14**, 565] soit utilisée. *P. Grisvard* (Nancy)

Guglielmino, Francesco 5099a

Su alcuni spazi di interpolazione. (French summary)

Boll. Un. Mat. Ital. (3) **18** (1963), 339–350.

Guglielmino, Francesco 5099b

A proposito di un teorema riguardante alcuni spazi di interpolazione. (English summary)

Boll. Un. Mat. Ital. (3) **19** (1964), 171–177.

Sur un ensemble Q mesurable de $R_x^m \times R_y^n$, on désigne par $L^{p_1,p_2}(Q)$ l'espace des (classes de) fonctions qui sont L^{p_1} en x et L^{p_2} en y. L'auteur étudie les espaces de traces [comme dans le rapporteur, Math. Scand. **9** (1961), 147–177; MR **28** #2429] entre $L^{p_1,p_2}(Q)$ et $L^{q_1,q_2}(Q)$; il donne diverses inclusions selon les valeurs des paramètres p, q et des paramètres des espaces de traces considérés. Comme conséquence on a un théorème d'interpolation pour les opérateurs linéaires entre ces espaces, théorème qui généralise le théorème classique de M. Riesz et qui complète, en autorisant les valeurs infinies pour les divers paramètres, un résultat (établi de façon complètement différente) de Benedek et Panzone [Duke Math. J. **28** (1961), 301–324; MR **23** #A3451]. Une difficulté de mesurabilité est résolue dans le second article.

J. L. Lions (Paris)

Hadžiivanov, Nikolaĭ 5100

On indecomposable elements of two cones. (Bulgarian. Russian summary)

Annuaire Univ. Sofia Fac. Sci. Phys. Math. Livre 1 *Math.* **56** (1961/62), 191–194 (1963).

The author finds the extreme elements of two cones of multiple sequences. The extreme elements of the cone K_n of the multiple sequences $\{a_{\nu_1\cdots\nu_n}\}_{\nu_i=0}^{\infty}$ which satisfy the equations

$$(1)\qquad (-1)^{\nu_1+\cdots+\nu_n}\Delta^{\nu_1}\cdots\Delta^{\nu_n}a_{k_1\cdots k_n} \geqq 0$$
$$(\nu_i = 0, 1, 2, \cdots;\ k_i = 0, 1, 2, \cdots)$$

are sequences of the type $\{cq_1^{\nu_1}\cdots q_n^{\nu_n}\}_{\nu_i=0}^{\infty}$; $c>0$; $0 \leqq q_i \leqq 1$; $i = 1, 2, 3, \cdots, n$.

The extreme points of the cone L_n are also found, where L_n is defined in the same manner as K_n, but with the condition that

$$\sum_{\nu_1=0}^{m_1}\cdots\sum_{\nu_n=0}^{m_n}(-1)^{\nu_1+\cdots+\nu_n} \times \binom{m_1}{\nu_1}\cdots\binom{m_n}{\nu_n}a_1^{\nu_1}\cdots a_n^{\nu_n}a_{k_1+m_1+\cdots+k_n+m_n} \geqq 0$$

instead of (1). {The paper contains some troublesome misprints.}

Bl. Sendov (Sofia)

Ishii, Jyun 5101

A remark on the space D_Φ^h.

Sci. Rep. Fac. Lit. Sci. Hirosaki Univ. **10** (1963), 1–4.

Let E be a measure space and $h(t)$ an unbounded positive measurable function such that if $E_\nu = \{t : h(t) \leqq \nu\}$, then $\mu(E_\nu) < \infty$. D_Φ^h is the totality of measurable functions $x(t)$ such that $\|x\|_\Phi + \|xh\|_\Phi < \infty$, where $\|\ \|_\Phi$ is a norm on the Orlicz space L_Φ and $xh = x(t)h(t)$. The properties of D_Φ^h are precisely what one would expect, and the author proves them by considering D_Φ^h as a modulared function space. Theorem 1: $D_\Phi^h \subseteq D_{\Phi_1}^h$ if and only if there exists $k>0$ such that $\Phi_1(\xi) \leqq \Phi(k\xi)$. Theorem 2: D_Φ^h is separable if and only if Φ satisfies the Δ_2 condition of Orlicz. Theorem 3: D_Φ^h is a regular (i.e., reflexive) Banach space if and only if Φ and $\bar\Phi$ satisfy the Δ_2 condition.

R. O'Neil (Houston, Tex.)

McLachlan, E. K. 5102

Extremal elements of the convex cone B_n of functions.

Pacific J. Math. **14** (1964), 987–993.

Let B_0 denote the set of non-negative continuous functions on $[0, 1]$; B_1 the set of $f \in B_0$ such that $\Delta_h f(x) = f(x+h) - f(x) \geqq 0$ for $h>0$ and $[x, x+h] \subset [0, 1]$; and B_n $(n>1)$ the set of $f \in B_{n-1}$ such that $\Delta_h^n f(x) \geqq 0$ for $h>0$ and $[x, x+nh] \subset [0, 1]$. Then B_n is a convex cone. The principal results of this paper specify exactly the extremal elements of B_n and (by using a general theorem of Choquet) give an integral representation of elements of $C_n = \{f \in B_n : f(1) = 1\}$ in terms of a positive measure on the pointwise closure in $B_n - B_n$ of the set of extremal points of C_n.

R. E. Edwards (Canberra)

Moeller, J. W. 5103

Translation invariant spaces with zero-free spectra.

Duke Math. J. **31** (1964), 99–108.

In $L_2(-\infty, +\infty)$ let $\hat H_2$ denote the subspace of functions which vanish to the left of zero and let H_2 denote the Fourier transform of $\hat H_2$. The subspace H_2 is characterized by the Paley-Wiener Theorem and consists of the vertical limits a.e. of functions regular in the upper half-plane which are of uniformly bounded L_2 norm along all lines parallel to the real axis. A subspace $\hat R$ of $\hat H_2$ is right translation-invariant, or briefly, an $\hat R$-space, if along with every function $\hat h(x)$ it also contains all functions $\hat h(x-\tau)$, $\tau>0$, and the Fourier transform of an $\hat R$-space is an R-space in H_2. Similarly, a subspace $\hat L$ of $\hat H_2$ is left translation-invariant, or an $\hat L$-space, if along with every function $\hat h(x)$ it also contains the projections onto $\hat H_2$ of all functions $\hat h(x+\tau)$, $\tau>0$, and the Fourier transform of an $\hat L$-space is an L-space in H_2. $\hat L$- and $\hat R$-spaces are orthogonal complements of one another in $\hat H_2$ and hence the same is true of L- and R-subspaces of H_2. The basic theorem in this context reads as follows: Every R-subspace of H_2 is of the form $G \cdot H_2$, where G is an inner function in H_2, i.e., the vertical limit a.e. of a regular function G in the upper half-plane which satisfies $|G(\lambda)| \leqq 1$ for $\operatorname{Im}\lambda > 0$ [see, e.g., Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, N.J., 1962; MR **24** #A2844]. If $\hat R$ is the $\hat R$-subspace of $\hat H_2$ whose Fourier transform is R and $\hat L = \hat H_2 \ominus \hat R$, then G is called the characteristic function of $\hat L$.

Now write T for the operator $T : \hat H_2 \to \hat H_2$ given by $\hat h(x) \to \hat h(x+1)$ followed by projection back onto $\hat H_2$. The $\hat L$-subspaces are all invariant under T, and for any such space $T|\hat L$ is a well-defined contraction on $\hat L$. A central problem is to relate the properties of $T|\hat L$ with those of the characteristic function G of $\hat L$. In an earlier paper [J. Math. Anal. Appl. **4** (1962), 276–296; MR **27** #588] the author determined the non-zero spectrum of $T|\hat L$ lying in the open disc; here the problem of the invertibility of $T|\hat L$ is solved. Main results: (1) A necessary and sufficient condition that $T|\hat L$ be invertible is that there exist $\gamma, \delta > 0$ such that $|G(\lambda)| \geqq \delta$ for $\operatorname{Im}\lambda \geqq \gamma$; (2) in this event we have

$$\|(T|\hat L)^{-1}\| \leqq \left[1 + \frac{e^{2\gamma}}{\delta^2(1-e^{-2\gamma})^2}\right]^{1/2}.$$

A. Brown (Ann Arbor, Mich.)

Benedek, A. 5104

Spaces of differentiable functions and distributions, with mixed norm.

Rev. Un. Mat. Argentina **22**, 3–21 (1964).

Spaces L_u^p $(0<u)$ of tempered distributions with mixed norm are introduced which are related to spaces H_u^p of Sobolev. Among the various results which are obtained by the author we mention an interpolation theorem for the family of spaces L_u^p; under certain hypotheses, which are too involved to repeat here, it is shown that the elements of L_u^p are equal almost everywhere to continuous functions, and a stronger result about the Hölder continuity of the functions belonging to L_u^p $(0 \leq u \leq 1)$ is given.

W. A. J. Luxemburg (Pasadena, Calif.)

Constantinesco, F. [Constantinescu, F.] 5105

Sur la valeur d'une distribution dans un point.

Studia Math. **24** (1964), 7–12.

By the theory of values of distributions as developed by Łojasiewicz [same Studia **16** (1957), 1–36; MR **19**, 433], a distribution f has the value 0 at the point $x=0$ if and only if there exist an integer $k \geqq 0$, a neighborhood N of 0, and a continuous function Φ such that $f=D^k\Phi$ on N and $\Phi(x)=o(x^k)$ as $x \to 0$. The author purports to show that the latter condition is equivalent to the condition $[0, x_1, \cdots, x_k] \to 0$ as $(x_1, \cdots, x_k) \to 0$, where $[0, x_1, \cdots, x_k]$ denotes the divided difference of Φ with base points $0, x_1, \cdots, x_k$. Counterexample: $\Phi(x)=x^{k+1} \sin 1/x$, $k \geqq 2$.

J. Korevaar (La Jolla, Calif.)

Bass, Jean 5106

Moyennes stationnaires et moyennes non stationnaires.

C. R. Acad. Sci. Paris **255** (1962), 232–234.

Author's summary: "On étudie de quelle façon les moyennes abstraites, définies dans une Note précédente [mêmes C. R. **254** (1962), 31–33; MR **27** #598], dépendent d'un paramètre auxiliaire t. On est ainsi conduit à distinguer les moyennes stationnaires, indépendantes du temps, compatibles avec l'existence d'une fonction de corrélation, et les moyennes non stationnaires, dépendant du temps, et relativement auxquelles la notion de fonction de corrélation est dépourvue de sens."

Birman, M. Š. 5107

On a test for the existence of wave operators. (Russian)

Dokl. Akad. Nauk SSSR **147** (1962), 1008–1009.

The following theorem is stated. If $\varphi(z)$ is a permissible function for the self-adjoint operators H_1 and H_2 and if $\hat{H}_k=\varphi(H_k)$ and $U_k=(\hat{H}_k-iI)(\hat{H}_k+iI)^{-1}$, $\hat{U}_l=(\hat{H}_l-iI)\times(\hat{H}_l+iI)^{-1}$ $(k, l=1, 2)$, where $(\hat{H}_2+iI)^{-1}-(\hat{H}_1+iI)^{-1}$ are nuclear operators, then there exist the following strong limits:

$$W_{\pm}(H_2, H_1) = \lim_{t \to \pm\infty} \exp(iH_2t)\exp(-iH_1t)P_1,$$

$$W_{\pm}(U_2, U_1) = \lim_{n \to \pm\infty} U_2{}^n U_1{}^{-n} P_1,$$

satisfying the relations:

$$W_{\pm}(H_k, H_l) = W_{\pm}(\hat{H}_k, \hat{H}_l) = W_{\pm}(\hat{U}_k, \hat{U}_l) = W_{\pm}(U_k, U_l).$$

S. Ciulli (Bucharest)

Kurepa, Svetozar 5108

On a triangular form of a family of commuting operators. (Serbo-Croatian summary)

Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hrvatske Ser. II **18** (1963), 39–42.

In this note the author proves the following two theorems. Theorem 1: Let X be a complex vector space, let F be a family of commuting linear operators defined on X with ranges in X, and suppose that for any A in F there exists a polynomial $p \neq 0$ such that $p(A)=0$. (The polynomial p may depend on A.) If X is finite-dimensional or if F is a finite family, then there exists an algebraic basic set which depends on F only and such that the matrix of A in this basic set is upper triangular for any A in F.

Theorem 2: If X is a complex Hilbert space, if F is a finite family of commuting and continuous linear operators on X into X such that for any A in F there exists an analytic function $f \neq 0$ such that $f(A)=0$, then an orthonormal basic set can be found such that the matrix of A in this basic set is upper triangular for any A in F.

These theorems generalize the well-known theorems that any square matrix of finite order is similar to an upper triangular matrix and any family of commuting normal matrices is similar to a family of diagonal matrices.

H. A. Gindler (Pittsburgh, Pa.)

Marek, Ivo 5109

A note on K-positive operators.

Comment. Math. Univ. Carolinae **4** (1963), 137–146.

This is a continuation of an earlier paper by the same author [same Comment. **3** (1962), no. 1, 20–30; MR **26** #1763]. The following notation is used: Y is a real Banach space, X is the complex Banach space of couples from Y, defined in a natural way, K is a cone in Y such that the closed linear span of K is Y, T is a bounded linear operator in Y such that $T(K) \subset K$. Spectral properties of T refer to the natural extension of T to become an operator in X. It is further assumed that T is strongly u_0-bounded. This means that there exists a nonzero element u_0 in K such that the following conditions are satisfied: (1) For some positive numbers α, β and some positive integer p, $\alpha u_0 - T^p x \in K$ and $T^p x - \beta u_0 \in K$ for each nonzero x in K; (2) to each x in Y correspond a positive integer p and a real number γ (depending on x) such that $u_0 - \gamma T^p x \in K$. Finally, it is assumed that for some analytic function f of class $\mathfrak{A}_\infty(T)$ (locally analytic on $\sigma(T)$ and at ∞) the operator $f(T)=U+V$ is a Radon-Nikol'skiĭ operator such that $|f(\lambda)| > r_e(V)$ if $|\lambda| = r_e(T)$. (Here U is to be compact and $r_e(V) < r_e(f(T))$.) It is then asserted that T has a simple, positive eigenvalue μ_0 such that $|\lambda| < \mu_0$ if $\lambda \in \sigma(T)$ but $\lambda \neq \mu_0$. There is a corresponding eigenvector x_0 in K. This eigenvalue is a simple pole of the resolvent. Moreover, μ_0 is a simple eigenvalue of the conjugate operator T', with an eigenvector x_0' such that $x_0'(x) > 0$ if x is a nonzero element of K. Finally, it is asserted that the spectral projection of X corresponding to μ_0 is strongly u_0-bounded. This is said to be very important for the construction of μ_0 and x_0 by the Kellogg iterative process [see the author, Czechoslovak Math. J. **12** (87) (1962), 536–554; MR **26** #6787].

A. E. Taylor (Los Angeles, Calif.)

Berens, Hubert; Butzer, P. L. 5110

Approximation theorems for semi-group operators in intermediate spaces.

Bull. Amer. Math. Soc. **70** (1964), 689–692.

Let $\{T_t : t \geqq 0\}$ be a holomorphic semi-group of class (C_0) with the infinitesimal generator A in a real or complex Banach space so that, for every $t>0$, $T_tX \subseteq D(A)$, the domain of A, and $\|AT_t\| \leqq M_0 t^{-1}$ as $t \downarrow 0$. Then $\|T_t f - f\| = O(\phi(1/t))$ (as $t \downarrow 0$) implies

$$\|AT_t f\| \leqq M_1 + M_2 t^{-1}\phi(1/t) + M_3 \int_1^{1/t} \phi(u)\,du \qquad (0 < t \leqq 1),$$

where $\phi(u)$ is a positive non-increasing function in $[1, \infty)$ and M_i $(i=0, 1, 2, 3)$ are constants. In particular, under the same hypothesis as above, $\|T_t f - f\| = O(t^\alpha)$ $(0<\alpha<1;$ $t \downarrow 0)$ if and only if $\|AT_t f\| = O(t^{\alpha-1})$ $(t \downarrow 0)$.

K. Yosida (Tokyo)

Arnautov, V. I. 5111

On the theory of topological rings. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 12–15.

A topological ring is an associative ring which is at the same time a Hausdorff space in which the ring operations are continuous. A two-sided ideal I (not necessarily closed) is called topologically nilpotent if for any neighbourhood V of zero there exists an integer n such that $I^n \subset V$. For a topological ring R, $\mathfrak{N}(R)$ designates the closure of the sum of all nilpotent ideals of R, and for any ordinal α, $\mathfrak{N}_\alpha(R)$ is defined as follows: $\mathfrak{N}_0(R)=0$; if $\beta=\alpha+1$, then $\mathfrak{N}_\beta(R)$ is the inverse image in R of $\mathfrak{N}[R/\mathfrak{N}_\alpha(R)]$; if β is a limit ordinal, then $\mathfrak{N}_\beta(R)$ is the closure of $\sum_{\alpha<\beta} \mathfrak{N}_\alpha(R)$. There exists an ordinal τ such that $\mathfrak{N}_\tau(R) = \mathfrak{N}_{\tau+1}(R)$, and $L(R) = \mathfrak{N}_\tau(R)$ is called a topological radical of Baire. A topological ring R is called an L-radical [L-semi-simple] ring provided $L(R) = R$ [$L(R) = 0$]. The author states that he has proved that $L(R)$ is the intersection of all closed ideals $N \subset R$ such that in R/N there are no non-zero topological nilpotent ideals, and that for any ideal $I \subset R$, $L(I) \subseteq I \cap L(R)$ (it is not mentioned whether or not these results have been published). The author gives an example of an ideal $I \subset R$ such that $L(I) = 0 \subset I \cap L(R) = I$ (this may be also given as an example in which $L(I) = I \cap L(R)$). He also gives an example of rings $\hat{R}$, R', R'' such that $R'+R''$ is a dense subring of $\hat{R}$ and $L(R'+R'') = 0 \subset R'+R'' \cap L(\hat{R}) = R'$, and an example of a commutative L-radical ring with unit element.

W. Żelazko (Warsaw)

Bauer, Heinz 5112

Darstellung von Bilinearformen auf Funktionenalgebren durch Integrale.

Math. Z. **85** (1964), 107–115.

Let X be a locally compact space, and let A be a separating subalgebra of continuous real functions which vanish at infinity. Let $\langle\ ,\ \rangle$ be a bilinear form on A with the properties $\langle f, g\rangle = \langle g, f\rangle$, $\langle f, f\rangle \geqq 0$, $\langle f, gh\rangle = \langle gf, h\rangle$, and $\langle fg, fg\rangle \leqq M_f\|g\|^2$, where M_f depends only on f. The author deduces from this that there exists a Radon measure m on X such that $A \subset L^2(m)$ and

$$\langle f, gh\rangle = \int f(x)g(x)h(x)m(dx)$$

for each triple f, g, h selected from A.

R. Arens (Los Angeles, Calif.)

Brown, Arlen; Pearcy, Carl 5113

Structure theorem for commutators of operators.

Bull. Amer. Math. Soc. **70** (1964), 779–780.

In the present paper, the authors announce an interesting result concerning commutators of operators. Here some indication of its proof is given. Details will appear elsewhere.

Let $\mathfrak{H}$ be a separable Hilbert space, and A a bounded operator on $\mathfrak{H}$; then A is a commutator if there exist bounded operators B and C on $\mathfrak{H}$ such that $A = BC - CB$. Wintner [Phys. Rev. (2) **71** (1947), 738–739; MR **8**, 580] and Wielandt [Math. Ann. **121** (1949), 21; MR **11**, 38] showed that no nonzero scalar multiple of the identity operator on $\mathfrak{H}$ is a commutator. Halmos [*Lectures on modern mathematics*, Vol. 1, pp. 1–22, Wiley, New York, 1963] generalized it as follows: No operator of the form $\lambda 1 + C$ is a commutator, where $\lambda \neq 0$ and C is a compact operator. Here the authors state the converse theorem as follows. An operator A on a separable Hilbert space $\mathfrak{H}$ is a commutator if and only if A is not of the form $\lambda 1 + C$, where $\lambda \neq 0$ and C is a compact operator.

Moreover, they state that results analogous with the above hold for an arbitrary infinite-dimensional Hilbert space. The reviewer thinks that the following problem is also interesting: Can we generalize the above theorem to a von Neumann factor of type II_∞ or III?

S. Sakai (Philadelphia, Pa.)

Gardner, L. Terrell 5114

A note on isomorphisms of C^*-algebras.

Bull. Amer. Math. Soc. **70** (1964), 788–791.

In the present paper, the author states two interesting results concerning isomorphisms of C^*-algebras. Their proofs are sketched. Full details will appear elsewhere. Definition: The atomic representation α of a C^*-algebra $\mathfrak{A}$ is the direct sum $\bigoplus_{\rho\in\mathscr{E}} \phi_\rho$ of the representations ϕ_ρ due to pure states ρ of $\mathfrak{A}$, where $\mathscr{E}$ is the set of all pure states of $\mathfrak{A}$.

Theorem 2: Let ψ be an algebraic isomorphism (not necessarily $*$-preserving) of the C^*-algebra $\mathfrak{A}$ onto the C^*-algebra $\mathfrak{B}$, and let α [β] be the atomic representation of $\mathfrak{A}$ [$\mathfrak{B}$] on the Hilbert space $\mathfrak{H}$ [$\mathfrak{K}$]. Then $\beta\psi\alpha^{-1}$ can be extended to an isomorphism of $L(\mathfrak{H})$ onto $L(\mathfrak{K})$ of the form $A \to SAS^{-1}$ for some S in $L(\mathfrak{H}, \mathfrak{K})$, where $L(\mathfrak{H})$ [$L(\mathfrak{K})$] is the algebra of all bounded operators on $\mathfrak{H}$ [$\mathfrak{K}$] and $L(\mathfrak{H}, \mathfrak{K})$ is the space of all bounded operators of $\mathfrak{H}$ into $\mathfrak{K}$.

This theorem implies that isomorphisms between C^*-algebras are in a certain sense "spatial" in nature. By means of Theorem 2, the author gives an affirmative solution to a fundamental problem in the theory of C^*-algebras [cf. the reviewer, "The theory of W^*-algebras", p. 1.53, Problem (i) (mimeographed notes), Yale Univ., New Haven, Conn., 1962] as follows. Theorem 3: If two C^*-algebras are algebraically isomorphic, then they are $*$-isomorphic.

S. Sakai (Philadelphia, Pa.)

Johnson, B. E. 5115

Centralisers on certain topological algebras.

J. London Math. Soc. **39** (1964), 603–614.

The author applies the results of his previous paper [Proc. London Math. Soc. (3) **14** (1964), 299–320; MR **28** #2450] to various topological algebras: $\mathscr{C}_0(\mathscr{T})$, $L_1(G)$, $\mathscr{C}_{00}(G)$, and $\mathscr{D}L_1$.

J. G. Wendel (Ann Arbor, Mich.)

Ingelstam, Lars 5116

Non-associative normed algebras and Hurwitz' problem.
Ark. Mat. **5**, 231–238 (1964).

Let A be an algebra (non-associative) over the real or complex field with an identity e. It is assumed that A is normed as a linear space, where the norm satisfies one or more of the following conditions. (1) A positive-definite inner product (x, y) is defined on A, where $\|x\|^2=(x, x)$; (2) $\|xy\|=\|x\|\,\|y\|$ for all x, y; (3) $\|e\|=1$ and $\|xy\|\leq\|x\|\,\|y\|$ for all x, y. In case (1) and (3) are satisfied, A is called a pre-Hilbert algebra with an identity. In case (2) is satisfied, A is called an absolute-valued algebra. The classical result of A. Hurwitz referred to in the title is that if A is finite-dimensional and (1) and (2) hold, then A is isomorphic to the real numbers R, the complex numbers C, the quaternions Q, or the Cayley numbers D.

It is shown that (a) if A is a power associative real [complex] absolute-valued algebra with identity, then A is isomorphic to R, C, Q or D (isomorphic to C), and (b) if A is a real [complex] alternative pre-Hilbert algebra with identity, then the same conclusion holds. Various related algebras are studied. In particular, it is shown that in (b) "alternative" cannot be replaced by "power associative". *B. Yood* (Eugene, Ore.)

Maltese, George 5117

Convex ideals and positive multiplicative forms in partially ordered algebras.
Math. Scand. **9** (1961), 372–382.

Un "coin" B^+ d'une algèbre sur $\mathbf{R}$ en est une partie stable pour l'addition, la multiplication par un réel positif et l'élévation au carré. La donnée d'un coin permet de définir un préordre partiel ($f\geq g \Leftrightarrow f-g\in B^+$). Une forme est positive si elle prend des valeurs réelles positives sur le coin. Un idéal I est convexe si $f\in I$, $g\in I$, $f\leq h\leq g$ implique que $h\in I$. Théorème: La condition nécessaire et suffisante pour qu'un idéal d'une algèbre de Banach réelle, commutative, munie d'un coin, soit le noyau d'une forme linéaire, multiplicative, positive et continue est qu'il soit convexe, régulier et maximal. Appliqué à $L^1(G, \mathbf{R})$ cela conduit à un résultat de Aubert: le seul idéal maximal régulier convexe de $L^1(G, \mathbf{R})$ est le noyau de la mesure de Haar [Aubert, Math. Scand. **6** (1958), 181–188; MR **21** #2918]. Théorème: Si V est une algèbre ordonnée sur $\mathbf{R}$ (multiplication notée $*$), dont V^+ est le cône positif stable pour la multiplication, contenant un élément 1 tel que $\forall f\in V$, $\exists\lambda_f\in\mathbf{R}$, $-\lambda_f f\leq 1\leq\lambda_f f$, et tel que $\forall f\in V^+$, $f\neq 0$, $\exists\alpha_f\in\mathbf{R}^+$ tel que $1*f\geq\alpha_f 1$, il existe une forme linéaire non nulle sur V qui soit positive et multiplicative. *A. Revuz* (Poitiers)

Naimark, M. A. 5118

Commutative operator algebras in the space Π_1. (Russian)
Dokl. Akad. Nauk SSSR **156** (1964), 734–737.

Let Π_1 denote a complex linear space with an indefinite scalar product (ξ, η) (see, for the definition and first properties of this concept, I. S. Iohvidov and M. G. Kreĭn [Trudy Moskov. Mat. Obšč. **5** (1956), 367–432; MR **18**, 320]). The present paper classifies (without proofs) all commutative self-adjoint algebras R of bounded linear operators on Π_1. The result is quite complicated. One considers various subcases depending on the properties of the family of all one-dimensional non-negative subspaces of Π_1 invariant under R. *J. M. G. Fell* (Seattle, Wash.)

Rudin, Walter; Schneider, Hans 5119

Idempotents in group rings.
Duke Math. J. **31** (1964), 585–602.

It is known that every idempotent in $L^1(G)$ for an abelian discrete group G has a finite support group. The authors show that this is also true for group rings of L^1-type over commutative Banach algebras. A purely algebraic analogue of this theorem and related discussions are given. *I. G. Amemiya* (Sapporo)

Taylor, Joseph L. 5120

The Tomita decomposition of rings of operators.
Trans. Amer. Math. Soc. **113** (1964), 30–39.

Let R be a separable C^*-algebra of bounded operators on a separable Hilbert space H and ξ_0 a vector in H which is cyclic with respect to R; then the positive functional $F(A)=(A\xi_0, \xi_0)$ for $A\in R$ may be written as an integral on a compact space M, i.e., $F(A)=\int_M f_m(A)\,d\mu(m)$, where μ is a positive regular Borel measure and the functionals f_m are pure for $m\in M-M_0$ and $\mu(M_0)=0$. Moreover, this decomposition induces a representation of R as a direct integral of ring R_m of operators on Hilbert spaces H_m and for almost all m, R_m is irreducible. This has been shown by many authors [cf. Segal, Mem. Amer. Math. Soc. No. 9 (1951); MR **13**, 472; Godement, Ann. of Math. (2) **53** (1951), 68–124; MR **12**, 421; Naimark, *Normed rings* (Russian), GITTL, Moscow, 1956; MR **19**, 870] as a refined form of Mautner's theorem [Ann. of Math. (2) **51** (1950), 1–25; MR **11**, 324] obtained by the help of the von Neumann reduction theory [ibid. (2) **50** (1949), 401–485; MR **10**, 548].

Tomita [Math. J. Okayama Univ. **3** (1954), 147–173; MR **15**, 968] studied the problem of extending this type of decomposition to C^*-algebras on an arbitrary Hilbert space, using neat techniques, and Naimark gave the elegant introduction of Tomita's theory in his book [loc. cit.].

However, unfortunately, certain parts of Tomita's development require a special measure-theoretic result which is not valid in general, as has been noticed by several mathematicians (Hewitt, in his review of Naimark's book, op. cit.; Lorch, in his review of the English transl. [Noordhoff, Groningen, 1959; MR **22** #1824] in Bull. Amer. Math. Soc. **69** (1963), 193–195). Consequently, it has been an interesting question whether or not this measure-theoretic difficulty could be circumvented.

In the present paper, the author gives a negative solution to this question, namely, he constructs a C^*-algebra where the Tomita decomposition fails to hold (Theorem 3.3). (The proofs of Lemmas 3.4 (b) and 3.5 are incomplete, and those lemmas are likely incorrect. However, by changing the discussions, we can easily see that Theorem 3.3 is still right. In his communications with the reviewer, the author wants to replace them by the following statements. Lemma 3.4 (b): If R_1 is the ring of operators (not the closed ring) generated by $C\cup T$, then $A\in R_1\Rightarrow AS\subset S$. Lemma 3.5: For each $A\in R_1$ and $m\in M$, there is a unique operator $A_m\in B(H)$ such that $(A\alpha)(g, m)=(A_m\alpha_m)(g)$, where $\alpha_m\in H$ and is the function determined by $\alpha_m(g)=\alpha(m, g)$ for $\alpha\in S$; finally, the correspondence

$A \to A_m$ may be extended to a symmetric, norm-decreasing representation of R onto a ring $R_m \subset B(H)$.}

Moreover, he shows that the Tomita decomposition holds if R is a W^*-algebra containing its commutant (Theorem 2.3).

Thus, it is impossible to extend Mautner's theorem to the general non-separable case in the above form. But this does not necessarily imply the impossibility of Mautner's theorem in the non-separable case; in fact, by the theorem of Bishop and de Leeuw [Ann. Inst. Fourier (Grenoble) **9** (1959), 305–331; MR **22** #4945], we may have a weaker form of Mautner's theorem in the non-separable case [cf. Loomis, Amer. J. Math. **84** (1962), 509–526; MR **26** #2575]. *S. Sakai* (Philadelphia, Pa.)

Wulfsohn, Aubrey 5121

Produit tensoriel de C^*-algèbres.

Bull. Sci. Math. (2) **87** (1963), *1ière partie*, 13–27.

The author begins by defining the C^*-tensor product $A \overset{*}{\otimes} B$ of two C^*-algebras A and B. For this he first considers A and B as concrete algebras of operators in Hilbert spaces H and K, respectively; he identifies the algebraic tensor product $A \otimes B$ in the natural way with a $*$-algebra of operators on the tensor product Hilbert space $H \otimes K$, and defines $A \overset{*}{\otimes} B$ as the completion of $A \otimes B$. Then he shows that $A \overset{*}{\otimes} B$ is independent (in the obvious sense) of the particular concrete realizations of A and of B, and so is defined for any two abstract C^*-algebras A and B.

Next he proves that $A \overset{*}{\otimes} B$ is of Type I if and only if A and B are of Type I. Finally, he shows that if either A and B is of Type I, then the dual space $(A \overset{*}{\otimes} B)^\wedge$ of $A \overset{*}{\otimes} B$ can be identified both setwise and topologically with $\hat{A} \times \hat{B}$. *J. M. G. Fell* (Seattle, Wash.)

Wulfsohn, Aubrey 5122

Le produit tensoriel de certaines C^*-algèbres.

C. R. Acad. Sci. Paris **258** (1964), 6052–6054.

Let A and B be two C^*-algebras. If A and B are both GCR (or both are GTC, or both have continuous trace), then $A \otimes B$ is GCR (or is GTC, or has continuous trace). For definitions see Dixmier [Ann. Inst. Fourier (Grenoble) **13** (1963), fasc. 1, 219–262; MR **26** #6807].

J. M. G. Fell (Seattle, Wash.)

Yamazaki, Saburo 5123

Normed ring and unconditional bases in Banach space.

Sci. Papers College Gen. Ed. Univ. Tokyo **14** (1964), 1–10.

An unconditional basis for a Banach space E is a sequence $e_1, e_2, \cdots$ of elements for which a sequence $f_1, f_2, \cdots$ exists in E' such that $\sum_{i=1}^\infty f_i(x)e_i$ converges unconditionally to x for each x in E. If $e_1, e_2, \cdots$ is such a basis, then one can make the space (m) act in E as follows. For $a = (a_1, a_2, \cdots) \in (m)$ let $A_a(\sum \lambda_i e_i) = \sum \lambda_i a_i e_i$. This gives rise to a Banach algebra M of operators in E. The following converse is one of several theorems proved involving these ideas. Suppose there is an algebra M of operators isomorphic $(a \to A_a)$ to (m), such that when $a = (0, 0, \cdots, 0, 1, 0, \cdots)$, the range of A_a is one-dimensional. Suppose that the linear envelope of these ranges is dense. Then E has an unconditional basis.

R. Arens (Los Angeles, Calif.)

Glicksberg, I.; Wermer, J. 5124

Errata: Remark on "Measures orthogonal to a Dirichlet algebra".

Duke Math. J. **31** (1964), 717.

The authors show that the lemma in § 5 of an earlier paper [same J. **30** (1963), 661–666; MR **27** #6150] can be proved without appeal to Proposition 6.

Minty, George J. 5125a

On the monotonicity of the gradient of a convex function.

Pacific J. Math. **14** (1964), 243–247.

Minty, George J. 5125b

On the solvability of nonlinear functional equations of 'monotonic' type.

Pacific J. Math. **14** (1964), 249–255.

Let X be a Banach space, Y its conjugate. A set E in $X \times Y$ is monotone if $\mathrm{Re}\langle x_1 - x_2, y_1 - y_2\rangle \geq 0$ for all (x_1, y_1), (x_2, y_2) in E. A nonlinear operator $F: X \to Y$ is monotone provided its graph in $X \times Y$ is monotone, i.e., $\mathrm{Re}\langle x_1 - x_2, F(x_1) - F(x_2)\rangle \geq 0$ for all x_1, x_2 in X.

In a recent paper [Duke Math. J. **29** (1962), 341–346; MR **29** #6319] the author exploited properties of monotone operators F to prove the existence and uniqueness of solutions of equations $(I + F)x = y$ in Hilbert spaces. The interest in such monotone operators is in their relationship to the (Fréchet) derivative of convex real-valued functions, and their generalization to the nonlinear case of the "dissipative" linear operators ($\mathrm{Re}\langle x, F(x)\rangle \geq 0$ [C. L. Dolph, Bull. Amer. Math. Soc. **67** (1961), 1–69; MR **25** #5612]) of mathematical physics.

Recall that "the derivative of a convex function of one real variable is monotone nondecreasing". The first paper generalizes this to a real-valued convex function φ defined on a real topological vector space X. But the requirement of differentiability is replaced by using the "generalized gradient" of φ, viz., a map assigning to x_0 all those y_0 in the conjugate space Y such that $r = \varphi(x_0) + \langle x - x_0, y_0\rangle$ is the equation of a hyperplane of support of the points above the graph of φ in $X + R$. Specifically, the graph (in $X \times Y$) of the generalized gradient of φ is a monotonic set. It is a maximal monotonic set if φ is continuous. (The paper has more general statements.) As a consequence, in Hilbert space, "if φ is, in addition, Fréchet differentiable, then the equation $x + \varphi'(x) = u$ is always solvable for x, and the solution depends continuously on u".

The second paper generalizes the "intermediate value theorem" to reflexive Banach spaces X with smooth unit ball. Notation: Y is the conjugate space; P is the projection of $X \times Y$ onto Y; and $K[S]$ is the convex hull of S. Theorem: If $E \subset X \times Y$ is a maximal monotonic set, then a sufficient condition that $\theta \in P(E)$ is that there exist a set $A \subset E$ with the properties (i) $\theta \in$ interior $K[P(A)]$, and (ii) $\{\mathrm{Re}\langle x, y\rangle : (x, y) \in A\}$ is bounded above. An application is given in Hilbert space H for a function $f: H \to H$ whose Fréchet derivative exists everywhere and is a dissipative linear operator, to conclude $f(x) = \theta$ has a solution. Several generalizing remarks are given.

R. H. Moore (Milwaukee, Wis.)

Amerio, Luigi 5126

Su un teorema di minimax per le equazioni differenziali astratte.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **35** (1963), 409–416.

On considère l'équation opérationnelle

$$(\mathrm{E})\quad \int_J \{(x'(t), h'(t))_Y - (A(t)x(t), h(t))_X + (B(t)x'(t), h(t))_X - (f(t), h(t))_Y\}\, dt = 0,$$

$J = (-\infty, +\infty)$, où X et Y sont deux espaces de Hilbert, $X \subset Y$, l'immersion étant continue, X est dense dans Y, $A(t) \in \mathfrak{L}(X, X)$ et $B(t) \in \mathfrak{L}(Y, X)$ étant supposées continues. Les dérivées sont considérées au sens généralisé et l'on admet $x(t), h(t) \in L^2(\Delta, X), x'(t), h'(t), f(t) \in L^2(\Delta, Y)$, quel que soit l'ensemble compact $\Delta \subset J$. La fonction arbitraire $h(t)$ est à support compact. Soit $\Delta_p = \{t : t \in J,\ |t| \leq p + \frac{1}{2},\ p = 0, 1, 2, \cdots\}$. Par W_p on désigne l'espace de Hilbert des fonctions $g(t)$, $t \in \Delta_p$, telles que $g(t) \in L^2(\Delta_p, X)$, $g'(t) \in L^2(\Delta_p, Y)$. Soit $x(t)$ W_0-bornée, c'est-à-dire, telle que $\sup_{t \in J} \|z(t)\|_{W_0} < +\infty$. Posons $\varphi(z\,;\tau) = \sup_{t \in J} \|z(t+\tau) - z(t)\|_{W_0}$. Par Λ_z on désignera l'ensemble $\{x(t)\}$ des fonctions W_0-bornées, telles que $\varphi(x\,;\tau) \leq \varphi(z\,;\tau)$ pour tout $\tau \in J$ et par $\Lambda_{z,Q,f}$ l'ensemble des solutions de l'équation (E), appartenant à Λ_z. Enfin, $\Lambda_{z,Q}$ désignera l'ensemble des solutions non-nulles de l'équation homogène associée à (E). En admettant l'existence d'une fonction W_0-bornée $z(t)$ telle que $\Lambda_{z,Q,f}$ ne soit pas vide et que $\inf_{t \in J} \|u(t)\|_{W_0} > 0$ pour tout $u(t) \in \Lambda_{z,Q}$, on démontre l'existence d'une fonction unique $\hat{x}(t) \in \Lambda_{z,Q,f}$, pour laquelle $\inf_{\Lambda_{z,Q,f}} \sup_{t \in J} \|x(t)\|_{W_0}$ est atteint. Sous diverses hypothèses on établit des propriétés de la fonction qui réalise le minimax (par exemple, la continuité uniforme, la presque-périodicité). *C. Corduneanu* (Iaşi)

Campanato, S. 5127

Proprietà di una famiglia di spazi funzionali.

Ann. Scuola Norm. Sup. Pisa (3) 18 (1964), 137–160.

The author defines the space $\mathscr{L}_k^{(q,\lambda)}(\Omega)$ to consist of all $u \in \mathscr{L}_q(\Omega)$ for which the seminorm

$$|||u|||_{k,q,\lambda} = \sup\left[\rho^{-\lambda} \inf \int_{\Omega(x_0,\rho)} |u(x) - P(x)|^q\, dx\right]^{1/q} < +\infty,$$

the sup being for all $x_0 \in \bar{\Omega}$ and $0 < \rho \leq d(\Omega) = \operatorname{diam} \Omega$ and the inf being for all polynomials P of degree $\leq k$. The norm is defined by $\|u\|_{k,q,\lambda} = \|u\|_{L_q} + |||u|||_{k,q,\lambda}$. The domain Ω is said to satisfy the condition (I) if and only if there exists an $A > 0$ such that $\operatorname{meas}[\Omega \cap B(x_0, \rho)] \geq A\rho^n$ for every x_0 in $\bar{\Omega}$ and each ρ, $0 \leq \rho \leq d(\Omega)$, A being independent of x_0. The principal theorem of the paper is that if $u \in \mathscr{L}_k^{(q,\lambda)}(\Omega)$ with $n + kq < \lambda \leq n + (k+1)q$, $q \geq 1$, and if Ω satisfies condition (I), then $u \in C_\alpha^k(\bar{\Omega})$ with $\alpha = (\lambda - n - kq)/q$; if $\lambda > n + (k+1)q$, then u is a polynomial of degree $\leq k$. *C. B. Morrey, Jr.* (Berkeley, Calif.)

Fedorjuk, M. V. 5128

The asymptotic behavior of the discrete spectrum of the operator $-w''(x) + \lambda^2 p(x) w(x)$. (Russian)

Dokl. Akad. Nauk SSSR **158** (1964), 540–542.

Consider the eigenvalue problem $w''(x) - \lambda^2 p(x) w(x) = 0$, $w(\pm\infty) = 0$. Suppose $p(z)$ (z a complex variable) is a polynomial with real coefficients, $p(x)$ has precisely two zeros $x_1 < x_2$ which are simple, and $p(\pm\infty) = +\infty$. Then for $n \to \infty$ we have the asymptotic expansion

$$\lambda_n \int_C [p(z)]^{1/2}\, dz \sim 2\pi n i + \pi i + \sum_{k=1}^{\infty} \lambda_n^{-k} \int_C \alpha_k(z)\, dz,$$

where $0 < \lambda_1 < \lambda_2 < \cdots$ are the eigenvalues of the problem and C is a certain contour encompassing the segment $[x_1, x_2]$ but not containing other zeros of p. Explicit formulas are given for the α_k in terms of p and its derivatives. Conditions are given under which the asymptotic expansion holds for more general polynomials, and also for entire functions p. In the case that p is an entire function, the expansion holds up to $O(\lambda_n^{-2})$.

R. C. Gilbert (Fullerton, Calif.)

Fil'man, K. M. 5129

On the equivalence of differential operators in the space of functions analytic in a circle. (Russian)

Uspehi Mat. Nauk **19** (1964), *no.* 5 (119), 143–147.

The structure problem is solved for a general class of linear differential operators in the space of functions analytic in a disk. The operators considered are of the form $D^s + \sum_{i=1}^{s} P_i(z) D^{s-i}$, where $D = d/dz$ and the coefficients are analytic in the disk. Any such operator is similar to D^s.

L. de Branges (Lafayette, Ind.)

Hildebrandt, S. 5130

Einige konstruktive Methoden bei Randwertaufgaben für lineare partielle Differentialgleichungssysteme und in der Theorie harmonischer Differentialformen. I.

J. Reine Angew. Math. **213** (1963/64), 66–88.

In this paper the author defines certain iteration schemes which he proves, under appropriate hypotheses, converge to a solution of the Dirichlet problem for a strongly elliptic system of order $2m$ in divergence form. These methods include the alternating method of Schwarz, the method of balayage of Poincaré, and the Ritz method. Let J be the totality of positive integers and let $J' = J$ or $J' = \{1, \cdots, k\}$ for some $k > 1$. The author considers regular partitions of J into sets J_m for $m \in J'$; the sets J_m are disjoint, their union is J, and the "regularity" requires additional restrictions. If $J' = \{1, \cdots, k\}$, one example of such a regular partition is obtained by taking $J_m = \{m, m+k, m+2k, \cdots\}$. Each of the author's methods depends on a theorem concerning the limit of a sequence of projections in a Hilbert space $\mathfrak{H}$ of which we quote two. Suppose P_m denotes the projection of $\mathfrak{H}$ onto $\mathfrak{A}_m$ ($m \in J'$), where $\bigcap \mathfrak{A}_m = \{0\}$; define the operators $L_i = P_m$ if $i \in J_m$, and $A_p = L_p \cdots L_1$. Then (1) If $J = \bigcup J_m$ is a regular partition, $x_0 = x$, and $x_i = L_i x_{i-1}$ for each $i \geq 1$, then $x_i \rightharpoonup$ (tends weakly to) 0 as i runs through each J_m; if J' is finite, the whole sequence converges. (2) If $J' = \{1, \cdots, k\}$ and $\mathfrak{A}_1, \cdots, \mathfrak{A}_k$ are such that each element $x = x_1 + \cdots + x_k$ with $x_m \in \mathfrak{H} \ominus \mathfrak{A}_m$ and $\|x_m\| \leq M\|x\|$ (M independent of x) and $J = \bigcup J_m$ is the particular partition above, then the operators A_p tend strongly to 0. The system of equations which is considered is of the form

$$D(y, v) = \int_T \sum_{i,j=1}^{N} \sum_{|\alpha|,|\beta| \leq m} a_{ij}^{\alpha\beta}(t) D^\alpha v^i D^\beta y^j\, dx = l(v),$$

$$v \in W_{20}^m,\ y - g \in W_{20}^m,$$

$W_{20}{}^n$ being the completion of $C_c{}^\infty(T)$ in the Sobolev space of vectors with nth derivatives in L_2. It is assumed that the form $D(x, x) \geqq \|x\|^2$ for $x \in W_{20}{}^n$, in which case we call $\mathfrak{H}_0{}^n$ the set of elements of $W_{20}{}^n$ with inner product $D(x, y)$. Then we let $\mathfrak{C}_m$ be subspaces of $\mathfrak{H}_0{}^n$, and define the operators C_m by the equation $C_m x = x + c_m$, where c_m is the unique element in $\mathfrak{C}_m$ such that $D(x+c_m, v) = l(v)$, $v \in \mathfrak{H}_0{}^n$. We define $B_i = C_m$ for $i \in J_m$, set $y_0 = g$, $y_i = B_i y_{i-1}$, $x_i = y_i - y_0$. Then x_i tends to zero. Several ways of setting up the spaces $\mathfrak{C}_m$ are exhibited.

C. B. Morrey, Jr. (Berkeley, Calif.)

Kaližanov, U. 5131

Holomorphic solutions of countable systems of differential equations in normed linear spaces. (Russian. Kazak summary)

Izv. Akad. Nauk Kazah. SSR Ser. Fiz.-Mat. Nauk **1963**, *no.* 3, 66–70.

The author gives the results (too complicated to be quoted here) concerning the existence of holomorphic solutions of the countable systems of differential equations

$$\frac{dx_s}{dt} = \varphi_s(t, x_1, x_2, \cdots), \qquad s = 1, 2, \cdots,$$

in normed linear spaces. A similar problem has been considered earlier by K. P. Persidskiĭ [Izv. Akad. Nauk Kazah. SSR Ser. Mat. Meh. **1959**, no. 7 (11), 52–71; ibid. **1961**, no. 9 (13), 11–34]. *B. S. Popov* (Skopje)

Kužel', A. V. 5132

On the spectrum of a regular quasi-differential operator. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 731–733.

Suppose that the operator A in $L^2(a, b)$ is defined by a regular quasi-differential operator

$$l(y) = \sum_{k=0}^{n} (-1)^k (p_{n-k} y^{(k)})^{(k)},$$

not necessarily self-adjoint, and the boundary conditions

$$\sum_{k=1}^{2n} a_{jk} y^{[k-1]}(a) + \sum_{k=1}^{2n} b_{jk} y^{[k-1]}(b) = 0, \qquad 1 \leqq j \leqq 2n,$$

where $y^{[k-1]}$ is a quasi-derivative. Suppose that the rank of the matrix $\|a_{jk}\|$ is r_a, and the rank of $\|b_{jk}\|$ is r_b, where $r_a r_b \neq 0$ and $r_a + r_b = 2n$. Set $n_1 = 2n - r_a$, $n_2 = 2n - r_b$, and let $\{u_s(x, \lambda) : 1 \leqq s \leqq n_1\}$ be a set of linearly independent solutions of $l(u) = \lambda u$ satisfying the boundary conditions $\sum_{k=1}^{2n} a_{jk} u_s{}^{[k-1]}(a, \lambda) = 0$, $1 \leqq j \leqq 2n$, and let $\{u_s(x, \lambda) : n_1 < s \leqq 2n\}$ be a set of n_2 linearly independent eigenfunctions satisfying the boundary conditions

$$\sum_{k=1}^{2n} b_{jk} u_s{}^{[k-1]}(b, \lambda) = 0, \qquad 1 \leqq j \leqq 2n.$$

Let $W(\lambda)$ be the Wronskian formed by using the quasi-derivatives of the functions in the set $\{u_s(x, \lambda) : 1 \leqq s \leqq 2n\}$. $W(\lambda)$ is independent of x, and the author announces that the zeros of $W(\lambda)$ are eigenvalues of the operator A. He also announces a partial converse to the effect that the non-real eigenvalues of A are zeros of $W(\lambda)$.

The operator A is called simple in case there is no non-trivial subspace of $L^2(a, b)$ which reduces A to a self-adjoint operator. In case A is simple, necessary and sufficient conditions are given, in terms of its eigenvalues, that the eigenelements of A are dense in $L^2(a, b)$.

A. Devinatz (St. Louis, Mo.)

Lax, Peter D.; Phillips, Ralph S. 5133

Scattering theory.

Bull. Amer. Math. Soc. **70** (1964), 130–142.

In this formulation of quantum scattering theory, the basic concept is a one-parameter group of unitary operators $U(t)$ and a pair of orthogonal closed subspaces D_+ and D_- of the Hilbert space, D_+ invariant under $U(t)$ for $t > 0$ and D_- invariant under $U(t)$ for $t < 0$. The scattering operator is an operator-valued analytic function which is bounded by 1 in the unit disk and which characterizes the situation within unitary equivalence. The Hilbert space considered consists of pairs (Df, g) of a square integrable vector field Df of zero curl and a square integrable function g, both defined in Euclidean 3-space. The operator $U(t)$ propagates the field through time t according to Maxwell's equations. If Ω is a given obstacle (i.e., region) of Euclidean space, the spaces D_+ and D_- consist of all fields which vanish in Ω at all later times and at all earlier times, respectively. It is shown that D_+ and D_- are orthogonal, so that a scattering operator exists. An integral representation of the scattering operator is then derived from the Fourier transformation. The scattering operator is said to determine the obstacle Ω. Special attention is given to star-shaped obstacles and a related compactness property of the scattering operator. Generalizations to Euclidean spaces of higher odd dimensions and to general hyperbolic differential operators are indicated. *L. de Branges* (Lafayette, Ind.)

Zaidman, Samuel 5134

Quasi-periodicità per una equazione operazionale del primo ordine.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **35** (1963), 152–157.

Soient V et H deux espaces de Hilbert tels que $V \subset H$, l'immersion étant algébrique et continue. De plus, V est dense dans H. Soit $a(u, v)$ une forme sesquilinéaire continue sur V, telle que $a(u, v) = \overline{a(v, u)}$, $a(u, u) + \lambda|u|_H{}^2 \geqq \alpha|u|_V{}^2$, où $\lambda, \alpha > 0$ sont des constantes. Il existe un opérateur auto-adjoint A, tel que $a(u, v) = (Au, v)_H$. Soient enfin $V^{1-\theta}H^\theta$ les espaces intermédiaires de Lions [Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **2** (50) (1958), 419–432; MR **27** #1812]. On a $V \subset V^{1-\theta}H^\theta \subset H$, au sens algébrique et topologique. Le résultat principal établi par l'auteur est un théorème du type Bohr-Neugebauer qui peut s'énoncer: Soit $f(t)$ une fonction presque-périodique sur $J = (-\infty, +\infty)$ à valeurs dans H. Si une solution $u(t)$ de l'équation $u'(t) + Au(t) = f(t)$ est telle que $u(t) \in L^2_{loc}(J, V)$, $u'(t) \in L^2_{loc}(J, H)$, $|u(t)|_H \leqq M$ pour $t \in J$, $M > 0$ étant une constante, alors $u(t)$ est presque-périodique comme fonction de $t \in J$, à valeurs dans $V^{1-\theta}H^\theta$, $0 < \theta \leqq 1$. *C. Corduneanu* (Iaşi)

Seidman, Thomas I. 5135

Linear transformation of a functional integral. II.

Comm. Pure Appl. Math. **17** (1964), 493–508.

Part I appeared in same Comm. **12** (1959), 611–621 [MR

22 #892]; correction, ibid. 16 (1963), 95 [MR 26 #6801]. Consider on R^∞ the countable direct product of real lines R, the product measure μ of the measures μ_1 on R^∞, where

$$\mu_1(E) = \frac{1}{\sqrt{(2\pi)}\sigma}\int_E \exp\{-x^2/2\sigma^2\}\,dx.$$

The real Hilbert space l_2 is imbedded in a natural manner in R^∞, and bounded linear operators on l_2 can be extended linearly to μ-almost all of R^∞ such that the extension of a linear combination of such operators is the corresponding linear combination of the separate extensions. The following main theorem is proved.

Let T be a bounded linear operator on l_2 of the form $T = V(I+A)$, where $I+A$ is nonsingular, A is Hilbert-Schmidt and $VV^* = I$. If the extension of T on R^∞ is also denoted by T, then the inverse image $T^{-1}(E)$ of every μ-measurable set $E \subset R^\infty$ is also μ-measurable, and the measure $\mu_T = \mu \circ T$ on the collection of these inverse images is μ-absolutely continuous. Denote $d\mu_T/d\mu$ by J_T. If f is μ-integrable over R^∞, then $g(x) = f(Tx)J_T(x)$ is also μ-integrable over R^∞, and both integrals are equal.

It is observed by the author that this result is similar to a result proved by I. E. Segal [Trans. Amer. Math. Soc. 88 (1958), 12–41; Theorem 3; MR 21 #1545]. The proof given here is more elementary. *A. C. Zaanen* (Leiden)

CALCULUS OF VARIATIONS

See also 4873, 5446, 5664.

Bolonkin, O. O. 5136

The calculus of variations and a functional equation of Bellman, and an interpretation of Lagrange's undetermined multipliers. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 1290–1293.

Author's summary: "In this article a simple inference from Bellman's equation in variational calculus is suggested. An interpretation is given for Lagrange's undetermined multipliers."

GEOMETRY

See also 4766, 4804, 4909, 5172.

Brossard, Roland 5137

Birkhoff's axioms for space geometry.

Amer. Math. Monthly 71 (1964), 593–606.

The author takes up Birkhoff's axioms of plane geometry [Ann. of Math. (2) 33 (1932), 329–345] and modifies them to build up space geometry on the model of Veblen [Trans. Amer. Math. Soc. 5 (1904), 343–384]. The system of axioms is based on coordinate functions. They are intuitively conceived as applications of a long graduated ruler and a protractor. Distance, angular measure and the betweenness relation are defined in terms of coordinate functions. Plane bundles of half-lines and axioms of continuity and similarity are used to develop this otherwise well-known euclidean geometry. The congruence of triangles is defined by similarity with a factor of proportionality K equal to one as was done by Birkhoff. Playfair's axiom of parallelism and Pasch's axiom of order are deduced as theorems.

S. R. Mandan (Kharagpur)

Corsi, Gabriella 5138

Sui sistemi minimi di assiomi atti a definire un piano grafico finito.

Rend. Sem. Mat. Univ. Padova **34** (1964), 160–175.

A. Barlotti [Boll. Un. Mat. Ital. (3) 17 (1962), 394–398; MR 26 #5473] considered minimal complete subsets of six conditions which are necessary and sufficient to determine a graphic plane. The author splits up each of these six conditions into two (thus the condition "there is exactly one line joining two points" is replaced by the two conditions "there is at least [at most] one line joining two points") and proceeds to determine all minimal complete subsets of these twelve conditions. There are 42 such sets.

J. A. Todd (Cambridge, England)

Bereis, Rudolf 5139

★Darstellende Geometrie. I.

Mathematische Lehrbücher und Monographien, I. Abt., Mathematische Lehrbücher, Band XI.

Akademie-Verlag, Berlin, 1964. xvi + 493 *pp.* *DM* 35.00.

Dieses 500 Seiten starke Buch stellt den ersten Band eines auf drei Bände geplanten Werkes dar. Der erste Band benutzt durchweg zugeordnete Normalrisse. Nach Erledigung von Grundaufgaben werden ebenflächige Körper ausführlichst behandelt. Es folgen die Kegelflächen zweiter Ordnung (vorbereitet durch ein Studium der Kegelschnitte), die Zylinder und die Kugel. Verfasser geht sodann zu allgemeinen Rotationsflächen über mit spezieller Berücksichtigung derer von zweiter Ordnung. Ein kurzes Kapitel über den Torus und ein ausführliches über gewisse Durchdringungsprobleme krummer Flächen beschliesst den Band. Im zweiten und dritten Band werden Schraubenflächen, ebene und sphärische Kinematik, kotierte Projektion, Axonometrie und Perspektive behandelt.

Wie man sieht, ein umfassendes und breit angelegtes Werk, solide und verlässlich. Es geht, wie der Verfasser auch hervorhebt, über den Rahmen einer Vorlesung hinaus und appeliert an des Lehrers Fähigkeit zu geeigneter Auswahl. Ziel ist die Erziehung zur Raumanschauung, zum geometrischen Verständnis und zum tatsächlichen Konstruieren. Doch gibt der Mathematiker Bereis auch interessante Aufgaben und Bemerkungen für den mathematisch interessierten Leser.

Ob ein Werk dieses Umfangs und dieser detaillierten Gründlichkeit heute noch Berechtigung hat, scheint vielleicht zweifelhaft, besonders, dass es in mancher Beziehung elementar ist. So werden 50 Seiten den ebenen Kegelschnitten, ein ausführliches Kapitel den Durchdringungen ebenflächiger Körper gewidmet, während auf eine allgemeine Diskussion und Betrachtung auch nur der Flächen zweiter Ordnung verzichtet wird. Der grossen Ausführlichkeit entspricht auch der Stil der Zeichnungen, die mit oft störender Gewissenhaftigkeit alle Konstruktionslinien bringen. Sicher brachten die ausgezeichneten werke von E. Müller und Th. Schmidt (der Verfasser bekennt sich zur Wiener Schule der darstellenden Geometrie) das nötige auf wesentlich geringerem Raum. Und in dem theoretisch und zeichnerisch gleich schönen Werk

von F. Rehbock wird auf 200 Seiten die ganze darstellende Geometrie geboten, wobei allerdings manche Einzelheiten—wohl mit Recht—dem intelligenten Leser überlassen werden. Eine ähnliche Entwicklung zur Knappheit ist in den U.S.A. noch ausgeprägter.

Doch betrachtet der Verfasser sein Werk wohl mehr als ein Handbuch, ein Nachschlagewerk, das dem Lehrer und Praktiker gleicherweise zur Verfügung steht, und es sollte als solches gewertet werden.

H. Geiringer (Cambridge, Mass.)

Salkowski, E. 5140

★Darstellende Geometrie.

Neunte bearbeitete Auflage. Bearbeitet von Walter Schulze. Mathematik und ihre Anwendungen in Physik und Technik, Reihe A, Band 3.

Akademische Verlagsgesellschaft Geest & Portig K.-G., Leipzig, 1963. x+210 *pp*. *DM* 9.50.

This edition does not differ significantly from the fifth edition (1955) which was reviewed earlier [MR **17**, 398].

Németh, Endre 5141

Grundeigenschaften des Dreieckes mit imaginären Winkeln.

Publ. Math. Debrecen **10** (1963), 131–144.

Die Zyklen der Ebene werden wie in der Zyklographie auf die Raumpunkte aufgebildet, nur dass für den Abstand zur Grundebene iR statt R gewählt wird. Der Abstand zweier Bildpunkte hängt dann mit der Länge der gemeinsamen Tangente zusammen. Der Verfasser findet in dieser Weise ein Bild für ein Dreieck mit reellen Seiten, die nicht der Dreiecksungleichung genügen.

O. Bottema (Delft)

Rossier, Paul 5142

Sur le théorème de Pythagore en géométrie affine.

Arch. Sci. Soc. Phys. Histoire Natur. Genève **16** (1963), 483–484.

Par le sommet A du triangle ABC, on mène une droite d; par B et C, on mène les parallèles à AC et AB qui coupent d en D et F; on construit les parallélogrammes $ABDE$ et $ACFG$, dont les côtés DE et FG se coupent en H. Par B et C, on mène BK et CJ équipollents à AH: L'aire du parallélogramme $BCKJ$ est égale à la somme des aires des parallélogrammes $ABDE$ et $ACFG$. La démonstration ne fait appel qu'à la congruence affine. C'est une généralisation du Théorème de Pythagore, dont l'auteur donne quelques cas particuliers.

B. d'Orgeval (Dijon)

Rossier, Paul 5143

Convergence de droites dans le triangle en géométrie affine.

Arch. Sci. Soc. Phys. Histoire Natur. Genève **16** (1963), 485–487.

Si l'on construit sur les trois côtés d'un triangle comme diamètres trois coniques homothétiques, elles se coupent deux à deux sur les supports des côtés, et les trois droites joignant ces points au sommet du côté opposé sont concourantes. Si l'un des points d'intersection de ces coniques est au milieu d'un côté, tous les trois le sont et l'on retrouve le théorème des médianes; si l'une des coniques est un cercle, toutes le sont et l'on obtient le théorème des hauteurs.

B. d'Orgeval (Dijon)

Hartmann, Heinrich; Lenz, Hanfried 5144

Quadriken und ihre Teilquadriken.

Arch. Math. **15** (1964), 302–309.

In einem projektiven Raum endlicher Dimension über dem Körper Q der rationalen Zahlen seien die Quadriken F, G (nicht notwendig gleicher Dimension) gegeben. Die Frage, wann G projektiv äquivalent zu einer Teilquadrik von F ist, wird (in die Sprache der Theorie der quadratischen Formen übersetzt) auf diejenigen Fragen zurückgeführt, die man bei Ersetzen von Q durch den Körper der reellen Zahlen bzw. durch die p-adischen Körper Q_p erhält.

G. Pickert (Giessen)

Degen, Wendelin 5145

Projektive Zielbewegungen und darin erzeugbare konjugierte Netze.

Abh. Math. Sem. Univ. Hamburg **27** (1964), 231–249.

Une famille de transformations d'un espace projectif en lui-même, dépendant d'un paramètre dit temps, constitue un mouvement projectif; les diverses images d'un point initial définissent sa trajectoire. Si à un instant donné les tangentes à toutes les trajectoires passent par un même point, on a affaire à un "Zielbewegung". Cette classe de mouvements jouit de propriétés remarquables, en particulier on peut les déterminer sans faire intervenir aucun signe d'intégration. A ces mouvements on peut associer, et à eux seulement, en partant d'une "trace" quelconque, un réseau conjugué de courbes; dans ce réseau, les transformées de Laplace d'un des systèmes de courbes sont planes, et la propriété réciproque caractérise ces "Zielbewegungen".

B. d'Orgeval (Dijon)

Krames, Josef 5146

Die windschiefen Flächen mit ebenen Fallinien.

Österreich. Akad. Wiss. Math.-Natur. Kl. S.-B. II **172** (1963), 159–172.

Von W. Wunderlich [J. Reine Angew. Math. **208** (1961), 204–220; MR **27** #651] wurde die Frage aufgeworfen nach allen jenen windschiefen Regelflächen, welche lauter ebene Fallinien besitzen. Diese Frage wird hier vollständig beantwortet. Die synthetisch gehaltene Untersuchung des Problems geht aus von der Bemerkung, daß alle in den Punkten einer Erzeugenden e einer windschiefen Regelfläche Φ gelegten Falltangenten von Φ eine gewisse Cayleysche Regelfläche dritten Grades bilden und gewinnt durch das eingehende Studium verschiedener Lagemöglichkeiten der Erzeugenden e das Ergebnis: Die einzigen windschiefen Regelflächen mit ebenen Fallinien sind (1) die einschaligen Drehhyperboloide mit vertikaler Drehachse, (2) die durch aufrechte Ellipsenbewegung erzeugbaren geraden Konoide (4. Grades VII. Art nach R. Sturm). Als aufrechte Ellipsenbewegung bezeichnet man dabei nach J. Krames [Monatsh. Math. Phys. **46** (1937), 38–50] jene räumlichen Bewegungen, bei denen eine Gerade a in sich verschoben wird und ein beliebiger Raumpunkt P eine a schneidende Ellipse beschreibt, welche einem Drehzylinder mit der Mantellinie a angehört.

K. Strubecker (Karlsruhe)

Levy, Harry 5147

★Projective and related geometries.

The Macmillan Co., New York; Collier-Macmillan, Ltd., London, 1964. x + 405 *pp*. $11.00.

This textbook on analytic geometry ("for undergraduate mathematics majors and beginning graduate students"), is restricted to two dimensions but makes a gallant attempt to be rigorous and consistent. In Chapter I the author introduces the Euclidean metric plane by means of ordered pairs of real numbers and a particular group of linear transformations. The crucial theorem that justifies this approach is stated on p. 49 without proof. Some geometers will not be happy to read (on p. 45) that "the processes of algebra and analysis are completely independent of geometry".

Chapter II (on projective spaces of dimension one) begins with a careful exposition of the idea of homogeneous coordinates over an arbitrary field in which $1+1\neq 0$. This is followed by a discussion of projectivities. The involution that interchanges two given pairs of points is described in terms of a pencil $\lambda Q_1+\mu Q_2$ of binary quadratic forms, with the conclusion that the invariant points (if any) of the involution are the roots of $\partial(Q_1, Q_2)/\partial(x_1, x_2)=0$.

In Chapter III (on projective spaces of dimension two), the author denotes a point by a column matrix $\mathbf{X}=(x_1, x_2, x_3)^T$ and a line by a row matrix $\mathbf{V}=(v_1, v_2, v_3)$, so that the condition for incidence is $\mathbf{VX}=0$ and the cross ratio of two points and two lines is

$$(VW, YZ) = \frac{\mathbf{VY}}{\mathbf{VZ}}\frac{\mathbf{WZ}}{\mathbf{WY}}.$$

The discussion of angular regions (called "sectors") could perhaps have been simplified by observing (on p. 134) that the four triangular regions formed by three nonconcurrent lines can be distinguished from one another by means of a fourth line that penetrates three of the regions while leaving the fourth untouched. In the proof of Desargues's two-triangle theorem (p. 144), the symbols $\lambda_i A_i$ and $\mu_i B_i$ might well have been abbreviated to A_i and B_i. On pp. 168 and 170, the author has interchanged the names Ceva and Menelaus.

Chapter IV provides a partially synthetic treatment of polarities and conics. The proof of Chasles's Theorem on p. 216 disregards the unpleasant possibility that the point C might lie on the line $A_1 \oplus B_1$.

Chapter V is on subgeometries of real projective geometry, namely, the affine plane (with a detailed discussion of sense), the hyperbolic line, the elliptic line, the Euclidean plane, the hyperbolic plane, the inversive plane, and the elliptic plane. He proves, for instance (on p. 297), that every equiaffine collineation is the product of two affine reflections. (Unhappily, he calls dilatations "dilations", disregarding the fact that, whereas "rotate" comes from the Latin "rotare", "dilate" comes from "dilatare".) In Theorem 8.4 (p. 339) he says that an opposite similarity is the product of a dilatation and three reflections; actually one reflection suffices. In Definition 9.4 (p. 361) he gives a parallel displacement the awkward name "infinite rotation", and defines it as the product of an even number of reflections in parallel mirrors; here two reflections suffice. The comparison of the projective and conformal models of the hyperbolic plane is neatly described on pp. 375–377.

There are numerous exercises (including some with hints and some with answers), excellent figures, a useful list of symbols, and an adequate index. {The only misprints noticed are "gives" for "give" in the middle of p. 62, and "i" for "it" on p. 157 (Theorem 9.3).}

H. S. M. Coxeter (Hanover, N.H.)

Börner, Walter 5148

Konstruktionen mit Lineal und Winkelübertrager in der hyperbolischen und euklidischen Ebene.

Publ. Math. Debrecen **10** (1963), 128–130.

Verfasser zeigt, dass die Systeme S_1 (Lineal mit Winkelübertrager) und S_2 (Lineal mit Streckenübertrager) in der hyperbolischen Ebene H äquivalente Instrumente sind, in der euklidischen Ebene E jedoch nicht. Man kann mit S_2 in E beliebige Winkel halbieren, aber nicht mit S_1.

O. Bottema (Delft)

Katz, Paul; Klee, Victor 5149

On the angle between two lines in a Minkowski plane.

Nieuw Arch. Wisk. (3) **12** (1964), 102–105.

Suppose E is a two-dimensional normed linear space with unit ball B, and L and M are two distinct lines through the origin in E. One of the two points of unit norm on L may be denoted by u_L. The distance from the point u_L to the line M is defined by

$$\delta(L, M) = \inf_{m\in M} \|u_L - m\|.$$

The authors investigate how the number $\delta(M, L)$ is related to the number $\delta(L, M)$ and find the following result:

$$\frac{\delta(L, M)}{1+\delta(L, M)} \leq \delta(M, L) \leq \min\left(1, \frac{\delta(L, M)}{1-\delta(L, M)}\right).$$

The condition under which equality holds is discussed.

H. R. Müller (Braunschweig)

Severi, Francesco 5150

★Funzioni quasi abeliane.

Seconda edizione ampliata. Pontificiae Academiae Scientiarum Scripta Varia, 20.

Pontificia Academia Scientiarum, Vatican City, 1961. 406 *pp*.

The first edition (1947) was reviewed in MR **9**, 578. The second edition consists of a reprint of the first (with reset type but apparently without any mathematical change of content), together with an appendix (improving the argument at one point) which contains (i) a reprint of the last part of a paper in Colloq. Fonctions de Plusieurs Variables (Bruxelles, 1953), pp. 9–20, G. Thone, Liège, 1953 [MR **15**, 521], (ii) a paper in Convegno Internaz. Geometria Differenziale (Italia, 1953), pp. 21–26, Edizioni Cremonese, Rome, 1954 [MR **16**, 163], and (iii) the text of the author's lecture in the Proc. Internat. Congr. Mathematicians, 1954 (Amsterdam), Vol. III, pp. 521–528, Noordhoff, Groningen, 1956 [MR **20** #5205].

Conforto's admirable review of the first edition, mentioned above, thus gives an adequate idea of the nature of the work. For developments, due in the main to Severi's pupils, we refer to the following review [#5151].

J. A. Todd (Cambridge, England)

Rosati, Mario 5151
★**Le funzioni e le varietà quasi abeliane dalla teoria del Severi ad oggi.**
Pontificiae Academiae Scientiarum Scripta Varia, 23.
Pontificia Academia Scientiarum, Vatican City, 1962. 194 *pp*.

This volume brings together in a convenient form extensions of the theory of quasi-abelian functions (see the preceding review [#5150]) due in the main to M. Benedicty and F. Conforto.

The first part of the book deals with the geometric theory of neutral domains on a curve, and extends to these, in a very complete way, the classical results of Hurwitz dealing with birational self-correspondences on an algebraic curve; this part of the theory is largely the creation of Benedicty. The remainder of the book is based on work, some of it posthumous, of Conforto, and is essentially a generalization of various aspects of the theory of Riemann matrices. This is by far the longer portion of the book, and contains a wealth of important and interesting results. There is a full bibliography of the subject at the end.

Those interested in the theory of quasi-abelian functions will be indebted to the author for this connected account of modern developments of the subject.

J. A. Todd (Cambridge, England)

Lüneburg, Heinz 5152
Finite Möbius-planes admitting a Zassenhaus group as group of automorphisms.
Illinois J. Math. 8 (1964), 586–592.

Let M be a finite Möbius plane [cf. Benz, Jber. Deutsch. Math.-Verein. **63** (1960), Abt. 1, 1–27; MR **22** #7012]. The only known types of such planes are the Miquelian planes (in some sense analogous to the Desarguesian planes of projective geometry) and the planes derived from the ovoid in Tits's presentation of the Suzuki groups [Tits, Arch. Math. **13** (1962), 187–198; MR **25** #3990]. The result of this paper is that M is in effect of one of these types if there is an automorphism group of M operating doubly transitively on the points of M and such that only the identity automorphism leaves three different points of M invariant. Any such group characterizes M up to isomorphism. (If the number of points on a circle of M is odd, then the finite Möbius plane M is of one of the above types. This result was recently announced by Dembowski [Bull. Amer. Math. Soc. **69** (1963), 850–854; MR **27** #5165].)

O. H. Kegel (Frankfurt a.M.)

Lüneburg, Heinz 5153
Charakterisierungen der endlichen desarguesschen projektiven Ebenen.
Math. Z. **85** (1964), 419–450.

Let $\mathscr{E}$ be a projective plane of finite order n and suppose that $\mathscr{E}$ is (X, PX)-transitive for some point P and all points X on some line g not containing P. (PX is the line joining P and X.) Then $\mathscr{E}$ is probably desarguesian. This conjecture is strongly supported by the paper under review. Failing to prove it in full generality, the author shows that it is true under any one of the following additional assumptions (here Δ denotes the group generated by all (X, PX)-collineations): (1) n is even, and all involutions of Δ are perspectivities (Satz 5); (2) n is even and not a square (Korollar 3, a simple consequence of Satz 5); (3) n is even, and $\mathscr{E}$ is (P, g)-transitive (Korollar 3); (4) n is odd, and Δ contains only one involution (Satz 6); (5) $n \equiv 3 \bmod 4$ (Korollar 5); (6) any collineation of Δ fixes either all or at most two points of g (Satz 7 (d)). {In the meantime the author has proved that the following condition also implies $\mathscr{E}$ to be desarguesian: $\mathscr{E}$ is (P, g)-transitive, and all involutions in the group generated by all (X, PX)- and (P, g)-collineations are perspectivities. This is an improvement of (1) and (4) above.}

Before proving these results, the author gives some preparations of independent interest. He begins with the following theorem (Satz 1): If the group H satisfies $PSL_2(q) \leq H \leq P\Gamma L_2(q)$ and if H possesses a doubly transitive permutation representation, then the degree of this representation is, with finitely many exceptions, equal to $q+1$. The exceptions (all for $q \leq 11$) are enumerated. This result is the main tool for the following characterizations of the finite desarguesian projective planes. Let $\mathscr{E}$ be a projective plane of prime power order q, and suppose that $\mathscr{E}$ has a collineation group Δ isomorphic to either (i) $SL_2(q)$ [here it is also postulated that, in case q even, every involution of Δ be a perspectivity] or (ii) $PSL_2(q)$ [in this case Δ is supposed to have an orbit of length $q+1$]. In both cases $\mathscr{E}$ is desarguesian (Satz 2 and Satz 3), and it is also determined how Δ must operate on $\mathscr{E}$ (Korollar 1 and Korollar 2). The proofs are highly nontrivial; they use some of the deepest recent results in finite group theory.

P. Dembowski (Frankfurt a.M.)

Barlotti, Adriano 5154
Sul gruppo delle proiettività di una retta in sè nei piani liberi e nei piani aperti.
Rend. Sem. Mat. Univ. Padova **34** (1964), 135–159.

The author considers the group G_r of projectivities (that is, finite products of perspectivities) on a line in free projective planes (in the sense of Marshall Hall). Among other results, the following are proved. (I) G_r contains no element, other than the identity, of finite period. (II) Any two elements of G_r which commute are contained in a cyclic subgroup. (III) G_r cannot be more than quintuply transitive on the points of the line. (In a footnote, the author mentions that, in a paper to be published, J. Joussen has proved that in fact G_r is triply transitive.) The author points out that these theorems are valid for a more extended class of projective planes.

J. A. Todd (Cambridge, England)

Dress, Andreas 5155
Eine geometrische Charakterisierung Desarguesscher Ebenen mit bewertetem Koordinatenkörper.
Abh. Math. Sem. Univ. Hamburg **27** (1964), 199–205.

The reviewer proved [Math. Ann. **132** (1956), 180–200; MR **18**, 667] that the existence of a homomorphism φ of a projective plane $P(K)$ over a field K onto the projective plane $P(\bar{K})$ of the field $\bar{K}$ implies that there is a valuation ring A in K, with maximal ideal I, such that A/I is isomorphic to $\bar{K}$ and such that φ is given as the composition of an isomorphism of $P(K)$ onto itself with the natural homomorphism of $P(A)$ (that is, the projective plane over the ring A as defined) onto $P(A/I) = P(\bar{K})$. In the

present paper the author attempts successfully a description of φ in the plane $P(K)$ itself, without referring to the image plane. For that purpose he defines a relation on the quadruples $(ABCD)$ of points of $P(K)$ as a property satisfying the following five conditions: (i) A, B, C, D are collinear and $A \neq D$ and $B \neq C$, (ii) for Menelaus configurations E, A_i, B_i, C_i $(i = 1, 2, 3)$ with collinear C_i, $(A_1C_2A_3B_3)$ implies $(A_1C_3A_3B_2)$ or $(A_3C_1A_2B_1)$, (iii) $(ABCD)$ and $(ADCB)$ do not hold simultaneously, (iv) $(ABCD)$ implies $(BACD)$, (v) there are three points A, B, C satisfying $(AABC)$. It turns out that the cross ratios of the points A, B, C, D satisfying the above relation form a maximal ideal of a valuation ring of K. The paper concludes with the description of a method to determine the homomorphism φ: $P(K) \rightarrow P(\bar{K})$ if one knows only the image of a projective frame in $P(K)$ and the associated valuation ring A in K.

W. Klingenberg (Mainz)

Dress, Andreas 5156

Metrische Ebenen und projektive Homomorphismen.

Math. Z. **85** (1964), 116–140.

Diese Arbeit stellt einen wichtigen Beitrag zur Strukturuntersuchung der metrischen Teilebenen E einer projektiv-metrischen Ebene $P(V)$ im Sinne von Bachmann [*Aufbau der Geometrie aus dem Spiegelungsbegriff*, Springer, Berlin, 1959; MR **21** #6557] dar. Das wesentliche Hilfsmittel dabei ist der Begriff des Homomorphismus von $P(V)$, wie er vom Referenten [Math. Ann. **132** (1956), 180–200; MR **18**, 667] eingeführt wurde. Wir erinnern daran, dass ein Homomorphismus φ einer projektiven Ebene $P(V)$, die aus einem Vektorraum V über dem Körper K hergeleitet ist, auf eine projektive Ebene $P(\bar{V})$, die aus einem Vektorraum über dem Körper $\bar{K}$ hergeleitet ist, Anlass gibt zu einem Bewertungsring R in K mit maximalem Ideal I so dass R/I isomorph zu $\bar{K}$ ist. Die zugehörige Primstelle von K werde mit $\mathfrak{p}$ bezeichnet. Seien nun überdies $P(V)$ und $P(\bar{V})$ projektiv-metrische Ebenen. Dann betrachten wir nur Homomorphismen φ, die mit dieser Struktur verträglich sind. Das Hauptergebnis der Arbeit gibt nun eine Charakterisierung der bezüglich $\mathfrak{p}$ pseudokonvexen metrischen Teilebenen E von $P(V)$ (zu diesem Begriff sei auf die Arbeit verwiesen) durch Eigenschaften ihrer Bilder unter dem Homomorphismus φ auf $P(\bar{K})$. Wie der Verfasser ankündigt, hat dieses Ergebnis weitreichende Folgerungen für den Fall, dass K nicht formal reell und perfekt für $\mathfrak{p}$ ist, da in diesem Falle jede Teilebene E von $P(V)$ bezüglich $\mathfrak{p}$ pseudokonvex ist. In diesem Falle ist also die Strukturuntersuchung der Teilebenen E von $P(V)$ reduziert auf die Untersuchung der Teilebenen in $P(\bar{V})$. Insbesondere wird hierdurch dann eine vollständige Bestimmung der metrischen Teilebenen E über lokalen Körpern gelingen, was in einer späteren Arbeit ausgeführt werden soll.

W. Klingenberg (Mainz)

Vittek, L'udovit 5157

A perfect decomposition of the square into 25 squares. (Slovak. English summary)

Mat.-Fyz. Časopis Sloven. Akad. Vied **14** (1964), 234–235.

Author's summary: "A dissection of the square into 25 mutually incongruent squares is shown in a figure."

Federico, P. J. 5158

A Fibonacci perfect squared square.

Amer. Math. Monthly **71** (1964), 404–406.

The author reports, inter alia, on an ingenious construction of a square dissected into 26 different squares.

C. J. Bouwkamp (Eindhoven)

CONVEX SETS AND GEOMETRIC INEQUALITIES

See also 4653, 5157, 5204, 5631.

Ewald, Günter 5159

Über die Schattengrenzen konvexer Körper. (Konvexe Körper und Konvexität auf Grassmann-Kegeln. VII).

Abh. Math. Sem. Univ. Hamburg **27** (1964), 167–170.

Part VI (by Shephard) appeared in J. London Math. Soc. **39** (1964), 307–319 [MR **29** #520]. Let K be contained in Euclidean n-space R^n. Let $\mathscr{R}$ be an arbitrary r-dimensional subspace of R^n through a fixed point z, $1 \leq r \leq n-1$. The $(n-1)$-dimensional supporting spaces of K which are at right angles to $\mathscr{R}$ form a cylinder $Z(K, \mathscr{R})$. The set $S(K, \mathscr{R}) = Z(K, \mathscr{R}) \cap K$ is called an r-shadow-boundary of K. It is sharp if every generator of $Z(K, \mathscr{R})$ has precisely one point in common with K. The main theorem is that, for all $\mathscr{R}$ except a set of measure zero (with a suitably defined measure), $S(K, \mathscr{R})$ is sharp.

A. M. Macbeath (Birmingham)

Grünbaum, B. 5160

A proof of Rogers' conjecture of pairs of convex domains.

J. London Math. Soc. **39** (1964), 697–702.

The following conjecture of Rogers is proved. Let K_1, K_2 be plane convex bodies; unless both bd K_1 and bd K_2 are ellipses, there is an affine transformation T such that $TK_2 \subset K_1$ and bd $K_1 \cap$ bd TK_2 has at least three connected components.

A. M. Macbeath (Birmingham)

Pliś, A.; Turowicz, A. 5161

On chords of convex bodies.

Colloq. Math. **12** (1964), 87–89.

The following (already well-known) result is proved: For any chord that passes through the centroid c of an n-dimensional convex body, let q be the ratio of the lengths of the two parts into which it is divided by c. Then $n^{-1} \leq q \leq n$. References to earlier proofs of this and related results will be found in a paper of B. Grünbaum [Proc. Sympos. Pure Math., Vol. VII, pp. 233–270, Amer. Math. Soc., Providence, R.I., 1963; MR **27** #6187].

G. C. Shephard (Birmingham)

Rubinšteĭn, G. Š. 5162

Theorems on the separation of convex sets. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 1098–1124.

The author's announcements [Dokl. Akad. Nauk SSSR **78** (1951), 213–215; MR **13**, 45; ibid. **102** (1955), 451–454; MR **17**, 185; ibid. **152** (1963), 288–291; MR **28** #1983] have stressed the geometric nature of his proofs. This claim is now made formal by doing everything in a "generalised vector space" whose structure consists merely of a family of copies of the ordered set of real numbers ("axes"), one

containing each ordered pair of distinct points x, y, reversing with (x, y), and satisfying the axiom of Pasch.

The question naturally arises whether such a space is really more general than a linearly open convex set in a real vector space. The author shows that it is, by giving a two-dimensional example in which the extremal parallel relation for rays is not symmetric. However, all such examples are maximal, at least in the sense that a generalized vector space not embeddable in a generalized vector space with unique parallels is neither a subspace with codimension >0 nor a proper factor space of any generalized vector space. {Generalized vector spaces with unique parallels seem to be vector spaces, but the author says nothing about it.} *J. R. Isbell* (New Orleans, La.)

Fejes Tóth, László 5163

Über eine Extremaleigenschaft der affin-regulären Vielecke. (Russian summary)

Magyar Tud. Akad. Mat. Kutató Int. Közl. **8** (1963), 299–302 (1964).

The author gives a simplified proof of a theorem of A. Rényi and R. Sulanke [Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **2** (1963), 75–85; MR **27** #6190]: Let T be a convex n-gon, in the plane, of area Δ, and let $t_1, \cdots, t_n$ be the areas of the triangles defined by three consecutive vertices of T. Then $x = t_1 t_2 \cdots t_n/\Delta^n$ attains its maximum when T is affine regular.

In the proof, we let T be a polygon for which x attains its maximum value (such exists by Weierstrass' Theorem), and consider an infinitesimal movement of one vertex of T. The author shows, by a geometrical and analytical argument, that x may be increased unless T is affine regular. *G. C. Shephard* (Birmingham)

Hajós, G. 5164

Über eine Extremaleigenschaft der affin-regulären Polygone. (Russian summary)

Magyar Tud. Akad. Mat. Kutató Int. Közl. **8** (1963), 293–297 (1964).

This paper presents another proof of the theorem of A. Rényi and R. Sulanke stated in the preceding review [#5163]. The author shows by a simple geometrical argument that if T is an n-gon for which x attains its maximum value, then for each pair of vertex-regions of T there is an "affine-symmetry" (a transformation of the type SRS^{-1}, where S is an affine transformation and R is a Euclidean reflection) which transforms one into the other. (A vertex-region is the cone of projection of T from a vertex of T.) The major part of the proof consists of showing that if this condition is satisfied, then T must be affine-regular.

The author also shows that if a polyhedron P, in three dimensions, has the property that each pair of vertex-regions are "affine-symmetric", then P must be either a tetrahedron or an affine-regular octahedron. *G. C. Shephard* (Birmingham)

Erdős, P.; Rogers, C. A. 5165

The star number of coverings of space with convex bodies.

Acta Arith. **9** (1964), 41–45.

If we are given a covering of a space by a system of sets, the star number of the covering is the supremum, over sets of the system, of the cardinals of the numbers of sets of the system meeting a set of the system. In view of the results of dimension theory, it is natural to conjecture that any covering of R^n by closed sets of uniformly bounded diameter has star number at least $2^{n+1}-1$. This has been proved, for $n=2$, by V. Boltjanskiĭ [Dokl. Akad. Nauk SSSR **75** (1950), 605–608; MR **13**, 149; errata, MR **13**, p. 1139].

In this paper only coverings of R^n by translates of a fixed convex body are considered, and the following are proved. Theorem 1: The star number of a lattice covering of R^n by translates of a closed centrally-symmetric convex body is at least $2^{n+1}-1$. Theorem 2: Provided n is sufficiently large, if K is a closed convex body with difference body DK, there exists a covering of R^n by translates of K with star number less than $(n \log n + n \log\log n + 4n + 1)(V(DK)/V(K))$, where the ratio of the volumes $V(DK)/V(K)$ is at most $\binom{2n}{n}$, in general, and equals 2^n if K is centrally-symmetric. The proof of Theorem 2 is a modification of the method used in a previous paper by P. Erdős and C. A. Rogers [Acta Arith. **7** (1961/62), 281–285; MR **26** #6863]. *G. C. Shephard* (Birmingham)

Leech, John 5166

Some sphere packings in higher space.

Canad. J. Math. **16** (1964), 657–682.

In the first part of this paper the densest lattice packings of equal spheres in four and eight dimensions are generalized to obtain a packing in 2^m dimensions in which each sphere is touched by $\prod_{r=1}^{m}(2^r+2)$ others.

In the second part of the paper analogues of the densest lattice packing in eight dimensions are generalised to obtain new packings in twelve and twenty-four dimensions. In the packing in twelve dimensions, each sphere is touched by 648 others, in twenty-four dimensions each sphere is touched by 98256 others. The lattices associated with these packings seem to be connected with the Steiner systems $S(5, 6, 12)$ and $S(5, 8, 24)$. Packings in spaces of lower dimension are derived by taking suitable linear sections.

The third part of the paper connects the number of spheres touching a given sphere of the packing, and the density of the packing, with the upper bounds determined by Coxeter [Proc. Sympos. Pure Math., Vol. VII, pp. 53–71, Amer. Math. Soc., Providence, R.I., 1963; MR **29** #1581] and Rogers [Proc. London Math. Soc. (3) **8** (1958), 609–620; MR **21** #847]. *J. A. Todd* (Cambridge, England)

DIFFERENTIAL GEOMETRY

See also 4651, 5084, 5257, 5260, 5424, 5599, 5603, 5610, 5614.

Finn, Robert 5167

On normal metrics, and a theorem of Cohn-Vossen.

Bull. Amer. Math. Soc. **70** (1964), 772–773.

Let S be a finitely connected surface with a Riemannian metric with respect to which it is complete and has finite

curvatura integra C. Then a neighborhood of each boundary component can be mapped conformally onto the punctured disk $0<|z|<1$. For $0<r_1<r_2<1$, let $L(r_1)$ be the length in the metric of $|z|=r_1$, and $A(r_1, r_2)$ the area in the metric of the annulus $r_1 \leq |z| \leq r_2$. The author considers a class of metrics for which he is able to prove the following theorem: The quantity $L^2(r_1)/A(r_1, r_2)$ tends to a finite limit as $r_1 \to 0$. If we denote this limit at the jth boundary component by $4\pi\nu_j$, then $C=2\pi(\chi-\sum\nu_j)$, where χ is the Euler characteristic of S.

Since obviously $\nu_j \geq 0$, this sharpens the classical result of Cohn-Vossen that $C \leq 2\pi\chi$ for the class of metrics considered. No proofs are given. {There is a misprint in the definition of ν_j in the paper. The numerator should be squared.} *R. Osserman* (Stanford, Calif.)

Giering, Oswald 5168

Über das Krümmungsverhalten der einer Fläche umschriebenen Torsen.

Monatsh. Math. **68** (1964), 193–212.

This paper, written in the style of classical differential geometry, uses the duality between space curves and developables as a heuristic method for the discovery of theorems about the curvature of developables circumscribing a surface. The main notion, corresponding to that of normal curvature, is the conical curvature of a developable in a ruling; this is the limit of the ratio of the angle of neighboring tangent planes to the angle of the corresponding rulings. The dual of the theorems of Meusnier and Euler for the normal curvatures is the theorem of Mannheim [E. Müller, Akad. Wiss. Wien Math.-Natur. Kl. S.-B. Abt. IIa **126** (1917), 311–318]. The dual of the fact that the curvature properties of curves on a surface can be studied on plane sections is that all developables circumscribing a surface along curves with a common osculating plane in some point have the same point of striction (or regression) and the same conical curvature for the common ruling, therefore the curvature properties of the developables can be studied on the surfaces circumscribing the sections of osculating planes. This leads to very detailed theorems, in particular, to a formula for the radius of conical curvature in dependence on certain normal curvatures, and to the proof that the conical curvature is invariant in the Gaussian map. Finally, the connection with previous constructions by E. Kruppa and C. Segre is indicated.

H. W. Guggenheimer (Minneapolis, Minn.)

Godeaux, Lucien 5169

La géométrie différentielle des surfaces considérées dans l'espace réglé.

Acad. Roy. Belg. Cl. Sci. Mém. Coll. in-8° (2) **34**. *no.* 6, 84 *pp.* (1964).

Table of contents: (I) Suite de Laplace associée à une surface; (II) Suites de quadriques associées aux points d'une surface; (III) Surfaces associées à des suites de Laplace périodiques; (IV) Surfaces associées à une suite de Laplace terminée; (V) Congruences W.

Gorowara, Krishan K. 5170

A problem in rectilinear congruences using tensor calculus.

Riv. Mat. Univ. Parma (2) **4** (1963), 159–165.

Une surface S étant regardée comme la surface de référence d'une congruence rectiligne, l'auteur, utilisant le calcul tensoriel, donne deux expressions générales de la courbure K d'une courbe quelconque C de S en l'un quelconque (x) de ses points, mettant en jeu la position du rayon de la congruence issu de (x) et la courbure normale de S suivant la direction de C. Diverses applications sont faites de ces formules, concernant les cas où les rayons de la congruence envisagée sont normaux à S ou bien situés dans l'une des faces du trièdre de Frenet de la courbe C.

P. Vincensini (Marseille)

Minkwitz, G. 5171

Über das Verhalten der Hauptkrümmungsdifferenz in der Umgebung einer Nabelpunktslinie.

Monatsb. Deutsch. Akad. Wiss. Berlin **5** (1963), 411–414.

The object of study is a surface in E^3 having a plane curve of symmetry C consisting of umbilics. One result is the following: If k_1, k_2 are the principal curvatures at each point, and if we let $A=|k_1-k_2|$ be the "astigmatism" of the surface at each point, then along C we have $\partial A/\partial n = \pm 2dk/ds$, where k and s are the curvature and arc length, respectively, of C. *R. Osserman* (Stanford, Calif.)

Vogler, Hans 5172

Die auf einer Torse verlaufenden Linien konstanten Gratabstandes als duale Seitenstücke zu den pseudorektifizierenden Torsen einer Raumkurve.

Österreich. Akad. Wiss. Math.-Natur. Kl. S.-B. II **172** (1963), 173–187.

In einer früheren Note [dieselben S.-B. **171** (1962), 173–188; MR **27** #1884] hat der Verfasser die pseudorektifizierenden Torsen einer Raumkurve k studiert, d.h. jene Torsen, auf denen k pseudogeodätische Linien im Sinne von W. Wunderlich [Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa **158** (1950), 61–73; MR **12**, 742; ibid. **158** (1950), 75–105; MR **12**, 742] sind, d.h. solche Linien, deren Schmiegebenen gegen die Torse unter festen Winkeln geneigt sind. In der vorliegenden Arbeit wird die dazu duale Figur untersucht, nämlich jene Kurven $\mathfrak{y}$ auf einer Torse, deren Punkte von der Gratlinie $\mathfrak{x}$ der Torse Θ einen festen, auf den Tangenten der Gratlinie gemessenen Abstand (konstanten Gratabstand) haben. Bei der Bewegung des begleitenden Dreibeins D der Gratlinie der Torse längs der Gratlinie $\mathfrak{x}$ beschreibt jeder Punkt der Tangente eine der möglichen Linien $\mathfrak{y}$ konstanten Gratabstandes der Torse. Die Bewegung des begleitenden Dreibeins D längs der Gratlinie ist in jedem Augenblick eine euklidische Schraubung (begleitende Schraubung S). Die Bahntangenten dieser Schraubung S bilden einen quadratischen Kollineationskomplex K mit der Segreschen Charakteristik [(11)(22)]. Dessen Grundfiguren (Komplexparabeln, Komplexkegel, kubische Ordnungskreise, kubische Ordnungstorsen usw.) spielen naturgemäß bei der Untersuchung jener Gebilde eine große Rolle, welche in jedem Augenblick von den Bahntangenten, Bahnschmiegebenen usw. der Punkte, Kanten und Ebenen des begleitenden Dreibeins D erzeugt werden. Natürlich gehören auch die Krümmungsachsen der Bahnkurven der begleitenden Schraubung S dem gleichen Komplex K an, woraus sich eine Reihe entsprechender Sätze ergibt. Als Beispiele der erzielten Ergebnisse erwähnen wir die folgenden Sätze: Sind die

Tangenten einer Kurve $\mathfrak{y}$ festen Gratabstandes einer Torse Θ mit dem begleitenden Dreibein D der Gratlinie $\mathfrak{x}$ von Θ starr verbunden, so ist die Gratlinie $\mathfrak{x}$ von Θ ein kubischer Kreis. Ermittelt man in den Punkten einer Erzeugenden e für die einzelnen Linien festen Gratabstandes der Torse Θ (1) die Hauptnormalen, (2) die Binormalen, (3) die Streckebenen, so erhält man (1) eine Regelfläche 4. Grades (mit der Erzeugenden e als einfacher und der Krümmungsachse a der Gratlinie als dreifacher Leitgeraden), (2) eine Regelfläche 3. Grades (mit der Erzeugenden e als einfacher und der Krümmungsachse a als doppelter leitgeraden), (3) eine Torse vierter Klasse.

K. Strubecker (Karlsruhe)

Pinl, M. 5173

Über einen Satz von G. Ricci-Curbastro und die Gauss'sche Krümmung der Minimalflächen. II.

Arch. Math. **15** (1964), 232–240.

Part I appeared in same Arch. **4** (1953), 369–373 [MR **15**, 554]. The theorem referred to in the title gives an intrinsic characterization of the first fundamental form of a minimal surface in E^3. Namely, given a Riemannian metric g_{ij} with corresponding Gauss curvature K, this metric is realized (locally) by a minimal surface in E^3 if and only if the metric $g_{ij}^* = \sqrt{(-K)}g_{ij}$ is flat (i.e., $K^* \equiv 0$). This property does not hold in general for minimal surfaces in E^n, $n > 3$. The author investigates conditions on a surface so that it will hold, together with related properties.

R. Osserman (Stanford, Calif.)

Gol'dberg, V. V. 5174

Pairs P of Laplace sequences of an N-dimensional projective space. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 783–787.

Un couple de suites de Laplace $\{A\}$, $\{B\}$ est dit couple $P_{k,l}$ si le plan à k dimensions $\Sigma_k = [A_h A_{h+1} \cdots A_{h+k}]$ et le plan à l dimensions $\Sigma_l' = [B_h B_{h+1} \cdots B_{h+l}]$ ont un point commun quel que soit h. L'auteur démontre que pour que le couple soit $P_{k,l}$, il faut et il suffit que les plans Σ_l et Σ_l' se coupent, ainsi que Σ_{l-1} et Σ_{l-1}'. Un tel couple dépend de deux fonctions arbitraires de deux arguments.

M. Decuyper (Lille)

Hejný, M. 5175

Construction of the relative normal in P_2. (Slovak. Russian and English summaries)

Acta Fac. Natur. Univ. Comenian. **9**, 95–98 (1964).

Author's summary: "Let $C: R(t)$ be a projective plane curve. Its tangent and the relative normal in the point R we denote by RT and RN. It is possible that the t is a projective arc. In this case the relative normal is changed into a projective normal whose construction is known. In the article we give the construction of the relative normal in the case when t is not necessarily an arc. It is clear that this construction depends on a parametrisation of the curve C."

Jůza, Miloslav 5176

Le système monoparamétrique des plans dans l'espace S_6.

Comment. Math. Univ. Carolinae **4** (1963), 99–101.

On considère dans l'espace projectif S_6 les variétés $V_{2,6}$, formées par un système monoparamétrique de plans $[y_0(t), y_1(t), y_2(t)]$, dont les courbes directrices $y_i(t)$ sont définies par certains systèmes d'équations différentielles. On donne ensuite, sans démonstrations, les conditions pour que la courbe $x = \sum a_0{}^i y_i(t)$ sur $V_{2,6}$ soit une quasi-asymptotique d'ordre 1, 2, ou 3.

P. Drăgilă (Timişoara)

Jůza, Miloslav 5177

Les monosystèmes d'espaces projectifs avec les asymptotiques formées par les droites.

Comment. Math. Univ. Carolinae **4** (1963), 103–104.

On considère dans l'espace projectif S_{2n+1} des sous-espaces S_n, auxquels on fait correspondre un autre espace projectif P_N, de dimension $N = \binom{2n+2}{n+1} - 1$.

L'auteur appelle monosystème le système des espaces $[x_0(t), \cdots, x_n(t)]$, $x_i(t)$ étant des courbes dans l'espace S_{2n+1}, et ensuite il donne des définitions nouvelles pour les asymptotiques, les points flecnodaux, et les pseudoregulus, qui furent étudiés dans un travail antérieur [Czechoslovak Math. J. **10** (85) (1960), 440–456; MR **22** #8451].

P. Drăgilă (Timişoara)

Picasso, E. 5178

Superficie differenziabili dello spazio proiettivo a cinque dimensioni.

Rend. Mat. e Appl. (5) **22** (1963), 392–415.

Ce travail est consacré à l'étude locale d'une surface V_2 de l'espace projectif S_5, dans une région à comportement régulier, tant pour l'espace 2-osculateur au point générique (x) que pour le système quintuple des lignes principales. L'auteur apporte à ce sujet une intéressante contribution personnelle relative à la généralisation de certains éléments fondamentaux de la géométrie des surfaces de l'espace projectif ordinaire, tels par exemple le repère normal de Fubini ou la quadrique de Lie des V_2 de S_3. La surface V_2 étant représentée par le système fondamental $[x_{rs} = c_{rs}^{\lambda\mu} x_{\lambda\mu},\ r, s, t = 1, 2, 3;\ \lambda, \mu = 0, 1, 2]$, il montre comment en associant à chacun de ses points un hyperplan tangent ξ et en envisageant la variété tangentielle W engendrée par ξ, on peut, en utilisant la condition en coordonnées locales d'appartenance d'un point et d'un hyperplan, former un système d'équations linéaires (indépendantes si ξ est générique) invariantes par rapport aux deux variétés V_2 et W, auquel se trouvent rattachés un repère local de V_2 et un repère local de W. Un tel rattachement n'est pas intrinsèque; les repères précédents dépendent, en dehors des équations fondamentales, des coordonnées de la droite intersection de ξ avec le plan des trois points dérivés seconds de (x). Si ξ est l'un des cinq hyperplans cupsidaux en (x), les deux repères corrélatifs attachés à V_2 et à W dépendent des équations fondamentales et de la direction dv/du de la section de V_2 par ξ, et en choisissant pour cette direction l'une des cinq directions principales on obtient, pour chaque point (x), des repères et des hypersurfaces du $2^{\text{ième}}$ ordre associées, à partir desquels il est possible de construire un repère et une hypersurface du $2^{\text{ième}}$ ordre intrinsèquement liés à V_2. L'auteur montre l'usage que l'on peut faire de ces deux éléments intrinsèques pour la construction géométrique de certains invariants infinitésimaux de contact.

P. Vincensini (Marseille)

Stanilov, G. 5179

Minimal lines in a bi-axial geometry. (Bulgarian)

Fiz.-Mat. Spis. Bŭlgar. Akad. Nauk. 7 (40) (1964), 51–53.

The author gives an elementary proof of the property shown by Ivanov [C. R. Acad. Bulgare Sci. 15 (1962), 697–698; MR 26 #4270] that the geodesic lines of the first kind, defined as those surface curves whose osculating planes contain the corresponding previously defined first normal of the surface, minimize a certain lineal element.

B. S. Popov (Skopje)

Terracini, Alessandro 5180

Elementi curvilinei composti e trasformazioni puntuali.

Math. Notae 19 (1964), 11–21.

L'auteur rappelle la définition d'un élément curviligne composé $E_{2,m}$, de support $Oo\omega$, commun à toute une famille de courbes passant par O, tangentes en ce point à une droite o issue de O, et admettant pour plan osculateur en O le plan ω passant par o. Il cherche s'il est possible qu'une transformation ponctuelle T change un élément composé en un élément composé. Il montre: (a) Soient O, O' deux points homologues dans T, o et o' deux directions homologues issues de O, O'; si aux courbes issues de O tangentes à o et ayant en O un plan osculateur donné ω, la transformation T fait correspondre des courbes issues de O' tangentes à o' et ayant en O' un plan osculateur donné ω', alors à tout élément composé $E_{2,m}$ de support $Oo\omega$ correspond par T un élément composé $E'_{2,m}$ de support $O'o'\omega'$. (b) Étant donné T, O, O', o, o', si les droites o, o' ne sont pas droites caractéristiques, il existe un seul couple de plans ω, ω' se trouvant dans les conditions de (a) et si ces droites o, o' sont caractéristiques, le plan ω est indéterminé. (c) Tout couple d'éléments composés qui se correspondent dans une transformation ponctuelle s'obtient nécessairement par le procédé de (a).

M. Decuyper (Lille)

Vasenin, V. V. 5181

On the existence of solutions of a system of exterior differential equations. (Russian)

Sibirsk. Mat. Ž. 5 (1964), 774–777.

The author proves a theorem concerning the existence of a solution of a special system of exterior differential equations. The systems of equations of the type under discussion occur if certain problems from the theory of line complexes, of pairs of complexes and of triples of non-holonomic surfaces are solved.

A. Urban (Prague)

Krjukov, M. S. 5182

On the inertial motion of a rod in a Lobatchevsky space. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika 1964, no. 4 (41), 86–98.

The title adequately describes the content. The Klein model of hyperbolic geometry is used and the formulas become very involved.

H. Busemann (Los Angeles, Calif.)

Mikan, Milan 5183

Möbius geometry and non-Euclidean geometry of one-parameter sets. (Czech. Russian and French summaries)

Rozpravy Československé Akad. Věd 73 (1963), no. 18, 90 pp.

L'ensemble (x) à un paramètre de sphères non-singulières x dans un espace M_3 de Möbius est défini comme un système de sphères dont les coordonnées pentasphériques x_i $(i=1,\cdots,5)$ sont des fonctions réelles d'un paramètre réel invariant s qui admettent dans l'intervalle $\langle 0, s_1\rangle$ des dérivées finies de tous les ordres par rapport à s. Ces coordonnées sont interprétées comme des coordonnées de Weierstrass des points x d'un espace hyperbolique P_4 à quadrique fondamentale elliptique réelle K à trois dimensions. L'image de l'ensemble (x) de sphères est une courbe (x) dans P_4 qui n'est pas situé dans un hyperplan de P_4 si et seulement si le Wronskien W_x de (x) est de rang $h_x=5$. Dans ce cas seulement l'image de l'ensemble (y) de sphères y orthogonales aux sphères x, $x^{\cdot}$, $x^{\cdot\cdot}$, $x^{\cdot\cdot\cdot}$ $(x^{\cdot}=dx/ds$, etc.) des ensembles (x), $(x^{\cdot})$, $(x^{\cdot\cdot})$, $(x^{\cdot\cdot\cdot})$ est une courbe. On a $h_x=5$ implique $h_y=5$ et on ne considère que le cas en question. Soient $y'=dy/du$, etc., les dérivées de y par rapport au paramètre invariant u de (y). On construit une sphère $\xi\equiv\eta$ telle que les sphères x, $x^{\cdot}$, y, y', $\xi\equiv\eta$ sont orthogonales deux à deux. Leurs images forment un simplex polaire par rapport à K. On l'appelle simplex conjoint des ensembles (x), (y) et on dit que les ensembles en question sont réciproques. Les relations entre les sphères singulières dans les faisceaux déterminés par x, $x^{\cdot}$ et x, y' permettent de distinguer trois cas spéciaux. On établit le système complet des invariants de l'ensemble (x), ainsi que les invariants de l'ensemble réciproque (y), ceux-ci étant déterminés par les invariants de (x) et inversement. En supposant que tous les invariants de (x) et (y) soient différents de zéro on examine leurs relations aux Wronskiens des deux ensembles et on établit les formules de Frenet, ainsi que les équations naturelles des deux ensembles réciproques. Les invariants en question sont en même temps invariants des enveloppes X, Y des ensembles (x), (y), les images de X, Y étant des surfaces réglées dont les droites, dans le cas de l'enveloppe réelle, ne coupent pas la quadrique K. On explique aussi la signification de l'annulation des invariants mentionnés. Les formules de Frenet ont, pour chaque position initiale du simplex conjoint une solution. Les résultats généraux sont appliqués ensuite au cas d'ensembles (x) à invariants constants. Dans ce cas, les courbes $(x^{\cdot})$, (y') se trouvent situées dans un même hyperplan C. Les formules de Frenet deviennent maintenant le système d'équations différentielles de d'Alembert. L'équation caractéristique de ce système a cinq racines dont une est toujours nulle, tandis que les autres forment deux couples de racines qui ne diffèrent que par le signe. On discute en détail la solution du système formé par les formules de Frenet. Une transformation infinitésimale qui reproduit une courbe (x) donnée à invariants constants s'appelle transformation G. On déduit les conditions nécessaires et suffisantes pour que les coordonnées des points de la courbe (x), reproduite par G, soient normalisées et pour que le paramètre sur (x) soit invariant, ainsi que les conditions nécessaires et suffisantes pour que G soit un mouvement hyperbolique dans P_4. L'ensemble de toutes les courbes (x) à invariants constants dépend de 13 paramètres, l'ensemble des transformations G est à 10 paramètres. Une transformation G reproduit un ensemble à trois paramètres de courbes (x) dont une seule passe par un point donné. On construit ensuite un groupe à un terme engendré par la transformation G reproduisant un ensemble de courbes (x) comme un sous-groupe de mouvements hyperboliques dans P_4. La transformation G

reproduit aussi les courbes (y) réciproques aux courbes (x) reproduites par la même transformation G, ainsi que les autres courbes engendrées par les sommets du simplex conjoint. On définit et on étudie encore les courbes osculatrices à invariants constants à une courbe donnée.
K. Svoboda (Brno)

Rozenfel'd, B. A.; Karpova, L. M.; Andreeva, L. P. 5184
Metric invariants and covariants of pairs of planes in a quasi-elliptic space. (Russian. Lithuanian and English summaries)
Litovsk. Mat. Sb. 4 (1964), 241–253.
Authors' summary: "In this article is found an effective method of determination of metrical invariants and covariants of couples of planes in a space with projective metric—the quasielliptic space, that is, the space with an absolute consisting of an imaginary cone of second order and a non-degenerate quadric in a summit plane of this cone. The metrical covariants of couples of planes are analogous to the perpendiculars of couples of planes in non-Euclidean spaces. The metrical invariants are analogous to the lengths of these common perpendiculars. The points of intersection of the metrical covariants of two planes with these planes are determined by means of eigenvectors of some matrices formed by the matrix coordinates of the planes. The matrix invariants are determined by means of eigenvalues of the matrices. These matrices are formed by various ways for the metrical covariants being in lines not intersecting with the summit plane of the absolute, intersecting with this plane and entirely being in this plane." *M. Kawaguchi* (Sapporo)

Rozenfel'd, B. A.; Ežova-Guseva, L. M.; Semenova, T. A. 5185
Metric invariants and covariants of pairs of planes in a flagspace. (Russian. Lithuanian and English summaries)
Litovsk. Mat. Sb. 4 (1964), 255–259.
Authors' summary: "In this article is found an effective method of determination of metrical covariants of couples of planes in a space with projective metric—the flag space, that is, the space with an absolute consisting of the planes of all dimensions (from hyperplane to point) enclosed one into another. Also given is a new basis of a method of determination of metrical invariants of couples of planes in the flag space proposed by the same authors. The metrical covariants of couples of planes are analogous to the common perpendiculars of planes in non-Euclidean spaces. The metrical invariants are analogous to the lengths of these common perpendiculars. The coordinates of the points of intersection of metrical covariants of two planes with these planes are determined by means of matrix coordinates of the planes, the metrical invariants of these planes coincide with some elements of these matrices."
M. Kawaguchi (Sapporo)

Frank, B. 5186
The theory of curves in Minkowski space. (Russian. English summary)
Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. 19 (1964), no. 2, 46–56.
Author's summary: "A special form of the method of the moving repère is given, which allows one to study the curves in Minkowsky space, including the case when one of the principal vectors (tangent, normal, etc.) is isotropic."

Širokov, P. A. [Широков, П. А.] 5187
★Tensor calculus [Тензорное исчисление].
Second edition.
Izdat. Kazan. Univ., Kazan, 1961. 447 *pp.* 1.94 *r.*
The only change from the first edition [Kazan, 1934] consists of a change in the notation of vectors.

Butler, T.; Carroll, R. 5188
Some remarks on vector analysis.
Math. Z. 86 (1964), 1–11.
As is well known, the vector operators "grad", "div" and "curl" are subsumed under exterior differentiation. The authors investigate these operations from the point of view of Hilbert spaces, and relate them to the corresponding operators on differential forms which occur in the $\bar{d}$-Neumann problem [see, e.g., P. E. Conner, Mem. Amer. Math. Soc. No. 20 (1956); MR 17, 1197] restricted to a subdomain of 3-space. *D. C. Spencer* (Stanford, Calif.)

Koszul, J. L. 5189
Notions de calcul différentiel extérieur.
Univ. Nac. Ingen. Inst. Mat. Puras Apl. Notas Mat. 1 (1962/63), 106–116.
An expository article emphasizing vector analysis for dimension 3 as a special case.
K. Nomizu (Providence, R.I.)

Libermann, Paulette 5190
Sur la géométrie des prolongements des espaces fibrés vectoriels.
Ann. Inst. Fourier (Grenoble) 14 (1964), *fasc.* 1, 145–172.
Various authors (Ehresmann, Atiyah, the reviewer, and others) have introduced differential constructions of differentiable fibre bundles ("jets of sections", "prolongations", "derivatives", etc.) for various purposes. The present paper explores the relationships of these and clarifies some points of the separate theories. Some previous results of the author are summarized in what is, to the reviewer's mind, a simpler form. The paper also contains a generalization to higher order of the theory of "sprays" of Palais, Ambrose, and Singer.
W. F. Pohl (Minneapolis, Minn.)

Edelen, Dominic G. B. 5191
On set functionals that preserve a given geometry class.
J. Math. Mech. 13 (1964), 927–937.
The well-known averaging process for scalar functions associated with

$$\mathscr{A}(\phi, d)(x) = \int_{d(x)} \phi(y)\, d\mu(y)$$

(where $d(x)$ is a neighborhood of x) is generalized for linear homogeneous geometric objects, which includes the tensor fields. The result of action on such a geometric object has the same transformation law as the original.
A. Nijenhuis (Philadelphia, Pa.)

Bilimovitch, A. D. [Bilimović, Anton D.] 5192

★**On a general phenomenological differential principle.**
Edited by Tatomir P. Angelitch (Anđelić). The Serbian Academy of Sciences and Arts Monographs, Vol. CCCLXVIII, the Section of Natural and Mathematical Sciences, Vol. 32.
Naučno Delo, Belgrade, 1964. ix + 141 *pp*. *Din.* 1500.00.

The linear differential form $\Phi = \sum_{i=1}^{N} X_i dx_i$ is called Pfaff's linear form or Pfaff's expression after Johann Friedrich Pfaff, who founded the theory of such differential forms. Pfaff's expression and Pfaff's equations are closely related to other mathematical concepts, which occupy an important place in modern mathematics. Applications of integral invariants were made by H. Poincaré, E. Cartan and G. Birkhoff for the first time, and the relationships between the theory of integral invariants and Pfaff's expression underlines the importance. On the other hand, the so-called Pfaff method in mechanics was presented for the first time by E. T. Whittaker in 1904. The relation between Pfaff's expression and differential equations in canonical form was established by G. Prange and E. Schering.

In his preface the author first gives the phenomenological interpretation of a principle of mechanics which is called Pfaff's principle and the substance of which, in a different form, has been known for quite a long time. The applications of that principle have proved to be very convenient, its theory being, especially in recent time, evaluated very intensively. Then, he uses that principle for the formulation of a general phenomenological differential principle. The phenomenological derivation of that principle and the phenomenological interpretation of its procedure, i.e., its algorism, is the main purpose of this treatise.

In Chapter 1, he speaks of the history of the Pfaff principle, and Chapter 2 is devoted to the phenomenological interpretation of Pfaff's principle in mechanics. Chapter 3 is concerned with the same interpretation in celestial mechanics. An interpretation in the mechanics of variable masses is considered in Chapter 4. Chapter 5 is devoted to an interpretation in physics. In Chapter 6 he makes general remarks on the phenomenological principle. The bibliography at the end of this treatise is extensive.

K. Takano (Tokyo)

Misner, Charles W. 5193

Differential geometry and differential topology.
Relativité, Groupes et Topologie (*Lectures, Les Houches*, 1963 *Summer School of Theoret. Phys., Univ. Grenoble*), *pp.* 881–929. *Gordon and Breach, New York*, 1964.

This highly mathematical, perfectly readable, article begins by defining certain differentiable manifolds. The second chapter deals with covariant vectors, in particular, with tangent vectors, vector fields, Lie brackets, Frobenius' theorem, and fibre bundles. In the next section tensors are introduced, and the Lie derivative is defined. Chapter 4 is about exterior differential forms. Following this, a sketch of Riemannian geometry is given, calling attention to Cartan methods with Chevalley concepts.

While the material so far presented is essentially a survey of differential geometry from a modern viewpoint, the second part of the lectures is devoted to an introduction to algebraic topology. Chapter 6 introduces the reader to cohomology, various homotopy theorems, and to cohomology computations. The last chapter presents suggestions of cellular cohomology and deals briefly with combinatorial topology, relative cohomology, and cohomology of spheres.

P. Roman (Boston, Mass.)

Goetz, A. 5194

On induced connections.
Fund. Math. **55** (1964), 149–174.

A detailed exposition of the results given in two previous papers [Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **10** (1962), 29–34; MR **25** #2543; ibid. **10** (1962), 277–283; MR **25** #2544].

K. Nomizu (Providence, R.I.)

Ngô Van Quê 5195

De la connexion d'ordre supérieur.
C. R. Acad. Sci. Paris **259** (1964), 2061–2064.

From the author's summary: "Dans cette note, on dégage quelques notions fondamentales de la théorie de la connexion d'ordre supérieur."

W. F. Pohl (Minneapolis, Minn.)

Kikkawa, Michihiko 5196

On affinely connected spaces without conjugate points.
J. Sci. Hiroshima Univ. Ser. A-I Math. **28** (1964), 33–38.

Let M^n be a C^∞ manifold with affine connexion ∇. A Jacobi equation for a vector field along a geodesic is derived from a geodesic deviation, allowing definitions of Jacobi fields and conjugate points as usual. If $n = 2$, ∇ is symmetric, and the Ricci curvature of a vector is always positive semi-definite (or negative semi-definite if one wants the Ricci curvature on a sphere to be positive), then there are no conjugate points. An example shows there exists a non-flat symmetric connexion on the 2-torus with no conjugate points, a situation shown to be impossible in the Riemannian case by E. Hopf [Proc. Nat. Acad. Sci. U.S.A. **34** (1948), 47–51; MR **9**, 378].

N. J. Hicks (Ann Arbor, Mich.)

Cenkl, Bohumil 5197

Les variétés de König généralisées. (Russian summary)
Czechoslovak Math. J. **14** (89) (1964), 1–21.

The author considers a differentiable manifold X_n and associates a projective space $P_N(\xi)$ and a distinguished subspace $P_p(\xi)$ of $P_N(\xi)$ with each point ξ of X_n. The subscripts n, N, p denote dimension. The spaces $P_N(\xi)$ are projectively related along differentiable curves by means of a connection, and the resulting structure is called a generalised König manifold with projective connection. The paper is mainly concerned with the development of a tensor calculus and curvature theory for this structure.

F. Brickell (Southampton)

Cenkl, Bohumil 5198

L'équation de structure d'un espace à connexion projective. (Russian summary)
Czechoslovak Math. J. **14** (89) (1964), 79–94.

The first part of this paper is a summary of the theory of a generalised König manifold with projective connection due to the author [#5197 above]. With the notation of that

paper (or the review) the author considers the particular case $n=2$, $N=3$, $p=0$, and introduces geometrical notions generalising those of the classical projective differential geometry of surfaces. The integrability conditions for the structure equations are interpreted in terms of these notions. *F. Brickell* (Southampton)

Švec, Alois 5199

Au sujet de la définition des variétés de König. (Russian summary)

Czechoslovak Math. J. **14** (**89**) (1964), 222–234.

Denote by G^n the general affine group in n variables and by U_r a domain in r-dimensional real number space. The author calls the product $P_n{}^r = U_r \times G^n$ a König manifold and, regarding $P_n{}^r$ as a principal fibre bundle with group G^n, he defines a connection in a standard way using a differential 1-form ω. Now let G^p be the subgroup of G^n leaving invariant a p-plane; then, using the curvature form of the connection, a 2-form is defined with values in the quotient space of the Lie algebras of G^n and G^p. This form is called the torsion form of the König manifold $P^r_{p,n}$. There is a natural injection of $U_r \times G^p$ into $P_n{}^r$ and, because G^n/G^p is reductive for $p=0$, the form ω can be used to define a connection form on $U_r \times G^0$. The curvature form of this connection is called the curvature form of $P^r_{0,n}$. The author calculates the various curvature and torsion forms in terms of local coordinates.

F. Brickell (Southampton)

Levine, Jack 5200

Liouville spaces admitting groups of motions.

Tensor (*N.S.*) **15** (1964), 243–257.

Defining a Liouville space, L_n, as a Riemannian space whose fundamental form is expressible as

$$\phi = (X_1+X_2+\cdots+X_n)\sum_{i=1}^{n} e_i(dx^i)^2,$$

where $X_i = X_i(x^i)$ is a function of the coordinate x^i only and $e_i = \pm 1$, the author determines all L_n, for $n \geqq 3$, admitting groups of motions, and obtains the complete group of motions in each case. *M. C. Chaki* (Calcutta)

Raghunathan, M. S. 5201

Addendum to "Deformations of linear connections and Riemannian manifolds".

J. Math. Mech. **13** (1964), 1043–1045.

A generalization of the results (Theorems 8.1, 9.1, and 9.2) and a correction to the proof of Proposition 10.1 in a previous paper [same J. **13** (1964), 97–123; MR **28** #4484]. The generalization consists in considering a variation of a tensor field along with the connection and gives a unified treatment of the deformation of Riemannian metrics and linear connections treated separately in the previous paper.

K. Nomizu (Providence, R.I.)

Roter, W. 5202

Some remarks on second order recurrent spaces.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 207–211.

A non-flat riemannian space with positive definite metric is recurrent if the components of the curvature tensor satisfy $R_{hijk,l} = c_l R_{hijk}$. The space is 2-recurrent if $R_{hijk,lm} = a_{lm} R_{hijk}$ for some a_{lm}. Lichnerowicz proved [Proc. Internat. Congr. Mathematicians (Cambridge, Mass., 1950) Vol. 2, pp. 216–223, Amer. Math. Soc., Providence, R.I., 1952; MR 13, 492] that a compact 2-recurrent space with scalar curvature $R \neq 0$ is either Cartan-symmetric or else recurrent. The author claims to prove this result without making the assumptions of compactness or that $R \neq 0$. Compactness is certainly unnecessary, but in his proof the author makes use of a vector which is defined to be the gradient of the logarithm of $|R|$, and here surely the assumption $R \neq 0$ is implicitly made.

Necessary and sufficient conditions are found for a space to be 2-Ricci-recurrent, i.e., the Ricci tensor satisfies $R_{ij,lm} = a_{lm} R_{ij}$. *T. J. Willmore* (Liverpool)

Efimov, N. V. 5203

Generation of singularities on surfaces of negative curvature. (Russian)

Mat. Sb. (*N.S.*) **64** (**106**) (1964), 286–320.

The author gives a detailed proof for the old conjecture that a complete surface F in E^3 with Gauss curvature $K \leqq c < 0$ and without singularities does not exist. The result was announced in Dokl. Akad. Nauk SSSR **150** (1963), 1206–1209 [MR **27** #694]; we retain the notation of the latter review. The idea of the proof and a fundamental lemma are discussed, and it is stated that further details are necessary to see how the lemma yields the result.

The lemma appears in the present paper essentially as Lemma A; the missing steps are provided by Lemmas B and C. The first states that the boundary of the spherical image F_1 of F cannot be anywhere concave, i.e., does not possess a subarc whose concave side lies (locally) in F_1. Concavity is to be understood in a certain generalized sense. Lemma C shows that a domain on the unit sphere, which is nowhere concave in this sense, is convex. The above review shows why this yields the theorem.

H. Busemann (Los Angeles, Calif.)

Šor, L. A. 5204

On the flexibility of convex polygons with a slit. (Russian)

Ukrain. Mat. Ž. **16** (1964), 513–520.

Let $\bar{P}$ be a closed convex polyhedron or a complete unbounded polyhedron with total curvature 2π. Let P originate from $\bar{P}$ by cutting $\bar{P}$, in the same sense as in classical complex variable theory, along a simple polygon L which may be unbounded if $\bar{P}$ is, but does not decompose $\bar{P}$. P is monotypic if no convex polyhedron intrinsically isometric, but not congruent, to P exists. P is bendable if it can be imbedded in a continuous family of intrinsically isometric convex polyhedra which are not all congruent.

A. D. Alexandrov proved long ago that P is monotypic if L is bounded and does not contain a vertex. It is shown here that this remains true when L contains at most one vertex. If the endpoints of L are vertices of $\bar{P}$, then P is, in general, bendable. The same is true for unbounded L beginning at a vertex and containing no other vertex. But examples are also given of such P which are monotypic.

H. Busemann (Los Angeles, Calif.)

Avez, André 5205

Essais de géométrie riemannienne hyperbolique globale. Applications à la relativité générale.

Ann. Inst. Fourier (*Grenoble*) **13** (1963), *fasc.* 2, 105–190.

This work is primarily concerned with the two problems: (1) Is a complete riemannian 4-manifold with hyperbolic metric and zero Ricci curvature locally euclidean? (2) Classify those complete riemannian 4-manifolds with hyperbolic metric and elliptic Ricci tensor. The essential difficulty of solving these problems is due to the absence of global theorems on riemannian manifolds with non-elliptic signature.

The first chapter describes a method for transforming the problem to an analogous problem involving an elliptic metric. This method shows that Mach's principle remains valid for a stationary model of space-time without any additional assumption other than compactness. Moreover, a stationary and compact model of the universe is constructed without using the cosmological constant, thus showing that the existence of global sections of space is essential to the theorem of Aufenkamp.

The second chapter examines a "temporal" analogue of the Hopf-Rinow theorem with a hyperbolic signature. An argument analogous to that used by Myers shows that every complete space-time whose density of matter exceeds a positive number is closed and homotopic to zero in the time. By means of theorems of Nash and Morrey it is proved that on every periodic closed space-time there is a global section of the space which is maximal in the sense of the calculus of variations, and from this is deduced the validity of Mach's principle for closed periodic space-times.

The final chapter attempts to extend to metrics of arbitrary signature the theory of Hodge and de Rham; here the situation differs considerably from the elliptic case, and there are essentially two different types of decomposition in the sense of Hodge. Techniques previously used by Bochner and Lichnerowicz are then applied to the corresponding decompositions. *T. J. Willmore* (Liverpool)

Dautcourt, G. [Dautcourt, Georg] 5206

Zum Anschluss von Lösungen der Einsteinschen Feldgleichungen.

Monatsb. Deutsch. Akad. Wiss. Berlin **5** (1963), 416–420.

The author discusses the nature of the solutions of the algebraic equations relating the discontinuities in the first derivatives of the metric tensor across a hypersurface S with those of the stress energy tensor for space-times satisfying the generalization of the Einstein field equations to the case where such singular hypersurfaces are present. Both the case when the normal to S is time-like and when it is null are considered. In the latter case it is shown that only special material tensors are allowed. The null geodesics whose tangents are the normals to the hypersurface are shown to be the same curves on both sides of the singular null-hypersurface. *A. H. Taub* (Berkeley, Calif.)

Dautcourt, Georg 5207

Über Flächenbelegungen in der allgemeinen Relativitätstheorie.

Math. Nachr. **27** (1963/64), 277–288.

The author considers as given two solutions of the Einstein field equations in vacuum. He raises the question as to whether these can be fitted together across a hypersurface in space-time in such a way that the metric tensor is continuous across this hypersurface but its derivatives need not be. He shows that this problem may be reduced to the problem: When are two metric tensors in a three- or two-dimensional space related by a coordinate transformation? The two-dimensional problem occurs when the hypersurface has a null-vector as a normal vector.

A. H. Taub (Berkeley, Calif.)

Treder, H. 5208

Statische Einstein-Räume mit veränderlicher Signatur.

Monatsb. Deutsch. Akad. Wiss. Berlin **6** (1964), 88–91.

The author gives a canonical form for the metrics of static space-times of signature -2 which satisfy the Einstein vacuum field equations. The space-times in question have a metric tensor whose determinant vanishes on a space-like hypersurface. It is shown that in the neighborhood of this surface this canonical form may be used to define a Riemannian space for which the Ricci tensor vanishes and for which the metric tensor has signature of -2 on one side of the hypersurface, -3 on the hypersurface, and -4 on the other side of the hypersurface. On the hypersurface the determinant of the metric tensor vanishes and across it the metric tensor and its first derivatives are continuous. *A. H. Taub* (Berkeley, Calif.)

Hlavatý, V. 5209

Infinitesimal deformation applied to a congruence of minimal curves.

Univ. Nac. Tucumán Rev. Ser. A **14** (1962), 91–103.

Let V_n be an n-dimensional pseudo-Riemannian manifold with indefinite metric. Let V and a be vector fields on V_n, let μ denote a transformation in the one-parameter family determined by V, and let $\mathscr{C}$ denote an integral curve of a. The author calls $\mathscr{C}$ "ε-embedded" in $\{\mathscr{C}\}$ if $\mu\mathscr{C} \in \{\mathscr{C}\}$ for μ sufficiently near to the identity. In this paper he studies conditions for ε-embedding of minimal geodesics in V_n.

F. A. E. Pirani (Waltham, Mass.)

Cattaneo-Gasparini, Ida 5210

Sulle G-strutture di una V_n definite da una 1-forma complessa a valori vettoriali.

Ann. Mat. Pura Appl. (4) **65** (1964), 81–96.

A discussion of complex almost product structures, based on, and generalizing somewhat, a large quantity of literature, among which are parts of Chapter V of Schouten's *Ricci-calculus* [second edition, Springer, Berlin, 1954; MR **16**, 521], Legrand [C. R. Acad. Sci. Paris **242** (1956), 335–337; MR **19**, 766] and Chapter V of Lichnerowicz's *Théorie globale des connexions et des groupes d'holonomie* [Edizioni Cremonese, Rome, 1957; MR **19**, 453]. The presentation is predominantly expository.

A. Nijenhuis (Philadelphia, Pa.)

GENERAL TOPOLOGY

See also 4712, 4841, 4855, 4898, 5111.

Heuchenne, C. 5211

Images directe et inverse d'une relation entre parties d'un ensemble.

Bull. Soc. Roy. Sci. Liège **33** (1964), 151–160.

C'est la continuation d'un article précédent de l'auteur [même Bull. **32** (1963), 636–652; MR **28** #2058]. Dans la première partie, il ajoute quelques résultats supplémentaires à l'article cité. Dans la seconde, il généralise deux opérations introduites par le rapporteur en posant: $f: E \to E'$ une application quelconque, $\mathscr{P} = \mathscr{P}(\mathscr{P}(E) \times \mathscr{P}(E))$, $\mathscr{P}' = \mathscr{P}(\mathscr{P}(E') \times \mathscr{P}(E'))$, $A'f^+(\alpha)B'$ si et seulement si $f^{-1}(A')\alpha f^{-1}(B')$ pour $\alpha \in \mathscr{P}$ quelconque, $Af^-(\alpha')B$ si et seulement si il existe $A', B' \subset E'$ tels que $A \subset f^{-1}(A')$, $A'\alpha'B'$, $f^{-1}(B') \subset B$ pour $\alpha' \in \mathscr{P}$. Il étudie en détail ces opérations en examinant leur rapport aux treillis $\mathscr{R}$, $\mathscr{S}$, $\mathscr{T}$, $\mathscr{Q}$, $\mathscr{B}$ et l'effet de l'injectivité ou de la surjectivité de f sur les propriétés de f^+ et f^-. *Á. Császár* (Budapest)

Franklin, Stanley P. 5212

Quotient topologies from power topologies.

Arch. Math. **15** (1964), 341–342.

The author shows that if R is an equivalence relation on a topological space X, then the quotient topology on X/R is the restriction to X/R of either of two topologies T_1 and T_2, which can be defined on the collection of all non-empty subsets of X. It should be noted that both T_1 and T_2 are defined in terms of R.

E. Michael (Seattle, Wash.)

Ikenaga, Shogo 5213

Product of minimal topological spaces.

Proc. Japan Acad. **40** (1964), 329–331.

Solving a problem of Berri [Trans. Amer. Math. Soc. **108** (1963), 97–105; MR **27** #711], the author shows that every product of minimal Hausdorff spaces is minimal Hausdorff. The corresponding question for regular spaces remains open; the author shows that the answer is affirmative if there is only one non-compact factor.

J. R. Isbell (New Orleans, La.)

Pickert, Günter 5214

Nachbarschaftsfilter und Verbandshalbgruppen.

Arch. Math. **13** (1962), 151–159.

The author obtains theorems on representations of filters and neighborhood filters. An example of the former is the following. Each filter $\mathfrak{F}$ in a set M is of the form (1) $\mathfrak{F} = \{K_\alpha | \alpha \in \mathbf{M}_*\}$ with (2) $K_\alpha = \{x | d(x) \leq \alpha\}$, where $\mathbf{M}$ is a completely distributive lattice with order relation $\geq$, $\mathbf{M}_0$ is an ideal in $\mathbf{M}$ (l.u.b.'s in $\mathbf{M}_0$ are the same as in $\mathbf{M}_*$) and d is a function from M to $\mathbf{M}$ with the following property: (3) To each $\alpha \in \mathbf{M}_0$ there is $x \in M$ such that $d(x) \leq \alpha$. Conversely, if $\mathbf{M}$ is a set with a transitive relation $\leq$ and $\mathbf{M}_0 \subset \mathbf{M}$ is directed downward and $\mathfrak{F}$ is defined by (1) and (2) with (3) holding, then $\mathfrak{F}$ is a filter base.

A neighborhood filter in a space R is given in terms of a "distance" function from $R \times R$ into a complete lattice-semigroup. The complete statement of this theorem is quite long. *R. W. Bagley* (Coral Gables, Fla.)

Bose Majumder, N. C. 5215

A study of certain properties of the Cantor set and of an (SD) set.

Bull. Calcutta Math. Soc. **54** (1962), 8–20.

The author proves the following theorems. Any point in the unit cell in the p-dimensional Euclidean space is the center mean position of at least $(p+1)^{p-1}$ sets, each of $(p+1)$ points, all belonging to the product set of p Cantor sets defined in the unit cell.

Theorem: A necessary and sufficient condition that a linear set E whose elements are distributed symmetrically in the unit interval be a set whose distance set fills up an interval of length equal to the diameter of the set is that any point in the unit interval is the midpoint of at least one pair of elements of E. A similar theorem is given for p-dimensional sets. These results are related to results of Randolph [Bull. Amer. Math. Soc. **46** (1940), 888; J. London Math. Soc. **16** (1941), 38–42; MR **3**, 226]. The methods are elementary. *L. K. Barrett* (Knoxville, Tenn.)

Efimov, B. 5216

Unimodular weight functions and a problem of P. S. Aleksandrov and P. S. Urysohn in the theory of bicompacta. (Russian)

Dokl. Akad. Nauk SSSR **158** (1964), 1260–1263.

The principal results are that if there exists a counterexample to the Aleksandrov-Urysohn conjecture that first countable bicompacta have at most c points, it cannot be a continuous image of either a product of bicompacta of weight $\leq c$ or of a bicompact subset of a Σ-product of intervals. *J. R. Isbell* (New Orleans, La.)

Heath, R. W. 5217

Separability and $\aleph_1$-compactness.

Colloq. Math. **12** (1964), 11–14.

A space is said to be $\aleph_1$-compact provided that every uncountable subset of it has a limit point. F. B. Jones has shown [Bull. Amer. Math. Soc. **43** (1937), 671–677] that if $2^{\aleph_0} < 2^{\aleph_1}$, then every separable normal space is $\aleph_1$-compact. The author constructs, under the assumption that $2^{\aleph_0} = 2^{\aleph_1}$, an example of a separable normal T_1-space which is not $\aleph_1$-compact. Two metrization theorems, which generalize some theorems of McAuley and Bing, are also proved: Every separable, strongly complete, regular semi-metric space and every separable Moore space, every open covering of which has a point-countable, open refinement, is metrizable. To prove these theorems the author shows that the spaces of the above classes are $\aleph_1$-compact, and he applies the theorem of F. B. Jones [loc. cit.] which asserts that every $\aleph_1$-compact Moore space is metrizable.

R. Engelking (Warsaw)

Treybig, L. B. 5218

Concerning continuous images of compact ordered spaces.

Proc. Amer. Math. Soc. **15** (1964), 866–871.

The main result is the following interesting theorem. Let X and Y be two infinite compact Hausdorff spaces and let the direct product $X \times Y$ be the image of a totally ordered compact space L under a continuous mapping $f: L \to X \times Y$. Then both X and Y are metrizable. This extends essentially results of P. Papić and the reviewer who obtained the same conclusion under stronger assumptions: (i) L is connected [Mardešić, Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hrvatske Ser. II **15** (1960), 85–89; MR **24** #A535], or (ii) X and Y are locally connected [Mardešić and Papić, ibid. **17** (1962), 3–25; MR **28** #1591]. The

proof is based on a sequence of elementary but delicate arguments. The same result was also obtained independently by A. J. Ward (unpublished).

S. Mardešić (Zagreb)

Duda, R. 5219

On compactification of absolute retracts.

Colloq. Math. **12** (1964), 1–5.

Let L_n be the segment (in the plane) with the ends $(0, 1)$ and $(1/n, 0)$ and let $N = \bigcup_{n=1}^{\infty} L_n$. It is proved that the space N is an AR (absolute retract) for normal spaces, but no compactification N^* of N is an AR for any class τ of T_2-spaces which has the following property: (i) For every space $X \in \tau$ there exists a locally connected space $Z \in \tau$ such that X is a closed subset of Z. From a result of Wojdysławski [Fund. Math. **32** (1939), 184–192] it follows that the class of metric and separable metric spaces satisfies (i).

{The validity of the remark before Lemma 1 is by no means obvious; hence the lemma cannot be considered proved. The same applies to the assertion that the class of normal spaces satisfies (i). Fortunately, a well-known specialized version of Lemma 1 is all that is needed to prove the theorem.}

R. Engelking (Warsaw)

Forge, A. B. 5220

Dimension preserving compactifications allowing extensions of continuous functions.

Duke Math. J. **28** (1961), 625–627.

The well-known Hurewicz compactification theorem [W. Hurewicz, Akad. Wetensch. Amsterdam Proc. Sect. Sci. **30** (1927), 425–430] asserts that for every separable metric space X there exists a compact metric space X^* such that (i) X^* contains X as a dense subset, (ii) $\dim X = \dim X^*$. The author shows that if, in addition, a countable family $\{f_i\}_{i=1}^{\infty}$ of continuous functions from X to the unit interval I is given, then there exists a space X^* which also satisfies (iii) the function f_i can be extended to a continuous function $F_i : X^* \to I$ for $i = 1, 2, \cdots$.

{This result was obtained simultaneously by the reviewer [Fund. Math. **48** (1959/60), 321–324; MR **22** #A1347].}

R. Engelking (Warsaw)

Dolčinov, D. 5221

A unified theory of topological spaces, proximity spaces and uniform spaces. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 21–24.

A theory is developed, unifying the theories of topological spaces, proximity spaces and uniform spaces. This theory differs considerably from the methods of Á. Császár [*Fondements de la topologie générale*, Akad. Kiadó, Budapest, 1960; MR **22** #4043; German transl., Akad. Kiadó, Budapest, 1963; MR **26** #6917; second edition (in English), Macmillan, New York, 1963; MR **28** #575], and is closer to the treatment of uniform spaces by A. Weil. Let X be a set, and let $\mathfrak{M}$ be a collection of subsets of X which contains in any case all singletons $\{x\}$, $x \in X$. A generalized topology (g.t.) Σ on X with respect to $\mathfrak{M}$ is defined as a set of mappings of $\mathfrak{M}$ in the power set of X satisfying certain conditions. (If $A \in \mathfrak{M}$ and $U \in \Sigma$, $U(A)$ is to be thought of as a neighbourhood of A.) Now if $\mathfrak{M} = \{\{x\} : x \in X\}$, the g.t.'s with respect to $\mathfrak{M}$ are in a natural one-to-one correspondence with the topologies (in the usual sense) on X. If $\mathfrak{M}$ is the full power set $\{A : A \subset X\}$, the g.t.'s on X with respect to $\mathfrak{M}$ satisfying a certain symmetry condition are in one-to-one correspondence with the proximity relations in X, again in some natural way. Finally, the uniform structures are obtained by uniformizing the conditions in the definition of a g.t. (switching of existential and universal quantifiers).

P. C. Baayen (Amsterdam)

Okuyama, Akihiro 5222

On metrizability of M-spaces.

Proc. Japan Acad. **40** (1964), 176–179.

K. Morita [Math. Ann. **154** (1964), 365–382; MR **29** #2773] introduced the class of M-spaces, and showed that it contains the class of all spaces topologically complete in the sense of Čech. The author proves that a topological space X is metrizable if and only if it is a paracompact M-space with a G_δ diagonal. {Reviewer's note: This generalizes Theorem 5.5 of J. Ceder [Pacific J. Math. **11** (1961), 105–125; MR **24** #A1707], which asserts that X is completely metrizable if and only if it is a paracompact, topologically complete space with a G_δ diagonal.}

E. Michael (Seattle, Wash.)

Saadaldin, Mohammed Jawad 5223

A generalized Lebesgue covering theorem.

Duke Math. J. **29** (1962), 539–542.

Suppose X is a compact metric space. A sequence G of subsets of X is called a contracting sequence if for each $\varepsilon > 0$ there are only a finite number of elements of G having diameter greater than ε. The following theorem is proved. Suppose $\dim X \geqq n$. There exists an $\varepsilon > 0$ such that if G is a contracting sequence of closed sets covering X with $\operatorname{diam}(g) < \varepsilon$ for each $g \in G$, then there exists a point common to at least $n+1$ elements of G. It is also shown that if $\varepsilon > 0$ there exists a countable closed cover G' of X such that $\operatorname{diam}(g) < \varepsilon$ for each $g \in G'$ and such that no three elements of G have a point in common.

E. Nunnally (Baton Rouge, La.)

Reátegui C., José 5224

On the topology of manifolds. (Spanish)

Univ. Nac. Ingen. Inst. Mat. Puras Apl. Notas Mat. **1** (1962/63), 35–40.

The author determines conditions under which two topologies on a space, one induced by a given family of mappings and the other by the family of inverse mappings, are in fact identical.

H. R. Gluck (Cambridge, Mass.)

Segal, Jack 5225

Mapping norms and indecomposability.

J. London Math. Soc. **39** (1964), 598–602.

Suppose X and Y are metric spaces and $f : X \to Y$. $N(f) = \operatorname{lub} \{\operatorname{diam}[f^{-1}(y)] : y \in Y\}$. f is a wrapping if for any two subcontinua A and B of X such that $A \cup B = X$, either $f(A) = Y$ or $f(B) = Y$. W is the set of wrappings of X onto itself. The author obtains several results, e.g., Theorem: If X is locally connected, then there exists r such that $0 < r < \infty$ and $N(f) \geqq r$ for every $f \in W$. Theorem: If X is a continuum, then X is indecomposable if and only if no positive constant r exists such that $N(f) \geqq r$ for every $f \in W$.

P. Porcelli (Bloomington, Ind.)

Hofmann, K. H.; Mostert, P. S. 5226

Totally ordered D-class decompositions.

Bull. Amer. Math. Soc. **70** (1964), 765–771.

This paper gives the theorems containing the following one without detailed proof. Let S be a compact semigroup which satisfies: (a) $S^2=S$; (b) The union of all subgroups of S is a subsemigroup; (c) S/D, namely, the natural ordering of the D-classes of S (i.e., $D(x)\leqq D(y)$ if and only if $SxS\subseteq SyS$) is connected and totally ordered. Then (1) the relation D is a congruence, (2) the mapping $S\to S/D$ is a homomorphism, and (3) S/D is an I-semigroup, which means a semigroup on an arc such that one endpoint is an identity of the semigroup and the other is a zero. The converse is also true, that is, (1), (2) and (3) imply (a), (b) and (c). Further, from one additional condition, together with (a), (b) and (c), several properties (besides (1), (2) and (3)) of S are derived. A few corollaries are given as immediate consequences of the main theorem. The proofs of the theorems need Clifford semigroups (i.e., the union of groups) with connecting homomorphisms, the structure of the semigroups over gaps, and the concept of admissible spaces. *T. Tamura* (Davis, Calif.)

Hudson, Sigmund N. 5227

Transformation groups in the theory of topological loops.

Proc. Amer. Math. Soc. **15** (1964), 872–877.

The author continues his studies of topological loops by means of the groups of translates, a technique introduced in his earlier paper [Trans. Amer. Math. Soc. **109** (1963), 181–190; MR **27** #5237]. In this paper, he is concerned with necessary and sufficient conditions for a space to admit a loop structure, particularly with a left-invariant uniform structure. Let G be a topological group acting transitively on a space X and $p\in X$. If the map $\pi:G\to X$ defined by $\pi(g)=g(p)$ is open, G is said to act reasonably on X. A cross-section $\delta:X\to G$ is said to be strongly transitive if $y\in X$, and if $\delta(x)(y)=\delta(x')(y)$, then $x=x'$. The author shows that if G acts reasonably on X and there is a strongly transitive cross-section, then X admits a loop structure with a left-invariant uniformity, but possibly lacking continuity for solutions to the equations $ax=y$, $xa=y$. One additional assumption which will ensure their continuity is the compactness of the isotropy group G_p. If X is locally compact and connected, these three conditions are also necessary, where G is the group of left translates. Several other equivalent conditions are given.
P. S. Mostert (New Orleans, La.)

Paalman-de Miranda, Aida Beatrijs 5228

★Topological semigroups.

Academisch Proefschrift ter Verkrijging van de Graad van Doctor in de Wiskunde en Natuurwetenschappen aan de Universiteit van Amsterdam.

Mathematisch Centrum, Amsterdam, 1964. 174 pp.

This dissertation is the longest and most detailed account of the structure of (topological) semigroups extant, and is an excellent exposition of most results involving only set-theoretic topology, elementary measure theory and some parts of transformation groups. It omits those results which require algebraic topology, manifold-theory or relation-theory (e.g., Koch's arc theorem).

Despite these lacunae it is very welcome and will be helpful both to the beginner and as a reference until the appearance of the projected books by Mostert and Hofmann and by Št. Schwarz. The bibliography is fairly complete through 1963 with the exception of papers in Eastern European languages, and the reviewer mentions Gluskin [Itogi Nauki (Algebra. Topology. 1962), pp. 33–58, Akad. Nauk SSSR Inst. Naučn. Informacii, Moscow, 1963; MR **30** #176], which cites 274 papers and fills many gaps.

Among the new results are an exhaustive treatment of compact 0-simple semigroups, a careful analysis of compact interval semigroups satisfying $S^2=S$ (which improves many results in the literature), some other useful improvements and extensions and a few shorter and more perceptive proofs.

The dissertation is meritorious as a scholarly document but no less for its contribution to current research and the reviewer regrets that the complexities of the latter make infeasible a detailed account of what is new.
A. D. Wallace (Gainesville, Fla.)

Selden, John 5229

A note on compact semirings.

Proc. Amer. Math. Soc. **15** (1964), 882–886.

The author defines a topological semiring as a Hausdorff space with two continuous associative operations, one distributing across the other, that is to say, $x(y+z)=xy+xz$ and $(y+z)x=yx+zx$. Let $E[+]$ and $E[\cdot]$ denote, respectively, the additive and multiplicative idempotents. The theorem proved is the following: If S is a compact semiring such that $E[\cdot]S\cup SE[\cdot]=S$, then each additive subgroup is totally disconnected. As a corollary, it follows that a compact connected additive subsemigroup with identity is acyclic and arcwise connected. Another corollary: Let S be an additively commutative semiring such that $SE[\cdot]\cup E[\cdot]S=S$. If $e\in E[+]$, and $H[+]$ denotes the union of the additive subgroups of S, then the sets $E[+]$, $E[+]+S$, $e+S$, $H[+]$ and $e+H[+]$ are arcwise connected. Finally, if S is a compact connected additively simple semiring, then $E[\cdot]\subseteq E[+]$.

A number of examples are given and a question is raised concerning the contractibility of certain additive subsemigroups. *R. P. Hunter* (University Park, Pa.)

Doyle, P. H. 5230

On the embedding of complexes in 3-space.

Illinois J. Math. **8** (1964), 615–620.

The author proves that a 2-complex imbedded in E^3 is tame if and only if the image of the 1-skeleton is tame and the image of each 2-simplex is tame.
L. Neuwirth (Princeton, N.J.)

Griffiths, H. B. 5231

Some elementary topology of 3-dimensional handlebodies.

Comm. Pure Appl. Math. **17** (1964), 317–334.

The author gives accurate definitions and treatment of the 3-dimensional n-handlebody T_n, described loosely as a "3-dimensional solid sphere with n solid handles". A standard model is constructed and it is shown that each handlebody is in fact homeomorphic with its appropriate standard model. Involutions on S^3 which turn these bodies T_n inside out are set up and their effect on the boundaries of the T_n is explored, especially relative to the generators of their first fundamental groups.
G. T. Whyburn (Charlottesville, Va.)

Černavskiĭ, A. V. 5232

Geometric topology of manifolds. (Russian)

Itogi Nauki (Algebra. Topology. 1962), pp. 161–187. *Akad. Nauk SSSR Inst. Naučn. Informacii, Moscow,* 1963.

The author gives a detailed expository account of developments up to 1962. The article is subdivided into sections on topological embeddings, homeomorphisms and mappings, the topology of combinatorial 3-manifolds and other manifolds. There is a bibliography of 244 items.

Michael, E. 5233

Cuts.

Acta Math. **111** (1964), 1–36.

Though some of the results apply more generally, we suppose throughout that all spaces referred to are completely regular. If A is a subset of a space X, we say "A nowhere cuts X" if (i) A is "thin" (has empty interior), (ii) for each $a \in A$ and each neighborhood U of a, $U-A$ is not expressible as $V_1 \cup V_2$, where V_1, V_2 are disjoint and open relative to $U-A$, and where $a \in \mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2)$ (Cl = closure in X). (If X is locally connected, (ii) is equivalent to saying that each $a \in A$ has arbitrarily small neighborhoods V in X for which $V-A$ is connected.) An "(X, A) cut" consists of a space **X**, a subset **A** of **X** which nowhere cuts **X**, and a (continuous) proper (i.e., closed and compact) light map $p: \mathbf{X} \to X$ which maps $\mathbf{X}-\mathbf{A}$ homeomorphically onto $X-A$, and maps **A** onto A. In simple cases, an (X, A) cut is just the result, intuitively speaking, of cutting X along A (as with scissors and paper). The principal results include the following. Given any thin $A \subset X$, an (X, A) cut exists and is essentially unique. It can be constructed by extending the injection map of $X-A$ in X to $j: \beta(X-A) \to \beta X$; p is then the light factor of a monotone-light factorization of $j|j^{-1}(X)$. If X is separable metric, its (X, A) cut will be (separable) metric if and only if A "nowhere scatters" X in the following sense: for each $a \in A$, and for each neighborhood U of a, if $\mathscr{V}$ is any family of pairwise disjoint sets, open relative to $U-A$, which covers $U-A$, then $\mathscr{V}$ is locally finite (in X) at a. If X is metric and A is thin and nowhere scattering, and if $X-A$ is locally connected, the (X, A) cut is also obtainable by taking **X** to be a suitable subset of the completion of $X-A$ in a suitable metric. If A is a thin sub-polyhedron of a locally finite polyhedron X, one can make **A** a sub-polyhedron of a polyhedron **X**, and p simplicial. If X is paracompact, locally connected and unicoherent, and A is closed, non-empty, thin and connected, and if $X-A$ has n components (where n is finite), then so has **A**. The case in which X is a mapping cylinder with "base" A is considered in detail, and the results are applied to study the following generalization of M. Brown's theory of collared sets [Ann. of Math. (2) **75** (1962), 331–341; MR **24** #A3637]. A thin set $A \subset X$ is said to be "multicollared" in X if **A** is collared in **X**; it is "locally multicollared" if it is covered by open sets U of X such that $U \cap A$ is multicollared in U (or, equivalently, in X). Theorem: If X is metric, every locally multicollared set is multicollared. The paper concludes by comparing the notion of an (X, A) cut with a similar notion due to Fox [*Algebraic geometry and topology*, pp. 243–257, Princeton Univ. Press, Princeton, N.J., 1957; MR **23** #A626] and expressed in terms of "completing a spread". Fox's definition is modified [compare the author, Nederl. Akad. Wetensch. Proc. Ser. A **66** (1963), 629–633; MR **28** #1589] without altering it if $X-A$ is locally connected, to produce what is here called a "Fox (X, A) cut", and it is shown that the (essentially unique) Fox (X, A) cut is imbeddable in the (X, A) cut, and coincides with it under mild conditions (though not in general), and is metrizable if X is metrizable and $X-A$ is locally connected.

There are many subsidiary results concerning proper maps, monotone-light factorizations (in this paper, "monotone" means "quotient (i.e., quasicompact) and monotone"), nowhere cutting and nowhere scattering sets and related notions. Sample results: In a first countable space, nowhere cutting implies nowhere scattering. If $(\mathbf{X}, \mathbf{A}, p)$ is an (X, A) cut, then **A** nowhere scatters **X** if and only if A nowhere scatters X. There is also an application to the Freudenthal compactification [compare Fox, loc. cit.].

{The author has informed the reviewer of the following corrections. In Proposition 2.2, it should be required that h is onto. In Proposition 11.2(b), the map f should be light as well as proper. In Theorem 13.1 it suffices that X/A (instead of X) be unicoherent. There are also a number of misprints, especially in the cross-references; the only seriously misleading misprint detected is in the definition of "nowhere separating", p. 31, which should presumably read "... such that $W-A$ is contained in some $V \in \mathscr{V}$". In the example just before Proposition 10.3, "$<$" should be "$\leqq$".} *A. H. Stone* (Rochester, N.Y.)

Ellis, Robert 5234

Global sections of transformation groups.

Illinois J. Math. 8 (1964), 380–394.

From the author's introduction: "Let (X, R) be a transformation group with phase space X and phase group R, the additive group of real numbers. Suppose further that (X, R) is minimal. Then what can be said about X? Various answers have been given to this question [see, for example, Chu, Proc. Amer. Math. Soc. **13** (1962), 503–508; MR **25** #1553; Chu and Geraghty, Bull. Amer. Math. Soc. **69** (1963), 377–381; MR **26** #4334; Ellis, Michigan Math. J. **10** (1963), 97–104; MR **27** #4217; Schwartzman, Ann. of Math. (2) **66** (1957), 270–284; MR **19**, 568; Proc. Nat. Acad. Sci. U.S.A. **48** (1962), 786–791; MR **25** #1543]. Schwartzman [loc. cit.; MR **25** #1543] shows that if, in addition, X is compact, locally pathwise connected, and if (X, R) admits a global section, then X is the base of a covering space with discrete fibers. This allows him to say something about the homotopy groups of X. In particular, he shows that $\pi_1(X) \neq 0$. Recently, Chu and Geraghty [loc. cit.] showed that if X is compact, locally pathwise connected, and if (X, R) is minimal but not totally minimal, then $\pi_1(X) \neq 0$.

"The first part of this paper is devoted to generalizing the notion of global section. The above results are considered in a more general setting, and the relation between them is studied. They are generalized to the case where R is replaced by any topological group whose underlying space is R^n.

"The second part of the paper is concerned with the following problem. Suppose X is a manifold which is minimal under R; need X be orientable? This question is answered in the negative by exhibiting an action of R on the cartesian product X of the torus with the Klein bottle such that (X, R) is minimal. The flow is constructed by first producing a homeomorphism f of $S^1 \times K$ (the circle cross

the Klein bottle) such that $S^1 \times K$ is minimal under the resulting discrete flow, and then R is allowed to act on $(S^1 \times K \times I)/f$ in the standard way; here I is the unit interval and $(S^1 \times K \times I)/f$ is obtained from $S^1 \times K \times I$ by identifying $(z, 0)$ with $(f(z), 1)$ $(z \in S^1 \times K)$. Since f turns out to be isotopic to the identity, the resulting space is homeomorphic to the cartesian product of the torus with the Klein bottle. This flow may be lifted to a flow on the four-torus, T^4. From a result of Auslander and Hahn [Bull. Amer. Math. Soc. **68** (1962), 614–615; MR **25** #4037], this flow does not come from a one-parameter subgroup of T^4." *Deane Montgomery* (Princeton, N.J.)

ALGEBRAIC TOPOLOGY

See also 4651, 4781, 4782a–b, 5193, 5231, 5251, 5255.

Zykov, A. A. 5235

The theory of graphs. (Russian)

Itogi Nauki (*Algebra. Topology.* 1962), *pp.* 188–223. *Akad. Nauk SSSR Inst. Naučn. Informacii, Moscow,* 1963.

An extensive expository account of the theory of graphs up to 1962. The algebraic and combinatorial developments are mentioned, as are the topological, geometric and probabilistic aspects. A bibliography of 223 recent papers is given, including papers with applications to network theory.

Aarts, J. M. 5236

The four-color problem. (Dutch)

Euclides (*Groningen*) **39** (1963/64), 193–199.

This is an introductory article of a very elementary sort. Kempe chains are introduced, and a few simple cases of reducible maps are discussed. It is shown that all maps with 14 or fewer countries are 4-colorable. Mention is made of the recent theorem by the author and de Groot [Nieuw Arch. Wisk. (3) **11** (1963), 10–18; MR **26** #5561] that every map in which every country has at most five neighbors is 4-colorable. *G. Sabidussi* (Hamilton, Ont.)

Corradi, K. A. 5237

A note on finite graphs.

Acta Sci. Math. (*Szeged*) **25** (1964), 169–171.

The author proves the following. "Let G be a finite graph. Suppose that A, B, and C are independent complete subgraphs of G such that every vertex of G belongs to exactly one of them. Suppose that the number of vertices in A, B, and C is $v(A) = a$, $v(B) = b$, and $v(C) = c$, respectively; further, that the inequality $c \geqq a \geqq b$ holds. If the number of edges connecting a vertex of C with a vertex of $A \cup B$ is at least $(a+b)c - 2b + 1$, then there exist two subgraphs D and E in G, both complete, having no vertex in common, such that $v(D) = c$, $v(E) \geqq a + 1$." An example shows that the theorem cannot be improved by a reduction of the number $(a+b)c - 2b + 1$.

A. R. Bednarek (Gainesville, Fla.)

Ljubič, Ju. I. 5238

A remark on a problem of C. Berge. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 961–962.

Verfasser bestätigt eine von M. P. Schützenberger au[f]gestellte Vermutung [C. Berge, *Théorie des graphes et s[es] applications*, p. 251, Dunod, Paris, 1958; MR **21** #1608]. Es sei $\alpha(G)$ die "innere Stabilität" (= Maximalzahl paa[r]weise nicht verbundener Knotenpunkte) eines endliche[n] Graphen G und $\Theta(G) = \sup_n \sqrt[n]{\alpha(G^n)}$. Dann gilt $\Theta(G) = \lim_{n\to\infty} \sqrt[n]{\alpha(G^n)}$.

Zum Beweis setze man $a_n = -\ln \alpha(G^n)$; wegen $\alpha(G \times H) \geqq \alpha(G)\cdot\alpha(H)$ [Berge, loc. cit., p. 38] gilt dann $a_{m+n} \leqq a_m + a$[$_n$] $(m, n = 1, 2, 3, \cdots)$, und nach einem Satz von Fekete [vgl. Pólya und Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Band I, zweite Auflage, p. 17, Nr. 98, Springer, Berlin 1954; MR **15**, 512] folgt hieraus $\inf_n (a_n/n) = \lim_{n\to\infty} (a_n/n)$ daraus ergibt sich die Behauptung.

Nach einer Bemerkung von Berge [loc. cit., p. 251] is[t] überdies $\Theta(G)$ stets endlich. *H. Sachs* (Ilmenau

Robertson, Neil 523

The smallest graph of girth 5 and valency 4.

Bull. Amer. Math. Soc. **70** (1964), 824–825.

A graph S is constructed which has 19 vertices, valence and girth 5. S is proved to be, up to an isomorphism, th[e] only graph with this valence and girth and with less tha[n] 20 vertices. The automorphism group of S is the dihedra[l] group of order 24. Three colors can be assigned to the ver[]tices such that each edge has different colored ends in [a] single way under the automorphisms.

R. Artzy (Princeton, N.J.

Yoeli, Michael; Ginzburg, Abraham 524

On homomorphic images of transition graphs.

J. Franklin Inst. **278** (1964), 291–296.

Authors' summary: "A simple method is derived for ob[]taining all homomorphic images of a transition graph, i.e. a finite, directed graph with at most one edge issuing fron each vertex. The method consists of the successive applica[]tion of elementary steps, corresponding to four types o[f] 'elementary' congruences. It is also shown that the numbe[r] of elementary steps required to derive a given homomor[]phic image is constant if the original transition graph i[s] complete and connected. The applicability of this study t[o] sequential machine decompositions is outlined."

J. Hartmanis (Schenectady, N.Y.

Crowell, R. H. 524

The annihilator of a knot module.

Proc. Amer. Math. Soc. **15** (1964), 696–700.

Let G be a knot group, and Zt the integral group ring o[f] the infinite cyclic group. The author proves that the an[]nihilating ideal of the Zt module G'/G'' is principal an[d] generated by Δ_1/Δ_2, where Δ_i is the ith Alexander poly[]nomial. *L. Neuwirth* (Princeton, N.J.

Bauer, Friedrich-Wilhelm 524

Eine Charakterisierung des Čechschen Kohomologie[]funktors.

Abh. Math. Sem. Univ. Hamburg **27** (1964), 85–96.

Sei L eine volle Unterkategorie einer Kategorie K vo[n] topologischen Räumen, seien $\mathfrak{P}$, $\mathfrak{M}$ kontravariante Funk[]toren von L, bzw. K, in die Kategorie der Mengen (oder der G-Moduln). $\mathfrak{M}$ heisse maximal bezüglich $\mathfrak{P}$, wenn fol[]gendes gilt: (1) die Restriktion von $\mathfrak{M}$ auf L ist $\mathfrak{P}$, (2) fü[r]

alle $\zeta \in \mathfrak{W}(X)$, $X \in K$ gibt es ein $Y \in L$, $\eta \in \mathfrak{B}(Y)$, $f: X \to Y$, sodass $\mathfrak{W}(f)\eta = \zeta$, (3) wenn $\mathfrak{T}$ irgendein (1), (2) erfüllender Funktor ist, so gibt es einen Homomorphismus $\varphi: \mathfrak{W} \to \mathfrak{T}$, dessen Restriktion auf $\mathfrak{B}$ die Identität ist. Satz: Zu vorgegebenem $L \subset K$, $\mathfrak{B}$ gibt es (bis auf Isomorphie) genau ein maximales $\mathfrak{W}$. Beispiel: Sei L die Kategorie der abzählbaren CW-Komplexe, K die Kategorie der parakompakten Räume (mit Homotopieklassen als Morphismen), $\mathfrak{B}$ die simpliziale Kohomologietheorie auf L. Die Čechsche Kohomologietheorie $\mathfrak{W}$ auf K ist charakterisiert durch die Eigenschaft, maximal bezüglich $\mathfrak{B}$ zu sein.

P. J. Huber (Zürich)

Adem, José 5243

On secondary cohomology operations. (Spanish)

Bol. Soc. Mat. Mexicana (2) **7** (1962), 95–110.

The author describes and applies a way of representing by functional cohomology operations a secondary cohomology operation Φ, associated with a relation of degree r: $\sum \alpha_k\beta_k = 0$ (written for brevity $\alpha\beta = 0$) in the Steenrod algebra on Z_p ($p \geq 2$). For $v \in H^q(X; Z_p)$ with $\beta_k(v) = 0$, take the map $f: X \to K = K(\pi, q)$ of the space X into the Eilenberg-MacLane space $K(\pi, q)$ such that $f^*(\gamma) = v$ (where γ is the fundamental class of K). Then, by a formula similar to the second formula of Peterson and Stein [Amer. J. Math. **81** (1959), 281–305; MR **23** #A1366], $\Phi(v) = (-1)^{r+1}\alpha_f\beta(\gamma)$, modulo a certain subgroup. However, the total indeterminacy here is, in general, larger than for the usual expression of Φ as a secondary operation. But conditions can be given in order that $f^*H^{q+r}(K) = 0$, which ensures that the additional indeterminacy vanishes; the functional representation is used to assure that some secondary cohomology operations vanish on low-dimensional classes. This is applied to some of the secondary operations that were introduced by Adams [Ann. of Math. (2) **72** (1960), 20–104; MR **25** #4530]. A Cartan formula is given for functional operations, and yields a product formula, by which the action of the secondary operation Φ on the cup product $u \cup v$ equals $\Phi(u) \cup v + (-1)^{ru} u \cup \Phi(v)$ (modulo the suitable indeterminacy), provided u and v satisfy some conditions.

G. Hirsch (Brussels)

Arkowitz, Martin 5244

Commutators and cup products.

Illinois J. Math. **8** (1964), 571–581.

Let j: $A_1 \vee A_2 \to A_1 \times A_2$ be the inclusion. If j is considered to be a fibre map, the fibre is, by definition, $A_1 \flat A_2$, the flat product of A_1 and A_2. Let X be an H'-space. Then the author constructs a flat product $\pi(X, A_1) \times \pi(X, A_2) \to \pi(X, A_1 \flat A_2)$. If $A_i = K(G_i, m_i + 1)$, he constructs a map $\theta: A_1 \flat A_2 \to K(G_1 \otimes G_2, m_1 + m_2 + 1)$. Hence, composition gives a cohomology flat product

$$\langle\ ,\ \rangle: H^{m_1+1}(X; G_1) \times H^{m_2+1}(X; G_2) \to H^{m_1+m_2+1}(X; G_1 \otimes G_2)$$

when X is an H'-space. Let $\tau: \pi(\Sigma Y, Z) \to \pi(Y, \Omega Z)$ be the adjoint isomorphism. Then the author's main theorem is as follows. Let $\alpha_i \in H^{m_i+1}(\Sigma Y; G_i)$. Then $\tau\langle\alpha_1, \alpha_2\rangle = \tau\alpha_1 \cup \tau\alpha_2$. He uses this theorem to prove corollaries concerning $\cup$-long and conil and a modified distributive law.

F. P. Peterson (Cambridge, Mass.)

Maunder, C. R. F. 5245

On the differentials in the Adams spectral sequence.

Proc. Cambridge Philos. Soc. **60** (1964), 409–420.

The Adams spectral sequence [J. F. Adams, Comment. Math. Helv. **32** (1958), 180–214; MR **20** #2711] is a spectral sequence with initial term $\mathrm{Ext}_A(H^*(X, Z_p), H^*(Y, Z_p))$, where A is the Steenrod algebra, and which converges to the stable track group $[Y, X]$. In general, little is known about the differentials in this spectral sequence. The author shows how one may define a set of stable, higher-order cohomology operations, whose action gives the action of the differentials. As an application in the case of stable homotopy groups of spheres, he gives a proof of the formula $d_2(h_m) = h_0h_{m-1}^2$ ($m \geq 4$), which is originally due to Liulevicius.

D. W. Kahn (Minneapolis, Minn.)

Recillas, Félix 5246

On the formulas of Peterson and Stein. (Spanish)

An. Inst. Mat. Univ. Nac. Autónoma México **1** (1961), 55–60.

The author gives proofs for Shimada's generalizations of the Peterson-Stein formulas relating secondary cohomology operations to primary and functional primary operations [Peterson and the reviewer, Amer. J. Math. **81** (1959), 281–305; MR **23** #A1366].

N. Stein (Haverford, Pa.)

Eckmann, B.; Hilton, P. J. 5247

Unions and intersections in homotopy theory.

Comment. Math. Helv. **38** (1964), 293–307.

In an earlier paper [Math. Ann. **151** (1963), 150–186; MR **27** #3681] the authors defined notions of kernel, coker, union, etc., for categories which are not necessarily abelian. For example, the kernel ideal, ker(f), of a map $f: A \to B$ is the set of all $g: D \to A$ such that $fg = 0$. Ker(f) is principal and is generated by c if $g = uc$ for all $g \in \ker(f)$.

These ideas, together with a definition of homotopy in an abstract category, are combined to give very simple proofs of some results in the homotopy theory of CW-complexes. For example, Lemma 2.6 of the reviewer's paper [Ann. of Math. (2) **75** (1962), 467–484; MR **25** #1551] and the Mayer-Vietoris sequence are obtained in this context.

E. H. Brown (Waltham, Mass.)

Iwata, Koichi 5248

On the realizability of Whitehead products.

Tôhoku Math. J. (2) **16** (1964), 203–208.

This paper is devoted to the realizability of quasi-Lie rings (in the sense of Hilton) as Whitehead rings of spaces, i.e., the quasi-Lie ring associated with a space by taking the homotopy groups $\pi_i(X)$ of the space and the Whitehead products $\pi_p(X) \times \pi_q(X) \to X_{p+q-1}(X)$ as the multiplication. This problem has previously been studied by Meyer [Amer. J. Math. **82** (1960), 271–280; MR **26** #6965] and Miyazaki [Tôhoku Math. J. (2) **12** (1960), 1–30; correction, ibid. (2) **12** (1960), 480; MR **22** #5971; errata, MR **22**, p. 2546]. In the present paper the author uses results of Peterson and the reviewer on cohomology operations and their relations with Whitehead products to derive necessary and sufficient conditions for realizability of the following problem: Given three abelian groups π_2, π_3, π_4 such that

$$\mathrm{Ext}(H_5(\pi_2, 2); \pi_4) = 0,$$

find homomorphisms $\pi_2 \times \pi_2 \to \pi_3$ and $\pi_2 \times \pi_3 \to \pi_4$.

The general problem seems as far from solution as is the

analogous problem of realizability of associative, commutative, graded rings as cohomology rings of spaces.

{The proof of the main theorem seems to have a serious gap. The statement on p. 207 that because ${}^1\theta(\iota-x')=0$ we can conclude that $\iota-x'=2x''$ for some $x''\in H^3(F;\pi_3)$ is not justified and seems to the reviewer to be false. Therefore the element $k_0\in H^5(E;\pi_4)$ may not exist and the proof depends crucially on this element.}

N. Stein (Haverford, Pa.)

Postnikov, M. M. 5249

Homotopy theory. **(Russian)**

Itogi Nauki (*Algebra. Topology.* 1962), *pp.* 107–133. *Akad. Nauk SSSR Inst. Naučn. Informacii, Moscow,* 1963.

This survey article is based on papers on homotopy theory reviewed in RŽMat. in 1961 and 1962. The sections of the survey are devoted to cohomology operations, duality principles, homology groups and homotopy groups and their computation, the homotopy classification of mappings, simplicial sets, fibre spaces, transformation groups, and H-spaces. There is a bibliography of 241 items.

Atiyah, M. F.; Bott, R.; Shapiro, A. 5250

Clifford modules.

Topology **3** (1964), *suppl.* 1, 3–38.

According to the authors, "the purpose of the paper is to undertake a detailed investigation of the role of Clifford algebras and spinors in the KO-theory of real vector bundles. On the one hand the use of Clifford algebras throws considerable light on the periodicity theorem for the stable orthogonal group. On the other hand the use of spinors seems essential in some of the finer points of the KO-theory which centre round the Thom isomorphism".

Part I is entirely algebraic, and studies Clifford algebras and spinor groups. The material is essentially known, but its presentation is improved. In particular, the authors emphasise the grading (over Z_2) of the Clifford algebra. They can thus write formulae which involve signs given by the standard "anticommutative law" of algebraic topology; in this way the algebra becomes simpler and more natural. A feature of this approach is that the spinor group with two pathwise-components, which is a double covering of $O(k)$, arises as naturally as the spinor group with one pathwise-component, which is a double covering of $SO(k)$. § 4 gives the structure of the Clifford algebras, and §§ 5, 6 discuss their representation theory.

In Part II the authors give a complete treatment of the "difference bundle construction" in K-theory. This includes a Grothendieck-type construction for the relative groups $K(X, Y)$ (9.1) and a discussion of the products in these groups (10.3, 10.4).

In Part III, §§ 11, 12, the authors set up the Thom isomorphism φ for real K-theory. Their construction of φ is clearly good; in particular, if φ is applied in a Whitney sum bundle, it satisfies a product formula. [Since it is one of the main objects of the paper to prove this formula, it is remarkable that the formula is not formally stated; the reader is left to deduce it from 11.3.] However, the authors seem less satisfied with their proof that φ is an isomorphism. In fact, the crucial step (11.5) amounts to case-by-case checking, using the results of R. Bott [Bull. Soc. Math. France **87** (1959), 293–310; MR **22** #A3577], that φ has the correct effect when the base-space is a point.

An alternative construction of the Thom isomorphism has been given in R. Bott [Bull. Amer. Math. Soc. **68** (1962), 395–400; MR **27** #2966]. This approach is convenient for computing the effect of representations, but does not lead to the product formula. It is therefore desirable to prove that the two constructions coincide. This is done in §§ 13, 14, by studying the sphere as a coset space of the spinor group.

{The reviewer remarks that the work of D. W. Anderson on $KO(BG)$ apparently yields a third approach to the Thom isomorphism for real K-theory.}

Having shown in the course of the work that Clifford algebras are related to the Bott periodicity theorem, the paper ends (§ 15) by showing that they are related to certain questions about vector bundles over stunted projective spaces and about vector-fields on spheres.

Although the main interest of the paper lies in real K-theory, it is a feature of the method that the real and complex cases can be treated in parallel throughout.

J. F. Adams (Manchester)

TOPOLOGY AND GEOMETRY OF DIFFERENTIABLE MANIFOLDS

See also 4651, 4653, 4657, 4762, 4912, 4913, 5250.

Novikov, S. P. 5251

Differential topology. **(Russian)**

Itogi Nauki (*Algebra. Topology.* 1962), *pp.* 134–160. *Akad. Nauk SSSR Inst. Naučn. Informacii, Moscow,* 1963.

The author gives a detailed expository account of recent developments in differential topology, mainly between 1960 and 1962. The article is subdivided into sections on homotopy groups of spheres, the Hopf invariant and the Steenrod algebra, Thom spaces and cobordism, homotopy properties of the classical Lie groups, immersions and regular mappings of smooth manifolds, critical-point theory and the generalized Poincaré conjecture, Mazur's theorem and the "Hauptvermutung", Milnor spheres, the smoothing problem for combinatorial manifolds, the diffeomorphism problem for simply connected manifolds, and other questions. There is a bibliography of 295 items.

Novikov, S. P. 5252

Foliations of co-dimension 1. **(Russian)**

Dokl. Akad. Nauk SSSR **157** (1964), 788–790.

In this paper, the author generalizes his previous work [same Dokl. **155** (1964), 1010–1013; MR **29** #2821] so as to give more information about oriented smooth foliations of codimension one of three manifolds, as well as some information in higher dimensions. His tools are the notion of component, introduced in the preceding note, and the notion of limiting epimorphism. Let $N_j(A)$, $j=1, 2$, be the set of classes α in the fundamental group of the leaf A such that the corresponding holonomy element (which belongs to the group of germs of orientation-preserving diffeomorphisms of the line leaving the origin fixed) is the identity on the positive [negative] side of the origin. If $\alpha\in N_j(A)$ or $\alpha\in\pi_k(A)$, $k>1$, then α may be pushed off to any nearby leaf on the appropriate side. The limiting epimorphism is the quotient map of which the kernel consists of all α which are contractible in all nearby leaves.

Theorem: If one of the following conditions is satisfied:

(a) there is a closed transversal, homotopic to zero in the manifold; (b) for some leaf A, $\pi_1(A) \to \pi_1(M)$ is not one-one; (c) $\pi_2(M) \neq 0$, but for every leaf A, $\pi_2(A) = 0$, then there is a leaf A_0 and a j such that the limiting epimorphism defined on $N_j(A_0)$ is not one-one.

If M is three-dimensional, this implies that if a foliation is analytic or has no Reeb component, either $M = S^2 \times S^1$, $M = S^2 \times I$, or the universal covering is contractible. On the other hand, an unpublished result of Zieschang is quoted, according to which every closed oriented three-manifold can be foliated.

Details of the proofs are not given. Nonetheless, it is safe to say that the methods introduced here will have far-reaching consequences.

B. L. Reinhart (College Park, Md.)

Conner, P. E.; Floyd, E. E. 5253

The SU-bordism theory.

Bull. Amer. Math. Soc. **70** (1964), 670–675.

For each family of classical groups (O, SO, Spin, U, SU, etc.) one can construct the cobordism groups of Thom (in the notation of the reviewer and the authors, these are called bordism groups of a point). For O, SO, U these groups have been explicitly determined. In particular, it has been shown that the SO-bordism groups of a point contain no odd torsion and no elements of order 4. In the present work the authors show that the same is true for the SU-bordism groups of a point. They remark that Lashof and Rothenberg have established this result independently.

The method of dealing with SO is to compare it with O. The reviewer showed [Proc. Cambridge Philos. Soc. **57** (1961), 200–208; MR **23** #A4150] how, using real projective space, one could obtain certain exact sequences connecting the SO- and O-bordism groups. The method of the authors is to derive analogous exact sequences connecting SU- and U-bordism using complex projective space. The first Stiefel-Whitney class is replaced by the first Chern class and dimensions get doubled. Thus the authors get the following exact sequence

$$\Gamma_k(X, A) \to \mathrm{U}_k(X, A) \to \Gamma_{k-2}(X, A) \oplus \mathrm{U}_{k-4}(X, A),$$

where (X, A) is a finite CW-pair, U_k denote the U-bordism groups and Γ_k denote the SU-bordism groups.

M. F. Atiyah (Oxford)

Graver, Jack E. 5254

An analytic triangulation of an arbitrary real analytic variety.

J. Math. Mech. **13** (1964), 1021–1036.

A local analytic variety is a subset V of an open set $U \subset E^n$, where V consists of the common zeros of finitely many real power series convergent in U. An analytic variety $V \subset E^n$ is characterized by the property that each $p \in V$ has a neighborhood $U_p \subset E^n$ on which $V \cap U_p$ is a local analytic variety. A triangulation $\tau: K \to S$ of a set $S \subset E^n$ is analytic if $\tau|s$ is analytic with analytic inverse for each open simplex s of K. The main theorem is that an open set $O \subset E^n$ containing finitely many analytic varieties $V_1, \cdots, V_n$ closed in O can be analytically triangulated so that each V_i is a subcomplex. The work appears to be independent of efforts by others who have investigated analytic triangulations.

S. S. Cairns (Urbana, Ill.)

Adem, José; Gitler, Samuel 5255

Secondary characteristic classes and the immersion problem.

Bol. Soc. Mat. Mexicana (2) **8** (1963), 53–78.

Let $\xi = (E, B, \pi)$ be an n-plane bundle over B and assume certain Stiefel-Whitney classes of ξ vanish. Then one may define certain secondary cohomology operations Φ on the Thom class $U \in H^n(E, E_0)$. These give rise to secondary characteristic classes $\xi^*(\Phi)$ analogous to the definition of the Stiefel-Whitney classes. These characteristic classes are studied more carefully for the tangent bundle and the normal bundle to a manifold. Next, the authors study some particular secondary characteristic operations Φ_{2j} in complex projective spaces. Φ_{2j} is the operation associated to the relation $(Sq^2Sq^1)Sq^{4k} + Sq^{4k+2}Sq^1 = 0$ if $2j = 4k+2$ or $(Sq^2Sq^1)Sq^{4k-2} + Sq^{4k}Sq^1 + Sq^1Sq^{4k} = 0$ if $2j = 4k$. For example, they prove that $\Phi_{2(a+b-c)}(w^{(2h+1)a-b-c}) = hw^{2(ha+a-c)}$, where $w \in H^2(CP^\infty)$ is the generator, $a = 2^r$, $b = 2^s$, $c = 2^t$ and $r > s \geq t \geq 0$ and $h > 0$. They also prove a theorem which shows that $\Phi_{2j}(u) = 0$ for dimensional reasons under certain hypotheses. Putting all this together, they prove the following geometric result. CP^n does not immerse in R^{4n-5} if $n = 2^r + 2^s$ and $r > s \geq 0$. In order to prove theorems about real projective spaces, the authors study "multiple" secondary cohomology operations. In this case, their geometric result is the following theorem. RP^n does not immerse in R^{2n-9} if $n = 4k+3$ and $k = 2^r + 2^s$ with $r > s \geq 0$.

F. P. Peterson (Cambridge, Mass.)

Reátegui, José [Reátegui C., José] 5256

A theorem on raising the class in a mapping of manifolds. (Spanish)

Univ. Nac. Ingen. Inst. Mat. Puras Apl. Notas Mat. **1** (1962/63), 147–151.

In this brief note the author shows that the differentiability class of a differentiable immersion of a differentiable manifold M into a differentiable manifold N cannot exceed the class of M.

H. R. Gluck (Cambridge, Mass.)

Brieskorn, Egbert 5257

Ein Satz über die komplexen Quadriken.

Math. Ann. **155** (1964), 184–193.

A theorem of Hirzebruch and Kodaira [J. Math. Pures Appl. (9) **36** (1957), 201–216; MR **19**, 1077] states the following. Let X be a Kähler manifold of dimension n which is diffeomorphic to complex projective space P_n. Then: (i) if n is odd, X is biholomorphically equivalent to P_n; (ii) if n is even, $c_1 = \pm(n+1)g$ (g = generator of $H_2(P_n, \mathbf{Z})$, c_1 = first Chern class), and if $c_1 = (n+1)g$, then X is biholomorphically equivalent to P_n. The purpose of the present paper is to prove the same statement with Q_n, the complex n-quadric, replacing P_n. Here one must make the exception $n \neq 2$ since $Q_2 = P_1 \times P_1$ carries infinitely many inequivalent Kähler structures (these are the Hirzebruch surfaces [cf. Hirzebruch, Math. Ann. **124** (1951), 77–86; MR **13**, 574]). The proof is similar to the proof of the aforementioned Hirzebruch-Kodaira theorem; the assumption $n > 2$ comes about in order to make a calculation from the general Riemann-Roch theorem. In fact, for $n > 2$, $H_2(X, \mathbf{Z}) \simeq \mathbf{Z}$ with generator g and $c(Q_n)(1+2g) = (1+g)^{n+2}$; for $n = 2$, $H_2(X, \mathbf{Z}) \simeq \mathbf{Z} \oplus \mathbf{Z}$, and this forces the assumption $n > 2$.

Phillip A. Griffiths (Berkeley, Calif.)

Dolbeault, Pierre 5258

Une généralisation de la notion de diviseur.

Atti Convegno Internaz. Geometria Algebrica (Torino, 1961), *pp.* 125–150. *Rattero, Turin,* 1962.

Il n'y a pas en général assez de diviseurs sur une variété analytique complexe pour assurer l'extension à ces variétés du théorème de Lefschetz et Hodge: "Pour qu'une classe de cohomologie entière de la variété algébrique projective complexe V, sans singularité, soit la classe de cohomologie d'un diviseur, il faut et il suffit que son image dans $H^2(V, C)$ par l'homomorphisme induit par l'inclusion $Z \subset C$, appartienne à $H^{1,1}(V, C)$".

L'auteur établit un théorème analogue sur une variété analytique complexe quelconque V en utilisant la variété analytique réelle sous-jacente V_r.

Soient respectivement $\mathscr{M}$, $\mathscr{M}_r$, $\mathscr{C}^*$, $\mathscr{F}^*$ les faisceaux de groupes multiplicatifs de germes de fonctions à valeurs complexes qui sont méromorphes sur V, méromorphes sur V_r, holomorphes non nulles sur V, holomorphes non nulles sur V_r. Un "pseudodiviseur" est une section de $\mathscr{M}_r/\mathscr{F}^*$. C'est un "pseudodiviseur spécial" s'il est l'image d'une section de $\mathscr{M}_r/\mathscr{C}^*$ par l'application: $\mathscr{M}_r/\mathscr{C}^* \to \mathscr{M}_r/\mathscr{F}^*$. Le groupe des diviseurs est un sous-groupe du groupe des pseudodiviseurs spéciaux.

On a alors la proposition suivante: tout élément de $H^2(V, Z)$ est la classe de cohomologie d'un pseudodiviseur tandis que la condition nécessaire et suffisante pour qu'un tel élément soit la classe d'un pseudodiviseur spécial est que son image dans $H^2(V, C)$ appartienne à $H^{1,1}(V, C)$. *R. Deheuvels* (Paris)

Fischer, Wolfgang 5259

Zur Deformationstheorie komplex-analytischer Faserbündel.

Schr. Math. Inst. Univ. Münster No. 30 (1964), ii + 49 *pp.*

The author constructs to a given holomorphic principal bundle P_0 over the compact complex manifold X a sheaf $A(P_0)$ whose elements are germs of holomorphic deformations of P_0. $H^1(X, A(P_0))$ turns out to be naturally isomorphic to the set of equivalence classes of local holomorphic deformations of P_0. Denoting by A_k the subsheaf of $A = A(P_0)$ consisting of all germs of order $\geq k$, one gets the short exact sequence of sheaves of groups $0 \to A_{k+1}/A_{k+2} \to A/A_{k+2} \to A/A_{k+1} \to 0$. Since A_{k+1}/A_{k+2} is canonically isomorphic to the sheaf $\mathcal{O}(\mathrm{Ad}\, P_0)$ of germs of holomorphic sections in $\mathrm{Ad}\, P_0$, the associated cohomology sequence permits one to define, for every "infinitesimal deformation of P_0 of order k" $q_k \in H^1(X, A/A_{k+1})$ and for every $l \geq k$, the l-obstruction $c_l(q_k)$ of q_k as the obvious subset of $H^2(X, \mathcal{O}(\mathrm{Ad}\, P_0))$. $c_l(q_k)$ is called trivial if it contains the neutral element of $H^2(X, \mathcal{O}(\mathrm{Ad}\, P_0))$. For principal $\mathrm{GL}(n, \mathbf{C})$ bundles, these obstructions can be given in fairly explicit form. If for the principal $\mathrm{GL}(n, \mathbf{C})$ bundle P_0 the relation $H^2(X, \mathcal{O}(\mathrm{Ad}\, P_0)) = 0$ holds, then every principal bundle P_1 whose defining 1-cocycle is sufficiently close to some defining 1-cocycle of P_0 can be obtained by a holomorphic deformation of P_0; moreover, every element in $H^1(X, \mathcal{O}(\mathrm{Ad}\, P_0))$ is an infinitesimal deformation of P_0. Analogous statements concerning constant principal $\mathrm{GL}(n, \mathbf{C})$ bundles are proved. Finally it is shown that, given the principal $\mathrm{GL}(n, \mathbf{C})$ bundle P_0 satisfying $H^1(X, \mathcal{O}(\mathrm{Ad}\, P_0)) = 0$, every other principal bundle P_1 whose defining 1-cocycle is sufficiently close to some defining 1-cocycle of P_0 is holomorphically equivalent to P_0. *H. Röhrl* (La Jolla, Calif.)

Hermann, Robert 5260

Convexity and pseudoconvexity for complex manifolds.

J. Math. Mech. **13** (1964), 667–672.

Let X be a complex manifold of dimension $n > 1$, $Z \subset X$ a relatively compact open submanifold defined by $\varphi < 0$, where φ is a real function on X. Let $H = \partial Z$ be the boundary of Z and assume $d\varphi \neq 0$ near H. Then the E. E. Levi form L is an Hermitian form in $n-1$ variables defined by restricting $\partial\bar{\partial}\varphi$ to complex tangent spaces to H. The form L is not an absolute invariant of the complex structure, but the number of positive and negative eigenvalues of L (= signature) is. The subdomain Z is said to be (strongly) pseudo-convex if L is positive definite, and the intuitive connection between ordinary convexity and pseudo-convexity is put in a general setting by the author's main result (Theorem A). Let X have a symmetric affine connexion and let J be the almost-complex structure tensor. Then

$$(1) \qquad L(u, v) = S(u, v) + S(Ju, Jv),$$

where S is the second fundamental form of H in X and u, v are complex tangent vectors to H ($Ju = \sqrt{(-1)}u$, $Jv = \sqrt{(-1)}v$).

It is perhaps worth remarking that the Levi form controls the (analytic) sheaf cohomology of Z in a manner similar to the way in which the second fundamental form controls the ordinary cohomology (compare the integral estimates in J. Kohn [Ann. of Math. (2) **78** (1963), 112–148; MR **27** #2999] and C. Hsiung [Math. Z. **82** (1963), 67–81; MR **27** #2943]). *Phillip A. Griffiths* (Berkeley, Calif.)

PROBABILITY

See also 4696, 4769, 4860, 4863, 4905, 5322, 5326, 5349, 5630, 5647, 5655, 5681.

★**Proceedings of the All-Union Conference on the Theory of Probability and Mathematical Statistics [Труды Всесоюзного Совещания по Теории Вероятностей и Математической Статистике].** 5261

Erevan, 19–25 September 1958.

Izdat. Akad. Nauk Armjan. SSR, Erevan, 1960. 292 *pp.* 1.16 *r.*

This volume contains the proceedings of the conference mentioned in the heading. Papers which are not abstracts will be reviewed or listed individually.

Castoldi, Luigi; Dolci, Alba 5262

Beyond Bertrand's paradox (What is "random choice"?). (Italian summary)

Rend. Sem. Fac. Sci. Univ. Cagliari **34** (1964), 1–4.

The well-known ambiguity of the expression "at random" is emphasized and illustrated. *I. J. Good* (Oxford)

Kappos, Demetrios A. 5263

Strukturtheorie der Räume von Zufallsvariablen.

Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (Liblice, 1962), *pp.* 359–375. *Publ. House Czech. Acad. Sci., Prague,* 1964.

The author continues his studies [cf. his *Strukturtheorie der Wahrscheinlichkeitsfelder und -Räume*, Springer, Berlin, 1960; MR **22** #9982; errata, MR **22**, p. 2547] on a (predominately algebraic) structural theory of the foundations of probability, which is based on a σ-Boolean ring with unit element (σ-W-field), equipped with a σ-additive strictly positive measure on the Boolean ring (W-field). He now introduces (real-valued) random variables (r.v.) and analyzes their properties within the context of the above structure. Countably-valued r.v. are discussed and related to the author's definition of "experiment". Various convergence modes for sequences of r.v. are investigated and compared. The final result concerns embedding of Banach lattices and is—like the other results of this paper—too involved to summarize in detail within the scope of this review. *F. J. Beutler* (Berkeley, Calif.)

Kellerer, Hans G. 5264

Allgemeine Systeme von Repräsentanten.

Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **2**, 306–309 (1964).

For each $i \in M = \{1, \cdots, m\}$ let $\bar{K}_i$ and $\underline{K}_i$ be two subsets of $N = \{1, \cdots, n\}$; and let $\underline{z}_i$, $\bar{z}_i$ ($i \in M$) and $\underline{s}_k$, $\bar{s}_k$ ($k \in N$) be integers. By a general representant system for these objects we mean a system of sets $\{K_i\}$ ($i \in M$) for which $\underline{K}_i \subset K_i \subset \bar{K}_i$ ($i \in M$), $\underline{z}_i \leq |K_i| \leq \bar{z}_i$ ($i \in M$), $\underline{s}_k \leq |I_k| \leq \bar{s}_k$ ($k \in N$). (Here $I_k = \{i : k \in K_i\}$ and $|\ |$ denotes cardinality.) Necessary and sufficient conditions for the existence of general representant systems are given, valid under certain extra hypotheses. *J. M. G. Fell* (Seattle, Wash.)

Miles, R. E. 5265

Random polygons determined by random lines in a plane.

Proc. Nat. Acad. Sci. U.S.A. **52** (1964), 901–907.

The author considers the convex regions formed by a homogeneous Poisson field of random lines in a plane and reports a series of new results. The distributions of the diameters of the in-circles are negative exponential and the distribution of the perimeters, conditional on the number of sides, is a χ^2 distribution. He also obtains the probability that such a region is a triangle and the lower moments and cross moments of the number of sides, the perimeter, and the area. The joint distributions of the orientations of some of the sides under certain conditions are also found. Generalisations are given for lines of finite thickness and anisotropic distributions. Methods of proof are to be given later. *P. A. P. Moran* (Canberra)

Ibramhalilov, I. Š. 5266

On a method of sharpening a bound for parameters. (Russian. Azerbaijani summary)

Akad. Nauk Azerbaĭdžan. SSR Dokl. **20** (1964), *no.* 6, 15–19.

The problem of finding an improved estimate of the value α which satisfies the equation $\sum_{i=1}^{n} \psi(\xi_i; \alpha) = 0$ is studied; here ξ_i, $1 \leq i \leq n$, are independent, identically distributed, random variables with distribution function $F(x; \alpha)$. Under certain regularity conditions concerning the partial derivatives of ψ with respect to α, and assuming the existence of an estimate of the form $\alpha_1^* = \alpha_0 + \eta_n/\sqrt{n}$, where η_n is an asymptotically normal random variable, a consistent and asymptotically normal estimate $\bar{\bar{\alpha}}$ of α is obtained. *R. Theodorescu* (Bucharest)

Čan Ving H'en [Tran Vinh-Hien] 5267

A central limit theorem for stationary processes arising from number-theoretic endomorphisms. (Russian. English summary)

Vestnik Moskov. Univ. Ser. I Mat. Meh. **1963**, *no.* 5, 28–34.

A. Rényi [Acta Math. Acad. Sci. Hungar. **8** (1957), 477–493; MR **20** #3843], among others, has considered the expansion of $x \in (0, 1)$ in the form $x = \varepsilon_0 + f(\varepsilon_1 + f(\varepsilon_2 + \cdots))$, showing this is possible for certain functions f. He has shown, moreover, that for certain f there exists an absolutely continuous measure carried on $(0, 1)$, of total measure 1, whose density is positive and bounded away from 0 and ∞, and such that with respect to the resulting measure space the remainders $r_n(x) = f(\varepsilon_{n+1} + f(\varepsilon_{n+2} + \cdots))$ constitute a stationary sequence of random variables. In this paper it is proved that for suitable functions f and g the limiting distribution of $g(r_1(x)) + \cdots + g(r_n(x))$ is Gaussian. This result had been proved earlier in two special cases by I. A. Ibragimov [Vestnik Leningrad. Univ. **15** (1960), no. 1, 55–69; MR **22** #11428].

D. A. Darling (Ann Arbor, Mich.)

Emel'janov, G. V. 5268

A local limit theorem for density functions. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 3 (40), 59–69.

Let $X_1, X_2, \cdots$ be independent random variables with means zero and let $S_n = X_1 + \cdots + X_n$, $E(S_n^2) = B_n^2$. Supposing that the distribution function of S_n has a density $q_n(x)$ for sufficiently large n, the author investigates under what conditions the expression

$$M_n = \sup_{-\infty < x < \infty} \left| B_n q_n(xB_n) - \frac{e^{-x^2/2}}{\sqrt{(2\pi)}} \right|$$

converges to zero. Assuming no further conditions on the moments of the X_i other than the finiteness of B_n^2, the author proves that $M_n \to 0$ if the usual Lindeberg condition prevails, and other conditions to center and stabilize the X_i. If a moment of order $2 + \delta$, $\delta > 0$, exists for the X_i and a strengthened form of Lyapounov's condition prevails, $M_n = O((\log B_n)^{-\delta/2})$. If moments of order 3 or higher exist, q_n has a Charlier-type expansion in inverse powers of $n^{1/2}$.

D. A. Darling (Ann Arbor, Mich.)

Guiasu, Silviu 5269

La répartition asymptotique des sommes aléatoires de variables aléatoires indépendantes non identiquement distribuées.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **34** (1963), 504–510.

The main result of the paper is the following. Let $\{X_n\}$ be a sequence of independent random variables with mean value 0 and finite variances, and suppose that

$$\lim_{n \to +\infty} P\left(\sum_{k=1}^{n} X_k \leq xW_n \right) = F(x)$$

for every point of continuity of $F(x)$, where W_n^2 denotes the variance of the sum $\sum_{k=1}^n X_k$ and $F(x)$ is a distribution function. If $\{N_n\}$ is a sequence of positive integer-valued random variables for which N_n/n converges in probability to a positive discrete random variable λ and if

(1) $W_{[n\lambda]}/W_{N_n}$ converges in probability to 1,

then

$$(2) \qquad \lim_{n\to\infty} P\left(\sum_{k=1}^{N_n} X_k \leq xW_{N_n}\right) = F(x).$$

Here $[n\lambda]$ denotes the integral part of the real number $n\lambda$. Statement (2) has been proved for the case when the X_n are identically distributed (in which case $W_n = D\sqrt{n}$ and (1) is satisfied) by the reviewer [Acta Math. Acad. Sci. Hungar. **9** (1958), 215–228; MR **20** #4623]; this result has been generalized for the case when the positive random variable λ has an arbitrary distribution by J. Mogyoródi [Magyar Tud. Akad. Mat. Kutató Int. Közl. 7 (1962), 409–424; MR **27** #1979] and independently by J. R. Blum, D. L. Hanson and J. I. Rosenblatt [Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 1 (1962/63), 389–393; MR **27** #5283]. *A. Rényi* (Budapest)

Ostrovskiĭ, I. V. 5270

Infinitely divisible laws with unbounded Poisson spectrum. (Russian)

Dokl. Akad. Nauk SSSR **152** (1963), 1301–1304.

Let $\mathscr{L}$ be the class of infinitely divisible laws with the characteristic functions whose logarithms are represented in the form

$$\log \varphi(t) = i\beta t - \gamma t^2 + \sum_{m=-\infty}^{\infty} \lambda_m^{(1)}(e^{i\mu_m t} - 1 - i\mu_m t/(1+\mu_m^2))$$
$$+ \sum_{m=-\infty}^{\infty} \lambda_m^{(2)}(e^{-i\nu_m t} - 1 + i\nu_m t/(1+\nu_m^2)),$$

where β is real, $\gamma \geq 0$, $\lambda_m^{(1)}, \lambda_m^{(2)} \geq 0$, $\mu_m, \nu_m > 0$ $(m = 0, \pm 1, \pm 2, \cdots)$ and

$$\text{(i)} \sum_{m=-\infty}^{\infty} \lambda_m^{(1)}\mu_m^2(1+\mu_m^2)^{-1} + \sum_{m=-\infty}^{\infty} \lambda_m^{(2)}\nu_m^2(1+\nu_m^2)^{-1} < \infty,$$

and (ii) $\mu_{m+1}\mu_m^{-1}$ and $\nu_{m+1}\nu_m^{-1}$ are integers. Such a class $\mathscr{L}$ of laws has been introduced by Ju. V. Linnik [*Decompositions of probability laws* (Russian), Izdat. Leningrad. Univ., Leningrad, 1960; MR **22** #A2900]. The present paper deals primarily with the factorization problem of the law of $\mathscr{L}$. The author obtains the result that if the i.d.l. of $\mathscr{L}$ satisfies $\lambda_m^{(1)} = O(\exp(-\kappa\mu_m^2))$, $\lambda_m^{(2)} = O(\exp(-\kappa\nu_m^2))$ for an arbitrary $\kappa > 0$, and its characteristic function $\varphi(t)$ admits the factorization $\varphi(t) = \varphi_1(t)\varphi_2(t)$, where $\varphi_1(t)$ and $\varphi_2(t)$ are entire functions such that (*) $|\varphi_i(\sigma+i\tau)| \leq |\varphi(i\tau)|$, $-\infty < \sigma, \tau < \infty$, $i = 1, 2$, then $\varphi_1(t)$ and $\varphi_2(t)$ are, up to constant multiples, the characteristic functions of i.d.l. As a matter of fact, the author gives a theorem of more general type. Let $f(t)$ be any entire function of the form $\sum_1^\infty \lambda_p^{(1)}e^{i\xi_p t} + \sum_1^\infty \lambda_p^{(2)}e^{-i\eta_p t} + l(it)$, where $\xi_p, \eta_p > 0$, $\lambda_p^{(1)}, \lambda_p^{(2)} = O(\exp(-\kappa p^2))$ and $l(z)$ is an entire function with certain properties. Let us suppose that $\exp(f(t)) = \varphi_1(t)\varphi_2(t)$, where $\varphi_i(t)$ are entire and satisfy (*), $i = 1, 2$. The author gives the analytic forms of $\log \varphi_i(t)$. The precise statement is not reproduced here. *T. Kawata* (Washington, D.C.)

Révész, P. 5271

On sequences of quasi-equivalent events. I. (Russian summary)

Magyar Tud. Akad. Mat. Kutató Int. Közl. **8** (1963), 73–83.

A sequence of events $\{A_n\}$ on a probability space $(\Omega, \mathcal{S}, P)$ is called quasi-equivalent if the ratio

$$P(A_{i_1}\cdots A_{i_k})/P(A_{i_1})\cdots P(A_{i_k})$$

depends only on k and does not depend on the indices $i_1, \cdots, i_k$. The paper extends to quasi-equivalent sequences of events the results of A. Rényi and the author [Publ. Math. Debrecen **10** (1963), 319–325; MR **30** #1543]. Further results, which we now state, are a law of large numbers and a tail σ-field law. Assume $\liminf P(A_n) > 0$. The quasi-density of a quasi-equivalent sequence of events is the Radon-Nikodým derivative of $\mu(B) = \lim_{n\to\infty} P(B/A_n)$ with respect to P. (It is shown that the limit exists; it then follows [e.g., from the Vitali-Hahn-Saks Theorem] that μ is a P-continuous measure.) Theorem: Let $\{A_n\}$ be a quasi-equivalent sequence of events with quasi-density λ and let α_n be the indicator functions of the A_n. Then

$$P\left[\frac{1}{n}\sum_{k=1}^{n} \frac{\alpha_k}{P(A_k)} \to \lambda\right] = 1.$$

Further, the tail σ-field of $\{A_n\}$ coincides (mod P) with the σ-field generated by λ. *L. Sucheston* (Columbus, Ohio)

Szynal, Dominik 5272

On the strong law of large numbers for random variables bounded by sequences of numbers. (Polish and Russian summaries)

Ann. Univ. Mariae Curie-Skłodowska Sect. A **16** (1962), 123–127 (1964).

The author attempts to give necessary and sufficient conditions for the strong law of large numbers for random sequences (X_n) such that there exist numerical sequences (L_n) with (1) $\sum_n P[|X_n| \geq L_n] < \infty$. However, for any sequence (X_n) there exists such a sequence (L_n) so that (1) is not a restriction; the author's theorem is obviously invalid, and the sequence he considers in his Remark 1 provides, in fact, a counterexample to his "theorem". The theorems used by the author are misquoted and misapplied, and confusion permeates the paper.

M. Loève (Berkeley, Calif.)

Delporte, Jean 5273

Prolongement par continuité d'une fonction uniformément continue sur l'ensemble dyadique $\mathscr{D}\times\mathscr{D}$ et construction de fonctions aléatoires de deux variables presque sûrement continues.

C. R. Acad. Sci. Paris **259** (1964), 1471–1474.

The author gives conditions under which a random function of two variables $x(t, s, \omega)$ is almost surely continuous on $[0, 1]\times[0, 1]$. He obtains as a corollary the following generalization of Kolmogorov's well-known result. Let $x(t, s, \omega)$ be a separable random function on $[0, 1]\times[0, 1]$, for which there exists an $r > 0$ such that $|h|$ and $|h'| \leq \delta$ imply $E|x(t+h, s+h') - x(t, s)|^r = O(\varphi(\delta))$, where $\varphi(\delta) = \delta^{2+\epsilon}$ or $\varphi(\delta) = \delta^2|\log \delta|^{-(1+\epsilon)/\beta}$ $(\beta = \inf(1, 1/r))$; then $x(t, s)$ is almost surely continuous on $[0, 1]\times[0, 1]$.

R. Getoor (Stanford, Calif.)

Fristedt, Bert 5274

The behavior of increasing stable processes for both small and large times.

J. Math. Mech. **13** (1964), 849–856.

The author investigates the sample behavior of the increasing stable processes which have no deterministic linear component, that is, right continuous, strictly increasing processes X on the real line with $X(0)=0$ and $E\{e^{-uX(t)}\}=\exp[-\beta 2^{1-\alpha}tu^{\alpha}]=\int_0^{\infty} e^{-ux}f(t,x)\,dx$, where $0<\alpha<1$ and $\beta>0$. The author gives conditions on h under which the events $\{X(t)\leq h(t)$ i.o. as $t\to 0\}$ or $\{X(t)\leq h(t)$ i.o. as $t\to\infty\}$ have probability zero or one. The following is one of his results. Let h be continuous and strictly increasing in a neighborhood of 0 with $h(0)=0$. For fixed α assume $t^{-1/\alpha}h(t)$ is monotone near 0 and consider the following conditions on h: (i) $t^{-1/\alpha}h(t)\to 0$ as $t\to 0$, (ii′c) $\int_0^1 f[t, ch(t)]\,dh(t)<\infty$, where $0<c\leq 1$ and f is given by its Laplace transform above. If conditions (i) and (ii′1) hold, then $P[X(t)\leq h(t)$ i.o. as $t\to 0]=0$, and if either (i) or (ii′c) for some $c<1$ fail to hold, then $P[X(t)\leq h(t)$ i.o. as $t\to 0]=1$. *R. Getoor* (Stanford, Calif.)

Karlin, Samuel 5275

Total positivity, absorption probabilities and applications.

Trans. Amer. Math. Soc. **111** (1964), 33–107.

An account of the beautiful connections between totally positive functions and 1-dimensional stochastic processes with continuous paths. A function $p(a,b)$ defined on a rectangle of $A\times B$ ($A, B=Z^1$ or R^1) is totally positive if for each choice of $n\geq 1$, $a_1\leq\cdots\leq a_n$, and $b_1\leq\cdots\leq b_n$, the determinant of $p(a_i,b_j)$ $(i,j\leq n)$ is non-negative. For totally positive functions of the special form $p(b-a)$, see I. J. Schoenberg [Bull. Amer. Math. Soc. **59** (1953), 199–230; MR **15**, 16] and I. I. Hirschman, Jr. and D. V. Widder [*The convolution transform*, Princeton Univ. Press, Princeton, N.J., 1955; MR **17**, 479].

The connection with stochastic processes, due to the author and J. McGregor [Pacific J. Math. **9** (1959), 1109–1140; MR **22** #5071; ibid. **9** (1959), 1141–1164; MR **22** #5072], comes about as follows. Consider a Markovian motion on a subinterval Q of Z^1 or R^1 with transition probabilities

$$P[x(t)\in db\,|\,x(s)=a]=P_a[x(t-s)\in db]=p_{t-s}(a,b)e(db)\qquad (t>s).$$

Under a mild technical condition $p_t(a,b)$ is totally positive for each $t>0$ if and only if the motion has continuous sample paths (meaning that the particle moves by unit jumps if $Q\subset Z^1$), and in this case the determinant of $p_t(a_i,b_j)$ $(i,j\leq n)$ is the transition density for $n\geq 1$ independent particles starting at $a_1<\cdots<a_n$ to come in time $t>0$ to $b_1<\cdots<b_n$ without crossing. The principal new result of this kind proved here is that the density $q_t(a,b)=\partial P_a[T_b<t]/\partial t$ for the passage time $T_b=\min(t\colon x(t)=b)$ is totally positive as a function of $t>0$ and a $(<b)$ and also as a function of $t>0$ and a $(>b)$.

Only part of the paper deals with total positivity as such. A number of related function classes are defined and studied. Applications to eigendifferential expansions, to the behaviour of ratios of transition probabilities as $t\uparrow+\infty$, to the unimodality of passage time densities and stable laws, etc., are included, together with a large number of instructive examples.

{A misapprehension spoils Theorem 6.3: A 1-dimensional diffusion with "non-negative drift" is simply translation at speed 1 relative to a suitable scale, so that the functions appearing in the statement do not exist.}

H. P. McKean, Jr. (Cambridge, Mass.)

Laha, R. G.; Lukacs, Eugene 5276

On identically distributed stochastic integrals.

Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (*Liblice*, 1962), *pp.* 467–474. *Publ. House Czech. Acad. Sci., Prague*, 1964.

Let $X(t)$, $t\in[a,b]\subset R$, be a process with independent increments, all moments finite, and mean value and variance functions of bounded variation on $[a,b]$. Form the integrals in quadratic mean $Y=\int_a^b a(t)\,dX(t)$, $Z=\int_a^b b(t)\,dX(t)$, where $a(t)$ and $b(t)$ are continuous in $t\in[a,b]$ with $\max_t|a(t)|\neq\max_t|b(t)|$. The authors prove that Y and Z are identically distributed if and only if $X(t)$ is a Wiener process and $\int_a^b[a(t)]^k\,dt=\int_a^b[b(t)]^k\,dt$ for $k=1$ and 2. Then they establish a similar theorem for $t\in[a,\infty)$. *M. Loève* (Berkeley, Calif.)

Mandl, Petr 5277

Diffusion processes with weakly absorbing boundary. (Russian. English summary)

Teor. Verojatnost. i Primenen. **9** (1964), 283–292.

Let $P(t,x;E)$ be transition probabilities corresponding to a homogeneous diffusion process on $[0,\infty)$. Let the corresponding semigroup of operators $T_tf=\int P(t,x;dy)f(y)$ on the space of continuous functions on $[0,\infty)$ have an infinitesimal generator which is a contraction of $\Omega f=d^2f/dx^2+b(x)\,df/dx$ with respect to a Feller boundary condition

$$(\tau+\varepsilon)f(0)=\tau\int_0^{\infty}f(x)\,dp(x)-\sigma\Omega f(0)+\gamma f'(0)$$

at $x=0$. Here, let $b(x)$ be continuous on $[0,\infty)$, $p(x)$ a distribution function continuous at $x=0$, and τ, σ, and γ non-negative constants with $\max(\tau,\sigma,\gamma)>0$. Let ε be a parameter >0, and $F(t,x;\varepsilon)$ the probability of absorption at 0 by time t starting from x at time 0. The author studies $\lim_{\varepsilon\to 0}F$ in terms of the distribution function

$$\Phi(t,\rho)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{t^{n\rho}}{\Gamma(n\rho+1)},\qquad 0<\rho\leq 1.$$

He proves four theorems, of which we state the first two.

Let $\Omega f=D_vD_uf$ be Feller's representation; $u(x)=\int_0^x Q(y)^{-1}\,dy$ and $v(x)=\int_0^x Q(y)\,dy$, where $Q(x)=\exp\int_0^x b(s)\,ds$. Theorem 1: If

$$M=\tau\int_0^{\infty}\int_0^{x}(v(\infty)-v(y))\,du(y)\,dp(x)+\gamma v(\infty)+\sigma<\infty,$$

then $\lim_{\varepsilon\to 0}F(tM\varepsilon^{-1},x;\varepsilon)=\Phi(t,1)=1-e^{-t}$.

Let $\alpha(x)$ and $\beta(x)$ be continuous on $[0,\infty)$. Theorem 2: If $\tau\neq 0$ and $b(x)=-B+\beta(x)$, where $0<B<\infty$, if $\int_0^{\infty}e^{(B+\delta)x}|\beta(x)|\,dx<\infty$ for arbitrary $\delta>0$, and if $\int_0^t(1-p(y))\,dy\sim At^{\gamma}$ for some $\gamma>0$, then for $M=A\Gamma(\gamma+1)B^{\gamma-1}$,

$$\lim_{\varepsilon\to 0}F(t[M\varepsilon^{-1}]^{(1-\gamma)^{-1}},x;\varepsilon)=\Phi(t,1-\gamma).$$

J. Chover (Madison, Wis.)

Pitcher, T. S. 5278

Likelihood ratios of Gaussian processes.

Ark. Mat. **4**, 35–44 (1960).

The author considers a real Gaussian stochastic process $x(t)$ on a finite time interval and considers the measure m_f on the set of sample functions as a functional of the mean value function $f(t)$. He shows that if m_f and m_0 are mutually absolutely continuous, then so are $m_{\lambda f}$ and m_0, and

$$\log \frac{dm_{\lambda f}}{dm_0}(x) = \lambda\varphi(x) - \tfrac{1}{2}\lambda^2 C,$$

where $\varphi(x)$ obeys conditions which amount to linearity almost everywhere, and C is a constant. Further properties of the φ functional are deduced, particularly for the case when $f(t)$ has a representation in terms of the covariance kernel of the process, and the results are extended to the case of an infinite time interval.

P. Whittle (Manchester)

Seregin, L. V. 5279

Conditions for continuity of stochastic processes. (Russian. English summary)

Teor. Verojatnost. i Primenen. **6** (1961), 3–30.

Author's summary: "Let x_t, $0 \leq t \leq c < \infty$, be a separable stochastic process in metric space X. The main purpose of this paper is to derive conditions under which almost all sample functions of the process x_t are continuous. We designate $\rho(x, y)$ the distance between the points $x, J \in X$. Let $\mathrm{P}(\cdots)$ be a Markov transition function, satisfying for each $\varepsilon > 0$

$$\sup_{x,s,t} \mathrm{P}(s, x, t, V_\varepsilon(x)) = o(1), \qquad h \downarrow 0,$$

where $x \in X$; $s, t \in [0, c]$, $0 < t - s \leq h$ and $V_\varepsilon(x) = \{y : \rho(x, y) > \varepsilon\}$. Then almost all sample functions of the Markov process x_t are continuous if and only if for each $\varepsilon > 0$

$$\int_0^{c-h} \mathrm{P}\{\rho(x_t, x_{t+h}) > \varepsilon\}\, dt = o(h), \qquad h \downarrow 0.$$

Almost all sample functions of a martingale [semimartingale] x_t are continuous if and only if for $h \downarrow 0$

$$\int_0^{c-h} \mathrm{P}\{x_t < a, x_{t+h} > b\}\, dt = o(h),$$

$$\int_0^{c-h} \mathrm{P}\{x_t > b, x_{t+h} < a\}\, dt = o(h)$$

for each a and b $(a < b)$."

Sytaja, G. N. 5280

On a multiple stochastic integral. (Russian. English summary)

Ukrain. Mat. Ž. **16** (1964), 351–364.

Let $w(t)$ be a Brownian motion process defined in the interval $[0, T]$. Then K. Itô [Mem. Amer. Math. Soc. No. 4 (1951); MR **12**, 724] has defined the stochastic integral $\int_0^T f(t)\,dw(t)$ for a certain class of functions $f(t)$. In the present paper, the author discusses an analogous definition of the multiple stochastic integral

$$\int_0^T \cdots \int_0^T f(w(t_1), \cdots, w(t_m))\, dw(t_1) \cdots dw(t_m).$$

R. G. Laha (Washington, D.C.)

Čerkasov, I. D. 5281

Criteria for coordinate-homogeneity for continuous Markov processes. (Russian. English summary)

Teor. Verojatnost. i Primenen. **5** (1960), 229–237.

Author's summary: "Let us consider a Markov process with an infinitesimal operator

$$(*) \qquad A_t = A^i(t, x)\frac{\partial}{\partial x^i} + B^{ij}(t, x)\frac{\partial^2}{\partial x^i \partial x^j},$$

where $x = (x^1, \cdots, x^n)$ is a point in Riemann space V_n with a metric $g_{ij}(t, x)$. The necessary and sufficient conditions for the existence of a transformation $x^{i'} = x^{i'}(t, x^1, \cdots, x^n)$ which transforms the operator $(*)$ into the well-known operator $A_t^0 = B^{i'j'}(t)\,\partial^2/\partial x^{i'}\partial x^{j'}$ are given. At the end of the paper an example is given from statistics, in which these conditions are applied for establishing the density of the probabilities $f(t, x, \tau, \xi)$ of a certain Markov process."

Hanen, Albert 5282

Théorèmes limites pour une suite de chaînes de Markov.

Ann. Inst. H. Poincaré **18**, 197–301 (1963).

Let P_n, $n = 1, 2, \cdots$, be a sequence of $r \times r$ substochastic matrices converging to a stochastic limit P. Let ${}^{(n)}\xi_j(k)$ be the number of visits at j of the corresponding sub-Markov chain ${}^{(n)}X$ (with initial state i_0) in the first k steps. The paper is devoted to studying the limit laws for the vectors ${}^{(n)}\xi_j(k)$, $1 \leq j \leq r$, as n and $k \to \infty$. The methods are analytic and depend upon the expression of the characteristic function in the form

$$(1) \quad E\left[\exp \sum_{j=1}^{n} iu_j{}^{(n)}\xi_j(n) \,\Big|\, \{{}^{(n)}X_0 = i_0\}\right] = (P_n D_u)^n V_0(i_0),$$

where D_u is the diagonal matrix with terms $\exp(iu_j)$ and V_0 is the vector with all entries 1. By writing the operators $P_n D_u$ in the spectral representation, and considering them as perturbations of PD_u, application is made of the functional calculus of operators to obtain the limit characteristic functions in a neighborhood of the origin.

In Chapter 2, the fundamental case $P_n \equiv P$ is considered. A central limit theorem of Kolmogoroff and Mihoc is extended to allow P to contain transient states, and a (weak) law of large numbers is proved for general P. In Chapter 3, the sequence $P_n \to P$ is treated under the restriction that P contains no transient states. It is shown, for example (Theorem 3.3), that if P has only one ergodic class, and if the P_n are strictly stochastic, then the random variables $Y_{j,n} = ({}^{(n)}\xi_j(n) - n\Pi_j{}^*)/\sqrt{n}$, where Π^* is the P-invariant distribution, converge jointly in the weak sense to the normal limit law corresponding to P as in Chapter 2. If P has several ergodic classes E_i, and if (a) $n(1 - {}_i\lambda_{0,n}) \to q_i$, where ${}_i\lambda_{0,n}$ is the maximal positive eigenvalue of P_n restricted to $E_i \times E_i$, and (b) $nE({}_i\lambda_0)\,{}_{i,j}S_n V_0{}^j \to q_i{}^j$, where $E({}_i\lambda_0)$ is the projection associated with ${}_i\lambda_0(=1)$, ${}_{i,j}S_n$ is the restriction of P_n to $E_i \times E_j$, and $V_0{}^j$ is the restriction of V_0, then a law of large numbers (Theorem 3.4) for the vectors ${}^{(n)}\xi_j(n)/n$ is proved. This is used to show that the transitions among the ergodic classes of P are "asymptotically Markovian" with a limit process having Q as infinitesimal generator (Q has diagonal entries $-q_i$ and $q_i{}^j$ elsewhere). In Chapter 4 the principal results of

Chapter 3 are extended to cover the case in which P has transient states.

The last chapter studies the consequences of two alternative sets of hypotheses. In the first, P has one ergodic class but may have transient states. Then if $n^{\alpha}(P_n - P) \to S$ for some α, $0 < \alpha < 1$, and if $n^{\alpha}(Q_i(n) - Q_i) \to B_i(u) > 0$ for i transient, where $Q_i(n)$ is the minor relative to (i, i) of the determinant of $[P_n D_u - \lambda_{0,n}(u)I]$, $\lambda_{0,n}$ being the eigenvalue which approaches 1 as $u \to 0$, then a normal limit law is obtained. In the second situation, P has two acyclic ergodic classes but no transient states. It is assumed that $n(1 - \underset{1}{\lambda}_n) \to \infty$ and $n(1 - \underset{2}{\lambda}_n) \to q$, where $\underset{i}{\lambda}_n$, $i = 1$ or 2, are the maximal positive eigenvalues for the two classes. It is then shown (Theorem 5.2) that $(1 - \underset{1}{\lambda}_n)^{(n)}\xi_{1,n,i_0}$ (where ${}^{(n)}\xi_{1,n,i_0}$ is the number of visits in the first class up to the nth step) converges weakly to the limit with characteristic function $\exp[iuq/(1 - iu)]$, in accordance with a result of Dobrušin.

F. B. Knight (Urbana, Ill.)

Il'in, A. M.; Has'minskiĭ, R. Z. 5283

On the equations of Brownian motion. (Russian. English summary)

Teor. Verojatnost. i Primenen. **9** (1964), 466–491.

Let $X_\mu(t)$, $Y_\mu(t)$ be the position and velocity, respectively, of a particle of mass m, moving on a line. The surrounding linear medium consists of particles of mass μ. A force field acts on the particle. Moreover, at times $\tau_1, \tau_2, \cdots$ (Poisson process, rate a) the particle has an elastic collision with a particle of the medium. The successive collisions are with particles having velocities $\eta_1, \eta_2, \cdots$, where these velocities are random variables with a common distribution and are independent of each other and of $\{\tau_n\}$. The (X_μ, Y_μ) process is a Markov process with stationary transition probabilities. If $u(x, y, t)$ is the expectation of $f[X_\mu(t), Y_\mu(t)]$ for a particle having initial position x and initial velocity y, it is shown that, under suitable restrictions on f and the force field, the function u satisfies an integro-differential equation. It is natural to identify $a\mu = A/2$ with viscosity, the expected value of $\mu\eta_k^2$ with temperature. If these two quantities and m are held constant while $\mu \to 0$, and if η_k has expectation 0 and third absolute moment $o(\mu^{-2})$, the equation satisfied by μ becomes a (phase space) diffusion equation. It is shown that the solutions of the original equation converge to those of the limit equation. The Green's function of the limit equation is the transition density of the limiting diffusion process. Possible generalizations are discussed briefly: The treatment can be extended to n-space. The particle can be supposed restricted to an open set whose boundary has suitable absorbing or reflecting properties.

J. L. Doob (Princeton, N.J.)

Isaac, Richard 5284

A uniqueness theorem for stationary measures of ergodic Markov processes.

Ann. Math. Statist. **35** (1964), 1781–1786.

Consider a Markov chain with probabilities $P_a(B)$, let the non-negative mass distribution e be stable in the sense that $\int e(da)P_a[x_1 \in db] = e(db)$, and define $P_e(B) = \int e(da)P_a(B)$. If e is ergodic in the sense that for each tail event A with complement B either $P_e(A) = 0$ or $P_e(B) = 0$, then each stable mass distribution is a constant multiple of e. A sufficient condition that e be ergodic is that $e(Q) > 0$ entail $P_e[x_n \in Q \text{ i.o.}] > 0$.

H. P. McKean, Jr. (Cambridge, Mass.)

Kingman, J. F. C. 5285

On inequalities of the Tchebychev type.

Proc. Cambridge Philos. Soc. **59** (1963), 135–146.

Denote by $\mathscr{P}$ the set of random variables x satisfying

$$E[g_i(x)] = c_i \qquad (i = 0, 1, 2, \cdots, k)$$

(where $g_0(x) \equiv c_0 = 1$) and consider the evaluation of an upper bound for $E[f(x)]$, $x \in \mathscr{P}$. Isii [Ann. Inst. Statist. Math. Tokyo **10** (1959), 65–88; MR **21** #4492] has shown that the "method of Markov" gives a sharp upper bound; the author uses a much simpler convexity argument to establish the same result, enunciated in the paper as follows. Theorem 1: Suppose that $\mathscr{P}$ is non-empty, and that there exists no non-zero vector $\boldsymbol{\mu}$ such that, whenever $x \in \mathscr{P}$, $\boldsymbol{\mu} \cdot \mathbf{g}(x) = \boldsymbol{\mu} \cdot \mathbf{c}$ with probability one. Then either there exists a vector $\boldsymbol{\lambda} = (\lambda_0, \cdots, \lambda_k)$ such that, for all x,

$$(*) \qquad f(x) \leq \boldsymbol{\lambda} \cdot \mathbf{g}(x)$$

and such that

$$\sup_{x \in \mathscr{P}} Ef(x) = \boldsymbol{\lambda} \cdot \mathbf{c}$$

or no vector $\boldsymbol{\lambda}$ satisfies $(*)$ and $\sup_{x \in \mathscr{P}} Ef(x) = \infty$. The supremum is unaltered if $\mathscr{P}$ is replaced by the smaller class $\mathscr{P}_{k+1}$, the subset of $\mathscr{P}$ consisting of random variables taking at most $k+1$ different values.

Extensions and limitations of the method for the multi-variable situation are discussed.

The author devotes a final section to discussing bounds stronger than the Tchebychev bound, which, although not valid uniformly, are valid in the limit. In fact, he shows that if $\Phi_n = \sup_{x \in \mathscr{P}} Ef_n(x)$ and $f_n(x) = o(\Phi_n)$ for all x as $n \to \infty$, then, for each $x \in \mathscr{P}$,

$$Ef_n(x) = o(\Phi_n), \qquad n \to \infty.$$

P. Whittle (Manchester)

Knight, Frank; Orey, Steven 5286

Construction of a Markov process from hitting probabilities.

J. Math. Mech. **13** (1964), 857–873.

One of the most important problems in the current theory of Markov processes is that of constructing a process from given hitting probabilities (or harmonic measures, as the hitting probabilities are sometimes called). This problem is of special importance in studying the relationships between the axiomatic potential theory of Bauer or Brelot and the probabilistic potential theory of Hunt. It was shown by Blumenthal, the reviewer, and McKean [Illinois J. Math. **6** (1962), 402–420; MR **25** #5550] that under quite general conditions there can exist at most one process up to a change of time scale with given hitting probabilities. More recently, P. A. Meyer [Ann. Inst. Fourier (Grenoble) **13** (1963), fasc. 2, 357–372; MR **29** #260] has attacked the problem of constructing a Markov process corresponding to Brelot's axiomatic potential theory. The present authors approach the general problem

of constructing a Markov process with given hitting probabilities from a quite different direction. They show that under suitable conditions (which are not easily summarized) one can construct the desired Markov process as a limit of certain approximating random walks. As is usual in this type of problem, it is clear how the approximating random walks should be defined; the difficulty is in proving the convergence of the approximations and in obtaining the properties of the limit. The authors carry out this program in an elegant manner under reasonable assumptions on the hitting probabilities.

R. Getoor (Stanford, Calif.)

Matthes, Klaus 5287

Ergodizitätseigenschaften rekurrenter Ereignisse. I.
Math. Nachr. **24** (1962), 109–119.

The author considers a Markov chain $\{(i_n, X_n): n=0, 1, \cdots\}$ with state space the positive integers times the positive reals. (i_0, X_0) has some initial distribution. Given that $i_n = i$, the probability that $i_{n+1}=j$ is p_{ij}, independent of X_n. Given that $i_{n+1}=j$, the distribution of X_{n+1} is F_j, independent of everything else in the past of the process. One can think of $X_1, X_2, \cdots$ as the staying times in the successive states of the Markov chain $\{i_n: n=0, 1, \cdots\}$. $n_i(t)$ is the number of transitions into state i during $(0, t)$, $n(t)=\sum_i n_i(t)$, and $Z_t = i$ if the process is in state i at time t. Such processes are usually described as Markov renewal processes or semi-Markov processes (cf. R. Pyke [Ann. Math. Statist. **32** (1961), 1231–1242; MR **24** #A3712], where precise definitions can be found). Classical renewal theory considers the case where $i_1, i_2, \cdots$ can only take one value, i.e., the underlying Markov chain is trivial. For this case Blackwell [Pacific J. Math. **3** (1953), 315–320; MR **14**, 994] proved the existence of $\lim_{T\to\infty} E[N_i(T+t)-N_i(T)]$, and one also knows the limiting distribution $(T\to\infty)$ of Z_T and the time of the last transition before T [E. B. Dynkin, Izv. Akad. Nauk. SSSR Ser. Mat. **19** (1955), 247–266; MR **17**, 865]. The author generalizes these results to the above setup and phrases them in terms of weak convergence $(T\to\infty)$ of the stochastic processes $\{N_i(T+t)-N_i(T),\ Z_{T+t}: t\geq 0, i\geq 1\}$ to some stationary limiting process $\{\tilde{N}_i(t), \tilde{Z}_t: t\geq 0, i\geq 1\}$. Basically, these results have also been obtained by J. Th. Runnenburg [Thesis, Chapter 3, Univ. Amsterdam, Amsterdam, 1960; MR **26** #5649].

H. Kesten (Ithaca, N.Y.)

Matthes, Klaus; Nawrotzki, Kurt 5288

Ergodizitätseigenschaften rekurrenter Ereignisse. II.
Math. Nachr. **24** (1962), 245–253.

The notation is as in the preceding review of Part I [#5287]. In this part it is shown that for suitable initial distributions of (i_0, X_0) the convergence of the processes $\{N_i(T+t)-N_i(T), Z_{T+t}\}$ to $\{\tilde{N}(t), \tilde{Z}_t\}$ as $T\to\infty$ is actually of a stronger type than weak convergence. The key renewal theorem of W. L. Smith [Proc. Roy. Soc. Edinburgh Sect. A **64** (1954), 9–48; MR **15**, 722] is generalized and tied up with this stronger type of convergence for all initial distributions. {Remark: In the definition of $(F_i)\in\mathfrak{R}$ one should add on the last line of p. 248 "and $p_{i_1 i_2}p_{i_2 i_3}\cdots p_{i_{n-1} i_n}>0$".}

H. Kesten (Ithaca, N.Y.)

Moy, Shu-Teh C. 52

Generalizations of Shannon-McMillan theorem.
Pacific J. Math. **11** (1961), 705–714.

Let A be a non-empty set, $\mathscr{A}$ a σ-algebra of subsets of A, $X=\prod_{n\in Z} X_n$, where $X_n = A$ for all $n\in Z=\{\cdots, -1, 0, \cdots\}$. For each $n\in Z$ let pr_n be the projection of X on X_n. Denote by $\mathscr{B}$ and $\mathscr{B}_{n,m}$ ($n\in Z$, $m\in Z$, $n\leq m$) the σ-algebras generated by $\bigcup_{k\in Z}\mathrm{pr}_k^{-1}(\mathscr{A})$ and $\bigcup_{n\leq k\leq m}\mathrm{pr}_k^{-1}(\mathscr{A})$, respectively. For a probability λ on $\mathscr{B}$ denote by $\lambda_{n,m}$ the restriction of λ to $\mathscr{B}_{n,m}$ ($n\in Z$, $m\in Z$, $n\leq m$). Let now ν and μ be two probabilities on B such that (i) ν is stationary; (ii) μ is Markovian with stationary transition probabilities; (iii) $\nu_{n,m}$ is absolutely continuous with respect to $\mu_{n,m}$ for all $n\in Z$, $m\in Z$, $n\leq m$ (denote $f_{n,m}=d\nu_{n,m}/d\mu_{n,m}$). The main result of the paper is as follows. Assume that $\log f_{0,0}\in\mathscr{L}^1(\nu)$ and that there is a constant $M>0$ such that $\int(\log f_{0,n}-\log f_{0,n-1})d\nu\leq M$ for all $n\in N^*=\{1, 2, 3, \cdots\}$. Then the sequence $((\log f_{0,n})/n)_{n\in N^*}$ converges in $\mathscr{L}^1(\nu)$. This generalizes the mean convergence theorem in information theory of A. Perez [Trans. First Prague Conf. Information Theory, Statistical Decision Functions, Random Processes (Liblice, 1956), pp. 183–208, Publ. House Czech. Acad. Sci., Prague, 1957; MR **20** #6325; ibid., pp. 209–243; MR **2**[0] #6326; ibid., pp. 245–252; MR **20** #6327]. The paper contains also some other results of independent interest.

A. Ionescu Tulcea (Urbana, Ill.)

Moy, Shu-Teh C. 529

A note on generalizations of Shannon-McMillan theorem.
Pacific J. Math. **11** (1961), 1459–1465.

Let A, $\mathscr{A}$ and X, $\mathscr{B}$, $\mathscr{B}_{n,m}$ be as above [#5289]. Consider again two probabilities ν and μ on $\mathscr{B}$ satisfying the hypotheses (i), (ii) and (iii) and, as before, write $f_{n,m}=d\nu_{n,m}/d\mu_{n,m}$ ($n\in Z$, $m\in Z$, $n\leq m$). The author proves the following theorem. Assume that there is a constant $K>$[0] such that $\int(f_{0,n}/f_{0,n-1})d\nu\leq K$ for all $n\in N^*$. Then $\sup_{n\in N^*}|\log f_{-n,0}-\log f_{-n,1}|\in\mathscr{L}^1(\nu)$ and the sequence $((\log f_{0,n})/n)_{n\in N^*}$ converges almost everywhere with respect to ν. In a certain sense this theorem extends the pointwise convergence theorems in information theory of L. Breiman [Ann. Math. Statist. **28** (1957), 809–811; MR **19**, 1148; ibid. **31** (1960), 809–810] and K. L. Chung [ibid. **32** (1961), 612–614; MR **24** #A1630]. See also the reviewer's paper [Ark. Mat. 4 (1961), 235–247].

A. Ionescu Tulcea (Urbana, Ill.)

Pyke, Ronald; Schaufele, Ronald 529

Limit theorems for Markov renewal processes.
Ann. Math. Statist. **35** (1964), 1746–1764.

The authors extend the definition of a Markov renewal process of Pyke [same Ann. **32** (1961), 1231–1242; MR **2**[4] #A3712] to allow for the possibility of infinitely many transitions in a finite time. For a very general class of such processes, they study Doeblin ratio limit laws, the weak and strong law of large numbers and the central limit theorem. The results and tools are entirely analogous to well-known results for Markov chains [see, for instance, K. L. Chung, *Markov chains with stationary transition probabilities*, I, §§ 14–16, Springer, Berlin, 1960; MR **2**[2] #7176], although for the laws of large numbers the authors have weaker conditions than the usual ones.

H. Kesten (Ithaca, N.Y.)

Theodorescu, Radu 5292

Une propriété limite pour les chaînes à liaisons complètes.

C. R. Acad. Sci. Paris **259** (1964), 1941–1943.

The results are summarized in the author's abstract: "Condition nécessaire et suffisante pour l'ergodicité des chaînes homogènes à liaisons complètes à un ensemble quelconque d'états faisant intervenir une grandeur étroitement liée au coefficient d'ergodicité."

S. M. Berman (New York)

Jiřina, Miloslav 5293

Branching processes with measure-valued states.

Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (*Liblice*, 1962), *pp.* 333–357. *Publ. House Czech. Acad. Sci., Prague,* 1964.

The author studies branching processes where there are a continuum of different types of particles, distinguished by means of a continuous variable τ. The state space M is the set of finite measures μ on the real line, and the probability mechanism is determined by a transition function $P(s, \mu, t, U)$, $s \leq t \in T = \{0, 1, \cdots\}$, $\mu \in M$, U in $\mathcal{M}$ (an appropriate σ-algebra of subsets of M). The main results, which are too detailed to reproduce here, give necessary and sufficient conditions for the extinction with probability one of all particles in the case of time homogeneous processes.

G. Baxter (Minneapolis, Minn.)

Koutský, Z. 5294

Periodische Verzweigungsfolgen.

Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (*Liblice*, 1962), *pp.* 429–439. *Publ. House Czech. Acad. Sci., Prague,* 1964.

The author considers a modification of the Galton-Watson model and its vector-valued generalization, which he calls a periodic branching process. Specifically, a periodic branching process of period k is a sequence of random variables ξ_t, $t = 0, 1, 2, \cdots$, with

$$P\{\xi_{tk+j+1} = r \mid \xi_{tk+j} = 1\} = P\{\xi_{j+1} = r \mid \xi_j = 1\}$$

for $j = 0, 1, 2, \cdots, k-1$; $t = 0, 1, 2, \cdots$, and the conditional distribution of ξ_{tk+j+1} given $\xi_{tk+j} = r$ is the distribution of the sum of r independent random variables, each of which has the same distribution as ξ_{tk+j+1} given $\xi_{tk+j} = 1$.

Then, if this process is observed at interval k, the resulting sequence of random variables is easily seen to be a homogeneous branching process and the usual results follow immediately under the usual regularity conditions. A similar treatment of the periodic branching process in the vector-valued case is also carried out.

Bernard Harris (Madison, Wis.)

Skorohod, A. V. 5295

Branching diffusion processes. (Russian. English summary)

Teor. Verojatnost. i Primenen. **9** (1964), 492–497.

Der Verfasser behandelt Verallgemeinerungen der von B. A. Sevast'janov [Teor. Verojatnost. i Primenen. **3** (1958), 121–136; MR **20** #4332; ibid. **6** (1961), 276–286; MR **24** #A3005] definierten Verzweigungsprozesse mit Diffusion, wobei die Diffusionskoeffizienten vom Ort und der Zeit abhängen dürfen. Es wird eine Differentialgleichung angegeben für die erzeugende Funktion der Übergangswahrscheinlichkeiten $V(x, t, T, k_1, A_1, \cdots, k_r, A_r)$, d.h. der bedingten Wahrscheinlichkeiten, daß zur Zeit T in den Mengen $A_1, \cdots, A_r$ jeweils $k_1, \cdots, k_r$ Teilchen sind, unter der Annahme, daß sich zur Zeit t ein Teilchen im Punkt x des m-dimensionalen Raumes R befand.

In einer weiteren Verallgemeinerung wird R durch einen beliebigen topologischen Raum X ersetzt und ein Verzweigungsprozess ξ_t mit Diffusion auf dem Phasenraum $X_s^{(\infty)} = \bigcup_{l=0}^{\infty} X_s^{(l)}$ definiert, wobei $X_s^{(l)}$ der Raum der Punkte $(x_1, \cdots, x_l)$, $x_i \in X$, ist, in welchem diejenigen Punkte identifiziert werden, die sich nur in der Reihenfolge der Elemente $x_1, \cdots, x_l$ unterscheiden. Für die bedingte Erwartung

$$\mathbf{E}[f(\xi_t) \mid \xi_0 \in X_s^{(1)}; \xi_0 = (x)]$$

wo f eine stetige, durch 1 beschränkte Funktion auf X mit

$$f(\xi_t) = 1 \qquad \text{für } \xi_t \in X_s^{(0)},$$
$$= f(x_1) \cdots f(x_l) \text{ für } \xi_t \in X_s^{(l)}, \ \xi_t = (x_1, \cdots, x_l),$$

ist, wird eine Integralgleichung abgeleitet.

Bei der Erweiterung dieses Modells auf r Teilchentypen tritt ein System von Integralgleichungen auf.

K. Dietz (Heidelberg)

Vinogradov, O. P. 5296

An age-dependent branching process. (Russian. English summary)

Teor. Verojatnost. i Primenen. **9** (1964), 146–152.

Author's summary: "Let us consider the age-dependent branching process introduced by Bellman and Harris [Proc. Nat. Acad. Sci. U.S.A. **34** (1948), 601–604; MR **10**, 311]. Let $Z(t)$ be the number of particles existing at t. In this paper certain conditions are given under which the asymptotic behavior of $A(t)$ $[A(t) = M\{Z(t)\}]$ coincides with that of a usual branching process. Moreover, the asymptotic behavior of $p_0(t)$ $(t \to \infty)$ is given for certain processes."

Alekseev, V. G. 5297

New theorems on properties of "almost sure" realizations of Gaussian random processes. (Russian. Lithuanian and English summaries)

Litovsk. Mat. Sb. **3** (1963), *no.* 2, 5–15.

Certain quadratic functionals of increments of stationary Gaussian processes are shown to converge with probability 1 to known values as the interval is more finely divided. The results are analogous to those of G. Baxter [Proc. Amer. Math. Soc. 7 (1956), 522–527; MR **19**, 890] and can be applied to the singular case of detecting Gaussian random signals in the presence of Gaussian noise [see D. Slepian, IRE Trans. Information Theory IT-4 (1958), 65–68; MR **22** #9359].

C. W. Helstrom (Pittsburgh, Pa.)

Alekseev, V. G. 5298

Some methods of obtaining exact estimates for the parameter of a stationary Gaussian process. (Russian. English summary)

Teor. Verojatnost. i Primenen. **9** (1964), 516–519.

Author's summary: "The paper deals with stochastic processes $\xi(t)$ and $\eta(t)$ $(0 \leq t \leq T)$ having stationary Gaussian increments, zero means and spectral densities $f_\xi(\lambda)$ and $f_\eta(\lambda) = f_\xi(\lambda) + cf_\zeta(\lambda)$, respectively, where $f_\xi(\lambda)$ and $f_\zeta(\lambda)$ are known non-negative functions, and $c \geq 0$ is an unknown parameter. It is assumed that the Gaussian measures in the function space corresponding to the processes $\xi(t) - \xi(0)$ and $\eta(t) - \eta(0)$ are orthogonal for $c > 0$. We give the functionals of sample functions of the process $\eta(t)$ which could be used for the exact determination of the parameter c."

D. A. Darling (Ann Arbor, Mich.)

Blum, J. R.; Rosenblatt, Judah 5299

On random sampling from a stochastic process.

Ann. Math. Statist. **35** (1964), 1713–1717.

Let $\{X_n, n = 1, 2, \cdots\}$ be a stationary, ergodic sequence. All the finite-dimensional distributions of the X's can be consistently estimated by means of the observations $X_1, \cdots, X_n$ for $n \to \infty$. For a subsequence $\{k_n\}$ of the positive integers, the subsequence $\{X_{k_n}\}$ is not necessarily stationary. Even if it is stationary and ergodic, the finite-dimensional distributions of the original process may not be estimable from the observed subsequence.

The authors handle the case in which $\{k_n\}$ is not a fixed sequence but a sequence of positive, integer-valued random variables which are partial sums of elements of a stationary, ergodic sequence $\{Y_n\}$, i.e., $k_n = Y_1 + \cdots + Y_n$. The bivariate process $\{Y_n, Z_n\}$, where $Z_n = X_{k_n}$, is studied; for simplicity, the results are given for random variables X_n assuming only the values 0 and 1. It is shown that the process $\{Y_n, Z_n\}$ is stationary. The prime results are the following. $\{Y_n, Z_n\}$ is ergodic, and the finite-dimensional distributions of the $\{X_n\}$ process are consistently estimable from observations of the $\{Y_n, Z_n\}$ process under either of the following two hypotheses: (i) $\{X_n\}$ is stationary and ergodic; $\{Y_n\}$ is a sequence of independent, identically distributed random variables, distributed independently of the $\{X_n\}$ process; and $P\{Y_1 = 1\} > 0$; or (ii) $\{X_n\}$ is stationary and (uniformly strongly) mixing; $\{Y_n\}$ is stationary, ergodic, and independent of the $\{X_n\}$ process; and $P\{Y_1 = 1, \cdots, Y_m = 1\} > 0$, $m \geq 1$.

S. M. Berman (New York)

Grigor'ev, S. V. 5300

Linear prediction for a class of stationary processes. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 4 (41), 56–60.

The purpose of the paper is to show that one can obtain linear prediction formulas in explicit form also in the case of a stationary (wide sense) stochastic process with a non-rational spectral density. A Wiener process is studied in which some of the parameters converge to zero or infinity and linear prediction formulas are computed for the limiting process. *L. Sucheston* (Columbus, Ohio)

Leadbetter, M. R. 5301

On the normal stationary process—areas outside given levels.

J. Roy. Statist. Soc. Ser. B **25** (1963), 189–194.

Author's summary: "Let $x(t)$ be a normal stationary process, observed for $0 \leq t \leq T$, and c a given positive constant. Formulae are derived for the mean and variance of the area above the level $+c$ formed by the process $x(t)$ in $(0, T)$. Corresponding results are obtained for the sum and the difference of the areas above the level $+c$ and below the level $-c$."

Rozanov, Ju. A. 5302

On the equivalence of probability measures corresponding to a Gaussian stationary process. (Russian)

Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (Liblice, 1962), pp. 599–604. *Publ. House Czech. Acad. Sci., Prague,* 1964.

Let $\xi_0(t)$, $\xi(t)$, $0 \leq t \leq T$, be real Gaussian stationary processes and $P_0(d\omega)$, $P(d\omega)$, $\omega \in \Omega$, be, respectively, the corresponding probability measures on the space Ω of sample functions. As is well known [cf. J. Feldman, Pacific J. Math. 8 (1958), 699–708; MR **21** #1546], either $P_0 \sim P$ (mutually absolutely continuous) or $P_0 \perp P$ (singular). The author now proves the following theorem: Suppose ξ_0 has the spectral density $f_0(\lambda)$ subject to the condition that

$$\limsup_{\lambda \to \infty} \lambda^{2m} f_0(\lambda) < \infty, \qquad \liminf_{\lambda \to \infty} \lambda^{2n} f_0(\lambda) > 0$$

for certain natural numbers m, n $(n \geq m)$, let $B(t)$, $B_0(t)$ be, respectively, the correlation functions of ξ_0 and ξ, and put $\Delta(t) = B(t) - B_0(t)$; then for $P \sim P_0$, it is sufficient that $\Delta(t)$, $-T \leq t \leq T$, has an absolutely continuous $(2n-1)$st derivative satisfying

$$(1) \qquad \int_0^T \int_0^T |\Delta^{(2n)}(s-t)|^2 \, ds dt < \infty;$$

the condition (1) with n replaced by m is also necessary for $P \sim P_0$. This is a generalization of the author's previous result which corresponds to the special case $m = n$ [Teor. Verojatnost. i Primenen. 8 (1963), 241–250; MR **27** #5295]. *G. Maruyama* (Fukuoka)

Rozanov, Ju. A. 5303

Probability measures in functional spaces corresponding to stationary Gaussian processes. (Russian. English summary)

Teor. Verojatnost. i Primenen. **9** (1964), 448–465.

Die Arbeit beschäftigt sich mit der Totalstetigkeit zweier Wahrscheinlichkeitsmaße $P(dx)$ und $P_1(dx)$, die den Gaußschen stationären Prozessen $\xi(t)$ und $\xi_1(t)$, $0 \leq t \leq T$, entsprechen. Sie enthält neben einem Überblick über die bisherigen Resultate von M. S. Pinsker [*Information and informational stability of random variables and processes* (Russian), Izat. Akad. Nauk SSSR, Moscow, 1960], J. Feldman [Pacific J. Math. **10** (1960), 1211–1220; MR **24** #A3688], V. G. Alekseev [Dokl. Akad. Nauk SSSR **147** (1962), 751–754; MR **26** #4398], vom Verfasser [Teor. Verojatnost. i Primenen. 7 (1962), 84–89; ibid. 8 (1963), 241–250; MR **27** #5295], und anderen auch einige neue Ergebnisse. Ein einfacher Beweis wird dafür gegeben, daß die Wahrscheinlichkeitsmaße $P(dx)$ und $P_1(dx)$ singulär sind, wenn $\lim_{\lambda \to \infty} \lambda^\beta [f_1(\lambda) - f(\lambda)] = \infty$; hier sind f und f_1 die Spektraldichten von $\xi(t)$ und $\xi_1(t)$, $\beta = \alpha + \frac{1}{2}$, wobei für die Dichte f gilt: $\lim_{\lambda \to \infty} \lambda^\alpha f(\lambda) > 0$ für ein $\alpha > 0$. Da sich die Wahrscheinlichkeitsmaße $P(dx)$ und $P_1(dx)$ und die Korrelationsfunktionen $B(t)$ und $B_1(t)$ der entsprechenden Prozesse gegenseitig bestimmen, lassen sich die Kriterien

für die Totalstetigkeit dieser Maße natürlicher mit Hilfe von $B(t)$ und $B_1(t)$ formulieren. Als Beispiel sei folgender Satz zitiert: Die Bedingung $\int_0^T \int_0^T [A^{(2m)}(s-t)]^2 ds dt < \infty$, wobei $A(t) = B(t) - B_1(t)$, ist hinreichend für die Totalstetigkeit von $P(dx)$ und $P_1(dx)$, wenn $0 < \liminf_{\lambda\to\infty} \lambda^{2m} f(\lambda)$, und notwendig, wenn $\limsup_{\lambda\to\infty} \lambda^{2m} f(\lambda) < \infty$.

Ferner werden einige Eigenschaften der Korrelationsfunktion $B(t)$ angegeben, die mit Wahrscheinlichkeit 1 mittels eines endlichen Abschnittes einer Trajektorie $x(t)$, $0 \leq t \leq T$, bestimmt werden können.

K. Dietz (Heidelberg)

Seth, Asha 5304

The correlated unrestricted random walk.

J. Roy. Statist. Soc. Ser. B **25** (1963), 394–400.

Author's summary: "A particle performs an unrestricted correlated random walk along a straight line with a velocity constant in magnitude but variable in direction. This direction remains constant for every time interval of unit length, and with probability p it is the same as in the previous interval and with probability $1-p$ it is reversed. The object of this paper is to obtain probabilities for the first passage to position r, the jth return to the origin and cognate results."

Dhondt, André 5305

Solution transitoire de la file poissonienne à plusieurs stations.

Cahiers Centre Études Recherche Opér. **4** (1962), 161–178.

This paper is devoted to a study of the non-stationary behaviour of a queueing system with S servers, Poisson arrivals, exponential service time and first-come, first-served discipline. The approach is similar to that of S. Karlin and J. McGregor [Pacific J. Math. 8 (1958), 87–118; MR **20** #3611] but uses mathematically simpler techniques. Equations for the Laplace transforms of the probabilities that n people are in the system at time t are easily written down. The first S of these involve $S+1$ unknowns. A further equation is produced by combinations of the remaining equations, and the total of $S+1$ equations solved. The solutions of the remainder then follow easily. The inversions of the Laplace transforms depend upon the locations of the roots of certain equations, and these are discussed in detail. The final expressions for the probabilities are given as the sum of the residues at certain well-located points. *D. V. Lindley* (Aberystwyth)

Dhondt, André 5306

Sur les racines d'une équation irrationnelle rencontrée dans la théorie des files d'attente.

Simon Stevin **37** (1963/64), 38–54.

This paper contains further discussion of the roots of the nucleus that arises in the paper reviewed above [#5305].

D. V. Lindley (Aberystwyth)

Fabens, A. J.; Perera, A. G. A. D. 5307

A correction to "The solution of queueing and inventory models by semi-Markov processes".

J. Roy. Statist. Soc. Ser. B **25** (1963), 455–456.

Authors' summary: "An incorrect assumption of independence produced an error in the waiting-time distributions derived in same J. **23** (1961), 113–127 [MR **23** #A2959]. The correct distributions are derived here."

Prohorov, Ju. V. 5308

Transition phenomena in queueing processes. I. (Russian. Lithuanian and English summaries)

Litovsk. Mat. Sb. **3** (1963), *no.* 1, 199–205.

In a simple queue let the inter-arrival and service time distributions have, among other restrictions, respective means $\alpha > 1$ and 1. Setting $\alpha = (1-\delta)^{-1}$, $\delta > 0$, the author considers the limiting form of the distribution of W_n, the waiting time of the nth item to be serviced, when $\delta \to 0$. If $n\delta^2 \to \tau > 0$, it is shown that $\Pr\{W_n \leq x/\delta\} \sim \Pr\{\xi(t) < \frac{1}{2}(x+t), 0 \leq t \leq \tau\}$, where $\xi(t)$ is the standard Wiener process. If $n\delta^2 \to \infty$, $\Pr\{W_n \leq x/\delta\} \sim 1 - e^{-x}$. If $F_\delta(x)$ is the stationary distribution of W_n, one has $F_\delta(x/\delta) \to 1 - e^{-x}$.

D. A. Darling (Ann Arbor, Mich.)

Weiss, George H. 5309

An analysis of pedestrian queueing.

J. Res. Nat. Bur. Standards Sect. B **67B** (1963), 229–243.

Queue distributions are obtained for pedestrians who wait to cross through gaps in a traffic stream. The pedestrians have a Poisson arrival distribution; the traffic stream has independent gaps; and a pedestrian will accept a gap of length t with probability $\alpha(t)$. The author obtains the steady-state queue distributions using renewal type arguments. The $\alpha(t)$ apparently has the following more precise interpretation. At the start of a new gap of length t all queued pedestrians as a group either accept or reject the gap with probability $\alpha(t)$ or $1-\alpha(t)$, respectively, independent of any past history of previous rejections. Any pedestrian that arrives after the start of a gap, however, will accept a residual gap of length τ with probability $\alpha(\tau)$, independent of whether or not other pedestrians may have already rejected the original gap or any part thereof. It is difficult to imagine what kind of people or things might behave in this way. *G. Newell* (Providence, R.I.)

Wu, Fang 5310

On the queuing process $GI/M/n$.

Acta Math. Sinica **11** (1961), 295–305 (*Chinese*); *translated as Chinese Math.* **2** (1962), 333–343.

This paper is identical to another by the same author [Sci. Sinica 11 (1962), 1169–1182; MR **26** #1941].

Grigelionis, B. 5311

A central limit theorem for sums of renewal processes. (Russian. Lithuanian and English summaries)

Litovsk. Mat. Sb. **4** (1964), 197–201.

Der Autor betrachtet eine Folge $X_1(t), X_2(t), \cdots, X_n(t), \cdots$ von unabhängigen, identisch verteilten Erneuerungsprozessen. Die Summe $\sum_{k=1}^n X_k(t)$ kann man interpretieren als die Zahl der Erneuerungen im Zeitintervall $(0, t)$ in einem System, das aus n gleichartigen Elementen besteht.

Der zentrale Grenzwertsatz für diese Summen wird mit Hilfe der Methode der charakteristischen Funktionen und den Resultaten von W. L. Smith [Biometrika **46** (1959),

1–29; MR 21 #3055] über das asymptotische Verhalten der höheren Momente eines Erneuerungsprozesses $X(t)$ für $t \to \infty$ bewiesen. *K. Dietz* (Freiburg)

Bishir, J. 5312

A lower bound for the critical probability in the one-quadrant oriented-atom percolation process.

J. Roy. Statist. Soc. Ser. B **25** (1963), 401–404.

Author's summary: "For a percolation process on a one-quadrant lattice let p denote the probability that each point or atom, independently of all other atoms, allows fluid to pass. For an oriented process of this type a lower bound is established for the largest or critical value of p for which the process remains confined to a finite region of the lattice."

Watterson, G. A. 5313

The application of diffusion theory to two population genetic models of Moran.

J. Appl. Probability **1** (1964), 233–246.

The author proves that a time-dependent diffusion equation can be used to describe the limiting behaviour of two genetic population models involving diploid individuals and two sexes, which were introduced by the reviewer. Remarks are also made on the meaning and limitations of the concept of "effective population size". *P. A. P. Moran* (Canberra)

STATISTICS

See also 4706, 5313, 5367, 5629.

Manija, G. M. 5314

★Some methods in mathematical statistics. (Georgian)

Izdat. Akad. Nauk Gruzin. SSR, Tiflis, 1963. 352 pp. 2.11 r.

MacKinnon, William J. 5315

Table for both the sign test and distribution-free confidence intervals of the median for sample sizes to 1,000.

J. Amer. Statist. Assoc. **59** (1964), 935–956.

Consider the results of n independent trials of an event for which the probability of occurrence on a single trial is $\frac{1}{2}$. Let s and r, where $s+r=n$, be respectively the number of occurrences and non occurrences of the event in question, or vice versa, whichever satisfies the condition $s \leqq n/2 \leqq r$. The two-tailed probability that the number x, representing either the number of occurrences or the number of non-occurrences, will be less than or equal to s when the number of trials is n is given by $P(x \leqq s \mid n) = (\frac{1}{2})^{n-1} \sum_{x=0}^{s} n!/x!(n-x)!$, $x = 0, 1, \cdots, [n/2]$. Let h be the significance level for the two-sided test. Then the critical value of s for particular values of n and h is the largest value of s such that $P(x \leqq s \mid n) \leqq h$. Dixon and Mood [same J. 41 (1946), 557–566] published a table of critical values of s for $n = 1(1)100$ and four values of h. Such a table is useful not only in sign tests but also in distribution-free estimation of the median of a population. The author [ibid. 54 (1959), 164–172] published a sign-test table for twelve values of h, replacing n by $d = r - s$ and extending the range of sample sizes to 1,000. The latter table suffers the drawbacks that it is complicated to use and does not facilitate the determination of confidence intervals for the median. In the present paper the author gives a table, derived from the one in his previous paper but using n instead of d, for $n = 1(1)1000$ and six values of h, thus combining the best features of both the previous tables (the extent of his own earlier table and the convenience and versatility of the Dixon-Mood table). The method of derivation and several examples of the use of the table are given. *H. L. Harter* (Dayton, Ohio)

Danziger, L.; Davis, S. A. 5316

Tables of distribution-free tolerance limits.

Ann. Math. Statist. **35** (1964), 1361–1365.

The paper is concerned with probabilities K that N_0 or more independent observations $Y_1, \cdots, Y_N$ with a continuous density will lie between the r_1th and r_2th order statistic of some other independent sample $X_1, \cdots, X_n$ from the same distribution. These probabilities are shown to depend on $r = r_1 + n + 1 - r_2$ only. In the table are given N_0 corresponding to $K = 0{,}50$; $0{,}75$; $0{,}90$; $0{,}95$; $0{,}99$; $r = 1, 2, \cdots, 10$; N, $n = 5, 10, 25, 50, 75, 100$. Also $\lim N_0/N$ for $N \to \infty$ is tabulated. *J. Hájek* (Prague)

Burr, E. J. 5317

Small-sample distributions of the two-sample Cramér-von Mises' W^2 and Watson's U^2.

Ann. Math. Statist. **35** (1964), 1091–1098.

The upper tails ($P \leqq 0.1$) of the distributions of the above-mentioned statistics are investigated for $m, n \geqq 4$, $m + n \leqq 17$. Exact values of P are given, together with their ratio to the asymptotic ones. *J. Hájek* (Prague)

Cheng, Ping 5318

Minimax estimates of parameters of distributions belonging to the exponential family.

Acta Math. Sinica **14** (1964), 252–275 (*Chinese*); *translated as Chinese Math.* **5** (1964), 277–299.

Dąbek, Marian 5319

On the speed of convergence of sums and differences. (Polish and Russian summaries)

Ann. Univ. Mariae Curie-Skłodowska Sect. A **16** (1962), 101–106 (1964).

Let $\{\xi\}$ and $\{\xi_n'\}$ be two sequences of independent random variables each having moments of all orders and each satisfying Lyapounov's condition for the third central moment; moreover, for every n, ξ_n and ξ_n' are identically distributed. Let $S_N = \sum_{n=1}^{N} \xi_n$, $S_N' = \sum_{n=1}^{N} \xi_n'$, and $Z_N = S_N + S_N'$ and $Y_N = S_N - S_N'$.

In the present paper, the author compares the rates of convergence of the sequences of random variables $\{Y_N\}$ and $\{Z_N\}$. The author proves that (i) if all the random variables ξ_n are symmetric, then the sequence $\{Y_N\}$ converges as quickly as the sequence $\{Z_N\}$, but (ii) if at least one random variable ξ_n is not symmetric, then the sequence $\{Y_N\}$ converges more quickly than the sequence $\{Z_N\}$. *R. G. Laha* (Washington, D.C.)

Eagleson, G. K. 5320

Polynomial expansions of bivariate distributions.

Ann. Math. Statist. **35** (1964), 1208–1215.

Suppose that W_1, W_2, W_3 are independent random variables belonging to a family $\mathscr{F}$ which is closed under convolutions. Then $X = W_1 + W_2$ and $Y = W_2 + W_3$ belong to the same family but are correlated. Suppose also that the distributions of $\mathscr{F}$ have a generating function for their orthogonal polynomials $\{P_n(x)\}$ of the form

$$G(t, x) = f(t)e^{xu(t)} = \sum_0^\infty P_n(x)t^n/n!.$$

The author studies the properties of the bivariate distribution functions of X and Y which result from these conditions and are ϕ^2-bounded [H. O. Lancaster, same Ann. **29** (1958), 719–736; MR **21** #944]. He distinguishes five types of distributions, depending on the values of parameters in a recurrence relation connecting $\{P_n(x)\}$.

R. L. Plackett (Newcastle upon Tyne)

Ewens, W. J. 5321

Two asymptotic results in sequential analysis.

Austral. J. Statist. **5** (1963), 1–4.

The standard Wald likelihood ratio sequential test with respect to a random variable x is assumed: sampling is continued until $z_1 + \cdots + z_r + \cdots$ goes outside a fixed interval, where the $\{z_i\}$ are independent log-likelihood ratios. Asymptotic here means the two hypotheses tested are infinitesimally close (and the A.S.N. $\to \infty$); this implies z is an infinitesimal variable and the process becomes one of diffusion, with the consequence that it is independent of the original d.f. of x. (This result is due to Bartlett (1944) [cf. M. S. Bartlett, *An introduction to stochastic processes, with special reference to methods and applications*, Cambridge Univ. Press, Cambridge, 1955; MR **16**, 939].) As a corollary of this, the problem of the behaviour of the sequential test between two values of the variance in normal samples with known mean when, in fact, the population is non-normal, is examined. It is shown that the same factor, $1 + \frac{1}{2}\gamma$, governs the effect of non-normality in the sequential case, as was found by Box in the fixed sample case. (γ is the kurtosis of the actual non-normal distribution.)

D. E. Barton (London)

Ferguson, Thomas S. 5322

A characterization of the exponential distribution.

Ann. Math. Statist. **35** (1964), 1199–1207.

In the present paper the author proves the following theorem. Let X and Y be two independent random variables each having absolutely continuous distribution functions. Then the random variables $U = \min(X, Y)$ and $V = X - Y$ are independent if and only if both X and Y have exponential distributions with the same location parameter.

R. G. Laha (Washington, D.C.)

Finucan, H. M. 5323

A note on kurtosis.

J. Roy. Statist. Soc. Ser. B **26** (1964), 111–112.

Author's summary: "This note 'rediscovers' the original interpretation of kurtosis as an indicator of 'a prominent peak and a prominent tail' on the density curve. A formal proof that such curves have indeed relatively high fourth moments is given. A comment is made concerning the recent decline of this correct classical interpretation."

Hanson, D. L.; Koopmans, L. H. 5324

Tolerance limits for the class of distributions with increasing hazard rates.

Ann. Math. Statist. **35** (1964), 1561–1570.

Let $Y_1, \cdots, Y_n$ be a sample from a distribution $F(x)$, and $X_1, \cdots, X_n$ the same values arranged in order. Non-parametric tolerance limits for $F(x)$ have the disadvantage that there is a minimum sample size below which the tolerance limits cannot be satisfied. The authors construct and study tolerance limits of the form $X_{n-k-j} + b(X_{n-k} - X_{n-k-j})$ $(b > 1)$ which are always satisfactory for absolutely continuous distributions which are increasing hazard rate distributions, and more particularly, Pólya frequency functions of order two. Monte Carlo comparisons with parametric tolerance limits are discussed.

P. A. P. Moran (Canberra)

Watanabe, Yoshikatsu 5325

The Student's distribution for a universe bounded at one or both sides. IV.

J. Gakugei Tokushima Univ. **14** (1963), 1–53.

Parts I, II, and III appeared in same J. **11** (1960), 11–51 [MR **25** #1595]; ibid. **12** (1961), 5–50; ibid. **13** (1962), 1–42. Author's introduction: "In the present note the author treats the most general form for the volume-element of the first half, i.e., one-sided case of the proposed theme, and has gained its general information, at least theoretically. Naturally, to get actual solutions for concrete cases of several sizes $n = 6, 7, \cdots$, it requires a vast bulk of computations which could be accomplished only by making constant use of electronic computers, etc. Also the results reported in the preceding notes are now supplemented with possible improvements, sometimes rebuilt to get a better insight, or made to be more general and intelligible. Lastly, to reveal the general feature of its application, the T.N.D. is exemplified by the special case $n = 4$."

Kotlarski, Ignacy 5326

On bivariate random variables where the quotient of their coordinates follows some known distribution.

Ann. Math. Statist. **35** (1964), 1673–1684.

In the present paper the author derives a characterization of the class $\mathscr{X}$ of all distribution functions $F(x, y)$ of bivariate random variables (ξ, η), where the coordinates ξ and η are identically distributed, and, moreover, the quotient ξ/η follows the Cauchy law. A similar characterization is also obtained for the class $\mathscr{Y}$ of all distribution functions $F(x, y)$ of bivariate random variables (ξ, η), where ξ and η are positive random variables, and, moreover, the quotient ξ/η has an F-distribution. The method of Mellin transforms is the main tool of the present investigation of the author.

R. G. Laha (Washington, D.C.)

Pillai, K. C. Sreedharan 5327

On the moments of elementary symmetric functions of the roots of two matrices.

Ann. Math. Statist. **35** (1964), 1704–1712.

This paper presents two lemmas for evaluating the moments of elementary symmetric functions (esf's) of the non-null characteristic roots of a matrix, useful in certain problems of multivariate analysis. The preliminary lemma shows how to express the product of an sth-order Vandermonde determinant and the kth ($k \geq 0$) and lth ($l \geq 0$) powers of the rth and hth ($r, h \leq s$) esf's as a linear compound of determinants. This is extended to products of powers of any number of esf's up to the sth. Then, using the preliminary lemma and reduction formulae for certain Vandermonde determinants, the second lemma shows how the moments of the esf functions of the s non-null characteristic roots of a matrix can be derived from formulae for moments of corresponding esf's in s non-null roots of another matrix, and vice versa. Illustrations are given.

P. S. Dwyer (Ann Arbor, Mich.)

Zinger, A. A.; Linnik, Ju. V. 5328

Polynomial statistics for a normal law and those related to it. (Russian. English summary)

Teor. Verojatnost. i Primenen. **9** (1964), 547–550.

The authors consider a class of n-dimensional distributions, including all regular normal distributions and distributions with density $\prod_{i=1}^{n} f(x_i)$, where f satisfies a differential equation $f^{(\nu)}(x)+\sum_{j=1}^{\nu} P_j(x) f^{(\nu-j)}(x)=0$, with the P_j denoting certain polynomials. For this class they show that two polynomial statistics $Q_1=Q_1(\xi)$ and $Q_2=Q_2(\xi)$ have the same distribution if and only if $Eh_j(Q_1)=Eh_j(Q_2)$, where $h_1, \cdots, h_M$ is a finite set of functions. Similarly, Q_1 and Q_2 are independent if and only if $E(h_j(Q_1)g_k(Q_2))=Eh_j(Q_1)Eg_k(Q_2)$, where $h_1, \cdots, h_M$, $g_1, \cdots, g_M$ is a finite set of functions. The functions h_j and g_j are of the form $h_j(x)=x^j \exp(-\sigma(x))$, $\sigma(x)$ being a polynomial.

J. Hájek (Prague)

Bennett, B. M. 5329

On combining estimates of a ratio of means.

J. Roy. Statist. Soc. Ser. B **25** (1963), 201–205.

Author's summary: "For a bivariate normal distribution with variances and covariance given multiples of an unknown σ^2, methods are presented for the combination of several separate estimates of the ratio of means, and for assessing confidence limits when an estimate of σ^2 is available. A numerical example is presented of the application of this method in combining estimates of relative potency based on several parallel line assays of insulin."

Birch, M. W. 5330

Maximum likelihood in three-way contingency tables.

J. Roy. Statist. Soc. Ser. B **25** (1963), 220–233.

Author's summary: "Interactions in three-way and many-way contingency tables are defined as certain linear combinations of the logarithms of the expected frequencies. Maximum-likelihood estimation is discussed for many-way tables and the solutions given for three-way tables in the cases of greatest interest."

Edwards, A. W. F. 5331

Estimation of the parameters in short Markov sequences.

J. Roy. Statist. Soc. Ser. B **25** (1963), 206–208.

Author's summary: "A parametric representation of a two-state Markov process is given which enables the maximum-likelihood estimates of the two parameters to be found without iteration."

Herbst, Laurence J. 5332

A test for variance heterogeneity in the residuals of a Gaussian moving average.

J. Roy. Statist. Soc. Ser. B **25** (1963), 451–454.

Author's summary: "A large sample test for variance heterogeneity in the residuals of a Gaussian moving average is presented, based upon periodogram-ratios of the data. For long series, it is shown that neither the null-hypothesis nor alternative-hypothesis behaviour of the test statistic is dependent upon the coefficients of the moving average (and these need not be estimated in advance). The alternative-hypothesis behaviour of the test statistic is discussed in terms of the coefficients in a Fourier expansion of the non-constant variance function."

McCord, James R. 5333

On asymptotic moments of extreme statistics.

Ann. Math. Statist. **35** (1964), 1738–1745.

Let Y_n be the maximum of n independent observations of a random variable X with distribution function F such that if $F(a_2)=1$ and $F(x)<1$ for $x<a_2$, then F has a continuous derivative f on some open interval (a_1, a_2), let μ_n and σ_n^2 be the mean and variance of Y_n, let $\lambda_k=E(|X-a|^k)$, and let u_n be defined by $1-F(u_n)=1/n$. Then (1) If there are real constants a, $b>0$, and $c>0$ such that $F(a)=1$, $F(x)<1$ for $x<a$, and, as $x \to a^-$, $1-F(x) \simeq b(a-x)^c$, $f(x) \simeq bc(a-x)^{c-1}$, then $E[(a-Y_n)^k] \simeq (bn)^{-k/c} \times \Gamma[1+(k/c)]$ as $n \to \infty$ ($k>-c$, $\lambda_k<\infty$) [or $E[(a-Y_n)^k] \simeq (a-u_n)^k \Gamma[1+(k/c)]$ as $u_n \to a^-$], whence $\sigma_n^2 \simeq (bn)^{-2/c} \times \{\Gamma[1+(2/c)]-\Gamma^2[1+(1/c)]\}$ as $n \to \infty$ ($\lambda_2<\infty$); (2) If there are real constants a, $b>0$, and $c>0$ such that $F(x)<1$ for $x<\infty$ and, as $x \to \infty$, $1-F(x) \simeq b(x-a)^{-c}$, $f(x) \simeq bc(x-a)^{-c-1}$, then $E[(Y_n-a)^k] \simeq (bn)^{k/c} \Gamma[1-(k/c)]$ as $n \to \infty$ ($k<c$, $\lambda_k<\infty$) [or $E[(Y_n-a)^k] \simeq (u_n-a)^k \Gamma[1-(k/c)]$ as $u_n \to \infty$], whence $\sigma_n^2 \simeq (bn)^{2/c}\{\Gamma[1-(2/c)]-\Gamma^2[1-(1/c)]\}$ as $n \to \infty$ ($c>2$, $\lambda_2<\infty$); (3) If there are real constants a, $b>0$, $c>0$ and $r>0$ such that $F(x)<1$ for $x<\infty$ and, as $x \to \infty$, $1-F(x) \simeq r \exp[-b(x-a)^c]$, $f(x) \simeq bcr(x-a)^{c-1} \exp[-b(x-a)^c]$, then $E[(Y_n-a)^k] \simeq (b^{-1} \ln rn)^{k/c}$ as $n \to \infty$ ($\lambda_k<\infty$) [or $E[(Y_n-a)^k] \simeq (u_n-a)^k$ as $u_n \to \infty$]. The author conjectures that, in the third case, $\sigma_n^2 \simeq (1/6)(\pi/bc)^2(b^{-1} \ln rn)^{(2/c)-2}$ ($\lambda_2<\infty$), and gives examples for which the conjecture holds.

D. R. Barr (Dayton, Ohio)

Rice, John R.; White, John S. 5334

Norms for smoothing and estimation.

SIAM Rev. **6** (1964), 243–256.

The authors compare the variances of the median, mean, and the midrange in samples from various distributions by actual sampling. Their conclusion that, depending on the parent distribution, one or another statistic is more efficient is well known and could have been reached easily without making the computer work.

M. M. Siddiqui (Fort Collins, Colo.)

Walker, A. M. 5335

A note on the asymptotic efficiency of an asymptotically normal estimator sequence.

J. Roy. Statist. Soc. Ser. B **25** (1963), 195–200.

Author's summary: "This paper gives a rigorous yet quite elementary proof that under fairly general conditions, the asymptotic efficiency of an asymptotically normal estimator sequence does not exceed unity. Both estimation of a single parameter and simultaneous estimation of several parameters are considered."

Anscombe, F. J. 5336

Tests of goodness of fit.

J. Roy. Statist. Soc. Ser. B **25** (1963), 81–94.

Author's summary: "An explicit account is given of a procedure for assessing goodness of fit of some observations with a hypothesis, generally known as a 'test of significance'; the description is close in spirit to R. A. Fisher's original conception. The relation of this test procedure with Bayesian procedures and with the Neyman-Pearson theory of tests is discussed. Some very tentative suggestions are made regarding the choice of a test criterion and the use to which the result of the test can be put. The role of conditional sampling distributions is studied. Various examples are discussed. In particular, a brief exploration is made in the apparently virgin territory of testing goodness of fit when the observations have been taken according to a sequential rule."

Inselmann, Edmund H. 5337

Hypothesis testing of Gaussian processes with composite alternatives.

J. Soc. Indust. Appl. Math. **12** (1964), 370–385.

A radar signal $s(t)$ is defined on the unit circle, $0 \leq t < 2\pi$. It is supposed that $s(t) = n(t) + u(t)$, where $n(t)$ is white Gaussian noise and $u(t)$ is a signal of known form but unknown location on the unit circle, that is, it is equal to $u(t-\theta)$, where $u(\cdot)$ is known but θ is not. It is desired to test the simple null hypothesis that the signal is absent against the composite alternative that it is present. The statistics used are the observable coordinates

$$s_k = \int_0^{2\pi} s(t)\exp(ikt)\,dt$$

for $k \leq n$. The suggested test is to reject when

$$\prod_{k<n} \mathrm{I}_0(|s_k u_k|/2S_k^2) > c,$$

where $\mathrm{I}_0(\cdot)$ is the Bessel function of the first kind with pure imaginary argument, $u_k = \int_0^{2\pi} u(t)\exp(ikt)dt$, and S_k is the standard deviation of the noise. This test is shown to be optimum in three senses indicated by the following considerations: (i) least favourable distributions, (ii) Bayes solution with probability $\frac{1}{2}$ on the null hypothesis, (iii) invariance. Numerical methods of performing the test are indicated. The last part of the paper extends the problem to the case where several sweeps of the unit circle may be made and a decision has to be made whether to carry out a further sweep or decide the signal is present.

D. V. Lindley (Aberystwyth)

Moses, Lincoln E. 5338

One sample limits of some two-sample rank tests.

J. Amer. Statist. Assoc. **59** (1964), 645–651.

From the author's introduction and summary: "There are many familiar two-sample tests of composite hypotheses which reduce to corresponding one-sample tests of simple hypotheses as one of the two samples is allowed to become infinite. The same thing holds for a number of rank tests. This paper exhibits such a correspondence between a two-sample rank test of Lehmann's [E. Lehmann, Ann. Math. Statist. **24** (1953), 23–43; MR **14**, 888] and (1) Fisher's [R. A. Fisher, *Statistical methods for research workers*, twelfth edition, Hafner, New York, 1950] test for combining independent significance tests, or (2) a test proposed by Karl Pearson [E. S. Pearson, Biometrika **30** (1938), 134–148] (the choice depending on the value of a certain parameter). Useful one-sample limits of Wilcoxon's [F. Wilcoxon, Biometrics **1** (1945), 80–83] (two-sample) test and the Brown-Mood [G. W. Brown and A. M. Mood, Proc. Second Berkeley Sympos. Math. Statist. and Prob., 1950, pp. 159–166, Univ. of California Press, Berkeley, Calif., 1951] median test are presented. Although the point is not developed in this paper, it may be remarked that the one-sample and two-sample forms of the Kolmogorov-Smirnov tests are similarly related, as the reader familiar with them can verify at once."

C. Hayashi (Tokyo)

Rogers, Gerald S. 5339

An application of a generalized gamma distribution.

Ann. Math. Statist. **35** (1964), 1368–1370.

Let $\{x_{i1}, \cdots, x_{in_i}\}$ $(i = 1, 2, \cdots, k;\ k \geq 2,\ n_i \geq 2)$ be random samples from stochastically independent Gaussian populations with unknown means μ_i and unknown variances σ_i^2. Let the hypothesis H_0 be $\sigma_1^2 = \cdots = \sigma_k^2$ and λ the corresponding likelihood ratio. The author derives the probability distribution for λ.

Č. Masaitis (Aberdeen, Md.)

De Carolis, Linda Vittoria 5340

Su la regressione interamente lineare. (French, English, Spanish, and German summaries)

Giorn. Ist. Ital. Attuari **26** (1963), 118–139.

The author extends to n-dimensional random variables certain concepts earlier studied for the bivariate case by G. Pompilj [Industria (Milano) **1955**, 496–510; ibid. **1957**, 211–226; Fac. Sci. Statist. Demograf. Attuar. Ist. Statist. Ist. Calcolo Prob. Publ. No. 15 (1956); MR **20** #3586; ibid. No. 30 (1957); MR **21** #955]. One assumes throughout that moments of all orders exist and completely determine the distribution function, and that the covariance matrix is non-singular.

The regression of X_n on $X_1, \cdots, X_{n-1}$ is said to be pseudo-linear if the regression polynomials of all degrees coincide with one and the same hyperplane. G. Dall'Aglio [Rend. Mat. e Appl. (5) **15** (1956), 453–468; MR **19**, 69] showed that the linearity of $E(X_n \mid X_1, \cdots, X_{n-1})$ implies that the regression of X_n on $X_1, \cdots, X_{n-1}$ is pseudo-linear, but not the converse. Let $Y_1, \cdots, Y_n$ be any set of variates obtained from $X_1, \cdots, X_n$ by a non-singular affine transformation. If, for every such transformation, the regression of Y_n on $Y_1, \cdots, Y_{n-1}$ is linear or pseudo-linear, then one says that the regressions of the n-dimensional random variables $(X_1, \cdots, X_n)$ are entirely linear or entirely pseudo-linear, respectively.

The present author finds three sets of necessary and sufficient conditions for the regressions to be entirely pseudo-linear. Entirely pseudo-linear regression always

implies entirely linear regression, in contrast to the situation when "entirely" is dropped. If the distribution of $(X_1, \cdots, X_n)$ is absolutely continuous, their density has spherical symmetry about the mean when measured in the metric of the covariance matrix. The multivariate Student's t-distribution is an example of a distribution whose regressions are entirely linear.

S. W. Nash (Vancouver, B.C.)

Good, I. J. 5341

On the independence of quadratic expressions.

With an appendix by L. R. Welch.

J. Roy. Statist. Soc. Ser. B **25** (1963), 377–382.

Author's summary: "Let $y_1, y_2, \cdots$ be pairwise independent quadratic expressions in $\mathbf{x}$, not necessarily homogeneous, where $\mathbf{x}$ has a multinormal distribution. Then $y_1, y_2, \cdots$ are also mutually independent. Various conditions for independence are given. In the Appendix, it is shown that $\mathbf{x}$ is a linear transformation of a vector $\mathbf{z}$, where $\mathbf{z}$ has a spherically symmetric distribution, and where the y's depend on disjoint subsets of the components of $\mathbf{z}$."

Khatri, C. G. 5342

Further contributions to Wishartness and independence of second degree polynomials in normal vectors.

J. Indian Statist. Assoc. **1** (1963), 61–70.

Let $X = ((x_{ij}))_{p\times n}$ be normally distributed with $E(X) = \mu$ and $\operatorname{cov}(x_{ij}, x_{i'j'}) = v_{ii'}, w_{jj'}$, and let $V = ((v_{ii'}))_{p\times p}$ and $W = ((w_{jj'}))_{n\times n}$.

In the present paper, the author studies the conditions under which a second-degree polynomial in X has a Wishart distribution when the matrix W is singular. Finally, the author derives a condition for the stochastic independence of two such second-degree polynomials in X under a similar assumption. These results are some generalizations of the results of a previous paper [Ann. Math. Statist. **33** (1962), 1002–1007; MR **25** #4594].

R. G. Laha (Washington, D.C.)

Khatri, C. G. 5343

Joint estimation of the parameters of multivariate normal populations.

J. Indian Statist. Assoc. **1** (1963), 125–133.

In the present paper, the author extends a method of estimation suggested by Fields, Kramer and Clunies-Ross [J. Amer. Statist. Assoc. **57** (1962), 446–454; MR **26** #4438] to the case of parameters of multivariate normal populations having equal mean vectors but different covariance matrices. The author illustrates his method by an example.

R. G. Laha (Washington, D.C.)

Meredith, William 5344

Notes on factorial invariance.

Psychometrika **29** (1964), 177–185.

Author's summary: "Lawley's selection theorem is applied to subpopulations derived from a parent in which the classical factor model holds for a specified set of variables. The results show that there exists an invariant factor pattern matrix that describes the regression of observed on factor variables in every subpopulation derivable by selection from the parent, given that selection does not occur directly on the observable variables and does not reduce the rank of the system. However, such a factor pattern matrix is not unique, which in turn implies that if a simple structure factor pattern matrix can be satisfactorily determined in one such subpopulation the same simple structure can be found in any subpopulation derivable by selection. The implications of these results for 'parallel proportional profiles' and 'factor matching' techniques are discussed.

P. S. Dwyer (Ann Arbor, Mich.)

Meredith, William 5345

Rotation to achieve factorial invariance.

Psychometrika **29** (1964), 187–206.

Author's summary: "Under certain conditions it is reasonable to assume that the same factor pattern matrix will describe the regression of observed on factor scores in different populations. However, ordinary factoring procedures will not reveal in general the existence of such a factor pattern matrix. Two procedures for rotating any number of factor pattern matrices based on different populations to conform to a single 'best fitting' factor pattern matrix are developed in this paper. It is assumed that the same number of factors have been determined for each population. Both procedures will yield oblique results in the various populations. The procedures are illustrated with data taken from the 1939 Holzinger-Swineford monograph. Four groups of individuals are utilized."

P. S. Dwyer (Ann Arbor, Mich.)

Cox, D. R. 5346

Some applications of exponential ordered scores.

J. Roy. Statist. Soc. Ser. B **26** (1964), 103–110.

Author's summary: "There are simple standard tests for the comparison of samples assumed to arise from exponential distributions, but the properties of these tests are known to depend severely on the assumption of exponential form. However, the observations can sometimes usefully be ranked, and then replaced by the corresponding expected values of order statistics in sampling the unit exponential distribution before calculating the appropriate test statistic. Some special tests of this type are examined. The procedure is analogous to the use of Fisher and Yates's scores in normal theory. Savage [Ann. Math. Statist. **27** (1956), 590–615; MR **18**, 243] gave the test of this type for comparing two samples, in the course of a general study of rank tests for the two-sample problem."

Switzer, Paul 5347

Significance probability bounds for rank orderings.

Ann. Math. Statist. **35** (1964), 891–894.

Author's summary: "The problem of two-sample rank order tests is examined from the point of view of I. R. Savage [same Ann. **27** (1956), 590–615; MR **18**, 243; ibid. **28** (1957), 968–977; MR **20** #396; ibid. **30** (1959), 1018–1023; MR **22** #289; Tech. Rep. No. 15, Univ. Minnesota (1962)]. Under suitable restriction of the class of alternatives some rank orderings are always more probable than (dominated by) others. Hence, if the rejection region of a test contains the ordering c, it must also contain all orderings dominated by c if the test is to be admissible. Thus, by counting the number of orderings dominated by c we

arrive at a lower bound for the size of an admissible test which rejects when c is observed. Similar reasoning leads to an upper bound. The counting is achieved by putting all orderings in one-one correspondence with paths on a grid; all paths lying below or along the observed path correspond to the orderings dominated by the observed ordering. An expression for this number of paths is obtained. This expression is used to compute significance bounds for a pair of illustrative examples. Finally, the main result for the two-sample problem is extended to obtain upper and lower bounds for a one-sample problem."

S. S. Gupta (Lafayette, Ind.)

Lindley, D. V. 5348

★Introduction to probability and statistics from a Bayesian viewpoint. Part I: Probability.

Cambridge University Press, New York, 1965. xi + 259 *pp*. $6.00.

This book is an introduction, at a fairly elementary level, to the mathematics of probability. It is designed to lead into a Bayesian treatment of statistical inference, but this fact rarely manifests itself, and the development of the theory, though often refreshing and stimulating, is basically orthodox. It is a book for the student with some mathematical equipment, taking for granted for instance the elementary properties of limits, integrals, and differential equations, but stopping well short of measure theory. The level of rigour is lower than a cursory glance would suggest, difficulties whose resolution would require more mathematics than the reader is likely to have being skilfully ignored.

The first chapter contains a careful discussion of probability as limiting frequency, and uses this to motivate an informal discussion of the axioms. Probability is treated throughout as a function $p(A|B)$ of two arguments, the usual axioms being supplemented by the Rényi axiom $p(C|AB)p(A|B)=p(AC|B)$. After giving the simplest consequences of the axioms, the author discusses the idea of probability as degree of belief (this is really the only important concession to the inclusion of the name of Bayes in the title).

The remainder of the book consists of three chapters, dealing respectively with one random variable, several variables, and stochastic processes. Expectation is defined separately for discrete and continuous distributions, the use of Stieltjes integration being avoided. The usual elementary distributions are discussed, and the Poisson process makes a welcome entry quite early and is used to introduce the Poisson, exponential, and Γ distributions. The simple random walk also comes early in the book, and its diffusion approximation is derived heuristically to illustrate the central limit theorem. One of the attractive features of the author's presentation is the way in which the rather dull manipulations of probability distributions are shot through with applications to the simple stochastic processes which are perhaps the most exciting aspects of probability theory at this level.

The usual results about characteristic functions and sums of random variables are discussed, the deeper ones being left unproved, and there is a good treatment of functions (linear and otherwise) of several random variables. The book ends with an account of processes of birth-and-death type, simple queueing processes, renewal theory (with a glaringly incomplete proof of the Erdős-Feller-Pollard theorem), and Markov chains (discrete in space and time).

Each chapter ends with a good set of exercises, taken largely from examinations set in English universities.

There are, of course, many introductory texts on probability theory. This one has a flavour of its own, which should ensure it a distinctive place in the elementary literature.

J. F. C. Kingman (Cambridge, England)

Lindley, D. V. 5349

★Introduction to probability and statistics from a Bayesian viewpoint. Part II: Inference.

Cambridge University Press, New York, 1965. xiii + 292 *pp*. $6.50.

This volume, which is written on the same general level as the first [see #5348 above], is perhaps the first elementary account of statistical inference developed entirely from a Bayesian (but not decision-theoretic) point of view. Its starting point is the idea of probability as degree-of-belief which made a brief appearance in Part I, and which is taken for granted in this part.

The emphasis throughout is very much on the normal distribution. Thus the first chapter begins by computing the posterior distribution of the mean of a normal distribution with known variance, the prior being assumed normal. This result is extended, elaborated, and interpreted at some length. The case of unknown variance is treated, the χ^2- and t-distributions making their appearance as posterior distributions. A pleasantly simple discussion of sufficiency (including the introduction of the exponential family) leads to an account of the likelihood principle and of significance tests (in the Bayesian sense).

The second chapter pursues the analysis further to deal with the comparison of several normal populations. The F-distribution appears as the posterior distribution of the ratio of two estimates of variance, when means and logarithms of variance have uniform prior distributions. Under the same conditions the Behrens-Fisher problem is easily disposed of, and the chapter ends with a brief introductory discussion of the analysis of variance.

The third chapter is headed "Approximate Methods", which title is held to include maximum likelihood (in large samples the posterior distribution is concentrated about the maximum likelihood estimate), the normal approximation to binomial and Poisson distributions, and the χ^2 technique for goodness-of-fit tests and contingency tables. The final chapter deals with least squares methods, and gives the Bayesian approach to regression and the general linear hypothesis.

In practice the usefulness of this book (Part II, not Part I) will be limited by its Bayesian approach, which would cause complete confusion in the mind of an undergraduate learning statistics from a more "classical" standpoint. Where, however, the Bayesian has driven the classical approach from the field of undergraduate teaching, this book will provide a first-rate introductory text on the problems of inference.

J. F. C. Kingman (Cambridge, England)

Owen, D. B.; Craswell, K. J.; Hanson, D. L. 5350

Nonparametric upper confidence bounds for $\Pr\{Y<X\}$ and confidence limits for $\Pr\{Y<X\}$ when X and Y are normal.

J. Amer. Statist. Assoc. 59 (1964), 906–924.

The authors extend the Birnbaum-McCarty confidence bounds (based on the Wilcoxon statistic) to the case of possibly discontinuous observations. Then, of course, the cases $\Pr(Y<X)$ and $\Pr(Y\leq X)$ must be distinguished. The other parts of the paper deal with normal bivariate samples in situations classified as follows: (i) variances and covariance known; (ii) independence and equal variances, but $\sigma=\sigma_1=\sigma_2$ unknown; (iii) variances and covariance unknown. Six tables are provided. *J. Hájek* (Prague)

Sukhatme, Shashikala 5351

Some non-parametric tests for location and scale parameters in a mixed model of discrete and continuous variables.

J. Indian Soc. Agric. Statist. **14** (1962), 121–137.

The author considers rank statistics used in the median test, the Wilcoxon test, the Mood test, etc. He assumes that the size of the first sample, n, is a random variable with binomial distribution, and computes the overall distribution of the above statistics, accounting for the variability of n. Then he suggests basing the critical values of corresponding tests on these overall distributions. He shows that the tests so modified have smaller asymptotic efficiency than the tests where we regard n as a constant and do not utilize the information about its random origin.

J. Hájek (Prague)

Willke, T. A. 5352

General application of Youden's rank sum test for outliers and tables of one-sided percentage points.

J. Res. Nat. Bur. Standards Sect. B **68B** (1964), 55–58.

From the author's summary: "The rank sum test for outliers advanced by W. J. Youden provides a method for detecting if the measurement distribution of any one of a group of objects has a mean significantly different from the rest. This paper discusses a more general application of the rank sum procedure which permits a similar test on other parameters, such as the variance, with the same tables. Tables of the critical values of the extreme rank sum and the corresponding significance levels for one-sided tests are given in this paper to supplement similar tables for two-sided tests already published."

J. Hájek (Prague)

Nair, C. Ramankutty 5353

A new class of designs.

J. Amer. Statist. Assoc. **59** (1964), 817–833.

The author defines an association scheme which relaxes the condition of symmetry of association between two treatments as given by Bose and Shimamoto [same J. **47** (1952), 151–184; MR **14**, 67]. A careful reader who goes through this paper is bound to get confused with the results, as the paper lacks consistency and contains misstatements and inadequate proofs. The author, in many places, assumes certain properties of his association scheme which are neither contained in his definition nor are consequences of the definition. The model assumed by the author does not point out, in any way, the necessity of introducing his new class of designs.

S. S. Shrikhande (Bombay)

Natrella, Mary Gibbons 5354

★Experimental statistics.

National Bureau of Standards, Handbook 91.

Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1963. xxix + 522 *pp.* (*not consecutively paged*) \$4.25.

This Handbook contains, in a single volume, a collection of statistical procedures that are useful in the design, development, and testing of materials, the evaluation of equipment performance, and the conduct and interpretation of scientific experiments.

The Handbook is presented in five sections: Section 1, Basic concepts and analysis of measurement data; Section 2, Analysis of enumerative and classificatory data; Section 3, Planning and analysis of comparative experiments; Section 4, Special topics, deals with a number of important but as yet nonstandard statistical techniques, and various other special topics; and Section 5 contains the mathematical tables needed for application of the procedures given in the first four sections. These are followed by an index covering all five sections.

Bhat, B. R. 5355

Bayes solution of sequential decision problem for Markov dependent observations.

Ann. Math. Statist. **35** (1964), 1656–1662.

From the author's summary: "After discussing the sequential decision problem in the general case and its Bayes solution as given by Wald, LeCam and others . . ., we give the integral equation satisfied by the Bayes risk. . . . We specialise to Markov dependent observations and . . . give an illustrative example."

H. D. Brunk (Columbia, Mo.)

Bunke, Olaf 5356

Bedingte Strategien in der Spieltheorie: Existenzsätze und Anwendung auf statistische Entscheidungsfunktionen.

Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (*Liblice*, 1962), *pp.* 35–43. *Publ. House Czech. Acad. Sci., Prague*, 1964.

The results are proved among strategies with a given bound on their risk functions, using compactness and lower-semicontinuity conditions similar to those of LeCam [Ann. Math. Statist. **26** (1955), 69–81; MR **16**, 730].

J. Kiefer (Ithaca, N.Y.)

Connor, W. S. 5357

The conditional distribution of sets of tests on a system simulated from tests on its components.

Ann. Math. Statist. **34** (1963), 1585–1587.

Consider a system which is made up of components in such a way that failure of any component causes the failure of the system, and the system cannot fail unless some component fails. A trial for the system is simulated by drawing at random a result for each component. If all component results are successes, the system trial is a success; if not, the system trial is a failure. The process is continued without replacement of component results until n^* system trials have been generated.

In this note the conditional distribution of S, the

number of successful system trials, given n_{i1}, the number of successes in the n_i trials for the ith component $(i=1, 2, \cdots, k)$, has been derived.

The expected value of S is shown to be equal to the product of the $\hat{p}_i$'s, where $\hat{p}_i = n_{i1}/n_i$, the estimated reliability of the ith component. *A. K. Gayen* (Kharagpur)

Hudimoto, Hirosi 5358

On the classification. I. The case of two populations. (Japanese. English summary)

Proc. Inst. Statist. Math. 11 (1963/64), 31–38.

The author considers the use of Mann-Whitney statistics for classifying a sample into either of two populations π_1 and π_2. The distribution functions $F_1(x^{(1)})$ of π_1 and $F_2(x^{(2)})$ of π_2 are known, and $(X_1, \cdots, X_n)$ is a random sample of size n from π_1 or π_2. He uses the statistics $\hat{p}_1 = (1/n)\sum_{k=1}^{n} F_1(X_k)$ and $\hat{p}_2 = 1-(1/n)\sum_{k=1}^{n} F_2(X_k)$, where $\hat{p}_1$ is an unbiased estimate of $\int_{-\infty}^{\infty} F_1(t)\,dF_2(t)$ when $(X_1, \cdots, X_n)$ is a sample from π_2 and an estimate of $\frac{1}{2}$ when $(X_1, \cdots, X_n)$ is a sample from π_1, $\hat{p}_2$ is an unbiased estimate of $\frac{1}{2}$ when $(X_1, \cdots, X_n)$ is a sample from π_2 and an estimate of $\int_{-\infty}^{\infty} F_1(t)\,dF_2(t)$ when $(X_1, \cdots, X_n)$ is a sample from π_1.

The author presents the following classification procedure: If $F_1(t) > F_2(t)$ for all t, classify $(X_1, \cdots, X_n)$ into π_1 or π_2 according as $\hat{p}_1 < \hat{p}_2$ or $\hat{p}_1 > \hat{p}_2$, respectively. In this case, he gives an evaluation of the success rate and then shows that the method is also applicable for the non-parametric case where these distribution functions are estimated from samples. *C. Hayashi* (Tokyo)

Kincaid, W. M. 5359

The combination of $2 \times m$ contingency tables.

Biometrics 18 (1962), 224–228.

It is supposed that at each of k levels $(x_1, \cdots, x_k)$ of a stimulus x and for each of m categories of a qualitative factor y, a fixed number n of individuals are assayed in respect of a yes/no response. It is noted that the hypothesis that the stimulus-response pattern is the same in all y-categories may be tested by adding the appropriate chi-squares of the $2 \times m$ contingency tables for each level of stimulus but that the resulting test is not particularly sensitive to systematic differences of stimulus-response pattern as between the different y-categories. It is suggested that a more sensitive test may be obtained by testing the total number of favourable responses in each y-category for heterogeneity. Appropriate formulae are given for this, and the extension to cases where different numbers of individuals are assayed in each (x, y) category. A rather inconclusive discussion in general terms is given of the circumstances where the corresponding test statistic may be considered to have, approximately, a chi-squared distribution under the null hypothesis. There is no mention of the standard methods for comparison between stimulus-response curves which, on the face of it, would seem to be optimal in these circumstances [see, e.g., D. J. Finney, *Statistical method in biological assay*, Hafner, New York, 1952]. *D. E. Barton* (London)

Walker, A. M. 5360

Large-sample estimation of parameters for autoregressive processes with moving-average residuals.

Biometrika 49 (1962), 117–131.

The author is concerned with the maximum likelihood estimation of parameters in the model

$$\sum_{j=0}^{p} \alpha_j(x_{t-j}-\mu) = \sum_{k=0}^{q} \beta_k \varepsilon_{t-k},$$

where the ε_t are NID$(0, \sigma^2)$, and the α_j are such that the x_t process is stationary. He generalises the methods of a previous paper [Biometrika 48 (1961), 343–357; MR 25 #728] to this situation.

The quantities $\alpha_1, \alpha_2, \cdots, \alpha_p$; $\rho_1, \rho_2, \cdots, \rho_q$ are taken as the set of parameters to be estimated (ρ_s being the auto-correlation coefficient of $\log s$), and it is shown that estimation equations can be written asymptotically in the form

$$\hat{\rho}_i = \hat{\bar{r}}_i \qquad (1 \leqq i \leqq q),$$

$$\sum_{j=0}^{p} \hat{\alpha}_j \hat{\bar{r}}_{i-j} = 0 \qquad (q+1 \leqq i \leqq q+p),$$

where $\bar{r}_i$ is the residual obtained by subtracting from the ith sample auto-correlation coefficient its asymptotic regression on the set of variables $(S_{q+p+1}, S_{q+p+2}, \cdots, S_k)$ and $S_l = \sum_{i,j=0}^{p} \alpha_i \alpha_j r_{l-i-j}$.

The author demonstrates a convenient iterative procedure for solution of the estimation equations and determines several asymptotic properties of the estimates. *P. Whittle* (Manchester)

Hannan, E. J. 5361

The estimation of seasonal variation in economic time series.

J. Amer. Statist. Assoc. 58 (1963), 31–44.

The author assumes the usual additive model $y_t = p_t + s_t + x_t$, where p_t, s_t and x_t represent "trend", "seasonal trend" and "stationary component", respectively. He is principally concerned with the effect of trend-removing operators upon the seasonal component, and shows that the effect of such operations can be allowed for, so that the seasonal component of the original series can be deduced from that estimated for the trend-adjusted series. The computation is reduced to a routine, which is illustrated numerically. The estimation of a slowly evolving seasonal is also considered. *P. Whittle* (Manchester)

Herbst, Laurence J. 5362

Periodogram analysis and variance fluctuations.

J. Roy. Statist. Soc. Ser. B 25 (1963), 442–450.

Author's summary: "Two problems in the subject of time-series analysis are considered: (A) the effect of fluctuating variances on the periodogram of a set of independent Gaussian data with zero means; (B) the possibility of using the periodogram of squares of independent Gaussian data with zero means as a tool in the Fourier analysis of the variances of the data. The basic effect of fluctuating variances on the periodogram is to introduce correlations (expressible in terms of the Fourier coefficients of the variance) which do not decrease as the sample size increases. The periodogram of squares of independent Gaussian data with zero means is shown to be an appropriate tool in the Fourier analysis of the variances."

Comer, John P., Jr. 5363

Some stochastic approximation procedures for use in process control.

Ann. Math. Statist. **35** (1964), 1136–1146.

Let M be a measurable function on the real line. Assume that a chemical process, for instance, can be described as $Y_n = M(X_n) + Z_n$, $n \geq 1$, where X_n, Y_n, Z_n are random variables. Let Y_0 be a given real number. The random variables X_n, $n \geq 1$, describe a control procedure which is chosen in such a way as to keep $E(Y_n - Y_0)^2$ small. X_{n+1} may depend on the prior values X_i, Y_i, $1 \leq i \leq n$. Let $A < B$ and $0 < g < h$. Suppose that

$$g \leq (M(x) - M(x'))/(x - x') \leq h \quad \text{for} \quad A \leq x < x' \leq B.$$

Furthermore, assume that there exists a θ in the interval $[A, B]$ such that $M(\theta) = Y_0$. A control procedure is defined which is very similar to the Robbins-Monro standard approximation procedure [same Ann. **22** (1951), 400–407; MR **13**, 144]. Let

$$\begin{aligned} i(x) &= A, \qquad x < A, \\ &= x, \qquad A \leq x \leq B, \\ &= B, \qquad B > x. \end{aligned}$$

Specify a value of X_1 in $[A, B]$ and define $X_{n+1} = i(X_n - a_n(Y_n - Y_0))$, $n \geq 1$, where a_n is a sequence of positive numbers satisfying $\lim_{n\to\infty} a_n = a$, $\sum_{n=1}^{\infty} a_n = \infty$. Under the further condition $0 \leq a < 1/h$ and

$$E\{[E(Z_n | Z_1, \cdots, Z_{n-m})]^2\} = \sigma_m^{\ 2} \to 0 \quad \text{as} \quad m \to \infty,$$

upper bounds for

$$\limsup_{n\to\infty} E(X_n - \theta)^2 \quad \text{and} \quad \limsup_{n\to\infty} E(Y_n - Z_n)^2$$

are derived which converge to 0 as a goes to 0. Some more results of a similar kind are mentioned without proofs. For them the author refers to his Doctoral Dissertation [Columbia Univ., New York, 1962].

L. Schmetterer (Vienna)

Kubo, Ryogo 5364

Generalized cumulant expansion method.

J. Phys. Soc. Japan **17** (1962), 1100–1120.

Author's summary: "The moment generating function of a set of stochastic variables defines the cumulants or the semi-invariants and the cumulant function. It is possible, simply by formal properties of exponential functions, to generalize to a great extent the concepts of cumulants and cumulant function. The stochastic variables to be considered need not be ordinary c-numbers but they may be q-numbers such as used in quantum mechanics. The exponential function which defines a moment generating function may be any kind of generalized exponential, for example, an ordered exponential with a certain prescription for ordering q-number variables. The definition of average may be greatly generalized as far as the condition is fulfilled that the average of unity is unity. After statements of a few basic theorems, these generalizations are discussed here with certain examples of application. This generalized cumulant expansion provides us with a point of view from which many existent methods in quantum mechanics and statistical mechanics can be unified."

NUMERICAL METHODS

See also 4703, 4767, 4768, 4874, 4876, 4914, 4915, 4916, 4917, 4918, 4927, 4931, 4960, 5058, 5261, 5327, 5435, 5514, 5639, 5674.

★Problems in numerical mathematics and computing technology [Вопросы вычислительной математики и вычислительной техники]. 5365

Edited by L. A. Ljusternik.

Gosudarstv. Naučno-Tehn. Izdat. Mašinostr. Lit., Moscow, 1963. 432 *pp*. 1.55 *r*.

The contributions in this volume are divided into three main sections: Numerical methods and applied mathematics (Part I); Digital computers (Part II); Analog computers (Part III). The longer papers in Part I will be reviewed or listed individually.

Frank, Evelyn 5366

Newton's formula and approximants of continued fraction expansions.

Univ. Lisboa Revista Fac. Ci. A (2) **10** (1963), 75–89.

The present paper supplements the author's earlier work [Numer. Math. **4** (1962), 85–95; MR **25** #3603; ibid. **4** (1962), 303–309; MR **26** #2765], which generalised results of J. Mikusiński and the reviewer [Mikusiński, Ann. Polon. Math. **1** (1954), 184–194; MR **15**, 954; Sharma, ibid. **6** (1959), 295–300; MR **21** #5621]. Let $\sqrt{(C+L)}$ be a binomial quadratic surd (C, L rational, $C > 0$) with the continued fraction expansion of period p. Consider the extended Newton's formula

$$(*) \qquad x_n = \frac{x_k \cdot x_l + C - L^2}{x_k + x_l - 2L}, \qquad n > k, n > l,$$

and let A_r/B_r denote the rth approximant of the continued fraction expansion. The author shows that if in (*) x_k and x_l are the kth and lth approximants of the continued fraction, then x_n is, in general, a mixed approximant $(\alpha_r A_r + \alpha_{r-1} A_{r-1})/(\beta_r B_r + \beta_{r-1} B_{r-1})$. Explicit forms of α_r, β_r are obtained. The case for symmetric regular periodic continued fraction expansions is also considered.

A. Sharma (Calgary, Alta.)

Hoel, P. G.; Levine, A. 5367

Optimal spacing and weighting in polynomial prediction.

Ann. Math. Statist. **35** (1964), 1553–1560.

The problem is the determination of observation points in the interval $[-1, 1]$, and the proportion of observations at each point, in order to minimize the variance of the predicted value of a polynomial regression curve of degree k at the specified point, x, beyond the interval. The results indicate that the desired $k+1$ observation points are given by the Chebyshev points, $x_i = -\cos i\pi/k$, $i = 0, 1, \cdots, k$, and that the number of observations should be proportional to the absolute value of the Lagrange polynomial. This Chebyshev solution becomes the minimax solution for the interval $(-1, t)$ if $t > t_1 > 1$ and t_1 satisfies a certain equation. Under the assumption of normality, this solution can be used to construct a confidence bound (for the polynomial curve) which possesses minimum width at x.

P. S. Dwyer (Ann Arbor, Mich.)

Podolsky, Boris; Denman, Harry H. 5368

Conditions on minimization criteria for smoothing.

Math. Comp. **18** (1964), 441–448.

Given a set of data points $\{Y_i(x_i)\}$, $i=1, 2, 3, \cdots, n$, one often seeks the "smoothest" curve $y(x)$ passing through or near the points; the criterion for smoothness is that the integral of some function $f(y, y', y'')$ be a minimum. However, the choice of units and scale are often governed by circumstances having little to do with the intrinsic mathematics of the situation. Consequently, the smoothness criterion should be invariant under change of scale or shift of origin.

The authors find that a necessary and sufficient condition that the smoothest curve $y(x)$ become $ay+b$ when Y_i is replaced by aY_i+b is that f be of the form

$$f = W(y', y'') + \partial Q(y, y', x)/\partial x,$$

where W is a homogeneous function of y' and y'', not linear in y''. *W. Langlois* (San Jose, Calif.)

Ul'm, S. 5369

A majorant principle and the method of secants. (Russian. Estonian and English summaries)

Eesti NSV Tead. Akad. Toimetised Füüs.-Mat. Tehn.-tead. Seer. **13** (1964), 217–227.

Author's summary: "Using the principle of majorants, some theorems on the convergence of the secant method are proved in linear normed spaces. The results of J. W. Schmidt [Z. Angew. Math. Mech. **41** (1961), Sonderheft, T61–T63; ibid. **43** (1963), 1–8; MR **26** #5442] and A. S. Sergeev [Sibirsk. Mat. Ž. **2** (1961), 282–289; MR **24** #A377] are made more precise. The theorems proved may also be considered as generalizations of the theorems on the convergence of Newton's method."

Fil'čakova, V. P. 5370

On a numerical method for the conformal mapping of exterior simply connected domains. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 1127–1132.

Author's summary: "An effective numerical method is proposed for conformal mapping of the exterior of a unit circle into the exterior of a given simply connected domain. It is based on trigonometrical interpolation. The method makes it possible to find the coefficients of the mapping polynomial with any preassigned degree of accuracy. The contour may be given analytically, graphically or tabularly."

Oettli, W.; Prager, W. 5371

Compatibility of approximate solution of linear equations with given error bounds for coefficients and right-hand sides.

Numer. Math. **6** (1964), 405–409.

Given an approximate solution x^0 to the linear system $Ax=b$, and assigned tolerances $\Delta A \geq 0$ and $\Delta b \geq 0$, it is asked under what circumstances there exist δA and δb, with $|\delta A| \leq \Delta A$, $|\delta b| \leq \Delta b$, such that $(A+\delta A)x^0 = b+\delta b$ exactly (absolute value signs and inequalities are to be interpreted elementwise). The argument is unnecessarily involved, since if $r = Ax^0 - b$, then it is necessary and sufficient that $|r| \leq \Delta b + (\Delta A)|x^0|$. The question is relevant to J. H. Wilkinson's backward error analysis.

A. S. Householder (Oak Ridge, Tenn.)

Balogh, L.; Békésy, A.; Fáy, Gy. 5372

Use of a matrix factorization method to some problems of dimensional analysis. (German, French, and Russian summaries)

Acta Tech. Acad. Sci. Hungar. **48** (1964), 241–251.

Authors' summary: "An algorithm is given for determining criteria when using the tools of dimensional analysis. The algorithm is based on a paper by E. Egerváry [Z. Angew. Math. Mech. **35** (1955), 111–118; MR **16**, 1156]. The method presented is, in fact, a modified version of an earlier method for decomposing matrices into dyads. Examples."

La Budde, C. Donald 5373a

Two new classes of algorithms for finding the eigenvalues and eigenvectors of real symmetric matrices.

J. Assoc. Comput. Mach. **11** (1964), 53–58.

Kaiser, Henry F. 5373b

A method for determining eigenvalues.

J. Soc. Indust. Appl. Math. **12** (1964), 238–248.

The methods of both papers use a sequence of unitary transformations of a given hermitian matrix such that the (1, 1)-element is monotonically increasing (or decreasing). They are, in fact, closely related. Following Householder [*The theory of matrices in numerical analysis*, p. 201, problem 17, Blaisdell, New York, 1964], a unified treatment can be given as follows: Let a hermitian matrix be $(1, n-1)$-partitioned in the form $A = \begin{pmatrix} \beta & b^H \\ b & B \end{pmatrix}$. Let u be a unit vector of dimension $n-1$. The 2-by-2 matrix $\bar{A} = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} A \begin{pmatrix} 1 & 0 \\ 0 & u^H \end{pmatrix} = \begin{pmatrix} \beta & b^H u \\ u^H b & u^H B u \end{pmatrix}$ has a larger root $\bar{\lambda}_1$, which is greater than β, provided $b^H u \neq 0$ (and a smaller root $\bar{\lambda}_2$, which is less than β), $\beta < \bar{\lambda}_1 \leq \lambda_1(A)$. Therefore, there exists a unit vector r and a number β' such that $\beta < \beta' = r^H \bar{A} r \leq \bar{\lambda}_1 \leq \lambda_1(A)$. Selecting such an r and forming $v = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} r$, we obtain $\beta' = v^H A v$. Thus, if a unitary matrix V contains such a v as its first column, the (1, 1)-element of the transformed matrix $A^{(1)} = V^H A V$ equals β', $\beta' > \beta$.

The choice of u and r is at our disposal and also the computational process for forming $A^{(1)}$ from A, that is, the selection of V. Kaiser economizes, by the choice of u, the computational work involved in forming $b^H u$ and $u^H B u$, by defining $u_i = \operatorname{sign} b_i/\sqrt{(n-1)}$. La Budde follows the tactical advantage of maximizing $|b^H u|$ by choosing $u = b/\|b\|$. Kaiser chooses for r the eigenvector of $\bar{A}$ belonging to $\bar{\lambda}_1$ (or to $\bar{\lambda}_2$). La Budde does so, too, but mentions also that convergence holds even if this absolute maximum for β' is not reached, and considers even more economical choices of r. Kaiser's computation is straightforward as outlined above; $\bar{A}$, r, v are constructed, V is a single elementary Householder matrix (hermitian, unitary), which is uniquely determined from v, which is its first column, $V = I - (e_1 - v)(e_1 - v)^H/(1 - e_1{}^T v)$, where $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(in order to ensure stability, the sign of V should be chosen such that $e_1{}^T v<0$). La Budde performs the calculation in two steps. In a first step, with u proportional to b, he applies the usual Householder transformation $\begin{pmatrix}1 & 0\\ 0 & Q\end{pmatrix} A \begin{pmatrix}1 & 0\\ 0 & Q\end{pmatrix} = A^*$ which annihilates $b_{13}, b_{14}, \cdots, b_{1n}$. (By using the representation

$$Q = I+(u-\ell e)(u-\ell e)^H/\ell\cdot(e^H u-\ell),$$

where $\|e\|=1$ (in fact, e is the second axis vector) and

$$\ell = -\|u\| \quad \text{if } e^H u \geq 0,$$
$$\;\;= +\|u\| \quad \text{if } e^H u < 0,$$

only one square root is to be formed; see, for example, the reviewer [J. Soc. Indust. Appl. Math. 7 (1959), 107–113; MR 20 #6778].) Since $Qu=\ell e$ and $Qe=u/\ell$, $\begin{pmatrix}1 & 0\\ 0 & e^H\end{pmatrix} A^* \begin{pmatrix}1 & 0\\ 0 & e\end{pmatrix} = \tilde{A}$. Next, he performs a Jacobi rotation of the two axes $\begin{pmatrix}1\\ 0\end{pmatrix}$, $\begin{pmatrix}0\\ e\end{pmatrix}$ on A^*, which gives $A^{(1)}$.

Limited experience shows that the Kaiser-La Budde method may well compare favorably in convergence speed and computational work with the best-known methods for determining eigenvalues, especially if only a few eigenvalues are wanted. *F. L. Bauer* (Munich)

Martin, D. W.; Tee, G. J. 5374

Iterative methods for linear equations with symmetric positive definite matrix.

Comput. J. 4 (1961/62), 242–254.

Authors' summary: "The salient features of iterative methods commonly encountered in the literature are described, to provide a self-contained résumé of recent developments and to serve as an introduction to detailed study of particular topics. The discussion is organized under two main headings: (i) Stationary methods and their linear acceleration; (ii) Gradient methods and kernel polynomials. A tentative evaluation of the various methods is made on the basis of recent experiments."

Ortega, James M. 5375

Generation of test matrices by similarity transformations.

Comm. ACM 7 (1964), 377–378.

Author's summary: "A method for obtaining test matrices with a prescribed distribution of characteristic roots is given. The process consists of using particularly simple similarity transformations to generate full matrices from canonical foms. The matrices generated also have known characteristic vectors, inverses and determinants."

Stojaković, M. 5376

Inversion of matrices appearing in the application of the method of least squares. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. 4 (1964), 911–915.

More specifically, this has to do with fitting polynomials by least squares. The matrix of coefficients of the equations to be solved can be written $W_n=\sum X_i X_i'$, where $X_i'=(1, x_i, x_i^2, \cdots, x_i^m)$. In particular, W_{m+1} is the product of two Vandermonde matrices, and the author has treated the inversion of Vandermonde matrices in a previous paper [C. R. Acad. Sci. Paris 246 (1958), 1133–1135; MR 19, 1079]. He now proposes to use a recursion

$$W_{n-1}^{-1} = W_n^{-1}-g_n^{-1}W_n^{-1}X_n X_n' W_n^{-1}$$

[see, e.g., the reviewer's *Principles of numerical analysis* p. 79, McGraw-Hill, New York, 1953; MR 15, 470]. *A. S. Householder* (Oak Ridge, Tenn.)

Teterev, A. G. 5377

Positive matrix invertibility and the method of monotone approximations. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika 1964, no. 3 (40), 143–150.

Some conditions are given on a matrix A that guarantee that all elements of A^{-1} are positive (positive invertibility). Counterexamples are given to show the sharpness of the conditions, and finally the notion of positive invertibility is generalized in an interesting way. Let N be the set $\{1, \cdots, n\}$; let I, J be subsets of N. Let X_I be the set of vectors $x=(x_1, \cdots, x_n)$ such that $x_i>0$ for $i\in I$, $x_i\leq 0$ for $i\notin I$. The matrix A is of type (I, J) if $AX_I\subset X_J$. The type (N, N) corresponds to positive invertibility. An example is the matrix (a_{ij}) with dominant diagonal and $a_{ii}>0$, $a_{ij}\leq 0$ $(j\neq i)$.

Finally, the author gives conditions for the monotone convergence of (upper and lower) approximations to solutions of nonlinear sets of equations [see L. Collatz and J. Schröder, Numer. Math. 1 (1959), 61–72; MR 21 #1688]. The excessively condensed style partly conceals a logical, attractive theory. *J. L. Brenner* (Palo Alto, Calif.)

Ting, Thomas C. T. 5378

A method of solving a system of linear equations whose coefficients form a tridiagonal matrix.

Quart. Appl. Math. 22 (1964), 105–116.

The author describes a method of solving a system of equations $Ax=d$ when A is tri-diagonal. It is based on the decompositions $A=L_1U_1$ and $A=U_2L_2$, where L_1 is lower-triangular, U_1 is unit upper-triangular, U_2 is upper-triangular and L_2 is unit upper-triangular. It is shown that $A=L_1+U_2-W$, where W is a diagonal matrix. Hence $W=L_1+U_2-A=A(U_1{}^{-1}+L_2{}^{-1}-I)$, showing that $A^{-1}=(U_1{}^{-1}+L_2{}^{-1}-I)W^{-1}$. The matrices U_1 and L_2 have only one super-diagonal line and sub-diagonal line respectively; the matrices $U_1{}^{-1}$ and $L_2{}^{-1}$ are therefore not computed explicitly. The method has advantages when solutions corresponding to several right-hand sides d are required. *James H. Wilkinson* (Teddington)

Kizner, William 5379

A numerical method for finding solutions of nonlinear equations.

J. Soc. Indust. Appl. Math. 12 (1964), 424–428.

As the title states, this paper presents a numerical method for finding solutions of nonlinear equations. If z_1 is an approximation for a root z of $f(z)=0$, then

$$z = z_1+\int_{f(z_1)}^{0} \frac{dz}{df}\,df.$$

The described method obtains a new approximation for z by performing the indicated integration numerically.

using the Runge-Kutta method. It is proved that under certain conditions, the order of convergence of the method is 5. That is, if e_k is the error after the kth step, then e_{k+1}/e_k^5 tends to a constant as $k\to\infty$.

Making certain reasonable assumptions, one finds that, asymptotically, the accuracy gained per unit amount of work is about the same for this method as for Newton's method. The author states, however, that in practice this method yields better results in the initial step which starts with a crude approximate root. He suggests using one step of this method followed by the use of Newton's method.

The method is easily extended for obtaining solutions of systems of equations. No numerical results are given.

E. R. Hansen (Palo Alto, Calif.)

Lancaster, P. 5380

Convergence of the Newton-Raphson method for arbitrary polynomials.

Math. Gaz. **48** (1964), 291–295.

The author gives an elementary account "of the convergence properties of the sequence of iterates obtained by applying the Newton-Raphson process to an arbitrary polynomial". In order to keep the treatment elementary, it is necessary to confine attention to polynomials, but the coefficients may take complex values; furthermore, it is possible to deal with multiple roots. Although the results obtained are not original, the methods used to derive them are novel.

G. N. Lance (Canberra)

Barrucand, Pierre 5381

Quadratures numériques, fonctions elliptiques et facteur de convergence.

C. R. Acad. Sci. Paris **258** (1964), 2742–2744.

This note gives an accurate procedure for the numerical evaluation of integrals of the form

$$\int_0^1 F(x)(1-x^2)^{-1/2}\,dx$$

when $F(x)$ is even, analytic and has no singularities on $[-1,+1]$. The integration formula is a weighted sum involving elliptic functions.

R. C. MacCamy (Pittsburgh, Pa.)

Ermakov, S. M. 5382

Stochastic quadratures of increased accuracy. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 550–554.

In a previous paper [Teor. Verojatnost. i Primenen. **5** (1960), 473–476; MR **24** #889] the author and Zolotunin proposed a general method for approximating multiple integrals by a new Monte Carlo technique. The author now continues the research by considering quadrature formulas of a high degree of precision which use random nodes.

Suppose f belongs to the space of square summable functions in a domain D of k-dimensional Euclidean space, and let φ_m $(m=0,1,\cdots,N)$ be a system of orthonormal functions in D. For approximating the integral $\alpha_j=\int_D \varphi_j(Q)f(Q)dQ$ one uses the sum $K_{n,j}(f)=\sum_{i=0}^{n} A_i(Q_0,Q_1,\cdots,Q_n)f(Q_i)$, where $Q_i\in D$. Let the quadrature formula $\alpha_j=K_{n,j}(f)$ be exact for the functions $\varphi_m(Q)$ $(m=0,1,\cdots,N)$ when the nodes Q_i are random points in D with a given probability denstiy, and let the mathematical expectation of $K_{n,j}(f)$ be α_j. Using the Fourier expansion formula for the function $f(Q)$, one finds the variance $D[K_{n,j}(f)]$. The quadrature formulas with smaller variance present a special interest, i.e., those for which $n<N$ and $D[K_{n,j}(\varphi_m)]<1$. The coefficients $A_m(Q_0,Q_1,\cdots,Q_n)$, for $n=N$, can be uniquely determined by comparatively weak restrictions imposed on the dispositions of the points Q_i. In this case Q_i should satisfy the unique condition $\det\|\varphi_j(Q_i)\|_0^n\neq 0$. When $n<N$, the nodes Q_i must, in addition, satisfy the condition

$$\det\|\varphi_m(Q_i)\varphi_0(Q_i)\cdots\varphi_{j-1}(Q_i)\varphi_{j+1}(Q_i)\cdots\varphi_n(Q_i)\|_0^n = 0.$$

The corresponding quadrature formulas are similar to the Gaussian numerical integration formulas.

Several examples are presented in detail.

D. D. Stancu (Cluj)

Hara, I. S. 5383

On a method of constructing the Hermite interpolation formula and on quadrature formulas for solving boundary-value problems and integral equations. (Russian)

Dokl. Akad. Nauk SSSR **141** (1961), 822–825.

Levin, M. 5384

On best quadrature formulae with fixed nodes. (Russian. Estonian and English summaries)

Eesti NSV Tead. Akad. Toimetised Füüs.-Mat. Tehn.-tead. Seer. **13** (1964), 110–114.

Let $W_0^{(1)}L^{(2)}$ be the class of absolutely continuous functions $f=f(x)$ on $[0,1]$ such that $f(0)=0$ and $\|f'(x)\|\le M$ when $f'(x)\in L^{(2)}$. Consider the quadrature formulas

$$(1)\qquad I(f) = \sum_{k=0}^{n-1} p_k f(x_k)+r_n(f),$$

$$(2)\qquad I(f) = \sum_{k=0}^{n-1} p_k f(x_k)+\sum_{j=0}^{r-1} B_j f(y_j)+R_n(f),$$

where $I(f)=\int_0^1 f(x)dx$ and p_k, x_k and y_j are given numbers. Let $r_n=\sup|r_n(f)|$, $R_n=\sup|R_n(f)|$ when $f\in W_0^{(1)}L^{(2)}$.

The author is concerned with the problem of finding the best quadrature formula (2) when f belongs to the class of functions considered, i.e., the problem of determining the coefficients B_j so that R_n is the smallest. In an earlier paper G. Ja. Doronin [Sb. Naučn. Trudy Dnepropetrovsk. Inž.-Inst. No. 1–2 (1955), 210–217] has shown that the best quadrature formula (1) for this class of functions is obtained if $p_k=2/(2n+1)$, $x_k=(2k+2)/(2n+1)$, $r_n=M/(2n+1)\sqrt{3}$.

Let (1) be the best quadrature formula. The author shows that in the class of functions considered, the best quadrature formula (2), for which $r=n$, $y_k=0.5(x_{k-1}+x_k)$ $(k=0,1,\cdots,n-1;\ x_{-1}=0)$, is the following:

$$I(f) = \frac{2}{2n+1}\sum_{k=0}^{n-1} f\left(\frac{2k+2}{2n+1}\right)+\frac{1}{2(2n+1)}\,f\left(\frac{1}{2n+1}\right)+R_n(f),$$

with

$$R_n = \frac{M}{(2n+1)\sqrt{3}}\sqrt{\frac{8n+1}{8n+4}}.$$

A similar problem is considered when (1) is the iterated Simpson formula and when f is absolutely continuous on $[0, 1]$, whereas $\|f'(x)\| \leq M$ when $f'(x) \in L^{(2)}$.
D. D. Stancu (Cluj)

Day, J. T. 5385
A one-step method for the numerical solution of second order linear ordinary differential equations.
Math. Comp. 18 (1964), 664–668.
The author combines the Lobatto four-point quadrature formula with Hermite interpolation to obtain a one-step method of order six (local truncation error $O(h^7)$) for the numerical integration of linear second-order differential equations. In three examples the method is compared with less accurate methods of Runge-Kutta, Numerov, and the reviewer. *Walter Gautschi* (Lafayette, Ind.)

Hao Šou [Šou, Hao] 5386
A Sturm-Liouville difference problem for a fourth-order equation with discontinuous coefficients. (Russian)
Ž. Vyčisl. Mat. i Mat. Fiz. 3 (1963), 1014–1031.
The difference scheme previously studied by the author [same Ž. 3 (1963), 841–860; MR 28 #724] is used to obtain the eigenvalues and eigenfunctions of the problem $(k(x)u'')''-(p(x)u')'+q(x)u=\lambda r(x)u$, $u(0)=u'(0)=u(1)=u'(1)=0$. Estimates are derived for the error in the nth eigenvalue and corresponding eigenfunction. In the case of smooth coefficients the error is $O(h^2)$ (h is the grid size), and in the case of piecewise smooth coefficients the error is $O(h)$, except that in certain cases (an example is given) the error may be $O(h^2)$ also for piecewise smooth coefficients. *R. C. Gilbert* (Fullerton, Calif.)

Holt, James F. 5387
Numerical solution of nonlinear two-point boundary problems by finite difference methods.
Comm. ACM 7 (1964), 366–373.
Author's summary: "Solution of nonlinear two-point boundary-value problems is often an extremely difficult task. Quite apart from questions of reality and uniqueness, there is no established numerical technique for this problem. At present, shooting techniques are the easiest method of attacking these problems. When these fail, the more difficult method of finite differences can often be used to obtain a solution. This paper gives examples and discusses the finite difference method for nonlinear two-point boundary-value problems."

Maistrov'skiĭ, G. D. 5388
On the applicability of the sweep method. (Ukrainian. Russian and English summaries)
Dopovidi Akad. Nauk Ukraïn. RSR 1964, 449–451.
It is shown that the sweep method will be applicable to the computation of the solution of a certain difference analogue to the two-point problem

$$\{-(py')'+qy=f;\ y'-H_0y|_0=a,\ y'-H_1y|_l=b\}$$

for all sufficiently small h if and only if the operator A:

$$[Ay=-(py')'+qy \text{ for } y|_0=0,\ y'+H_1y|_l=0]$$

is definite. *T. I. Seidman* (Seattle, Wash.)

McGill, Robert; Kenneth, Paul 5389
Solution of variational problems by means of a generalized Newton-Raphson operator.
AIAA J. 2 (1964), 1761–1766.
Authors' summary: "This paper presents the development of an indirect method for solving variational problems by means of an algorithm for obtaining the solution to the associated nonlinear two-point boundary-value problem. The method departs from the usual indirect procedure of successively integrating the nonlinear equations and adjusting arbitrary initial conditions until the remaining boundary conditions are satisfied. Instead, an operator is introduced which produces a sequence of sets of functions that satisfy the boundary conditions but, in general, do not satisfy the nonlinear system formed by the state equations and the Euler-Lagrange equations. Under appropriate conditions, this sequence converges uniformly and rapidly (quadratically) to the solution of the nonlinear boundary-value problem. The computational effectiveness of the algorithm is demonstrated by three numerical examples."

Storey, C.; Rosenbrock, H. H. 5390
On the computation of the optimal temperature profile in a tubular reaction vessel.
Computing Methods in Optimization Problems (Proc. Conf., Univ. California, Los Angeles, Calif., 1964), pp. 23–64. *Academic Press, New York*, 1964.
Given the system $\dot{x}=Fx$, $x(t_0)=x_0$, and F is a 3×3 matrix with $F_{ij}=ae^{-(b/u+c)}$, where a, b, c are given constants, find the control program $u(t)$ (temperature profile) such that $x_3(T)$ is maximized. Five numerical methods were studied experimentally in connection with the solution of this problem: (i) discrete approximation and conversion to a straight hill-climbing problem of many variables, (ii) Rayleigh-Ritz method using ramps and exponentials, (iii) direct attempt at solving the two-point boundary-value problem by guessing the missing multipliers, (iv) gradient method, which is not really different from (i), and (v) dynamic programming. It was found that (i) and (iv) worked best. *Y.-C. Ho* (Cambridge, Mass.)

Tihonov, A. N.; Samarskiĭ, A. A. 5391
Homogeneous difference schemes. (Russian)
Ž. Vyčisl. Mat. i Mat. Fiz. 1 (1961), 5–63.
The authors summarize results obtained in their work between 1956 and 1960 [Dokl. Akad. Nauk SSSR 122 (1958), 562–565; MR 23 #B1656a; ibid. 122 (1958), 188–191; MR 20 #6651; ibid. 108 (1956), 393–396; MR 18, 938; ibid. 124 (1959), 529–532; MR 23 #B1656b; Uspehi Mat. Nauk 14 (1959), no. 3 (87), 185–188; Dokl. Akad. Nauk 124 (1959), 779–782; MR 25 #4642; ibid. 131 (1960), 514–517; MR 27 #6392; ibid. 131 (1960), 761–764; MR 27 #6393; ibid. 131 (1960), 1264–1267; MR 27 #6394], where they studied homogeneous schemes for the solution of the first boundary-value problem

$$\text{(1)}\qquad L^{(k,q,f)}u=[k(x)u']'-q(x)u+f(x)=0,$$

$$0<x<1,\ u(0)=\bar{u}_1,\ u(1)=\bar{u}_2,$$

whose coefficients k, q, f are piecewise continuous functions with $k(x)\geq M>0$, $q(x)\geq 0$. In § 1 they characterize the family of difference schemes for the differential equation (1) in the class $Q^{(0)}$ of piecewise continuous coefficients. In

§ 2 they study problems connected with the convergence and accuracy of homogeneous difference schemes in the class of smooth coefficients $C^{(m)}$. In § 3 they study the convergence of a certain solution of the difference equation to the solution of equation (1), and give a certain necessary condition for this convergence. In § 4 they introduce a perturbation norm for the coefficients of a scheme and define what they refer to as the co-stability of a difference scheme. They show that a necessary and sufficient condition for the co-stability of a canonical scheme is that it be conservative. In § 5 the authors discuss questions relating to convergence and accuracy of conservative difference schemes.

A. J. Lohwater (Providence, R.I.)

Tihonov, A. N.; Samarskiĭ, A. A. 5392

Homogeneous difference schemes on irregular meshes. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **2** (1962), 812–832.

The authors continue their work [#5391] on homogeneous difference schemes for the equation $(k(x)u')'-q(x)u+f(x)=0$, $0<x<1$. In the present paper an irregular mesh $x_0=0, \cdots, x_i, \cdots, x_N=1$ is considered. A typical result is the following. Let $[a_{i+1}(y_{i+1}-y_i)-a_i(y_i-y_{i-1})]-d_iy_i+g_i=0$ for $i=1, \cdots, N-1$, where $a_i=[\int_{x_{i-1}}^{x_i}\{dt/k(t)\}]^{-1}$, $d_i=\int_{(x_{i-1}+x_i)/2}^{(x_i+x_{i+1})/2}q(t)\,dt$ and $g_i=\int_{(x_{i-1}+x_i)/2}^{(x_i+x_{i+1})/2}f(t)\,dt$. If $u(x_0)=y_0$, $u(x_N)=y_N$, $k(x)>0$, $q(x)\geq 0$ and the first derivatives of k, q, f satisfy Lipschitz conditions in $[0, 1]$, then $\max_i|y_i-u(x_i)|\leq M\sum_{i=1}^N(x_i-x_{i-1})^3$, M being independent of the mesh points. *H. J. Norton* (Teddington)

Vasil'ev, F. P. 5393

A difference method of solving problems of Stefan type for a quasi-linear parabolic equation with discontinuous coefficients. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 1280–1283.

The author solves a rather complicated two-phase nonlinear free boundary diffusion problem by utilizing an implicit-difference scheme. He asserts that the resulting nonlinear algebraic equation can be solved iteratively and that the solutions converge to the solution of the original problem as the mesh size vanishes. No proofs are given in the paper. *W. T. Kyner* (Los Angeles, Calif.)

Volčenko, A. P.; Poddubnyĭ, E. P. 5394

A method of solving differential equations on an electronic computer. (Russian. English summary)

Bjull. Inst. Teoret. Astronom. **9** (1963/64), 587–600.

Authors' summary: "A quadratic method of numerical integration of differential equations is proposed, founded on Bessel's interpolation formula. The working formulas giving the possibility of solving the equations with the automatic choice of integration pitch by means of the electronic computer are obtained."

Birkhoff, Garrett; Varga, Richard S.; Young, David 5395

Alternating direction implicit methods.

Advances in Computers, Vol. 3, pp. 189–273. Academic Press, New York, 1962.

The authors survey the theory of the original two alternating direction iteration (ADI) methods due to Peaceman, Rachford and the reviewer, and some variants due to Wachspress, Habetler, and the authors. The first part treats the case of a single iteration parameter. The relations between the various methods are pointed out, and a reasonably complete convergence analysis is clearly presented. A method for the selection of the optimum or a nearly optimum parameter is derived. The second part of the paper is concerned with the case in which all operators involved commute. Again, a satisfactory analysis exists and is exhibited in an elegant fashion. The third part considers the relation between ADI methods and successive overrelaxation (SOR). Primarily, a nice summary of SOR methods is given, along with a few results involving ADI methods. It is pointed out that the ADI method of Peaceman and Rachford is an SOR method for a larger matrix. The fourth part presents the results of two series of numerical experiments with ADI and SOR methods on Dirichlet problems. In general, the ADI results compare favorably with the SOR results for sufficiently small mesh size. The appendices are concerned with specialized questions concerning the choice of optimum parameters and commutativity. Transient problems are not considered, nor are any of the iterative variants that have been invented in the last few years.

J. Douglas, Jr. (Houston, Tex.)

Çakiroğlu, Adnan; Kayan, İlhan 5396

Exact forms of finite difference equations for certain differential equations. (Turkish summary)

İstanbul Tek. Üniv. Bül. **16** (1963), 77–99.

The authors give exact finite-difference forms of the following differential equations: $W''=f(x)$, $W^{(\mathrm{iv})}=f(x)$, $\Delta W=f(x, y)$, $\Delta\Delta W=f(x, y)$. The left-hand sides of these exact equations are the first-order finite-difference relations given in the classical form, and the right-hand sides are integral expressions such as $\int f(x)\cdot K(x)\,dx$ or $\iint f(x, y)\cdot K(x, y)\,dx\,dy$. The authors determine the K functions by constructing elastic models and applying Betti's reciprocal theorem to the equivalent elastic problems. For the equation $W''=f(x)$ they obtain, in particular, the "funicular polygon formula" of F. Stussi [Z. Angew. Math. Phys. **1** (1950), 53–70; MR **11**, 405]. The results obtained with this method would be more accurate than those found by the classical finite-difference method. The authors consider also some boundary conditions which occur in elasticity theory. *A. Dou* (Madrid)

Greenspan, Donald 5397

Approximate solution of axially symmetric problems.

Comm. ACM **7** (1964), 373–377.

Author's summary: "A variety of physical problems in such diverse fields as electrostatic field theory, heat and ideal fluid flow, and stress concentration theory reduce, under the assumption of axial symmetry, to the study of the elliptic partial differential equation $\partial^2u/\partial x^2+\partial^2u/\partial y^2+k/y(\partial u/\partial y)=0$. Dirichlet-type problems associated with this equation are studied on regions whose boundaries include a nondegenerate portion of the x-axis, and exceedingly accurate numerical methods are given for approximating solutions."

Ljašenko, I. M. 5398

Remarks on the method of matrix sweeping. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 458–461.

From the author's summary: "Application of the matrix double-sweep method to boundary-value problems for $\Delta v - 2\lambda v = f$ in a rectangle involves cumbersome calculations due to the necessity of inverting nth-order matrices at each step. By a transformation involving certain special matrices it is possible to reduce the procedure to computations for which the only required matrix inversions are of diagonal matrices."

T. I. Seidman (Seattle, Wash.)

Položiĭ, G. N. [Положий, Г. Н.] 5399

★**Numerical solution of two- and three-dimensional boundary-value problems of mathematical physics and functions of a discrete argument [Численное решение двумерных и трехмерных краевых задач математической физики и функции дискретного аргумента].**

Izdat. Kiev. Univ., Kiev, 1962. 161 *pp*. 0.55 *r*.

In two dimensions some finite-difference approximations to linear partial differential equations may be written in the form

$$(1)\quad \sum_{p=0}^{m} a_{pk}[u_k(x-ph)+u_k(x+ph)] + \sum_{p=1}^{m} b_{pk}[u_{k+p}(x)+u_{k-p}(x)] = f_k(x),$$

where m is a small integer (often $m=2$) and $u_k(x) = u(x, y_0+kh)$, $k=0, \cdots, n+1$. By using the transformation of the $n\times n$ matrix $B\,(=b_{pk})$ into diagonal form ($B=PDP$), the author reduces equations (1) to the equations

$$(2)\quad \sum_{p=0}^{m} a_{pk}[\hat{u}_k(x-ph)+\hat{u}_k(x+ph)] = \phi_k(x).$$

The solution $\hat{u}_k(x)$ of (2) may then be written in explicit form.

In the first chapter the solution of equations (2) is discussed and also the reduction of B to diagonal form. The second chapter deals with the derivation and transformation of approximations (1) for Laplace's, Poisson's, and the biharmonic equations, and also equations of parabolic and hyperbolic type with constant coefficients. The extension of the method to curvilinear boundaries and three-dimensional problems is described. Brief consideration is given of equations with variable coefficients.

This method would appear to be useful in avoiding round-off errors when a small mesh, and consequently a large number of nodes, is required.

H. J. Norton (Teddington)

Ting, Thomas C. T. 5400

The maximum-minimum principles for a quasi-linear parabolic finite difference equation.

Quart. Appl. Math. **22** (1964), 47–55.

The author proves both strong and weak maximum principles for various finite-difference equations which approximate the quasi-linear parabolic differential equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial}{\partial t}F(u) = 0,$$

where $F(u)$ is a continuously increasing function of u with piecewise continuous first derivative. A proof of the existence and uniqueness of the solution of the finite-difference equations is presented.

J. R. Cannon (Upton, N.Y.)

Aĭzenštat, N. D.; Vaĭnštein, I. A.; Kreĭnes, M. A. 5401

On non-rectangular interlacings. (Russian)

Trudy Moskov. Mat. Obšč. **9** (1960), 537–561.

Introducing the notion of an interlacing by a topological mapping, and using the notion of three families of curves called the generators of the interlacing, the authors generalize their earlier work [Mat. Sb. (N.S.) **48** (**90**) (1959), 377–395; MR **22** #4125; Dokl. Akad. Nauk SSSR **131** (1960), 249–252; MR **23** #B2609] on the nomographability of a function $z=f(x, y)$ in certain non-rectangular regions.

A. J. Lohwater (Providence, R.I.)

Bowman, Howard S. 5402

A nomogram for computing $\dfrac{a+jb}{c+jd}$ and a nomogram for computing $\left|\dfrac{a+jb}{c+jd}\right|$.

Nat. Bur. Standards Tech. Note No. 250 (1964), ii + 13 *pp*.

Author's summary: "This report gives two nomograms designed and constructed for use in computing complex ratios of two-dimensional vector quantities. One nomogram can be used to compute the real and imaginary parts of a vector ratio, and the other to compute the absolute value of such a function. Whenever a large number of vector ratios are required in data reduction, the use of these nomograms saves time in manual processing."

Galajda, P. 5403

A nomogram for functions of the first nomographic class in the complex domain. (Slovak. Russian and German summaries)

Acta Fac. Natur. Univ. Comenian. **9**, 83–93 (1964).

Smirnov, S. V. 5404

Uniqueness of a nomogram of aligned points with one rectilinear scale. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 910–922.

An alignment nomogram is called degenerate if all three scales lie on one plane curve of third order. All nomograms projective among themselves form a projective class. Vaona proved that in the general case the number N of projective classes of nomograms is $N \le 11$.

Here the author considers nomograms with one rectilinear scale, and proves that, in the class of nomograms of this type, $N \le 1$.

It can be shown from this particular result that the complete solution of the problem of uniqueness in the general case may be obtained. This particular case forms the content of the present paper. The paper makes use of the so-called nomographic invariants introduced by Blaschke.

D. Mazkewitsch (Knoxville, Tenn.)

COMPUTING MACHINES

See also 4673, 4720, 5240, 5365, 5375, 5696.

Bolliet, L.; Gastinel, N.; Laurent, P. J. 5405

★Un nouveau langage scientifique: ALGOL. Manuel pratique.

Actualités Sci. Indust., No. 1310. *Hermann, Paris*, 1964. 196 *pp.* (2 *inserts*) 36.00 *F.*

This is a fine description of the ALGOL 60 language, containing an impressive amount of material. The authors have made very effective use of flow diagrams to illustrate the syntactical forms in the language, and there are many examples of the various constructions. It is a manual for the language, not a primer, so it would probably not be easy for someone to learn this on his own, but with some help available, this should be a very useful book. The authors also went far beyond a mere description of the language. Once the language is explained, there are concise but effective discussions of translation techniques and character representations on several computers, as well as the complete revised report on the ALGOL 60 language, an index to all ALGOL algorithms published in Comm. ACM, and a very complete bibliography of articles (and even published letters) relating to ALGOL. As a bonus, inside the back cover is a large syntactic chart of the entire language. The only disconcerting feature is the transliteration of English key words in ALGOL into French, such as "pour" instead of "for", but then, why not!

B. A. Galler (Ann Arbor, Mich.)

Brooks, Frederick P., Jr.; Iverson, Kenneth E. 5406

★Automatic data processing.

John Wiley and Sons, Inc., New York-London, 1963. xxv + 494 *pp.* $10.75.

Table of contents: Fundamentals of data processing (Chapter 1); Manual data processing equipment (Chapter 2); Punched card equipment (Chapter 3); Computer coding (Chapter 4); Computer organization (Chapter 5); Programming (Chapter 6); Searching and sorting (Chapter 7); Metaprograms (Chapter 8); System design (Chapter 9).

Gnedenko, B. V. [Гнеденко, Б. В.]; 5407
Koroljuk, V. S. [Королюк, В. С.];
Juščenko, E. L. [Ющенко, Е. Л.]

★Elements of programming [Элементы программирования].

Second printing.

Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1963. 348 *pp.* 0.62 *r.*

Table of contents: Principles of electronic computers (Chapter I); Principles of program control on automatic computers (Chapter II); Elementary programming (Chapter III); Operator programming (Chapter IV); Address programming (Chapter V); Programming automation (Chapter VI); Organization of routine on a digital computer (Chapter VII).

In an appendix the authors give the salient features of some of the major Soviet computers (BESM, STRELA, URAL, M-3, KIEV) in connection with the methods of programming discussed in the book.

Foxley, Eric 5408

Determination of the set of all four-variable formulae corresponding to universal decision elements using a logical computer.

Z. Math. Logik Grundlagen Math. **10** (1964), 302–314.

In order to determine all four-variable formulae corresponding to universal decision elements [cf. A. Rose, same Z. 4 (1958), 169–174; MR **21** #1924] a method is developed which is suitable for use on the Nottingham University logical computer. Mechanisms necessary for subtests are also discussed in detail. The total number of operations to be performed turns out to be approximately one million and the number of decision elements required less than 90. The result is that there are 263 essentially different (i.e., not obtainable from each other by mere permutation of variables) four-variable formulae for universal decision elements. *V. Devidé* (Zagreb)

Garwick, Jan V. 5409

Data storage in compilers.

Nordisk Tidskr. Informations-Behandling **4** (1964), 137–140.

Author's summary: "Arrays in ALGOL 60 are dynamic in the sense that an array with the same name may be declared several times but with different dimensions. In a compiler this is usually of no interest because the arrays are declared only once but the programme does not in advance know how big they should be. A storage system which dynamically changes the size of arrays is described. The extra time required to perform these changes of array size so the available store is used in an optimal way is very small, and the gains in efficiency of the programme can be considerable."

Gelernter, Herbert 5410

Machine-generated problem-solving graphs.

Proc. Sympos. Math. Theory of Automata (New York, 1962), *pp.* 179–203. *Polytechnic Press of Polytechnic Inst. of Brooklyn, Brooklyn, N.Y.*, 1963.

Author's introduction: "Problems for which no well-defined solution procedures exist generally present the problem solver, at several stages during his search for a solution, with the necessity for choosing one path (hopefully, the best one) from among many alternative possibilities that present themselves at a given point. The process of solution may be formally represented by a tree-like structure called the problem-solving graph. When the problem-solving agent is a machine, rather than human, the generation, manipulation, and traversal of the problem-solving graph must be specified with precision and care. A properly constructed graph will not only favor the discovery of more elegant and efficient problem solutions, but will also guard the machine against certain pitfalls that humans seem to avoid naturally."

A. P. Ershov (Novosibirsk)

Ginsburg, Seymour 5411

Abstract machines: A generalization of sequential machines.

Proc. Sympos. Math. Theory of Automata (New York, 1962), *pp.* 125–138. *Polytechnic Press of Polytechnic Inst. of Brooklyn, Brooklyn, N.Y.*, 1963.

A complete sequential machine is a 5-tuple $S = (K, \Sigma, \Delta,$

δ, λ), where (i) K is a finite set of "states"; (ii) Σ is a finite set of "inputs"; (iii) Δ is a finite set of "outputs"; (iv) δ is a "transition" function which maps $K \times \Sigma$ into K; (v) λ is an "output" function which maps $K \times \Sigma$ into Δ.

A quasi-machine is a 5-tuple $(K, W, Y, \delta, \lambda)$, where (1) K is a set of "states"; (2) W and Y are semigroups of "inputs" and "outputs", respectively; (3) δ is a mapping of $K \times W$ into K such that $\delta(q, IJ) = \delta(\delta(q, I), J)$ for each $q \in K$ and all I and J in W; (4) λ is a mapping of $K \times W$ into Y such that $\lambda(q, IJ) = \lambda(q, I)\lambda(\delta(q, I), J)$ for each $q \in K$ and all I and J in W.

An abstract machine is a quasi-machine in which the output semigroup satisfies the left cancellation law.

Main results: (I) abstract machines model many certain situations in data processing computers; (II) redundant (=equivalent) states may always be removed from an abstract machine, but not always from a quasi-machine; (III) equivalent inputs of an abstract machine can be removed; (IV) a synthesis procedure that is always successful is given for synthesis of an abstract machine with no equivalent inputs, which has a specified input structure.

A. P. Ershov (Novosibirsk)

Gram, Christian 5412

On the representation of zero in floating-point arithmetic.

Nordisk Tidskr. Informations-Behandling **4** (1964), 156–161.

Author's summary: "Two representations of zero in floating-point arithmetic are considered in relation to a summation with correction for rounding errors. The common representation with exponent zero is shown in this case to be better than the 'academic' representation where the exponent depends on the 'history' of the zero."

Lombardi, Lionello A. 5413

Mathematical models of file processes.

Atti Sem. Mat. Fis. Univ. Modena **12** (1962/63), 173–214.

The author develops what he calls a non-procedural language primarily designed for non-arithmetic processes, independently of the procedures they involve (Algebraic Data System Language). This language is based on what the author calls a theory of files. A file is a set of items of information which is logically orderable, and which is ordered by a mapping into the linearly ordered set of natural numbers or an isomorphic ordered set. Apart from computational elements similar to the basic part of ALGOL, table operations and so-called flow control expressions involving files are proposed.

The presentation is partly sketchy, partly vague; it does not discuss implementation nor is it precise enough to define implementation implicitly. How tentative the formulations are is illustrated by the sentence (p. 205): "this requires a careful analysis of the procedure description in order to detect tautologies and to determine the independence among expressions; this analysis has to be done by the system at compilation time", and a reference made at this point to the books of Church, Phister, Roth, Quine and Barankin, which indicates quite an enormous task.

F. L. Bauer (Munich)

Svoboda, A.; Čulík, K. [Čulík, K.] 5414

An algorithm for solving Boolean equations. (Russian. English summary)

Automat. i Telemeh. **25** (1964), 374–381.

The authors present a straightforward procedure for determining the set of solutions of a system of Boolean equations

$$F_i(x_1, \cdots, y_1, \cdots) = G_i(x_1, \cdots, y_1, \cdots), \qquad i = 1, \cdots, n,$$

where the x_j are independent variables and the y_k are the unknowns. A simple encoding of the algorithm requires about 2^L bits of memory for storing the graphs of functions defined by the F_i and G_i; here $L = 2m + r$, where m is the number of y_k and r is the number of x_j. The paper ends with a proof of a theorem stating a characterizing condition under which the solutions of the above system of equations exist and (when they do) the number of such solutions.

R. M. Baer (Berkeley, Calif.)

Trotter, H. F. 5415

A machine program for coset enumeration.

Canad. Math. Bull. **7** (1964), 357–368.

The author gives an algorithm and a few hints of an IBM 7094 program for determining the index of a subgroup of cosets H of a group K. Here K is specified by a finite set of generators and relations between them and H is specified as generated by a given finite set of words in the generators of K. The method is "much the same" as that of Leech [Proc. Cambridge Philos. Soc. **59** (1963), 257–267; MR **26** #4513]. In practice this problem tends to run out of storage space. The present program has features to alleviate this distress. The number of generators is limited to nine. As a test, the program found the 448 cosets of $\{A^2, A^{-1}B\}$ in the group generated by A and B, with the relations $A^8 = B^7 = A^2B^2 = A^{-3}B^3 = 1$, in six seconds.

D. H. Lehmer (Berkeley, Calif.)

GENERAL APPLIED MATHEMATICS

See also 4969.

Chisholm, J. S. R.; Morris, Rosa M. 5416

★Mathematical methods in physics.

Undergraduate Textbooks in Physics, Vol. 2.

North-Holland Publishing Co., Amsterdam, 1964. xviii + 719 *pp.* $10.00.

Starting with the elementary concepts of analysis, the authors lead the reader through the classical advanced calculus, linear algebra, functions of a complex variable, certain applied aspects of Fourier analysis, the classical special functions of mathematical physics, and statistics and probability.

Sommerfeld, Arnold 5417

★Vorlesungen über theoretische Physik. Band VI: Partielle Differentialgleichungen der Physik.

Fünfte Auflage. Bearbeitet und ergänzt von Fritz Sauter.

Akademische Verlagsgesellschaft Geest & Portig K.-G., Leipzig, 1962. xii + 298 *pp.* DM 16.50.

Table of contents: Fouriersche Reihen und Integrale (I); Allgemeines über partielle Differentialgleichungen (II); Randwertaufgaben bei der Wärmeleitung (III);

Zylinder- und Kugelprobleme (IV); Eigenfunktionen und Eigenwerte (V); Probleme der drahtlosen Telegraphie (VI).

The book also contains an interesting set of problems and solutions arranged by chapter.

MECHANICS OF PARTICLES AND SYSTEMS

See also 4650, 5145, 5182, 5192, 5621.

Edwards, Hiram W. 5418

★**Analytic and vector mechanics.**

Dover Publications, Inc., New York, 1964. x+428 *pp.* $2.00.

An unabridged and unaltered reprinting of the work first published in 1933 [McGraw-Hill, New York, 1933].

Kilmister, C. W. 5419

★**Hamiltonian dynamics.**

Mathematical Physics Series, 2.

John Wiley and Sons, Inc., New York, 1964. ix+146 *pp.* $4.75.

This booklet is an introduction to analytical mechanics intended, it appears, mainly for mathematicians. Most of the material can be found in other textbooks, but the emphasis and approach are often different from the more usual ones. After a brief introductory chapter reminding the reader of the Newtonian equations of motion, rigid body dynamics, constraints, tensor calculus, and so on, the author discusses in the other chapters the Lagrangian equations, both with the time as a parameter and with the time as an extra coordinate, and the Hamiltonian equations.

In the chapters on Lagrangian theory, Feynman's path integral method is discussed in some detail, a topic not usually found in a text on classical mechanics. On the other hand, in the chapters on Hamiltonian theory one misses a discussion of action and angle variables, of such great importance both in celestial mechanics and in modern plasma physics, and a discussion of classical perturbation theory which is such a delightfully elegant discipline. However, the author may have opted intentionally for a short, sweet account rather than a more comprehensive one. There are various historical notes, of which the longest and most delightful one is the story of Maupertuis' principle. *D. ter Haar* (Oxford)

Ferrarese, Giorgio 5420

Sulle equazioni di moto di un sistema soggetto a un vincolo anolonomo mobile.

Rend. Mat. e Appl. (5) **22** (1963), 351–370.

The author first considers a holonomic Lagrangian system of n degrees of freedom, subject to variable frictionless constraints. The equations of motion, with a separately postulated energy equation, are written as path equations in a Riemannian $(n+1)$-space. Next, the author considers the addition of a variable anholonomic constraint, linear in the velocities. This is interpreted geometrically, and it is shown how, by the introduction of $(n+1)$ reference congruences, the force due to the new constraint may be made to disappear from the path equations. There remain n equations for n unknowns, which are then restored to Lagrangian form. There appear, added to the generalized forces, new terms which vanish when (1) the constraint is in fact holonomic, or (2) the coordinates are suitably chosen. A large part of the article is devoted to the theory of congruences and to the explanation of the Ricci rotation coefficients. *R. H. Boyer* (Austin, Tex.)

Károlyházy, F. 5421

Mach's principle and general relativity. (Russian summary)

Acta Phys. Acad. Sci. Hungar. **17** (1964), 31–40.

The author gives a clear and eloquent exposition (almost without equations) of the implications of Mach's principle of the relativity of inertia for Newtonian mechanics, special relativity, and general relativity. He concludes with the interesting suggestion that the solutions of Einstein's field equations should be restricted by the following criterion, which he asserts could easily be made rigorous: "A physical solution S, that is a complete regular space-time with an everywhere acceptable (non-negative) mass density, etc., obeying equations $R_{ik}-\frac{1}{2}g_{ik}R=-\kappa T_{ik}$ is permissible if it is impossible to give a set $S(p)$ of other physical solutions, depending continuously on the parameter p, and a set $\Sigma(p)$ of space-time domains, where $\Sigma(p)$ is a suitably chosen domain of $S(p)$, in such a manner that $S(p_0)=\Sigma(p_0)=S$ and, for some other value p_1 of p, $\lim_{p\to p_1}\Sigma(p)=S_p$, where S_p describes a completely empty but otherwise complete and regular space-time." *F. A. E. Pirani* (Waltham, Mass.)

Osipov, P. M. 5422

Motor operators in curvilinear coordinates. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 322–327.

Author's summary: "The author discusses operations on biscalar functions and motor functions in curvilinear coordinates, and the formulae of Jacobi and Lamé are extended to motor and biscalar fields."

Osipov, P. M. 5423

Bi-scalar and motor fields. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 740–744.

Author's summary: "The concept of directional derivative is extended to biscalar and motor fields. An expansion of biscalar and motor fields into Taylor's series by means of directional derivatives of higher orders of three fields is given. Diad identities are obtained, expressed in terms of directional derivatives of motor fields. These identities are applied to the integration of motor equations."

Topencharov, V.; Cheshankov, B. 5424

On the kinematics of multi-dimensional Euclidean spaces. (Russian summary)

C. R. Acad. Bulgare Sci. **16** (1963), 573–576.

A kinematic matrix-operator is defined for n-dimensional Euclidean spaces, and is used to generalize the basic velocity and acceleration formulas. Known properties of

acceleration of any order in two- and three-space are explained in this general context.

For motion with one point fixed, the maximum and minimum dimensions of the linear varieties which correspond to the loci of zero velocity are determined according to whether n is even or odd.

B. Roth (Stanford, Calif.)

Kretzschmar, Horst 5425

Zur Berechnung der Eigenschwingungen zweiseitig gelagerter Kreuzwerke.

Wiss. Z. Techn. Univ. Dresden **13** (1964), 509–516.

Author's summary: "Die einwandfreie Gestaltung dynamisch beanspruchter Tragwerke setzt u.a. die Kenntnis der Eigenfrequenz voraus. In der vorliegenden Arbeit werden deshalb einige Ansätze zur Berechnung der freien Schwingungen zweiseitig gelagerter Kreuzwerke, die besonders im Brückenbau sehr oft Verwendung finden, mitgeteilt. Für das Matrizenpaar der Frequenzbedingung lassen sich einfache Bildungsgesetze angeben."

Grioli, Giuseppe 5426

Questioni di dinamica del corpo rigido.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **35** (1963), 35–39.

The author considers a rigid body rotating about a fixed point O, under the general hypothesis that the moment of the applied forces with respect to O is a function of the angular velocity ω and of a constant vector H. Various important physical problems are included as special cases. Conditions are given under which the equations of motion possess two integrals which are linear and quadratic, respectively, in the components of ω. The problem of the explicit determination of the motion is discussed briefly.

L. A. MacColl (New York)

Mayné, Georges 5427

Le problème de la séparation des variables pour les systèmes dynamiques à deux et trois degrés de liberté.

Acad. Roy. Belg. Cl. Sci. Mém. Coll. in-8° (2) **34**, *no.* 4, 125 *pp.* (1964).

Table of contents: Introduction; (Chapitre I) Solution singulière de classe 0: Étude générale des conditions de séparabilité de Levi-Civita; Cas "essentiellement géodésique"; Systèmes de Stäckel; (Chapter II) Solutions singulières de classe 1; (Chapitre III) Solutions singulières de classe 2; (Chapitre IV) Solutions singulières de classe 3; (Chapitre V) Séparation des variables pour les systèmes bidimensionnels; Conclusions générales.

Nahon, Fernand 5428

Sur une classe de fonctions de force qui généralisent les fonctions de force radiales.

C. R. Acad. Sci. Paris **259** (1964), 65–68.

The circular motions in a radial force field with potential $U(r)$ form a one-parameter family of solutions, the orbits of which are the level curves of $U(r)$. The author gives a procedure for the determination of all force fields which admit equipotential orbits and studies some of their properties.

E. Leimanis (Vancouver, B.C.)

Rózsa, M. 5429

Théorèmes de réciprocité dynamique relatifs à des systèmes ayant plusieurs degrés de liberté. (English, German, and Russian summaries)

Acta Tech. Acad. Sci. Hungar. **48** (1964), 395–399.

Author's summary: "L'étude présente deux théorèmes relatifs à la réciprocité dynamique entre une force généralisée agissant en un point quelconque d'un système holonome et une accélération généralisée se produisant en un autre point quelconque de ce système."

Schweiger, Fritz 5430

Bemerkungen zum Laplace-Lenzschen Vektor.

Acta Phys. Austriaca **17** (1963/64), 343–346.

The relationship between certain infinitesimal transformations, a conservation theorem, and a quantity called the "Laplace-Lenz vector" are discussed.

T. R. Kane (Stanford, Calif.)

Silven, Saul 5431

Application of operational methods to the analysis of the motion of rigid bodies.

J. Mathematical Phys. **5** (1964), 1424–1426.

This paper deals with an application of an extension of one-sided Laplace transforms to vector functions $\mathbf{f}(t)$. The transform of such a function is defined by the new vector function

$$\mathbf{F}(\mathbf{S}) = \int_0^\infty \exp(-\mathbf{S}t)\mathbf{f}(t)\,dt,$$

the tensor $\mathbf{S}$ of which corresponds to the scalar operational variable p or s of the conventional operational calculus. The rules based on this definition are completely analogous to those holding for scalar Laplace transforms. They are applied here to solve the equation of motion (in a linearized approximation) for the angular-velocity vector of a rigid body, the coordinate system being attached to this body. The transformed equation is solved first, with the aid of matrix analysis. The derived results refer to a few special cases for the torque generating the motion of the rigid body; the solutions can be interpreted in terms of precessional motions of the angular-velocity vector.

H. Bremmer (Eindhoven)

ELASTICITY, PLASTICITY

See also 4993, 5396, 5485, 5626.

Gol'denblat, I. I. 5432

★Some problems of the mechanics of deformable media.

Translation from the Russian by Z. Mróz; edited by J. R. M. Radok.

P. Noordhoff N. V., Groningen, 1963. xvi + 304 *pp.* *f.* 35.00.

This is a translation of the 1955 Russian edition [Fizmatgiz, Moscow, 1955; MR 17, 1152].

Kelly, P. D. 5433

A reacting continuum. (French, German, Italian, and Russian summaries)

Internat. J. Engrg. Sci. **2** (1964), 129–153.

The author formulates the theory of a heterogeneous reacting continuum from the conservation laws. The balance equations for mass, momentum, spin momentum, energy, entropy, electric charges and magnetic flux are given in integral equations from which the corresponding differential and jump equations are derived for both an individual constituent and for a reacting continuum.
S. I. Pai (College Park, Md.)

Chen, Yu 5434
Remarks on variational principles in elastodynamics.
J. Franklin Inst. **278** (1964), 1–7.
The author presents a derivation of the complementary extremum principle for general small vibrations of an elastic body. For the steady vibration case this was given earlier by Reissner [J. Math. and Phys. **27** (1950), 90–95; MR **12**, 301]. *J. Miklowitz* (Pasadena, Calif.)

Kupradze, V. D.; Aleksidze, M. A. 5435
The method of functional equations for the approximate solution of certain boundary-value problems. (Russian)
Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 683–715.
The same authors [Soobšč. Akad. Nauk Gruzin. SSR **30** (1963), 529–536; MR **27** #5039] have earlier discussed methods of approximation to the formal solutions of boundary-value problems of elliptic partial differential equations when these solutions are expressed as functionals. In the present paper, these methods are discussed further and are now applied to fundamental problems in harmonic function theory and elasticity theory, some simple illustrative numerical examples being given. It is stated that the new methods compare favourably in application with the more usual methods of the calculus of variations and of finite differences, and also there are applications to problems whose solution leads to singular integral equations. *H. G. Hopkins* (Sevenoaks)

Karni, Z.; Reiner, M. 5436
The general measure of deformation.
Second-order Effects in Elasticity, Plasticity and Fluid Dynamics (Internat. Sympos., Haifa, 1962), pp. 217–227. Jerusalem Academic Press, Jerusalem; Pergamon, Oxford, 1964.
Based on a general definition of deformation tensor (strain) it is found that other second-order measures of deformation exist besides the two named after Green and Almansi. These measures have the property of absence of cross terms in simple shear.
{Reviewer's comment: The two additional second-order measures of strain found in this paper do not appear to satisfy the proper invariance requirements.}
R. L. Fosdick (Chicago, Ill.)

Bašeleišvili, M. O. 5437
A property of the solution of the third and fourth boundary-value problems of the statics of anisotropic elastic bodies. (Russian. Georgian summary)
Soobšč. Akad. Nauk Gruzin. SSR **34** (1964), 283–290.
This paper concerns the solution of the third and fourth boundary-value problems of elasto-statics for bodies of anisotropic material under two-dimensional conditions. These problems refer to situations in which either (i) the normal displacement and the tangential stress, or (ii) the tangential displacement and the normal stress, respectively, are prescribed at the boundary of a region, certain analytical conditions being imposed. The analysis leading to the formal solution of the problems for anisotropic material is an extension of earlier, closely related analysis applying for isotropic material, and it is based similarly upon applications of potential theory and singular integral equations. Restriction is made to finite and simply-connected regions, but it is stated that this may be relaxed without difficulty so that the analysis includes cases of infinite and multiply-connected regions. Furthermore, it is stated that similar analysis can be developed for the theory of bending of anisotropic plates.
H. G. Hopkins (Sevenoaks)

Davies, G. A. O.; Buchwald, V. T. 5438
Plane elastostatic boundary value problems of doubly connected regions. II. The orthotropic ring.
Quart. J. Mech. Appl. Math. **17** (1964), 271–277.
The technique developed by the authors for obtaining solutions of the two-dimensional biharmonic equation for a doubly connected region consists of using combinations of the complex potentials to satisfy exactly the conditions on one boundary. On the other boundary the potential is supposed expanded as a Laurent series, the coefficients being determined exactly or approximately by the boundary conditions. This procedure can be extended to other fourth-order partial differential equations, in particular, to elastic problems with general anisotropy. Here the extension is, for simplicity, confined to orthotropy. The illustrative problem is that of an annulus under internal pressure, approximate calculations being made by the method of collocation with 22 coefficients. Graphs and a brief discussion are given.
L. M. Milne-Thomson (Tucson, Ariz.)

Spencer, A. J. M. 5439
Perturbation methods in plasticity. II. Plane strain of slightly irregular bodies.
J. Mech. Phys. Solids **10** (1962), 17–26.
Part I appeared in same J. **9** (1961), 279–288 [MR **24** #B1815]. Author's summary: "A perturbation method is formulated for the problem of plane plastic strain of a body B which differs slightly in shape from a second body B_0, when the solution for B_0 is known and similar boundary conditions obtain for the two bodies; the case in which B undergoes continuous deformation but can at all times be regarded as a perturbation of B_0 is also considered. Examples solved are the indentation of an almost plane surface, and the continued indentation of an initially plane surface by the curved surface of a parabolic cylinder. This last problem closely approximates the two-dimensional analogue of the Brinell hardness test."

Abenova, M. 5440
Construction of the integral equations of the first boundary-value problem of the mathematical theory of elasticity for a hemisphere. (Russian. Kazak summary)
Vestnik Akad. Nauk Kazah. SSR **1964**, no. 8 (233), 66–71.

Integral equations for dilatation and curl of displacement inside a hemisphere, for the equations of elasticity with prescribed displacements on the boundary, are written down in explicit form. *S. Drobot* (Columbus, Ohio)

Aida, Toshio; Terauchi, Yoshio 5441

Zahnfussfestigkeit geradverzahnter Stirnräder.

Mem. Fac. Engrg. Hiroshima Univ. **2**, *no.* 2, 15–55 (1964).

Authors' summary: "Wir haben die theoretische Untersuchung über die Größe der auf dem gefährlichen Querschnitt entstandenen Spannung angestellt und geklärt, daß die Spannung auf der Filetkurve immer größer als diejenige innerhalb des Zahns ist. Daraus hat es sich ergeben, daß die Zahnfußfestigkeit durch Anwendung der Berechnungsgleichung der maximalen Spannung entlang der an der Zugseite befindlichen Filetkurve berechnet werden kann. Aufgrund dieses Ergebnisses haben wir die maximalen Spannungen der verschiedenen Zahnprofile entlang der Filetkurve berechnet, und einen Vorschlag gemacht, daß der gefährliche Querschnitt durch die Hofersche Methode mit genügender Genauigkeit gesucht wird. Im Vergleich der maximalen Spannung, wenn die konzentrische Biegungsbelastung auf den Zahn wirkt, mit derjenigen, wenn das reine Biegungsmoment wirkt, ist es nachgeprüft worden, daß man die Schubspannung berücksichtigen muß, wenn man die Berechnungsgleichung der Fußfestigkeit erhalten will. Ferner haben wir gefunden, daß der Formfaktor bei Einwirkung der Druckkomponente von der Belastung auf den Zahn zu demjenigen bei Einwirkung des reinen Biegungsmoment im Verhältnis steht. Als Ergebnis dieser Untersuchungen haben wir folgende Berechnungsgleichung der maximalen Spannung geführt:

$$\sigma_t = \left(1+0.08\frac{S}{\rho}\right)(0.66\sigma_{Nb}+0.40\sqrt{(\sigma_{Nb}^2+36\tau_N^2)}+1.15\sigma_{Nc}).$$

Mit Anwendung der obigen Gleichung kann man die Biegungsspannung all der Evolventenstirnräder mit geraden Zähnen und der Zahnräder mit den ähnlichen Profilen berechnen." *R. M. Morris* (Cardiff)

Schile, R. D.; Sierakowski, R. L. 5442

On the axially symmetric deformation of a nonhomogeneous, elastic material.

J. Franklin Inst. **278** (1964), 327–336.

Authors' summary: "The problem of stress analysis of a solid of revolution, deformed symmetrically with respect to the axis, in which the elastic 'constants' are arbitrary functions of the radial and axial coordinates, is considered. The solution of the torsion-free problem is formulated in terms of two stress functions which are determined by two coupled partial differential equations with variable coefficients. The torsion problem is solved by means of a single stress function. In addition, the following three special cases of interest to engineers are considered: (1) Torsion of a circular bar of varying section, (2) Circular bar in tension, (3) Pressurized thick cylinder."

Sucher, M. 5443

On a certain generalization of the Michell problem. (Polish and Russian summaries)

Arch. Mech. Stos. **15** (1963), 645–657.

The generalised Michell problem consists in the determination of the stress in two different elastic half-planes of uniform thickness united along a common straight edge, and acted upon by a concentrated force applied to one of the planes. The solution is found here, when both planes are anisotropic, by the use of Fourier integrals, but the end results are expressed in closed form in terms of elementary functions. Various particular cases are discussed, including the solution by Mossakovski for the anisotropic half-plane with its edges free from stress. The case where the planes are isotropic is obtained by a limiting process. *L. M. Milne-Thomson* (Tucson, Ariz.)

Reissner, Eric 5444

Note on the problem of St. Venant flexure. (German summary)

Z. Angew. Math. Phys. **15** (1964), 198–200.

The St. Venant theory of flexure is applied to beams of transverse isotropic nonhomogeneous material, by assuming that elastic constants are functions of cross-sectional coordinates x, y and $\sigma_x = zs_x(x,y)$, $\sigma_y = zs_y(x,y)$, $\tau_{xy} = zt(x,y)$ instead of the classical (*) $\sigma_x=\sigma_y=\tau_{xy}=0\cdots$. The problem is reduced to a boundary-value problem for the stress function, which becomes homogeneous and satisfies (*) if the ratio $\nu_z E_z/E$ is constant. *J. Nowinski* (Newark, Del.)

Bufler, H. 5445

Die Torsion der dicken Platte mit stetig veränderlichem Schubmodul. (English and Russian summaries)

Z. Angew. Math. Mech. **43** (1963), 545–551.

Author's summary: "Das Torsionsproblem einer unendlich ausgedehnten Platte (Schicht) mit senkrecht zum Rand stetig veränderlichem Schubmodul $G(z)$ läßt sich in übersichtlicher Weise mit Hilfe der Hankel-Transformation behandeln; es wird von einer gewöhnlichen Differentialgleichung zweiter Ordnung mit im allgemeinen veränderlichen Koeffizienten beherrscht. Ihre Lösung wird für das Gesetz $G(z)=G_0(1+az)^\gamma$ mit beliebigem (positivem oder negativem) Exponenten γ angegeben und diskutiert." *R. P. Nordgren* (Houston, Tex.)

Di Pasquale, Salvatore [di Pasquale, Salvatore] 5446

Impostazione e risoluzione variazionale delle equazioni di Wlassow-Marguerre per le lastre curve ribassate.

Giorn. Mat. Battaglini (5) **10** (**90**) (1962), 26–43.

The author shows how variational methods can be applied to obtain and solve the equations of the title, and numerical applications are given. *U. D'Ambrosio* (Providence, R.I.)

Haberland, G. 5447

Die Auswertung spannungsoptischer Plattenversuche unter Berücksichtigung der Theorie von E. Reissner.

Monatsb. Deutsch. Akad. Wiss. Berlin **6** (1964), 401–407.

Die Anwendung der Kirchhoffschen Plattentheorie zur Auswertung spannungsoptischer Plattenversuche ist unbefriedigend, da die Annahmen dieser Theorie nicht ganz frei von Widersprüchen sind, die sich insbesondere an den Plattenrändern bemerkbar machen, also dort, wo die Randbedingungen zu erfüllen sind. Zur Vermeidung

dieser Mängel wird in der vorliegenden Arbeit zur Auswertung spannungsoptischer Versuche an Platten die verfeinerte Plattentheorie von E. Reissner herangezogen. Es wird gezeigt, wie in diesem Fall die Untersuchungen sowohl bei Anwendung des Zweischichtverfahrens als auch bei Verwendung des Reflexionsverfahrens auszuwerten sind. *F. Chmelka* (Vienna)

Langenbach, Arno 5448

Zur Lösung eines Minimum-Problems der nichtlinearen Plattentheorie.

Math. Nachr. **26** (1963/64), 339–351.

Es wird das Minimum-Problem für Platten untersucht, die sowohl physikalisch als auch geometrisch nichtlinear deformiert sind. Die Formulierung der Variationsgleichungen wird einer Monographie von W. W. Nowoshilow entnommen. Die Funktion des Verschiebungsvektors, deren Minimaleigenschaften untersucht werden, wird aus dem Prinzip der virtuellen Verschiebungen gewonnen. Für diese Funktion wird eine Reihe von charakeristischen Sätzen abgeleitet. *F. Chmelka* (Vienna)

Szmodits, K. 5449

Solution numérique simplifiée des problèmes de plaques. (English, German, and Russian summaries)

Acta Tech. Acad. Sci. Hungar. **48** (1964), 231–239.

Author's summary: "The paper deals with the numerical solution of the bipotential plate equation. The described method unifies the difference and Ritz's method, and determines the elastic surface $w(x, y)$ sought for having the same value-set as that which results on the base of the difference method. The advantage of the method dealt with in comparison to the difference method is, that for a simply supported plate, the value-set w sought for can be produced without the solution of difference equations and without a considerable loss in numerals. For fully clamped plates, difference equations comprising great transversal terms and being by iteration quickly solvable, can easily be set up."

De Silva, C. N. 5450

The effect of transverse shear deformation on the bending of elliptic toroidal shells.

J. Soc. Indust. Appl. Math. **12** (1964), 465–476.

The second-order complex differential equation derived for the title problem by Naghdi [Quart. Appl. Math. **15** (1957), 41–52; MR **20** #4959] is solved by asymptotic integration. The approximate solution, valid for large values of a shell geometry parameter μ, is given in terms of Hankel functions of orders 1/3 and $-2/3$, and reduces upon specialization to the solution for circular toroidal shells given by Clark [J. Mathematical Phys. **29** (1950), 146–178; MR **12**, 567] for the case of classical shell theory. *H. J. Weinitschke* (Hamburg)

Masur, E. F. 5451

On the consistency of linear structural shell theory.

J. Mécanique **3** (1964), 3–14.

The author discusses the equations of linear shell theory with emphasis on the strain-displacement relations. The techniques of introducing "modified" or "generalised" stress and strain measures, as has been done in some of the recent literature, leads to certain new difficulties when the theory is extended to include transverse shear strains. The author offers a solution to some of these questions and leaves others open. *J. L. Sanders, Jr.* (Cambridge, Mass.)

Payton, R. G. 5452

Bond stress in cylindrical shells subjected to an end velocity step.

J. Math. and Phys. **43** (1964), 169–190.

Concluding remarks: "We have considered the dynamic bond stress in a semi-infinite cylindrical shell composite resulting from a velocity step end impact. For the particular shell considered, e.g., an epoxy-plastic metal combination, it was found that the peak stress occurred during the early stage of the shell motion. If we had applied a velocity pulse, rather than a velocity step, to the end of the structure, the same bond-stress pattern would have developed along the bond for a distance, measured back from the leading front, equal to the pulse width. Also if the structure were finite in length instead of semi-infinite, the stress pattern would remain the same until reflections from the other end occurred. No attempt has been made to analyze the changes in the stress pattern caused by varying the parameters α and c. This, together with the applied-pressure boundary condition, we hope to consider in the future." *B. A. Boley* (New York)

Vol'mir, A. S.; Kil'dibekov, I. G. 5453

Non-linear acoustical vibrations of a cylindrical shell. (Russian. Armenian summary)

Izv. Akad. Nauk Armjan. SSR Ser. Fiz.-Mat. Nauk **17** (1964), *no.* 3, 65–70.

The forced vibrations of cylindrical panels supported in an acoustic field are considered. From the flexure equation, the equation of compatibility of the deformations (both nonlinear) and the equation for the initial bending $f(t)$, a second-order ordinary differential equation is derived by the Galerkin method for the parameter $\zeta = f(t)/h$. This furnishes a first approximation to the bending in terms of the independent variable $\tau = \omega t$, where ω is the forcing frequency. In the static case this equation leads to a cubic polynomial in ζ, the zeros of which give the values of ζ for which the shell is in equilibrium. Graphs of this function are plotted for square panels having various degrees of curvature. The effect of an initial irregularity in the form of the middle surface is shown. Amplitude-frequency characteristics are drawn for these cases and also for the case in which the panel is subjected to an additional longitudinal stress. *R. N. Goss* (San Diego, Calif.)

Woźniak, Cz. 5454

Finite deformation of shells (analysis of the strain geometry). (Polish and Russian summaries)

Arch. Mech. Stos. **15** (1963), 535–545.

The author suggests adoption of the affinors of the mapping of the undeformed shell space on the deformed shell space as measures of finite strain. These have, according to the author, certain advantages over the Cauchy-Green measures from the point of view of group

theory. The author does not discuss possible disadvantages from the point of view of applied mechanics.

J. L. Sanders, Jr. (Cambridge, Mass.)

Perkins, R. W.; Evan-Iwanowski, R. M. 5455

Mechanical dislocations and physical processes of anisotropic bodies.

J. Franklin Inst. **278** (1964), 337–351.

Authors' summary: "The necessary and sufficient conditions which must be satisfied by a physical process (e.g., the diffusion of heat through a body) in order that its mechanical effects can be represented by mechanical dislocations are derived for anisotropic media. It has been found that the only kind of physical processes which can be represented by mechanical dislocations are those which are steady-state processes with respect to time. Further, it has been found that in order for a steady-state physical process to be represented by mechanical dislocations in an anisotropic body, certain relationships between the coefficients which govern the physical process must be satisfied. Several examples of materials and processes for which the required relationships are satisfied have been presented."

Menditto, Giovanni 5456

Sull'instabilità dinamica flesso-torsionale delle travi alte.

Ist. Lombardo Accad. Sci. Lett. Rend. A **97** (1963), 796–816.

In the present paper the author discusses the bending and torsional dynamical instability of deep beams. These are thought to be stressed transversely and axially by pulsating loads which are either distributed or concentrated.

In particular, the cases are considered of beams with rectangular section and of double-T section, which are stressed either by a pulsating transverse load of arbitrary continuous distribution or by two pulsating eccentric axially concentrated loads. In each case the instability problem is reduced to the existence of unbounded solutions of linear periodic differential systems $x' = (A + \lambda B(t))x$, where A is a constant diagonal matrix, λ a small parameter, $B(t)$ a given $n \times n$ periodic matrix of period T. A theorem of boundedness of the solutions proved by the reviewer [Atti Accad. Italia Mem. Cl. Sci. Fis. Mat. Natur. (6) 11 (1940), 633–695; MR 3, 41; MR 8, 208] is used consistently, and practical criteria for stability are obtained.

L. Cesari (Ann Arbor, Mich.)

Ang, D. D.; Knopoff, L. 5457

Diffraction of vector elastic waves by a clamped finite strip.

Proc. Nat. Acad. Sci. U.S.A. **52** (1964), 201–207.

This paper is a straightforward extension of the authors' previous work on scalar diffraction [same Proc. 51 (1964), 471–476; MR 28 #4771; ibid. 51 (1964), 593–598; MR 28 #4772] to diffraction of vector elastic waves.

C. J. Bouwkamp (Eindhoven)

Mitra, M. 5458

Disturbance produced in an elastic half-space by impulsive normal pressure.

Proc. Cambridge Philos. Soc. **60** (1964), 683–696.

The displacement produced in an elastic half-space by a uniform impulsive pressure acting over a circular portion of the surface has been obtained in terms of definite integrals. For the surface, terms representing the P-waves and the Rayleigh-waves were found. The equations obtained are quite complex and appear to reproduce many of the features of seismic disturbances.

W. J. Pierson, Jr. (New York)

Subba Rao, V.; Nigam, S. D. 5459

Wave propagation in rotating elastic media.

Mathematika **11** (1964), 29–38.

Lighthill's method [Philos. Trans. Roy. Soc. London Ser. A 252 (1960), 397–430; MR 26 #5844] is used to study the propagation of waves due to a localised disturbance, oscillating with period $2\pi/\omega$, in an elastic medium rotating with angular velocity Ω. The application is straightforward and nothing essentially new emerges, but there is virtually no discussion of the basic equations of motion on which the whole theory rests. This is unfortunate because it is not immediately clear why the acceleration of the displacement vector $\mathbf{V}$ has been written in the form

$$(1)\qquad \frac{\partial^2 \mathbf{V}}{\partial t^2} + 2\mathbf{\Omega} \wedge \frac{\partial \mathbf{V}}{\partial t} + \mathbf{\Omega} \wedge (\mathbf{\Omega} \wedge \mathbf{r})$$

instead of

$$\frac{\partial^2 \mathbf{V}}{\partial t^2} + 2\mathbf{\Omega} \wedge \frac{\partial \mathbf{V}}{\partial t} + \mathbf{\Omega} \wedge (\mathbf{\Omega} \wedge (\mathbf{r} + \mathbf{V})).$$

In the paper $\mathbf{r}$ is undefined but presumably it is the undisturbed position of the element whose displacement vector is $\mathbf{V}$. Various arguments can be given justifying (1) but they require $\omega^2 \gg \Omega^2$, which invalidates many of the conclusions of the paper.

K. Stewartson (London)

Thurston, R. N. 5460

Wave propagation in fluids and normal solids.

Physical acoustics, Vol. I—Part A, pp. 1–110. *Academic Press, New York*, 1964.

This paper is tutorial in nature. It contains a discussion of nonlinear continuum mechanics followed by a treatment of small-amplitude waves in elastic solids and viscous fluids. The theory presented here does not disregard thermodynamics; there is a section on thermal relaxation in a perfect fluid and one on thermal losses in an elastic solid.

M. E. Gurtin (Providence, R.I.)

Meeker, T. R.; Meitzler, A. H. 5461

Guided wave propagation in elongated cylinders and plates.

Physical acoustics, Vol. I—Part A, pp. 111–167. *Academic Press, New York*, 1964.

In this paper, which is for the most part expository, is discussed the propagation of sinusoidal waves in isotropic elastic plates and cylinders.

M. E. Gurtin (Providence, R.I.)

Favre, Henry 5462

Sur la propagation des vibrations transversales le long d'une barre prismatique viscoélastique à comportement linéaire.

J. Mécanique **3** (1964), 251–276.

This paper deals with the theory of lateral vibrations of viscoelastic beams, where, besides the lateral inertia, the influence of the inertia of rotation and the influence of the shearing forces are taken into account, as in the elastic case treated by Lord Rayleigh and Timoshenko. In the case of a first-order viscoelastic law between stresses, strains and their time derivatives, the basic equations of the problem are two simultaneous linear partial differential equations, one of the fourth order, the other of the third order for the lateral displacement and the angle of rotation. For the applications, and especially for the integration, it is necessary to distinguish between long, mean and short waves on one hand, and to specialize the constitutive equations on the other hand, so that certain terms can be neglected and the angle of rotation can be easily eliminated. In all the cases the discussion of the problem turns around a biquadratic characteristic equation. Thus, the paper, as an earlier one of the author on the longitudinal waves along conical rods [same J. **2** (1963), 135–152; MR **27** #4426], gives an interesting and carefully studied view of the behavior of certain types of one-dimensional viscoelastic waves.

W. Schumann (Zürich)

Mișicu, M. 5463

Theory of viscoelasticity with couple stresses and some reductions to two-dimensional problems. I.

Rev. Méc. Appl. 8 (1963), 921–952.

The author generalizes the governing equations of the linearized theory of dynamic viscoelasticity to include the effects of couple stresses. The equations thus obtained are used to set up the boundary-value problems for plane strain and generalized plane stress. The latter are then reformulated in terms of complex variables.

W. S. Edelstein (Providence, R.I.)

Shinozuka, M. 5464

Stresses in an incompressible viscoelastic-plastic thick-walled cylinder.

AIAA J. **2** (1964), 1800–1804.

Author's summary: "Stresses and strains in an incompressible viscoelastic-plastic thick-walled cylinder subject to an internal pressure are determined under conditions of infinitesimal plane strain. The internal pressure is assumed to increase monotonically as a function of time approaching a finite final value. With the von Mises yield condition, which is identical with the Tresca condition under the assumption of plane strain and incompressibility, the problem is statically determined; the stresses can be determined without the knowledge of the strains, which in the present problem have to be evaluated by solving an integral equation. A numerical example is given for a material with deviatoric stress-strain relations characteristic of a standard solid with yield limit in which the integral equation is reduced to a differential equation."

Irmay, Shragga 5465

Réfraction d'un écoulement à la frontière séparant deux milieux poreux anisotropes différents.

C. R. Acad. Sci. Paris **259** (1964), 509–511.

Author's summary: "Les lois de réfraction d'une ligne de courant et d'une équipotentielle à la surface separant deux milieux poreux anisotropes différents, sont indépendantes."

R. M. Morris (Cardiff)

Borodačev, N. M. [Borodačov, M. M.] 5466

On the solution of a contact problem in thermo-elasticity in the case of axial symmetry. (Russian)

Izv. Akad. Nauk SSSR Otd. Tehn. Nauk Meh. i Mašinostr. **1962**, *no.* 5, 86–90.

Indentation of a die of revolution in an elastic half-space in the presence of a given temperature distribution on the boundary plane is investigated. Using the Hankel transformation, the thermo-elastic stresses in the half-space are determined for a cylindrical die with a flat base and for conical and paraboloidal dies.

J. Nowinski (Newark, Del.)

FLUID MECHANICS, ACOUSTICS

See also 5002, 5080, 5432, 5578.

Hilbig, Harald 5467

Existenzsatz für begrenzte Potentialströmungen mit Totwasser um ein vorgegebenes Hindernis.

S.-B. Sächs. Akad. Wiss. Leipzig Math.-Natur. Kl. **105**, *no.* 7, 52 *pp.* (1964).

The existence and uniqueness of plane steady flow of an incompressible inviscid fluid with dead-water wake about a certain class of obstacles was proved by Leray [Comment. Math. Helv. 8 (1935), 149–180; ibid. 8 (1935), 250–263]. The present report extends this proof to flows bounded by a straight wall. The flow region is mapped on the interior of a half-ring. The Dirichlet problem for the interior of the half-ring is solved in definite-integral form. Leray's method, involving application of a fixed-point theorem of functional equations in Banach space, is then followed, and the existence theorem is proved.

W. R. Sears (Ithaca, N.Y.)

Irmay, Shragga 5468

Solution générale des écoulements incompressibles potentiels du type laplacien $u=\vec{\nabla}\varphi$ et du type de Poisson $u=K(x,y,z)\vec{\nabla}\varphi$.

C. R. Acad. Sci. Paris **259** (1964), 295–296.

A relation is obtained between the potential φ and the metric coefficients of the orthogonal curvilinear coordinate system of which the surfaces $\varphi=\text{const}$ form one family for the case when the velocity u is solenoidal. The author is apparently unaware that an orthogonal curvilinear coordinate system cannot, in general, be constructed because φ must satisfy a certain third-order partial differential equation in order that the system exists.

P. G. Saffman (Pasadena, Calif.)

Marris, A. W. 5469

Generation of secondary vorticity in a stratified fluid.

J. Fluid Mech. **20** (1964), 177–181.

Author's summary: "The theory is presented for the generation of secondary vorticity in a stratified fluid. The analysis is a generalization of original work of Hawthorne

[Proc. Roy. Soc. London Ser. A 206 (1951), 374–387; MR 13, 177]. It is shown that for an incompressible, inviscid and non-diffusive flow, a flow-wise vorticity component will be generated in a curved stream when a density gradient exists in the direction of the bi-normal to the streamline."

Hayes, Wallace D. 5470

Rotational stagnation point flow.

J. Fluid Mech. **19** (1964), 366–374.

This paper presents a solution for the flow of an inviscid incompressible fluid bounded by the fixed plane $z=0$. The components of velocity parallel to the plane are assumed to vary linearly in x and y, and the author derives a set of differential equations to determine the dependence on z. In general, the solution of these equations is non-analytic at $z=0$, and as a consequence the tangential vorticity becomes infinite on this plane. The stagnation streamline approaches the plane tangentially, in contrast to the two-dimensional case in which it meets the plane at a finite angle. Boundary-layer equations corresponding to the inviscid flows are also given.

The author concludes that, in a general three-dimensional flow past a body, vorticity in the mainstream will cause the flow near a stagnation point to behave as described in his solution. The opinion of the reviewer is that this is doubtful, since a real flow will involve quadratic and higher-order terms in x and y, and these terms contribute to the author's differential equations. In fact, if the velocity is analytic, but rotational, near the stagnation point, then it is locally two-dimensional.

E. J. Watson (Manchester)

Petrov, A. A.; Popov, Ju. P.; Puhnačev, Ju. V. 5471

Calculation by a variational method of the eigenvibrations of liquids in fixed containers. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 880–895.

Chaudhury, Z. H. 5472

Laminar wall-jet mixing of two different gases. (German summary)

Mathematika **11** (1964), 19–28.

Author's summary: "Solutions of the boundary-layer equations governing the radial laminar flow of a mixture of two different gases forming a wall jet are obtained. Attention is concentrated on flow in which the concentration of one gas in the mixture is small. The stream function is expanded in terms of a parameter whose magnitude depends upon the concentration of this gas in the mixture."

L. J. Crane (Dublin)

Olmstead, W. E.; Raynor, S. 5473

Depression of an infinite liquid surface by an incompressible gas jet.

J. Fluid Mech. **19** (1964), 561–576.

The problem of small angle depressions in a liquid surface due to an impinging two-dimensional potential jet is treated by conformal mapping techniques and a method due to Birkhoff and Zarantonello [*Jets, wakes and cavities*, Academic Press, New York, 1957; MR 19, 486]. The real and imaginary parts of the Levi-Civita function are shown to be related by an integral transform on one part of the liquid-surface profile, and by straightforward substitutions this can be rewritten as a finite Hilbert transform. The second boundary condition along the gas-liquid interface, after some manipulation, reduces the problem to the solution of the following non-linear singular integral equation

$$\bar{\tau}(r) = -\frac{1}{3}\lambda(1-r)\frac{d}{dr}\,{}^{*}\!\!\int_{-1}^{+1}\left(\frac{1-r}{1-s}\right)^{1/2}\frac{\left[\dfrac{2^{1/2}-(1-s)^{1/2}}{2^{1/2}+(1-s)^{1/2}}\right]}{(s-r)}e^{3\tau(s)}\,ds,$$

where the asterisk denotes the Cauchy principal integral. An asymptotic solution for the case in which the liquid surface profile approaches a straight line is derived and is relevant to the situation of low jet velocity. The non-linear integral equation is approximated by a set of non-linear algebraic equations by using the trapezoid rule and finite differences. These equations are solved by a method of repeated linear corrections and the liquid surface profiles and the free stream-lines of the jet are calculated in four cases. The authors conclude that for the cases they consider the physical model of a potential flow gas jet impinging on a motionless liquid surface yields results which compare favourably with experimental work of Banks and Chandrasekhara [J. Fluid Mech. **15** (1963), 13–34], in particular, verifying the appearance of lips on the liquid surface which become more prominent and more slightly outward as the centre line depression increases.

K. B. Ranger (Toronto, Ont.)

Stewartson, K. 5474

Falkner-Skan equation for wakes.

AIAA J. **2** (1964), 1327–1328.

The Falkner-Skan equation

$$f''' + ff'' + \beta(1-f'^2) = 0$$

is considered, subject to the boundary conditions $f(0)=f''(0)=0$, $f'(\infty)=1$, appropriate for the flow in a wake. The author seeks the dependence of $f'(0)$ on β as $\beta\to 0-$. Following a conjecture that $f'\to f_0'$ as $\beta\to 0-$ apart from a shift of origin of the independent variable, where f_0 is the solution for $\beta=0$ satisfying $f_0'(\infty)=1$ and $f_0'(-\infty)=0$, a small-perturbation analysis is used to obtain the estimate

$$\beta \approx -q^3\left[\frac{-2f'(0)}{\pi}\right]^{1/2}\exp\left[-qb+\frac{q^2}{2f'(0)}\right],$$

where $q=0.876$ and $b=2.014$. *M. G. Hall* (Farnborough)

Radbill, John R. 5475

Application of quasilinearization to boundary-layer equations.

AIAA J. **2** (1964), 1860–1862.

This note describes a straightforward application of the so-called quasilinearization method (monotonically convergent generalized Newton method) to the Falkner-Skan equation $f'''+ff''+\beta(1-f'^2)=0$, $f(0)=f'(0)=0$, $f'(\infty)=1$. The computational results show that the process gives good results even when starting from relatively poor initial approximations. *W. C. Rheinboldt* (Silver Spring, Md.)

Morse, T. F. 5476

Secular behavior in an analysis of damped standing waves in a gas.

Phys. Fluids 7 (1964), 1691–1695.

Author's summary: "By considering multiple time scale behavior in the viscous damping of a one-dimensional wave, it has been shown how secularities (terms growing unbounded in time) present in the Hilbert solution of this problem may be eliminated. The method of time scales and the accompanying freedom to remove secular terms overcomes in part the major disadvantage of the Hilbert procedure which requires a periodic renormalization of the lowest order initial conditions. This method, which has proved most useful in nonequilibrium statistical mechanics, provides a link between the Hilbert and Chapman-Enskog treatment of a simple hydrodynamic problem."

Bush, William B. 5477

Local similarity expansions of the boundary-layer equations.

AIAA J. 2 (1964), 1857–1858.

H. J. Merk [J. Fluid Mech. 5 (1959), 460–480] developed asymptotic expansions for the incompressible boundary-layer equation

$$f_{\eta\eta\eta}+ff_{\eta\eta}+\beta(\xi)[1-(f_\eta)^2] = 2\xi[f_\eta f_{\eta\xi}-f_\xi f_{\eta\eta}],$$

$$f(\xi,0)=f_\eta(\xi,0)=0,\quad f_\eta(\xi,\infty)=1.$$

This note shows that an additional contribution of first order in ξ should be considered in these expansions. Using this fact, a new approximate solution for the shear function at the wall is developed. The resulting correction turns out to be most important for small values of β.

W. C. Rheinboldt (Silver Spring, Md.)

Devan, Leroy; Oberai, Madan Mohan 5478

Approximate solution of second-order boundary-layer equations.

AIAA J. 2 (1964), 1838–1840.

The second-order boundary-layer equation derived by M. Van Dyke [J. Fluid Mech. 14 (1962), 161–177; MR 26 #5824; ibid. 14 (1962), 481–495; MR 26 #5825] is solved by using the von Kármán-Pohlhausen method. The second-order solution is divided into three parts accounting for the effects of displacement, curvature, and external vorticity. Polynomials of fourth degree are used to approximate each of these second-order solutions as well as the first-order solution. As an illustrative example, the flow near a stagnation point is worked out, and reasonable agreement is obtained when compared with results from existing calculations.

S. H. Lam (Princeton, N.J.)

Rogers, M. H.; Lance, G. N. 5479

The boundary layer on a disc of finite radius in a rotating fluid.

Quart. J. Mech. Appl. Math. 17 (1964), 319–330.

Authors' summary: "The axisymmetric boundary layer on a fixed circular disc of radius a due to a rotating fluid has been examined numerically. A series expansion solution, starting at the outer edge of the disc, is found to match the similarity solution due to Bodewadt at $r=\frac{1}{2}a$. Numerical solutions, obtained by using the series expansion approach, are also given for cases in which the disc rotates in the opposite sense to that of the external flow. These make it appear likely that the boundary layer erupts at the axis of symmetry, although the possibility of separation before the axis is reached cannot be ruled out."

L. J. Crane (Dublin)

Lam, S. H. 5480

Interactions of heat transfer and hypersonic boundary layers under highly favorable pressure gradients.

Proc. 1963 Heat Transfer and Fluid Mech. Inst. (*Calif. Inst. Tech., Pasadena, Calif.*, 1963), *pp.* 44–57. *Stanford Univ. Press, Stanford, Calif.*, 1963.

Author's summary: "The boundary layer in a hypersonic nozzle is studied. The main interest lies in the interesting interaction between the wall thermal condition and the development of the boundary layer itself. It is shown that for reasonable nozzle shapes and for high values of γ, the Falkner-Skan pressure gradient parameter, β, is generally large. Taking advantage of this observation, the boundary layer on the nozzle walls is analyzed in details by means of the method of inner and outer expansions."

Duty, R. L.; Reid, W. H. 5481

On the stability of viscous flow between rotating cylinders. I. Asymptotic analysis.

J. Fluid Mech. 20 (1964), 81–94.

The stability of Couette flow, in the case that the gap divided by the mean radius is small, is considered for the mathematically difficult situation $\mu=\Omega_2/\Omega_1\to-\infty$ (here Ω_1 and Ω_2 are the angular velocities of the inner and outer cylinders, respectively). The analysis is based on an asymptotic method in which the Taylor number is treated as a large parameter. The solutions are expressed asymptotically in terms of the solutions of the comparison equation $y^{(vi)}=xy$. The limiting values of the critical Taylor number and the critical wave number as $\mu\to-\infty$ are determined. It is found that there exists an infinite number of cells between the cylinders, but that the amplitude of the secondary motion in all but the innermost cell is small.

R. C. DiPrima (Rehovot)

Harris, D. L.; Reid, W. H. 5482

On the stability of viscous flow between rotating cylinders. II. Numerical analysis.

J. Fluid Mech. 20 (1964), 95–101.

The classical problem of the stability of Couette fluid, mentioned in the previous review [#5481], is treated by direct numerical procedures. The eigenvalue problem is solved by choosing a fundamental set of solutions so that the boundary conditions at one end are satisfied; the eigenvalue is determined by trial and error so that after integrating the remaining solutions to the other end-point, the boundary conditions there are satisfied. Results are given for the small-gap problem for a range $-3<\mu<1$ for the first mode, and $-2<\mu<1$ for the second mode [see #5481 above].

R. C. DiPrima (Rehovot)

Plesset, Milton S.; Hsieh, Din-Yu 5483

General analysis of the stability of superposed fluids.

Phys. Fluids 7 (1964), 1099–1108.

Two compressible, inviscid fluids in a gravity field are

separated by a plane interface and move parallel to the interface. The stability analysis is made based on the small-perturbation approach. The dispersion relation or the characteristic equation is found to involve Whittaker functions. The familiar Rayleigh-Taylor and Kelvin-Helmholtz criteria for stability are obtained in the incompressible limits. In the subsonic case it is found that compressibility decreases the range of stability. In the supersonic case it is found that Landau's stability holds only for one disturbance mode and that stratification, in general, leads to instability of the interface. The paper concludes with a simple physical discussion of the Kelvin-Helmholtz stability problem. *L. N. Tao* (Chicago, Ill.)

Simon, R. 5484

Stability of step shocks by the normal-modes method.

J. Mécanique **3** (1964), 165–172.

A stationary normal shock in a uniform stream of homogeneous fluid with an arbitrary equation of state is considered to be perturbed from its undisturbed position, $x=0$, to $x=\operatorname{Re} C_0 \exp i(\sigma t+ly+mz)$, with corresponding perturbations to the up and downstream flow quantities.

When the flow is supersonic ahead and subsonic behind the shock, it is concluded that a necessary and sufficient condition for stability is that F_s, a certain function of the unperturbed downstream fluid quantities, should be positive, in agreement with Erpenbeck [Phys. Fluids **5** (1962), 1181–1187; MR **27** #5449]. The conclusion is also stated to hold for oblique shocks if the normal flow is considered, which seems obvious. As the equation of state is arbitrary, cases other than supersonic ahead, subsonic behind, may arise; these are all stated to lead to instability. *H. C. Levey* (Perth)

Thomas, R. H.; Walters, K. 5485

The stability of elastico-viscous flow between rotating cylinders. II.

J. Fluid Mech. **19** (1964), 557–560.

From the authors' summary: "The work of Part I [same J. **18** (1964), 33–43; MR **28** #1855] is extended to include highly elastic liquids. To facilitate this, use is made of the orthogonal functions used by Reid [Proc. Roy. Soc. London Ser. A **244** (1958), 186–198; MR **19**, 1119] in the associated Dean-type stability problem. It is shown that the critical Taylor number decreases steadily as the amount of elasticity in the liquid increases, until a transition is reached, after which the roots of the determinantal equation that determines the Taylor number as a function of the wave-number become complex. It is concluded that the principle of exchange of stabilities may not hold for highly elastic liquids." *D. W. Dunn* (Ottawa, Ont.)

Lettau, H. 5486

A new vorticity-transfer hypothesis of turbulence theory.

J. Atmospheric Sci. **21** (1964), 453–456.

After criticizing the momentum-transfer and vorticity-transfer hypotheses, the author proposes a new expression for $\mathbf{c}'$, the vorticity change caused by particle displacement $\mathbf{r}'$:

$$\mathbf{c}' = -\mathbf{r}'\cdot\nabla\bar{\mathbf{c}}+\bar{\mathbf{c}}\cdot\nabla\mathbf{r}',$$

where $\bar{\mathbf{c}}$ is the mean vorticity. The first term is that of the ordinary vorticity-transfer theory, while the second represents adjustment of the particle vorticity toward the local value. An equivalent form is $\mathbf{v}'=\mathbf{r}'\times(\nabla\times\bar{\mathbf{V}})$, from which Reynolds stresses can be calculated in terms of quantities such as $\overline{x'z'}$. The von Kármán constant appears as $|\overline{x_z'y_y'}|^{1/2}$. For flow in a constant-stress layer near a wall, consistency of the new hypothesis with the basic mechanics requires a logarithmic velocity profile. Brief mention is made of application to flow with bi-directional shear and to large-scale motion in the atmosphere. *A. A. Townsend* (Cambridge, England)

Pao, Yih-Ho 5487

Statistical behavior of a turbulent multicomponent mixture with first-order reactions.

AIAA J. **2** (1964), 1550–1559.

From the author's summary: "The dilute turbulent concentration fields of a multicomponent mixture with any type isothermal first-order reaction are investigated. For certain given initial conditions, the reacting concentration fields are related to the nonreacting case provided that the diffusivities of all species with respect to the main solution are equal. Known results of turbulent mixing are used to estimate the rate of decay and/or growth of the reacting turbulent concentration fields. For the case of stationary turbulence with first-order reactions and with nonequal diffusivities, where the Reynolds number and Peclet number are large, the small-scale structure of the turbulent concentration fields are investigated under the assumption of local isotropy and local homogeneity. A unified concept for spectral transfer at large wave numbers is proposed which, in essence, is a generalization of the Onsager-Corrsin spectral transfer concept. With this unified spectral transfer concept, as well as other concepts, the reacting concentration spectrum functions are deduced for three-wave number ranges: (1) the inertial-convective range, (2) the viscous-convective and viscous-diffusive range for large Schmidt numbers, and (3) the inertial-diffusive range for small Schmidt numbers." *P. G. Saffman* (Pasadena, Calif.)

Rotta, J. C. 5488

Incompressible turbulent boundary layers. (French summary)

Mécanique de la Turbulence (Marseille, 1961), pp. 255–285. (1 plate) Éditions Centre Nat. Recherche Sci., Paris, 1962.

Author's summary: "A brief outline of the present knowledge of turbulent boundary layers in two-dimensional incompressible flow is given. In particular, the attention is directed towards the different behaviour of the flow near the wall and in the outer part of the layer. The applicability of the two-layer conception is reviewed and the problem to predict the development of the turbulent boundary layer in an arbitrarily given pressure field is discussed."

Smyglevskiĭ, Ju. D. [Шмыглевский, Ю. Д.] 5489

★Some variational problems in gas dynamics [Некоторые вариационные задачи газовой динамики].

Vyčisl. Centr Akad. Nauk SSSR, Moscow, 1963. 142 *pp.* 0.71 *r.*

This monograph deals with variational problems in steady supersonic gas dynamics, e.g., minimal drag bodies, maximum thrust nozzles, etc. It is divided into six chapters.

Chapter 1 gives the general equations of gas dynamics and a detailed discussion of the steady flow characteristics. Chapter 2 considers variational problems for shockless flows in a fixed region; necessary conditions for extrema are derived, and solution methods for variational problems are presented.

Chapter 3 considers variational problems for flows with shocks, and Chapter 4 discusses minimal drag bodies; in both these chapters, however, only bodies with no shocks at the front-most point are dealt with. Bodies with attached nose shocks are studied in Chapter 6.

Chapter 5 deals with variational problems involving nozzles and jets, i.e., internal flows and axi-symmetric free-streamline flows.

The bibliography (25 items) contains Soviet works on the subject and many Western references on optimal nozzles, but none on minimal drag bodies.

C. K. Chu (New York)

Fisher, Donald D. 5490

Calculation of subsonic cavities with sonic free streamlines.

J. Math. and Phys. **42** (1963), 14–26.

This paper gives the solution of the compressible Riabuchinsky flow by means of numerical methods in the hodograph plane. Consider a two-dimensional, everywhere-subsonic flow, with given Mach number and parallel streamlines at infinity. A Riabuchinsky cavity consists of two flat plates orthogonal to the flow, with a cavity in between bounded by two free streamlines. The problem treated in this paper is to determine the cavity ratio such that the flow is everywhere subsonic, except on the free streamlines where it is required to be sonic.

The equations are transformed into the logarithmic hodograph plane. The resulting equations are linear, and there is no difficulty with the boundary conditions in the hodograph plane as is usually encountered. However, the sonic line is singular in that the elliptic differential equations become parabolic there. The solution is calculated in the hodograph plane by an appropriate finite-difference scheme which takes into account this singularity. The solutions obtained agree closely with those of Helliwell and Mackie [J. Fluid Mech. **3** (1957), 93–109; MR **19**, 914; Quart. J. Mech. Appl. Math. **12** (1959), 298–313; MR **22** #2234] using the Tricomi approximation, while the von Kármán-Tsien approximation is shown to yield inferior accuracy in this problem. This problem is of interest in the solution of the extremal problem of determining the maximum critical Mach number profile due to Gilbarg and Shiffman [J. Rational Mech. Anal. **3** (1954), 209–230; MR **15**, 756]. *C. K. Chu* (New York)

Borisov, V. M. 5491

On the optimal shape of bodies in a supersonic gas flow. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **3** (1963), 786–793.

Variational methods are used to determine the optimum shape of a flat-nosed body of revolution producing minimum wave resistance in a uniform supersonic stream. The problem is solved, firstly, for frozen flow and, secondly, for equilibrium flow. Meridian sections of the optimum body shapes are calculated in two representative cases.

M. Holt (Berkeley, Calif.)

Luboński, Jan 5492

Hypersonic, plane Couette flow in rarefied gas. (Polish and Russian summaries)

Arch. Mech. Stos. **14** (1962), 553–560.

Plane Couette flow of a rarefied gas is treated under the assumption that the walls slide relatively to each other with a speed much greater than the thermal speed of the molecules. The gas is treated as a mixture of three types of molecules; those having just left one wall, those having just left the other wall, and those having experienced multiple collisions. Under the assumption of small gap and large velocity, the collision integrals in the Boltzmann equation for each of the three species are either neglected or replaced by simple expressions. The three simultaneous equations are then solved iteratively, the zeroth order term being free molecular flow. The paper does not compare its results with the standard Couette flow results available in the literature in the limit of high-wall velocities. *C. K. Chu* (New York)

Sjagaev, V. F. 5493

A method of solving numerically a problem of the flow of a supersonic gas around conical bodies. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **3** (1963), 742–754.

The problem of supersonic flow past a yawed cone is solved by a numerical finite-difference process. In effect, the unknown conical shock wave is represented as one of a two-parameter family of surfaces. The calculation starts with an assumed position of shock wave, providing enough data for a Cauchy problem, the solution of which determines a certain body surface. The assumed shock position is adjusted successively until the body surface coincides with the actual cone to the desired degree of accuracy. Calculations for a circular cone agree well with other theoretical results and with experiment.

M. Holt (Berkeley, Calif.)

Korobeinikov, V. P. [Коробейников, В. П.]; 5494
Mel'nikova, N. S. [Мельникова, Н. С.];
Rjazanov, E. V. [Рязанов, Е. В.]

★Theory of point explosions [Теория точечного взрыва].

Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1961. 332 *pp.* 1.22 *r.*

This book is a sequel to *Similarity and dimensional methods in mechanics* by L. I. Sedov [Academic Press, New York, 1959; MR **21** # 6840]. The first two chapters recapitulate much of the basic theory set out in the earlier volume. The third chapter introduces the main topic, namely, the approximate treatment of non-self-similar problems as linear perturbations of self-similar problems. The remaining chapters deal with practical problems of this type, and these fall into two main categories. First, there are problems in which self-similarity is violated due to consideration of more than two independent physical parameters (e.g., point explosions when account is taken of the pressure in the undisturbed medium). In the second class of problems, additional independent variables are

introduced (e.g., point explosions in a medium with a density gradient or in a moving medium). A wide variety of problems of the first kind are considered in detail; these include not only unsteady flow problems but also, using the blast wave analogy, steady hypersonic flow problems. Fewer solutions of the second type are available. The book is lucidly written and is recommended to those active in research in gas dynamics.

M. Holt (Berkeley, Calif.)

Sichel, Martin 5495

Structure of weak non-Hugoniot shocks.

Phys. Fluids **6** (1963), 653–662.

Shock structure is investigated in cases when the shock thickness is comparable with its curvature, so that the effects of tangential gradients of velocity, pressure, etc., can no longer be neglected. The author investigates this problem in transonic flow, solving the Navier-Stokes equations by means of an expansion in terms of a transonic parameter. The governing equation combines the characteristics of Burgers' equation and the transonic small disturbance equation. The structure of weak shocks is determined by a similarity solution of this equation. Expressions for the velocity components inside the shock are derived.

M. Holt (Berkeley, Calif.)

Gerber, Henry 5496

Acoustic properties of fluid-filled chambers at infrasonic frequencies in the absence of convection.

J. Acoust. Soc. Amer. **36** (1964), 1427–1434.

Author's summary: "The general equation of heat conduction inside a fluid-filled chamber takes two forms, which correspond to zero and infinite acoustic impedance of the driving source, respectively. By means of a simple transformation, it is possible to convert this equation to the standard diffusion equation with simple initial and boundary conditions. Special solutions to this equation are used to derive transfer functions E, which relate the impressed temperature to the resultant average temperature. The equation of state of the fluid is then used to determine the relationships between pressure and volume changes in terms of E. These relationships take the form of two different acoustic impedances Z_p and Z_v, which correspond to zero and infinite source impedance, respectively. In general, the chamber impedance is a function of Z_p, Z_v, and the external impedances. The above results can be used to extend the validity of acoustic circuit diagrams to zero frequency. The infrasonic acoustic impedance is an R-C network, and numerical values of the components can be calculated in a simple manner."

Gerber, Henry 5497

Calculation of dynamic-pressure changes in the presence of free convection.

J. Acoust. Soc. Amer. **36** (1964), 1435–1441.

Author's summary: "If a piston compresses and heats gas in a chamber that is larger than a few cubic inches, then the heat exchange between the gas and the wall is governed by convection. The simultaneous, nonlinear partial differential equations of free convection are given. Application of dimensional analysis to these equations indicates that the convection process can be described in terms of three dimensionless variables: the Grashof number, the Prandtl number, and the thermal-diffusion number. A simplified model of the process of free convection is derived and it is shown that the geometrical constants of the model can be expressed in terms of the Grashof and Prandtl numbers. The chamber length l and the boundary-layer thickness l_b play a key role. Mathematical solutions for the chamber pressure that follows a step piston displacement are derived and they agree within 1% with the measured pressure."

Atabek, H. B. 5498

Start-up flow of a Bingham plastic in a circular tube.

Z. Angew. Math. Mech. **44** (1964), 332–333.

A solution is presented for the title problem for a step function pressure gradient loading. As is usual, the Bingham material is assumed rigid for stresses below the critical stress. This assumption leads to minor difficulties in the representation of the stress field in the rigid core of the material during acceleration.

No numerical results are given, which is unfortunate since the solution is given in a Fourier-Bessel series which is not readily evaluated numerically.

P. R. Paslay (Houston, Tex.)

DeSilva, Carl N.; Kline, Kenneth A. 5499

Rectilinear laminar flow of a visco-elastic fluid. (German summary)

Z. Angew. Math. Phys. **15** (1964), 557–560.

Authors' summary: "Eine inkompressible viscoelastische Flüssigkeit fliesst zwischen zwei parallelen, starren, unendlichen Wänden. Es wird unter der Voraussetzung, dass die eine Wand fest ist und die andere sich mit einer exponentiell abklingenden Geschwindigkeit bewegt, das zur Aufrechterhaltung einer speziellen laminaren Strömung nötige Kraftfeld bestimmt."

Rajeswari, G. K. 5500

On the effects of variable suction on the steady laminar flow due to rotating bodies of revolution in non-Newtonian fluid. (German and Russian summaries)

Z. Angew. Math. Mech. **44** (1964), 193–202.

Author's summary: "The effects of variable suction on the steady laminar flow within the boundary layer due to a rotating sphere, prolate and obalte spheroids in non-Newtonian fluid are investigated. The secondary flow patterns in a meridian section past these bodies of revolution are represented graphically. Displacement and momentum thicknesses along the bodies of revolution are evaluated."

Singh, Devi 5501

Flow of visco-elastic Maxwell fluid through concentric circular cylinders.

Z. Angew. Math. Mech. **44** (1964), 330–331.

A Maxwell fluid is a visco-elastic material in which the elastic stresses are characterized by a simple relaxation time. In this paper, the author treats the flow of such a fluid between two concentric rotating cylinders when the

applied pressure and angular velocity of the cylinders decay exponentially with arbitrary time constants. Solutions are obtained in terms of Bessel functions of first and second order for axial and circumferential velocity components, which are independent for axially symmetric flow. The solution is shown to have different form for large and small values of relaxation time. The interested reader should note that the time derivative of the pressure has been omitted from Equation 2.3.

{Reviewer's note: The change of form of the solution with increasing relaxation time demonstrates the non-uniform interaction of the viscous and elastic material properties. For large relaxation time, the asymptotic form of the solution has a boundary-layer character: the viscous flow is restricted to an exponentially thin boundary layer on the pipe wall; the bulk of the material behaves elastically, moving as a solid body in the central core. In contrast, for small relaxation time, viscous effects permeate the entire flow.} *O. R. Burggraf* (Columbus, Ohio)

★Magnetofluidodinamica. 5502

3° Ciclo, Varenna, Villa Monastero, 28 settembre–6 ottobre 1962. Centro Internazionale Matematico Estivo (C.I.M.E.).

Edizioni Cremonese, Rome, 1964. 330 *pp.* (*not consecutively numbered*). *L.* 4.000.

This collection contains nine papers, which will be reviewed individually. In addition to the unfortunate pagination, the publishers have failed to include the date of printing of the volume.

Agostinelli, Cataldo 5503

Forma assoluta spazio-temporale delle equazioni della magneto-fluidodinamica.

Univ. Nac. Tucumán Rev. Ser. A **14** (1962), 127–152.

L'auteur établit d'abord les équations intrinsèques dans une variété spatio-temporelle à quatre dimensions pour le mouvement d'un fluide conducteur soumis à la gravitation. Les détails donnés sur cette variété et sur la mise en équation permettent l'accès de l'article à des lecteurs peu initiés. A partir de ces équations, les composantes spatiales sont ensuite isolées pour revenir à la formulation tridimensionnelle. Puis un système d'équations approchées est obtenu en négligeant dans les équations précédentes les termes d'ordre supérieur à 1 par rapport à $1/c$; par exemple, l'équation habituelle de conservation de la masse se trouve maintenant remplacée par:

$$\frac{\partial\rho}{\partial t}+\operatorname{div}(\rho\mathbf{V})-\frac{1-\epsilon\mu}{2c\mu}\operatorname{div}(\mathbf{E}\wedge\mathbf{B})=0.$$

En négligeant enfin dans le système précédent les termes qui contiennent le facteur $(1-\epsilon\mu)/(c\mu)$, on retrouve les équations usuelles. *R. Thibault* (Sotteville-lès-Rouen)

Bhatnagar, P. L.; Kumar, Ram 5504

Propagation of small disturbances in viscous and electrically conducting liquids in the presence of magnetic field.

Indian J. Math. **6** (1964), 1–19.

This paper extends the authors' results [same J. **2** (1960), 41–66; MR **22** #5976] to a fluid that is viscous and electrically conducting. *C. S. Morawetz* (New York)

Caldwell, Douglas R. 5505

Oscillating boundary layer in magnetohydrodynamics. I. Nonconducting boundary.

Phys. Fluids **7** (1964), 1062–1070.

S. Lundquist [Phys. Rev. (2) **76** (1949), 1805–1809] and B. Lehnert [ibid. (2) **94** (1954), 815–824] have investigated the generation of Alfvén waves in liquid metals. The corresponding experimental results have not agreed at all with the theoretical results. According to the author of the present paper, this discrepancy is caused by the neglect of mechanical friction (i.e., neglect of kinematic viscosity term) compared to magnetic viscosity, i.e., in other words, the main dissipative mechanism in the fluid is considered to be Joule losses, not mechanical friction. In the present paper, the author takes into consideration both the above terms and obtains as a final result a fourth-order differential equation from the Navier-Stokes equation, Ohm's law and Maxwell's equations instead of a resulting second-order differential equation.

With this in view and applying relevant boundary conditions, the author solves the problem of the oscillating non-conducting disk immersed in a fluid of infinite extent, where the direction of a constant magnetic field H_0 coincides with the axis of symmetry of the oscillating disk. The experimental measurements of the damping of such a torsional pendulum in liquid like mercury in the presence of an axial magnetic field H_0 are in nice agreement with the theoretical results up to $H_0\sim 10$ kilogauss. *K. M. Ghosh* (Belgharia)

Caldwell, Douglas R. 5506

Oscillating boundary layer in magnetohydrodynamics. II. Conducting boundary.

Phys. Fluids **7** (1964), 1338–1348.

The problem discussed in this paper is a continuation of an earlier problem discussed by the same author [#5505]. In the earlier problem the author considered hydromagnetic motion caused in an infinite fluid by an oscillating non-conducting boundary in the presence of an axial magnetic field, both theoretically and experimentally. In the present paper he discusses the general problem of the generation of Alfvén waves by an oscillating electrically conducting boundary in a fluid of finite extent in the presence of an axial magnetic field where the oscillating plate and plates bounding the fluid have arbitrary electrical conductivity. Results are given specifically for cases where the fluid region is much larger than the viscous penetration depth, but much smaller than the wavelength of Alfvén waves. According to the author, these are necessarily the experimental conditions for this oscillating viscometer since the viscous penetration depth is less than 0.04 cm at usual frequencies in mercury and the Alfvén mode wavelength is greater than a meter. The experimental data as obtained here from the above oscillating disk viscometer have given somewhat good qualitative agreement with the theoretical calculations made from the infinite plate approximation. *K. M. Ghosh* (Belgharia)

Förste J. 5507

Ein Beispiel hydromagnetischer Induktion.

Monatsb. Deutsch. Akad. Wiss. Berlin **6** (1964), 561–565.

An incompressible, viscous and electrically conducting fluid is flowing under a time-dependent pressure gradient along the annular passage formed by two concentric,

straight circular tubes. It is assumed that the magnetic force is small compared to the dynamic force ($M^2/(\mathrm{Re\ Rm})$ small; M = Hartmann number, Re = Reynolds number, Rm = magnetic Reynolds number), so that the effect of the magnetic force on the fluid motion is negligible. The effect of the fluid motion on the magnetic field is studied under the assumption that at both surfaces of the wall the magnetic lines are perpendicular to the wall at all times ($t \geq 0$).

The author proves that the above boundary-value problem for the velocity and magnetic fields possesses a unique solution for a prescribed pressure function $p = f(t)$ and that the solution tends to a limit for $t \to \infty$ if $\lim_{t\to\infty} f(t)$ exists. Furthermore, it is shown that the limit satisfies the stationary-state differential equations and can be expressed in terms of elementary functions.

Finally, the author justifies his assumption regarding the effect of the magnetic field on the motion of the fluid by indicating that the solution obtained for the magnetic field only gives rise to magnetic forces of negligible magnitudes. *I-Dee Chang* (Stanford, Calif.)

Kapur, J. N. 5508

Characterisation of axially-symmetric self-superposable hydromagnetic flows.

Bull. Calcutta Math. Soc. **54** (1962), 59–66.

The author continues the discussion initiated in two earlier papers [Appl. Sci. Res. A 8 (1958), 198–208; ibid. 9 (1959), 139–147] to obtain equations characterizing axially-symmetric self-superposable hydromagnetic flows and axially-symmetric force-free fields.

L. M. Milne-Thomson (Tucson, Ariz.)

Nikol'skii, A. A. 5509

Hyperbolic problems for magnetohydrodynamic perfect fluid flows with "frozen-in" circular magnetic field lines. (Russian. Polish and English summaries)

Arch. Mech. Stos. **14** (1962), 675–682.

The paper deals with unsteady, axisymmetric flows of an infinitely conducting incompressible fluid, linearized about some equilibrium state for which the magnetic lines of force are azimuthal. Particular solutions are sought. The governing equations are seen to be either elliptic or hyperbolic in type. Some stability criteria are given.

C. K. Chu (New York)

Wells, Daniel R. 5510

Axially symmetric force-free plasmoids.

Phys. Fluids **7** (1964), 826–832.

The theory of force-free magnetic fields is applied to equilibrium configurations of an incompressible fluid with axisymmetric motions. It is demonstrated that solutions satisfying collinear force-free flow satisfy the vorticity equation for zero viscosity. Experimental measurements of the magnetic and flow fields of the plasmoids generated by a conical θ-pinch plasma gun are shown to be in agreement with the solutions to these equations.

J. N. Kapur (Kanpur)

Uberoi, C. 5511

Gravitational instability of an infinitely extending layer of finite thickness surrounded by non-conducting material in the presence of magnetic field and rotation.

J. Indian Inst. Sci. **46** (1964), 11–27.

Stability calculations are given for a rotating layer of conductive fluid surrounded by a non-conductive fluid of different density, subjected to a magnetic field normal to the axis of rotation and including gravitational effects. Although a small perturbation method is used, the actual expressions given for this perturbation are not bounded. In these expressions and in the dispersion equation a quantity G occurs which is not defined and which does not occur in the equations of motion or in the boundary conditions. The reviewer is therefore not convinced of the validity of the results. *L. J. F. Broer* (Eindhoven)

Gheorghitza, St. I. [Gheorghiţă, Şt. I.] 5512

A generalization of the circle theorem.

Bull. Calcutta Math. Soc. **54** (1962), 97–101.

The author generalizes the reviewer's circle theorem to flow in porous media in which the non-homogeneity occurs in crossing two concentric circles. The method can be extended (with increasing complexity) to the case of three or more concentric circles and, of course, to other analogous problems in mathematical physics.

L. M. Milne-Thomson (Tucson, Ariz.)

Gheorghitza, Stefan I. [Gheorghiţă, Şt. I.] 5513

Sur le mouvement dans un milieu poreux homogène ayant une cavité elliptique.

C. R. Acad. Sci. Paris **259** (1964), 2779–2780.

Author's summary: "On indique quelques propriétés du mouvement plan stationnaire linéaire dans un milieu poreux homogène illimité ayant une cavité elliptique, quand aux grandes distances il y a un courant uniforme."

OPTICS, ELECTROMAGNETIC THEORY, CIRCUITS

See also 4973, 5063, 5235, 5567, 5585.

Hahn, Dietrich; Metzdorf, Joachim; Schley, Ulrich; Verch, Joachim 5514

★Seven-place tables of the Planck function for the visible spectrum. (Text in English and German)

Friedr. Vieweg & Sohn, Braunschweig; Academic Press, New York-London, 1964. xxi + 135 pp. $5.50.

The main table gives $S_t = \lambda^{-5}[\exp(C_2^{(i)}/\lambda T) - 1]^{-1}$ for λ = 350(5)845nm, for various T in the range 973.15 to 15000°K and for two values of C_2: the present international standard, $C_2^{(1)} = 14380$ and $C_2^{(2)} = 14420\mu m$°K, close to an expected revised standard. There is also given, for λ = 385(5)845nm, and for the same T, the values of $S_t V(\lambda)$, where $V(\lambda)$ is a function, tabulated to 3D, which represents a certain aspect of the behavior of the human eye.

John Todd (Pasadena, Calif.)

Alvazjan, Ju. M.; Mergeljan, O. S. 5515

On the determination of the parameters of optically active media by optical methods. (Russian)

Izv. Akad. Nauk Armjan. SSR Ser. Fiz.-Mat. Nauk **17** (1964), no. 4, 125–126.

The authors show theoretically that the refractive index and the gyration constant of an optically active isotropic medium can be determined by observing the rotation of the plane of polarization of a linearly polarized light wave

when the wave is incident normally on a slab and propagates through it. The observations are performed twice: first with no external magnetic field and then with the field applied parallel to the wave vector so as to compensate for the rotation due to the natural optical activity.

Y. Shimizu (Tokyo)

Bardotti, Graziella; Bertotti, Bruno; Gianolio, Laura 5516

Magnetic configuration of a cylinder with infinite conductivity.

J. Mathematical Phys. 5 (1964), 1387–1390.

The azimuthal current density in an infinitely conducting closed cylindrical sheet has been calculated. This is done by numerical solution of an integral equation for various values of the radius to height ratio.

L. J. F. Broer (Eindhoven)

Geppert, Donovan V. 5517

Free-space bounded plane-wave solutions to the Maxwell-Lorentz equations.

Phys. Rev. (2) 134 (1964), B1407–B1409.

Author's summary: "A relativistic fluid concept of electric charge is introduced, from which free-space laterally-bounded plane-wave solutions to the inhomogeneous Maxwell-Lorentz equations can be obtained in harmony with the particle-like nature of the radiation field."

Hatfield, W. B.; Auld, B. A. 5518

Method of characteristics solution for electromagnetic wave propagation in a gyromagnetic medium.

J. Math. and Phys. 43 (1964), 34–37.

The authors purport to give conditions for simple waves to exist in what is known to the reviewer as a magnetically gyrotropic medium. They then purport to show that some of these may steepen to electromagnetic shocks.

In the opinion of the reviewer this paper should be considerably revised. In the first case, when dealing with transient phenomena, as in this paper, it is not valid to speak of characteristic frequencies in relation to the resonant frequency of the medium to derive the M-H relation. What is true is that if the approximate Landau-Lifshitz equation for a ferrite magnetised to saturation has been used, then the characteristic time for the phenomena under discussion must be long compared with the characteristic time, $(\gamma H)^{-1}$, occurring in that equation, and this condition would be violated ultimately for a steepening simple wave. The characteristic equations are indeed simply derivable, though the reviewer disagrees with the result obtained for the second set. Also, no hint is given as to how the auxiliary conditions necessary for the degeneracy to simple waves are obtainable from possible initial and boundary conditions. Some other conclusions with which the reviewer does not agree are also presented, but it seems useless to pursue them in the light of the points already raised.

H. C. Levey (Perth)

Macke, Wilhelm 5519

★Elektromagnetische Felder: Ein Lehrbuch der theoretischen Physik.

Akademische Verlagsgesellschaft Geest & Portig K.-G., Leipzig, 1960. xvi+393 *pp.* *DM* 29.50.

In the order of appearance this is after "Wellen" and "Quanten" [*Wellen*, 1958; *Quanten*, 1962; MR 27 #3275], the third volume out of six. In spite of the original announcement of the series, which intended to give common cross-sections through various special subjects, the present volume does not make a broad cross-section through classical and quantum theories from the point of view of the field concept. The slice which has been cut out after restricting the fields to electromagnetic ones is hardly different from that in conventional textbooks on classical electrodynamics. Owing to the original intention of the series, there are inevitable overlaps with other volumes. The density and current density of continuum mechanics, which are used as an illustration for the introduction of scalar and vector fields, might overlap with the announced volume on mechanics of particles, systems and continua. Various aspects of electromagnetic waves have been treated more extensively in the volume on waves and, e.g., the 4-dimensional tensor representation of the energy-momentum density-current density or the canonical formalism are only treated or mentioned there. Special relativity theory, which already had been introduced rather transparently in the volume on waves, is introduced once more along somewhat different lines, now making use of the Lorentz covariance of the electromagnetic part of the non-relativistic equations. It is not clear whether difficulties of overlap and spreading are partially due to deviations from the original intention, or that these deviations are due to difficulties in making the intended cross-sections. Anyhow, overlaps may also be instructive for the student. The presentation of the present volume is again very clear. Electrodynamics is transparently built up step by step via static and stationary systems. In every step the essential mutual relations of the generation of the electromagnetic field by discrete or continuous electric charges and currents and the action of the field on the charges and currents are carefully explained. The electromagnetic properties of matter are dealt with in a somewhat schematic way. As technical applications the modern linear and circular high-energy machines are considered. MKSA units are used. In the three volumes which have appeared up till now, the imaginary unit is denoted by j in the complex notation for electromagnetic and other classical waves, by i in the pseudo-euclidean notation of 4-dimensional Minkowski space in special relativity theory and also by i in quantum mechanics. It may make sense to use a separate notation for the pseudo-euclidean tensors, but then (contrary to what has been done in the volume on waves) the other notation should still be used in the exponentials, which describe classical waves. It seems difficult to keep friends with electrical engineers and quantum theorists all the time and to maintain consistently a different notation for classical and quantum waves, in particular, in a semi-classical treatment of, e.g., interaction between quantized electrons and classical electromagnetic waves. Such a treatment has not been given in any of the three volumes, but a justification of the different and difficult notation has neither been given. There are again many instructive problems with solutions.

H. J. Groenewold (Zbl 91, 208)

Morgan, Thomas A. 5520

Two classes of new conservation laws for the electromagnetic field and for other massless fields.

J. Mathematical Phys. 5 (1964), 1659–1660.

Recently D. M. Lipkin [same J. 5 (1964), 696–700; MR 28 #5683] studied a class of differential invariants of the free electromagnetic field which were defined in terms of a tensor of rank 3. Lipkin's invariants are bilinear in the field vectors, but involve their first partial derivatives with respect to the space-time coordinates.

The author extends Lipkin's analysis to show the existence of an infinite family of tensor differential invariants for the free electromagnetic field. A corresponding analysis is given also for free spinor fields of particles of zero rest-mass of the type studied by M. Fierz and W. Pauli [Proc. Roy. Soc. London Ser. A 173 (1939), 211–232; MR 1, 190]. {Since this work, like that of Lipkin, deals with free fields, its interpretation in terms of physical theory appears to depend on the physical content which can be attributed to the Cauchy initial-value problem as applied to these fields.} *E. L. Hill* (Minneapolis, Minn.)

Petit, Roger; Cadilhac, Michel 5521

Étude théorique de la diffraction par un réseau.

C. R. Acad. Sci. Paris **259** (1964), 2077–2080.

Die Autoren behandeln das Problem der Beugung einer ebenen, senkrecht einfallenden elektromagnetischen Welle, deren $\mathfrak{E}$-Vektor parallel zur z-Achse gerichtet ist, an einer leitenden, periodisch verformten Ebene $y=f(x)$, $-\infty<x<\infty$, $-\infty<z<\infty$, mit $f(x+d)=f(x)$. Schreibt man die z-Komponente des Gesamtfeldes in der Form $E=E^i+E^d$ mit $E^i(x,y)=e^{-iky}$, so ergibt sich für $E^d(x,y)$ die folgende mathematische Randwertaufgabe. Gesucht ist: $E^d(x,y)$ in $y>f(x)$, $-\infty<x<\infty$, mit

$$\Delta E^d+k^2\cdot E^d=0,\quad E^d[x,f(x)]=-E^i[x,f(x)]=g(x),\quad -\infty<x<\infty,$$

E^d enthalte für $y\to+\infty$ nur auslaufende, beschränkte Wellen. Wegen $E^d(x+d,y)=E^d(x,y)$ läßt sich $E^d(x,y)$ in eine Fourierreihe entwickeln, die jedoch im gewöhnlichen Sinne nur für $y>\max_{-\infty<x<\infty}f(x)$, $-\infty<x<\infty$, Gültigkeit hat. Die Autoren geben ein Verfahren an, die Lösung zu $\tilde{E}(x,y)$ in das Gebiet $y<f(x)$, $-\infty<x<\infty$, fortzusetzen. Längs der deformierten Ebene $y=f(x)$ findet wegen der Randbedingung ein stetiger Anschluß statt, aber es muß zugelassen werden, daß $\partial\tilde{E}/\partial y$ dort Sprünge besitzt, deren Höhen vorerst unbekannt sind. Demzufolge soll die periodische Funktion $\tilde{E}(x,y)$ der inhomogenen Wellengleichung $\Delta E+k^2\cdot E=2ik\cdot\varphi(x)\cdot\delta[y-f(x)]$ genügen, wobei $\delta(u)$ das Dirac-Maß bezeichne. $\tilde{E}(x,y)$ hat dann die Fourierentwicklung

$$\tilde{E}(x,y)=\int_0^d R[x-x',y-f(x)]\varphi(x')\,dx',$$

$$R(x,y)=\frac{kK}{2\pi}\sum_{n=-\infty}^{\infty}\frac{\exp[inKx+i\chi_n|y|]}{\chi_n},$$

$$\chi_n=\sqrt{(k^2-n^2K^2)},\qquad K=\frac{2\pi}{d}.$$

$\varphi(x)$ muß dann der Integralgleichung

$$\tilde{E}[x,f(x)]=\int_0^d R[x-x',f(x)-f(x')]\varphi(x')\,dx'=g(x),\quad 0<x<d,$$

genügen, welche einen symmetrischen Kern besitzt. Es werden dann noch einige Spezialfälle der Entwicklung von $\tilde{E}(x,y)$ diskutiert. *E. Meister* (Saarbrücken)

Wirgin, Armand 5522

Considérations théoriques sur la diffraction par réflexion sur des surfaces, quasiment planes, applications à la diffraction par des réseaux.

C. R. Acad. Sci. Paris **259** (1964), 1486–1488.

Der Autor diskutiert kritisch eine Formel von Lord Rayleigh [Proc. Roy. Soc. London Ser. A 79 (1907), 399–416], die die Reflektion einer ebenen, elektromagnetischen Welle an einer periodisch deformierten, leitenden Ebene $z=f(x)$, $-\infty<x<+\infty$, $f(x+a)=f(x)$, in der Form

$$(1)\quad U(x,z)=U_i(x,z)+\sum_{n=-\infty}^{\infty}C_n\exp\left\{i\left[\left(k\sin\theta+\frac{2n\pi}{a}\right)x+\kappa_nz\right]\right\}$$

mit $\kappa_n=\sqrt{(k^2-(k\sin\theta+2n\pi/a)^2)}$ und

$$U_i(x,z)=\exp[ik(x\sin\theta-z\cos\theta)]$$

beschreibt. Unter Verwendung der Kirchhoff-Helmholtzschen Integraldarstellung für das Streufeld $U_s(x,z)$ in $z>f(x)$, $-\infty<x<+\infty$, ergibt sich die Formel (1) nur für $z\geqq\max_{0\leq x\leq a}f(x)$ und

$$(2)\quad U(x,z)=U_i(x,z)+\sum_{n=-\infty}^{\infty}\left(B_n^+\exp\left\{i\left[\left(k\sin\theta+\frac{2n\pi}{a}\right)x+\kappa_nz\right]\right\}+B_n^-\exp\left\{i\left[\left(k\sin\theta+\frac{2n\pi}{a}\right)x-\kappa_nz\right]\right\}\right)$$

für $f(x)\leqq z<\max_{0\leq x\leq a}f(x)$ mit

$$(3)\quad B_n^\pm=\frac{-i}{2a\kappa_n}\int_{0\,(\pm)}^{a}\exp\left\{-i\left[\left(k\sin\theta+\frac{2n\pi}{a}\right)x'\pm\kappa_n\cdot f(x')\right]\right\}\times\frac{\partial}{\partial n}U(x',z')|_{z'=f(x')}\,dx',$$

worin $\partial/\partial n$ die Differentiation bzgl. der in das Innere von $z>0$ gerichteten Normalen auf $z=f(x)$, $-\infty<x<+\infty$, bedeutet. Formel (2) enthält mithin auslaufende und einlaufende ebene Wellen. ${}_{(\pm)}\int_0^a$ bedeutet hierbei, daß über alle x' zu intergrieren ist, für die $z>f(x')$ bzw. $z<f(x')$ ist. *E. Meister* (Saarbrücken)

Erma, Victor A. 5523

Perturbation approach to the electrostatic problem for irregularly shaped conductors.

J. Mathematical Phys. **4** (1963), 1517–1526.

Author's summary: "A simple and straightforward perturbation method for treating the electrostatic problem of a charged, irregularly shaped conductor is presented. The perturbation solution is generated starting from the zero-order solution for a charged sphere. The method consists of expanding the boundary condition in a Taylor series, which in effect transforms the boundary condition at the irregular boundary into a succession of boundary conditions to be satisfied at the surface of a sphere. The simplicity of the formalism consists further in applying, in a consistent manner, sufficient rather than necessary conditions on the successive correction potentials. First- and second-order expressions for the potential, surface charge density, and capacitance of irregularly shaped conductors, are derived explicitly, and an elementary theorem for the

first-order capacitance is obtained. A perturbation expansion for the capacitance valid to all orders is presented. The application of the method is illustrated by calculating the capacitance of several irregularly shaped conductors. Possible generalizations to more complicated boundary-value problems are indicated."

Heins, Albert E. **5524**

On diffraction by a half-plane.

J. Math. Pures Appl. (9) **43** (1964), 59–66.

A Green's function for the equation

$$u_{xx}+u_{yy}-k^2u=0$$

is determined for the half-plane. This is essentially the Sommerfeld diffraction problem with line-source excitation. An integral equation formulation is given, and this equation is solved by function-theoretic methods similar to those of the author and the reviewer [Quart. J. Math. Oxford Ser. (2) 9 (1958), 132–143; MR 21 #1508]. The integral equation is shown to be equivalent to a Hilbert problem for analytic functions in a slit plane. The solution is complicated by the necessity of considering exponential behavior at infinity. *R. C. MacCamy* (Pittsburgh, Pa.)

Evangelisti, Giuseppe **5525**

Sopra le trasformate di Laplace dei problemi di propagazione.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **35** (1963), 211–217.

The propagation properties of a non-uniform electrical transmission line are studied by means of the Laplace transform method. Assuming the parameters of the line to be continuous functions of the position x, the general nature of the Laplace transforms $V(x,s)$ and $I(x,s)$ of the potential difference and current are investigated. It is found that the line can be characterized by four entire analytic functions of the complex variable s from which V and I can be obtained by means of rational operations. *K. Schram* (Utrecht)

Kron, Gabriel **5526**

The frustrating search for a geometrical model of electrodynamic networks.

Tensor (N.S.) **13** (1963), 111–128.

CLASSICAL THERMODYNAMICS, HEAT TRANSFER

See also 5390.

Rosenbrock, H. H. **5527**

The approach to a stable steady state far from the equilibrium.

Proc. Phys. Soc. **80** (1962), 962–970.

A proposal of Prigogine's is here taken up in the form of a description of the approach to a steady state, far from equilibrium, through the use of a Lyapunov function. The equations of motion of the variables are derived by the method used by Cox [Rev. Modern Phys. 22 (1950), 238–248; MR 12, 467] to derive Onsager's reciprocity relations, and are subject to the same assumptions as used in that derivation. The Lyapunov function is then shown to exist, and to reduce to the rate of entropy production relative to the rate in the steady state when the steady state is near equilibrium. An interesting consequence is deduced, restricting the oscillations of a system under certain conditions, which the authors propose as a test of the assumptions used. *A. Siegel* (Boston, Mass.)

Bueckner, H.; Horvay, G. **5528**

Exponential solutions of a boundary value problem of the Helmholtz equation related to change in phase.

J. Math. and Phys. **43** (1964), 227–233.

The boundary-value problem here is one in which $T_{xx}+T_{yy}+2T_y=0$ in an angular domain $\alpha<\theta<\pi/2$, $r>0$, and T is to satisfy the conditions $T=-1$ when $\theta=\alpha$ and $T_x=0$ when $\theta=\pi/2$, together with order conditions at $r=0$ and $r=\infty$. It is a simplified case of a problem of determining steady temperatures T in a bath of liquid metal through which a heat sink in the shape of a slab travels and freezes out metal. The problem is solved by introducing the functions $S(r,\theta)=e^yT(x,y)$ and representing S by a series of terms $c_i\exp[r\sin(\theta+\beta_i)]$, then using various properties of periodic functions. Here $\nabla^2S-S=0$. In particular, if $\alpha=\pi/4$, then $T=e^{-2y}-e^{-x}e^{-y}-e^{x}e^{-y}$; if $\alpha=\pi/3$, then $T=-e^{-2y}-4e^{-y}\sinh(y/2)\cosh(x\sqrt{3}/2)$. The measure of surface heat transfer T_θ/r at the boundary $\theta=\alpha$ is examined. Uniqueness of the solution T is established. *R. V. Churchill* (Ann Arbor, Mich.)

Lefur, Bernard; Bataille, Jean; Aguirre-Puente, Jaime **5529**

Étude de la congélation d'une lame plate dont une face est maintenue à température constante, l'autre face étant soumise à une température variable en fonction du temps (problème de Stéfan unidimensionnel).

C. R. Acad. Sci. Paris **258** (1964), 1483–1485.

The one-dimensional Stefan problem is treated with the additional new feature that the temperature of the (plane) surface of solidification is assumed a given function of time. Solutions in terms of confluent hypergeometric functions are presented for the "quasi-stationary" case that the local variation of temperature with time is negligible compared to the variation of temperature with distance. *K. Forster* (Los Angeles, Calif.)

Sidljar, M. M. **5530**

Approximate solution of non-stationary heat conduction problems in the case of perturbation of the shape of the body. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukrain. RSR **1964**, 1136–1138.

Author's summary: "A method is given for finding the approximate solution of the plane problem of nonstationary heat conductivity for weakly strained bodies in the case when the solution of the unperturbed problem is known in the form of orthonormal functions. Using the variational method the solution of the problem is reduced to finding the temperature field in the form of a series $\sum_k a_k(\tau)\varphi_k$, the coefficients of which are determined from differential equations of the form

$$\frac{da_k}{d\tau}+k^2a_k=\frac{\varepsilon}{N_k}\Phi(\tau)."$$

QUANTUM MECHANICS

See also 4652, 4837, 4915, 4921, 4956, 5107, 5133, 5519, 5576, 5587, 5590, 5592, 5598.

★Schrödinger-Planck-Einstein-Lorentz: Briefe zur Wellenmechanik. 5531
Herausgegeben im Auftrage der Österreichischen Akademie der Wissenschaften von K. Przibram.
Springer-Verlag, Vienna, 1963. vi + 68 *pp*. (4 *plates*) $2.50.

An excellently printed and edited series of letters covering the period 1926–1927 between Schrödinger and Planck and the period 1926–1950 between Schrödinger and Einstein, together with three letters between Schrödinger and Lorentz from 1926.

Granzow, Kenneth D. 5532
N-dimensional total orbital angular-momentum operator. II. Explicit representations.
J. Mathematical Phys. **5** (1964), 1474–1477.

This is a continuation of an earlier paper by the author [same J. **4** (1963), 897–900; MR **27** #4517]. The author gives a systematic method of constructing polar coordinate systems in N-dimensional spaces.

D. ter Haar (Oxford)

Ball, J. S.; Frazer, W. R.; Nauenberg, M. 5533
Scattering and production amplitudes with unstable particles.
Phys. Rev. (2) **128** (1962), 478–494.

The aim of the paper is to find a procedure to include unstable particles in the S-matrix theory. Unstable particles are treated as resonances in the scattering of their decay products, integral equations for production amplitude being written with the help of the generalized unitarity which relates the discontinuity in the s (total energy, squared) variable of the production amplitude to the amplitudes T_{11} (scattering), T_{21} (production) and $T_{22}{}^c$, the connected three-three particle amplitude:

$$T_{21}(s_+, t, \omega_+) - T_{21}(s_-, t, \omega_+) = 2i \sum T_{21}(s_+, t', \omega_+) \times T_{11}(s_-, t'') + T_{22}{}^c(s_+, t', \omega_+, \omega_+{}'')T_{21}(s_-, t'', \omega_-{}'')$$

(t the momentum transfer, ω the pion energy). The conventional unitarity gives a relation for the simultaneous discontinuity of the amplitudes in both s and ω variables across their physical cut, and therefore cannot be used directly in deriving integral equations.

To illustrate the general procedure, the authors confine themselves to S-waves (spin and isospin neglected). Dispersion relations are written in the s variable, keeping ω fixed below its physical threshold. ω is raised afterwards, by analytical continuation, to its physical domain. Finally, the resulting coupled integral equations are reduced by an extension of the N/D method to nonsingular Fredholm integral equations, which can be solved by straightforward numerical methods.

S. Ciulli (Bucharest)

Mittleman, Marvin H. 5534
Coupled equations for rearrangement collisions.
Ann. Physics **28** (1964), 430–434.

Author's summary: "An exact system of equations which describe the scattering of a system into a rearranged channel is derived. The method is based on a technique invented by Feshbach for deriving an equivalent Hamiltonian (or optical potential) for the scattering of a particle by a system of particles identical with it."

Newton, Roger G. 5535
★The complex j-plane. Complex angular momentum in nonrelativistic quantum scattering theory.
The Mathematical Physics Monograph Series.
W. A. Benjamin, Inc., New York-Amsterdam, 1964. viii + 235 *pp*. $9.00.

This monograph begins with a short introduction, in which the author outlines the motivation of the current interest in complex angular momenta and stresses the scope of this monograph ("the bulk of this book deals only with the nonrelativistic domain, where results are well-grounded and, starting from known dynamics, can be proved"). In a subsequent section the Watson-Sommerfeld-Regge transformation is introduced, and its possible relevance to high-energy scattering emphasized. Then the various properties of the S-matrix as a function of complex angular momentum are analyzed in detail from a rigorous point of view in the framework of potential scattering (Sections 3 to 15). Several examples of numerically computed Regge trajectories are also given (in Section 12), which will be quite interesting for the reader, be he a beginner or an expert in this field. Section 16 is devoted to the discussion of the scattering of two particles of spin $\frac{1}{2}$. Finally, the last two sections deal with the three-body problem, where the possibility of different analytic continuations is emphasized. The book is completed by three appendices and a very extensive bibliography which includes more than three hundred papers (almost all of them published within the last three years).

This book will be very useful to all workers in this field, because it provides a comprehensive summary of our present knowledge about the angular momentum plane in potential scattering (except for the treatment of particles with spin, where only the "spin $\frac{1}{2}$, spin $\frac{1}{2}$" case is considered). It is perhaps worth emphasizing that most of the material covered is now completely settled, so that this book is likely to remain valid and useful for some time. The major exception is probably the three-body problem, whose treatment, although quite thorough, is likely to be superseded by present and future developments.

The book is well-written and well-edited. It contains also the following reprints: S. C. Frautschi, M. Gell-Mann and F. Zachariasen [Phys. Rev. (2) **126** (1962), 2204–2218; MR **25** #3728]; K. Bardakci [ibid. (2) **127** (1962), 1832–1836; MR **25** #4847]; G. F. Chew, S. C. Frautschi and S. Mandelstam [ibid. (2) **126** (1962), 1202–1208; MR **25** #3697]; P. T. Matthews [Proc. Phys. Soc. **80** (1962), 1–12].

F. Calogero (Rome)

Roberts, M. J. 5536
Negative energy moments and low energy approximations to phase shifts.
Proc. Phys. Soc. **83** (1964), 503–517.

Relations between sums of inverse powers of the bound state energies and corresponding moments of the energy derivative of the phase shifts in non-relativistic scattering by potentials of at least exponential decrease at infinity

are derived by an extension of a method due to Buslaev and Faddeev [Dokl. Akad. Nauk SSSR 132 (1960), 13–16; MR 22 #11171]. Similar relations involving positive powers were employed by the author [M. J. Roberts, Proc. Phys. Soc. 82 (1963), 594–604] to obtain high-energy approximations to the phase shifts. By employing also the present relations, improved approximations at low energies should be obtained. In the example of s-wave scattering by a Gaussian potential, they are shown to lead to better results than the effective range approximation. However, in contrast to the case of positive moments, the present relations involve coefficients that have to be determined from the asymptotic behaviour of solutions to the radial equation at low energies, so that they are more difficult to apply than the positive-moment relations.

H. M. Nussenzveig (New York)

Vedrinskiĭ, R. V. 5537

An estimate for the wave function which is analytic with respect to the energy. (Russian)

Vestnik Moskov. Univ. Ser. III Fiz. Astronom. **1964**, no. 4, 81–82.

The negative-energy solution of the S-state Shrödinger equation is expanded in the form

$$u(E, r) = \sum_{n=0}^{\infty} (-E)^n u_n(r).$$

It is shown that

$$u_n(r) = u_0(r) \int_0^r \int_0^{r_1} \cdots \int_0^{r_{2n-1}} \left(\frac{u_0(r_2)}{u_0(r_1)}\right)^2 \times \left(\frac{u_0(r_4)}{u_0(r_3)}\right)^2 \cdots \left(\frac{u_0(r_{2n})}{u_0(r_{2n-1})}\right)^2 dr_1 \cdots dr_{2n}.$$

Upper and lower bounds are obtained for the wave function and its logarithmic derivative in terms of the zero-energy solution $u_0(r)$.

K. Kumar (Canberra)

Živopiscev, F. A. 5538

Non-elastic scattering of nucleons on nuclei and the method of quantum Green's functions. (Russian)

Vestnik Moskov. Univ. Ser. III Fiz. Astronom. **1964**, no. 4, 45–51.

Non-elastic scattering of nucleons on nuclei of the type magic plus a nucleon (or hole) at low energies is considered. In particular, the author investigates the nuclear reactions (n, n'), (p, p'), (n, p), (p, n) on a nucleus of the type magic plus a nucleon (a hole) and the nuclear reactions (d, n), (d, p), (d, np) on a magic nucleus when the energy of the incoming particles is less than that of the magic nucleus excitation. It is shown that the problem of computing the S-matrix elements is connected with finding the two-particle Green's function. No assumptions about the interaction weakness or the nuclear process mechanism are used. An approximation for the T-matrix is obtained; it allows one to find the cross-sections of the nuclear reactions without using the usual perturbation theory (the Born series). In the first approximation the non-elastic scattering of nucleons on a nucleus with one particle outside a full nuclear shell is determined by the free nucleons scattering amplitude and by the magic nucleus optical potential. The theory is compared with the zero range approximation of nuclear forces.

M. Blažek (Bratislava)

Acharya, R. 5539

Some consequences of "minimal" analyticity and unitarity in field theory. (Italian summary)

Nuovo Cimento (10) **27** (1963), 1151–1155.

Author's summary: "A simple proof of the incompatibility of a field theory with a finite number of processes with the principles of unitarity, minimal analyticity and positive definiteness is presented. It is further argued that such a theory also violates local commutation relations."

{The reviewer believes that the author's arguments are inconclusive on three grounds: (1) The assumption of "minimal analyticity" is not stated, so that one cannot assess its consequences; (2) The terms in Equation (2), which are written in detail above Equation (2), are not, in general, positive definite as the author states; and (3) It does not seem reasonable that CTP invariance implies "minimal analyticity" as the author states, since whatever "minimal analyticity" means, as used by the author, it has consequences for many different "crossing relations" while CTP invariance has consequences only for the single "crossing relation" in which the order of the fields is completely reversed.}

O. W. Greenberg (College Park, Md.)

Blankenbecler, R.; Goldberger, M. L. 5540

Behavior of scattering amplitudes at high energies, bound states, and resonances.

Phys. Rev. (2) **126** (1962), 766–786.

Although the Mandelstam representation displays the analyticity of the scattering amplitude in an admirable fashion, the unitarity condition is extremely awkward to deal with, especially in the high-energy, large-momentum transfer region. On the other hand, the diffractional behaviour of the scattering amplitude at high energies suggests the use of the eikonal approximation.

Starting from these points, the authors introduce an exact Fourier-Bessel representation for the scattering amplitude

$$M(s, t) = \int_0^{\infty} b\, db \{J_0(b\sqrt{-t}) H_1(b^2, s) + J_0(b\sqrt{-u}) H_2(b^2, s)\}$$

which satisfies unitarity exactly in the high-energy limit, even in the many-channel situation. If one writes $H_{\pm}(b^2, s) = H_1(b^2, s) \pm H_2(b^2, s)$ for large s, the elastic unitarity takes the simple form

$$\operatorname{Im} H_{\pm}(b^2, s) = r(s) |H_{\pm}(b^2, s)|^2$$

which permits the use of the N/D formalism.

On the other hand, this representation reduces to the eikonal approximation if one replaces H by the well-known expression $\{\exp i\chi - 1\}$, where

$$\chi_l(b^2, k) = -\frac{1}{2k} \int_{-\infty}^{+\infty} dz\, V[(z^2 + b^2)^{1/2}].$$

It is shown that the Mandelstam representation is contained as a special case. The simple form of the asymptotic unitarity permits analytical continuation to unphysical sheets, and resonance and bound states discussion is similar to the partial wave case [see also R. Oehme, Z. Physik 162 (1961), 426–437; R. Blankenbecler, M. L. Goldberger, S. W. MacDowell, and S. B. Treiman, Phys. Rev. (2) 123 (1961), 692–699].

S. Ciulli (Bucharest)

Brander, O. 5541

A Mandelstam representation in hard-core potential scattering. (Italian summary)

Nuovo Cimento (10) **32** (1964), 1059–1066.

Author's summary: "The difference between the scattering amplitude for scattering from a potential with a hard core and the scattering amplitude for scattering from the hard core alone is studied. Provided the potential outside the hard core is a superposition of Yukawa potentials, it is shown that this difference satisfies a Mandelstam representation of conventional form."

W. M. Frank (Rehovot)

Cohn, J. H. E. 5542

On the number of bound states of a certain potential.

Acad. Roy. Belg. Bull. Cl. Sci. (5) **49** (1963), 1195–1205.

The one-dimensional Schrödinger equation is studied for a certain two-parameter class of potentials $V(x)$, parabolic for small x and decreasing exponentially for large x. Upper and lower bounds are placed on the number of bound states which may be formed in such a potential.

J. M. Charap (London)

Cutkosky, R. E. 5543

Expansion of the scattering matrix in terms of multiple impulsive peripheral interactions.

Nuclear Phys. **37** (1962), 57–77.

Author's summary: "A new perturbation expansion is developed for the S-matrix. Each term of the expansion corresponds to a Feynman graph which describes a multiple peripheral collision. The complete effects of the short range forces are included in the first approximation. The peripheral interaction expansion is based on the analytic properties of the ordinary perturbation expansion and may be looked upon as a rearrangement of the terms of that expansion. It provides a simple perturbation-theory picture for all the concrete results that have been derived by formulating dynamical theories in terms of dispersion relations. The Feynman graphs encountered in the peripheral interaction expansion are those of the ordinary expansion with all self-energy parts omitted. This omission is possible because each line of a graph represents a set of quanta which correspond not only to the discrete states, but also to all states lying in the continuum which have a limited angular momentum. Each vertex corresponds to an elementary interaction which has a strength equal to the appropriate exact S-matrix element. Renormalization of the successive terms of the peripheral interaction is the most complicated problem which is encountered, although it is similar to renormalization of the ordinary expansion. In the treatment of this problem one encounters all possible successive graphical contractions. With the aid of these topological constructions it is possible to state the general renormalization rule quite concisely."

Eftimiu, C. 5544

On the redundant zeros of the S-matrix. (Romanian. Russian and English summaries)

An. Univ. C. I. Parhon Ser. Şti. Nat. Mat.-Fiz. No. **25** (1960), 215–228.

An excellent review article on the analytic properties of the Jost functions. The author performs a thorough analysis of the singularities of the S-matrix not related to bound states, placed beyond the Bargman strip.

S. Ciulli (Bucharest)

Glauber, Roy J. 5545

Photon correlations.

Phys. Rev. Lett. **10** (1963), 84–86.

The quantum theory of coherence is formulated; the central idea is the identification of the coherence function with the expectation value of a normal ordered operator product. The use of the (right) eigenvectors of the annihilation operators in discussions of coherence is pointed out. Since then, the reviewer has demonstrated that this quantum theoretic treatment is equivalent to the semiclassical treatment provided the probability distribution function is replaced by a suitable distribution [same Lett. **10** (1963), 277–279; MR **27** #3315].

E. C. G. Sudarshan (Syracuse, N.Y.)

Garrido, L. M.; Sancho, F. J. 5546

Degree of approximate validity of the adiabatic invariance in quantum mechanics.

Physica **28** (1962), 553–560.

The time-dependent Schrödinger equation with a time-dependent Hamiltonian $H(t)$ determines the evolution operator $U(t)$. When the time interval $T = t_1 - t_0$ is made to grow indefinitely while the total change of $H(t)$ is fixed, $U(t)$ admits a certain asymptotic expansion in $1/T$. This paper gives such an expansion which is valid up to the $(m+1)$st order in $1/T$. The basic assumptions made are that the first m derivatives of $H(t)$ are zero at the initial and final times, and that the eigenvalues of $H(t)$ are discrete and non-degenerate and depend smoothly on t as well as on the associated one-dimensional projections (which implies, in particular, that there is no crossing between eigenvalues).

T. Kato (Berkeley, Calif.)

Garrido, L. M. 5547

Generalized adiabatic invariance.

J. Mathematical Phys. **5** (1964), 355–362.

The mathematical problem for the quantum-mechanical adiabatic theorem considered here is to find an asymptotic formula as $T \to \infty$ for the unitary operator $T_T(\tau)$ determined by the differential equation $i\hbar(d/d\tau)U_T(\tau) = TH(\tau)U_T(\tau)$, where $H(\tau)$, $0 \leq \tau \leq 1$, is a family of self-adjoint operators depending smoothly on τ. The author shows that $U_T(\tau) = R^{(1)}(\tau) \cdots R^{(l)}(\tau)\phi^{(l)}(\tau)(1 + O(T^{-l}))$, where $R^{(1)}(\tau)$, independent of T, is the "adiabatic transformation" identical with the operator deduced by the reviewer [J. Phys. Soc. Japan **5** (1950), 435–439], the $R^{(k)}(\tau)$ are determined successively by solving linear differential equations with coefficients that can be computed by formal perturbation theory (the perturbation parameter is $1/T$) from $H(\tau)$ and the previous $R^{(k)}(\tau)$, and where $\phi^{(l)}(\tau)$, a unitary operator involving rapidly oscillating phase factors, can also be computed by perturbation theory. The essential assumption is that $H(\tau)$ has a pure discrete spectrum and there is no "crossing" of eigenvalues for $0 \leq \tau \leq 1$. Various consequences of this basic formula are discussed. In particular, it is shown that $U_T(1)$ sends an eigenvector of $H(0)$ into an eigenvector of $H(1)$ within an error of $O(T^{-l})$ if the derivatives of $H(\tau)$ up to order $l-1$ are zero

for $\nu = 0$ and 1, a result obtained by the author and F. J. Sancho in an earlier paper by a less general method [#5546]. *T. Kato* (Berkeley, Calif.)

Hostler, Levere 5548

Nonrelativistic Coulomb Green's function in momentum space.

J. Mathematical Phys. **5** (1964), 1235–1240.

In a previous paper [same J. **5** (1964), 591–611] the author has given an integral representation for the non-relativistic Coulomb Green's function in coordinate space. In the present paper, the author computes the Fourier transform of his previous result. After some calculations he finds that the Fourier transform of his previous formula can conveniently be written as a sum of three terms. The first term is the non-relativistic Coulomb Green's function for a free particle. The second term is the Born approximation, i.e., the result obtained as a first correction term in a power series expansion in Z_α, where α is the fine structure constant and Z is the nuclear charge. The third term is considerably more complicated than the others. It is exhibited both in terms of an integral representation and also evaluated explicitly in terms of hypergeometric functions. In this way, the author obtains a closed form for the momentum-space Green's function. At the end, the author verifies that his result has the correct analyticity properties as a function of the complex energy. *G. Källén* (Lund)

Kenschaft, R. P.; Amado, R. D. 5549

Solution of a singular integral equation from scattering theory.

J. Mathematical Phys. **5** (1964), 1340–1342.

In the field-theoretical model originally proposed by T. D. Lee [Phys. Rev. (2) **95** (1954), 1329–1334; MR **16**, 317] and later studied by many authors, one encounters a singular integral equation describing the scattering of the "V-particles" by the "θ-particles" [the reviewer and W. Pauli, Danske Vid. Selsk. Mat.-Fys. Medd. **30** (1955), no. 7; MR **17**, 927]. So far, the explicit solution of this integral equation has not been known completely. A few years ago one of the present authors [R. D. Amado, Phys. Rev. (2) **122** (1961), 696–704; MR **25** #2827] succeeded in extracting the energy shell contribution to the scattering matrix by using dispersion-theoretic methods. Evidently, such an approach cannot be immediately generalised to off-shell quantities like the complete scattering amplitude. In the present paper, the authors find a particular solution of the scattering equation in question. This solution is found by heuristic arguments and is afterwards verified by direct substitution to be a solution of the equation in question. Consequently, the authors have found also the off-shell scattering amplitude in the $V-\theta$ sector.

The authors do not claim that their solution is the most general solution of the integral equation under investigation. However, they state (and the reviewer agrees) that their solution most probably is the unique solution which is of physical importance to describe $V-\theta$ scattering in the Lee model. Finally, it should be mentioned that a very similar result has been obtained independently by A. Pagnamenta [Bull. Amer. Phys. Soc. **9** (1964), 449, FE5]. *G. Källén* (Lund)

Lurçat, F.; Mazur, P. 5550

Statistical mechanical evaluation of phase-space integrals. (Italian summary)

Nuovo Cimento (10) **31** (1964), 140–153.

It is observed that the computation of phase-space integrals for many-particle processes, with conservation of total energy, is related to the evaluation of microcanonical partition functions in statistical mechanics. In the latter case, extension to a canonical ensemble simplifies the integrals, and it is shown that the same is true in the former. Some numerical examples are given. *H. W. Lewis* (Madison, Wis.)

Malik, F. B. 5551

On the variational method for nonrelativistic scattering. I. Differential form.

Ann. Physics **20** (1962), 464–478.

Two alternative variational procedures for the solution of scattering problems using the differential form of the Schrödinger equation are examined, one due to Hulthén [Kungl. Fysiogr. Sällsk. i Lund Förhandl. **14** (1944), no. 8; MR **6**, 110; Ark. Mat. Astronom. Fys. **35A** (1948), no. 25; MR **10**, 120], the other due to Kohn [Phys. Rev. (2) **74** (1948), 1763–1772]. The criticism made by Schwartz [Ann. Physics **16** (1961), 36–50; MR **24** #B269], that the trial parameters do not vary smoothly with energy, but become infinite at certain energies, is reviewed and another question is raised: How can there be two different variational procedures using the same functional? The author shows that both of these difficulties are connected with a more fundamental problem. Hulthén's, as well as Kohn's, equations for the former's functional are not enough to satisfy the Euler equation of the variational problem because the phase shift found in either way usually fails to reflect the asymptotic behavior of the wave functions. The author suggests a set of variational equations to replace the usual self-consistency procedure. This insures that the Schrödinger equation becomes the Euler equation of the variational problem. With this modification, Hulthén's and Kohn's methods become equivalent, and choices of trial wave functions giving rise to discontinuous parameters are automatically excluded. Finally, the author points out that in all those cases already calculated, where Hulthén's or Kohn's method gave good results, his prescription has, in fact, been applied. *H. Lustig* (New York)

Malik, F. B. 5552

On the variational method of the nonrelativistic scattering. II. The integral form.

Ann. Physics **21** (1963), 1–7.

As in his discussion of the differential form of the variational approach to scattering problems [see #5551 above], the author is here concerned with making the variational equations consistent with the boundary conditions at large distances. An appropriate modification to insure this consistency is introduced in the integral variational procedure of Schwinger [Mimeographed lecture notes, Vol. II, Harvard, 1947; see also Blatt and Jackson, Phys. Rev. (2) **76** (1949), 18–37]. As a second approach, the author develops an expression for the variational functional which corresponds to the integral form of the functional used in the differential procedure. This

second type of functional looks somewhat simpler than Schwinger's, and since the presence of a Green's function makes the application of the integral version difficult at times, this may be an advantage. *H. Lustig* (New York)

Martin, J. L. 5553

Covariant quantum field theories with indefinite metric, and the Lorentz covariant Lee model.

Proc. Roy. Soc. Ser. A **272** (1963), 231–240.

This is an important paper and describes an attempt at the construction of a covariant finite quantum field theory of relativistic particle interactions. The author introduces creation and destruction operators for particles with both positive and negative energies. For spin zero particles a natural scalar product may be defined which associates positive [negative] definite quadratic forms with positive [negative] frequency amplitudes for a single particle. A Lorentz-invariant version of the Lee model that was considered before by the author [same Proc. **271** (1963), 332–356; MR **26** #5874] can be given an operator formulation, and this is carried out here. This paper raises the important question: Is it possible to construct a complete relativistic field theory with non-trivial crossing symmetry and no explicit infinities?

E. C. G. Sudarshan (Syracuse, N.Y.)

Weaver, D. L.; Hammer, C. L.; Good, R. H., Jr. 5554

Description of a particle with arbitrary mass and spin.

Phys. Rev. (2) **135** (1964), B241–B248.

Authors' summary: "A Lorentz covariant description of a particle and antiparticle with spin $s=0, \frac{1}{2}, 1, \cdots$ and finite rest mass is given in this paper. The wave function has $2(2s+1)$ components so that no auxiliary conditions are needed. The basic idea is to postulate the rest-system Hamiltonian and make the Lorentz transformation to the laboratory system. An algorithm which is a generalization of the Foldy-Wouthuysen transformation is found for constructing the laboratory-system Hamiltonian and polarization operators." *C. G. Bollini* (Buenos Aires)

Yamasaki, Hisaiti 5555

An extension of Feynman-Bunge-Corben's relation regarding the position-operator of Dirac electron to arbitrary operators. I.

Progr. Theoret. Phys. **31** (1964), 322–323.

Given a Dirac operator Q, the author introduces the operator $\tilde{Q}=Q+(i\beta/2m)dQ/dt$ and investigates its time derivative. *F. Calogero* (Rome)

Yamasaki, Hisaiti 5556

An extension of Feynman-Bunge-Corben's relation regarding the position-operator of Dirac electron to arbitrary operators. II.

Progr. Theoret. Phys. **31** (1964), 324–325.

The physical meaning of certain operators $\tilde{Q}$ [see #5555 above] is discussed. This letter and the one reviewed above summarize the work reported in Soryusiron Kenkyu **16** (1957), 351; ibid. **27** (1963), 540. *F. Calogero* (Rome)

Grashin, A. F. [Grašin, A. F.] 5557

Solution of the linear equations of the dispersion method in the two-particle approximation.

Ž. Èksper. Teoret. Fiz. **43** (1962), 277–286 (*Russian. English summary*); *translated as Soviet Physics JETP* **16** (1963), 198–204.

A solution of the linear equation

$$F(t) = \tilde{F}(t)+\frac{t^N}{\pi}\int_{4\mu^2}^{\infty}\frac{\operatorname{Im} F(t')\,dt'}{(t'-t)t'^N},$$

$$\operatorname{Im} F(t) = F(t)e^{-i\delta(t)}\sin\delta(t)+f(t), \quad f(t) = 0 \text{ for } t < 16\mu^2,$$

is sought, which requires stability (for $t<T\approx 16\mu^2$) against perturbations of $f(t)$ at infinity. The solution $F(t)$, calculated in the vicinity of $|t|\leq T$ with an accuracy $\sim|t/T|$, is unique and has the form

$$F(t) = \tilde{F}(t)+\varphi(t)\frac{t^N}{\pi}\int_{4\mu^2}^{\infty}\frac{\tilde{F}(t)\exp\{i\Delta\}\sin\Delta}{(t'-t)t'^N\varphi(t')}\,dt',$$

$$\varphi(t) = \exp\left\{\frac{t-t_0}{\pi}\int_{4\mu^2}^{\infty}\frac{\Delta(t')\,dt'}{(t'-t-i0)(t'-t_0)}\right\},$$

where the auxiliary phase $\Delta(t)$ has a simple, integrable form, which approximates the input phase $\delta(t)$ with an accuracy $\sim t/T$ for $|t|<T$. *S. Ciulli* (Bucharest)

Morey, F. [Morey Terry, F.] 5558

Definition of nuclear potentials from double dispersion relations in field theory and potential scattering. (Italian summary)

Nuovo Cimento (10) **24** (1962), 585–605.

Charap and Fubini [Nuovo Cimento (10) **14** (1959), 540–559; MR **22** #6478] have shown that one may define a potential for the scattering of bosons interacting through a charged pseudo-scalar meson, consistent with the results of the field theory. In the present note these results are generalized to take into account the fermionic nature of the nucleons. In order to reproduce the field theory results, five direct and five exchange potentials are introduced. Finally, a method for the explicit calculation of the potentials is given. *S. Ciulli* (Bucharest)

Parasjuk, O. S. 5559

Feynman integrals and the method of Poincaré. (Russian)

Ukrain. Mat. Ž. **15** (1963), 320–321.

Recently the analyticity properties of Feynman amplitudes as functions of invariants have been studied extensively. For a survey of this work, see, for example, J. C. Polkinghorne [*Lectures in theoretical physics*, Vol. 1, pp. 102–167, Benjamin, New York, 1962; MR **27** #2290]. The author points out that the methods used were already employed by Poincaré in the perturbation theory of celestial mechanics. Both Poincaré and modern authors base their technique on deformation of contours of integration and the concepts of coinciding singularities and pinching of contours. (Poincaré even used the English word "pinch" at one point.) The works cited by the author are the following. [H. Poincaré, Bull. Astronom. **14** (1897), 353–354; reprinted in *Œuvres de Henri Poincaré*, Tome VIII, pp. 110–111, Gauthier-Villars, Paris, 1952; MR **15**, 277; Bull. Astronom. **15** (1898), 449–464; reprinted in

Œuvres de Henri Poincaré, Tome VIII, pp. 33–47, Gauthier-Villars, Paris, 1952; MR 15, 277; *Les méthodes nouvelles de la mécanique céleste*, Vol. I, Gauthier-Villars, Paris, 1892.] *R. L. Warnock* (Chicago, Ill.)

Parasjuk, O. S. 5560

Dual dispersion relations. (Russian)

Ukrain. Mat. Ž. **13** (1961), *no.* 3, 100–103.

Nakanishi [Progr. Theoret. Phys. **25** (1961), 155] showed that the sufficient and necessary condition that a function $M(s, t)$ defined by

$$M(s,t) = \int_0^1 dz \int_0^\infty d\alpha \frac{\Psi(z,\alpha)}{\alpha - zs - (1-z)t - i\epsilon}$$

satisfies the Mandelstam representation $M(s, t) = \int_0^\infty ds' \int_0^\infty dt' \varphi(s', t')/(s'-s-i\varepsilon)(t'-t-i\epsilon)$ is

$$\Psi(z,\alpha) = \int_0^\infty ds' \int_0^\infty dt' \varphi(s',t')\delta'(\alpha - zs' - (1-z)t').$$

For

$$\Psi(z,\alpha) = \frac{\delta(\alpha - (1-z)m_2{}^2 - zm_1{}^2)}{(1-z)m_2{}^2 + zm_1{}^2 - z(1-z)M^2}$$

the Nakanishi condition is fulfilled only if $|m_1 - m_2| < M < m_1 + m_2$.

The author gives a very short proof of the above result by investigating Nakanishi's condition with the help of a Fourier transform. His Fourier method seems to provide a general powerful tool in the investigation of similar problems. *S. Ciulli* (Bucharest)

Parasyuk, O. S. [**Parasjuk, O. S.**] 5561

A generalization of a theorem of T. Regge.

Dokl. Akad. Nauk SSSR **147** (1962), 571–572 (*Russian*); *translated as Soviet Physics Dokl.* 7 (1963), 976–977.

The author discusses the connection between a result of the reviewer [Nuovo Cimento (10) **14** (1959), 951–976; MR **26** #1088] on expansion in Legendre polynomials, a theorem of Le Roi-Lindelöf [E. Lindelöf, *Le calcul des résidus et ses applications à la théorie des fonctions*, Gauthier-Villars, Paris, 1905], and Faber's theorem [O. S. Parasjuk, Dokl. Akad. Nauk SSSR **145** (1962), 1247–1248; MR **26** #7350]. The possibility of extending the reviewer's result is also discussed. *T. Regge* (Turin)

Mestvirišvili, M. A.; Teplickiĭ, È. S. 5562

Quasi-stationary levels in a cylindrical magnetic field. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **35** (1964), 293–298.

An application of the Regge-pole method [T. Regge, Nuovo Cimento (10) **14** (1959), 951–976; MR **26** #1088] to a problem with cylindrical symmetry is presented. The problem discussed is that of quantum-mechanical scattering by a homogeneous magnetic field restricted to the interior of the infinite cylinder. The Sommerfeld-Watson transformation on the S-matrix is performed, and Regge trajectories in some special cases are found.

I. Bialynicki-Birula (Warsaw)

Klotz, A. H. 5563

On the mathematical foundations of relativistic quantum mechanics of particles with variable mass. (Italian summary)

Nuovo Cimento (10) **32** (1964), 1191–1201.

Author's summary: "The mathematical formulation of the relativistic quantum theory of particles with variable mass is discussed. Some difficulties arising from it are removed and it is pointed out that there exists a generalization within the framework of its assumptions. The transformation properties of the space of the new spin operators η are considered. It is shown that the second-order wave equation of the theory necessarily involves terms linear in the displacement operators. A suggestion is made concerning an introduction of the isobaric spin space into the theory."

Królikowski, W. 5564

Remarks on SU_3 triplets.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 339–341.

The author amends his previous SU(4) idea [Nuclear Phys. **52** (1964), 342–344; MR **28** #5748] by introducing hypothetical baryon SU(3) triplets instead of the suggested leptons. This then becomes identical with the theory previously put forward by P. Tarjanne and V. L. Teplitz [Phys. Rev. Lett. **11** (1963), 447–448].

Y. Ne'eman (Pasadena, Calif.)

Kurşunoğlu, Behram; Perlmutter, Arnold (Editors) 5565

★**Coral Gables Conference on Symmetry Principles at High Energy.**

January 30–31, 1964, University of Miami.

W. H. Freeman and Co., San Francisco, Calif.-London, 1964. viii + 171 *pp.* $4.75.

The publisher and the editors have to be strongly praised for bringing out these conference proceedings in a relatively short time and in a very satisfactory format. The first of the editors also added a further touch of his personality by procuring on the cover an amusing picture of Nasreddin Hoja sitting on his donkey and adorned with symbols of elementary particles.

Symmetry principles in leptonic weak interactions are discussed by R. Marshak. A possible merger of SU_3 and the Lorentz group is proposed by B. Kurşunoğlu. The first experiments to find quarks are reviewed by R. Adair. Departures from the eightfold way are discussed by S. Glashow. L. Biedenharn analyzes operator structures in SU_3 with an application to triplets. Certain aspects of Cabibbo's suggestion for SU_3 symmetry in weak interactions are commented on by P. Matthews. The possibility of the existence of an exact higher symmetry encompassing both internal and space time symmetry is proposed in the contribution by A. Barut. Y. Ne'eman reports on interesting speculations concerning the origin of internal symmetries in the light of some cosmological possibilities. Experimental consequences of SU_3 are analyzed by S. Meshkov. J. Schwinger proposes the W_3 symmetry and the field theory based on it. In the closing talk Ne'eman presents a thorough review of the then-current status of internal symmetries. *P. Roman* (Boston, Mass.

Noack, C. 5566
"Quark" particles and unitary symmetry.
Z. Physik **179** (1964), 1–3.
Author's summary: "The possibility of including the triplet representation of SU_3 in the usual octet model without losing the integrity of charges is discussed."
K. Maurin (Warsaw)

Jan Ši 5567
On collective perturbations of certain magnetic states. (Russian. Armenian summary)
Akad. Nauk Armjan. SSR Dokl. **39** (1964), 73–79.
The variational method of Bogoljubov [N. N. Bogoljubov, Uspehi Fiz. Nauk **67** (1959), 549–580; MR **22** #1395] is applied to a model of magnetic material in order to find the spectrum of the elementary excitations above the ground state. Because only normal and not superconducting states occur, the equations are rather simpler than in the original applications of the method. The author shows that there are two types of ferromagnetic states, one corresponding to transverse and the other to longitudinal spin waves, and in each case there is both a continuous and a discrete branch of the spectrum. The case of Overhauser spiral spin waves is also considered.
C. A. Hurst (Adelaide)

Kraichnan, Robert H. 5568
Stochastic models for many-body systems. I. Infinite systems in thermal equilibrium.
J. Mathematical Phys. **3** (1962), 475–495.
Author's summary: "Some model Hamiltonians are proposed for quantum-mechanical many-body systems with pair forces. In the case of an infinite system in thermal equilibrium, they lead to temperature-domain propagator expansions which are expressible by closed, formally exact equations. The expansions are identical with infinite subclasses of terms from the propagator expansion for the true many-body problem. The two principal models introduced correspond, respectively, to ring and ladder summations from the true propagator expansion, but augmented by infinite classes of self-energy corrections. The latter are expected to yield damping of single-particle excitations. The eigenvalues of the ring and ladder model Hamiltonians are real, and they are bounded from below if the pair potential obeys certain conditions. The models are formulated for fermions, bosons, and distinguishable particles. In addition to the ring and ladder models, two simpler types are discussed, one of which yields the Hartree-Fock approximation to the true problem. A novel feature of all the model Hamiltonians (except the Hartree-Fock) is that they contain an infinite number of parameters whose phases are fixed by random choices. Explicit closed expressions are obtained for the Helmholtz free energy of all the models in the classical limit."

Kraichnan, Robert H. 5569
Stochastic models for many-body systems. II. Finite systems and statistical nonequilibrium.
J. Mathematical Phys. **3** (1962), 496–521.
Author's summary: "In a preceding paper [#5568], some model Hamiltonians were proposed for quantum-mechanical many-body systems with pair forces. For infinite systems in thermal equilibrium, they led to temperature-domain propagator expansions which were formally summable and expressible by closed equations. These expansions were identical with infinite subclasses of terms from the propagator expansion for the true many-body problem. The two principal models corresponded to ring- and ladder-diagram summations from the true propagator expansion, augmented by infinite classes of self-energy corrections. The model Hamiltonians were called stochastic because they contained parameters whose phases were fixed by random choices. In the present paper, more general models are formulated which yield formally summable propagator expansions for finite systems. The analysis is extended to correlation and Green's functions defined for nonequilibrium ensembles. The nonequilibrium treatment is developed in the Heisenberg representation in such a way that unlinked diagrams do not arise. A basic convergence question associated with the formal closed equations for the model propagators and correlation functions is examined by means of finite-difference integration of the Heisenberg equations of motion. This procedure appears to converge independently of whether the perturbation expansions for the propagators and correlation functions converge. It yields substantial support for the validity of the formal closed model equations."

STATISTICAL PHYSICS, STRUCTURE OF MATTER
See also 5364, 5494, 5550.

Green, H. S.; Hurst, C. A. 5570
★Order-disorder phenomena.
Monographs in Statistical Physics and Thermodynamics, Vol. 5.
Interscience Publishers [John Wiley & Sons, Ltd.], London-New York-Sydney, 1964. x + 363 *pp.* $15.00.
About two-thirds of this book is devoted to a detailed review of the Ising problem. The other third is a general discussion of the theory of order-disorder phenomena but apparently with the main object of showing that the Ising problem is the key to the understanding of nearly all order-disorder transitions. This general discussion gives only a very brief account of experimental results or approximate calculations but enough background so that a mathematician with limited knowledge of physics will learn why the Ising problem has attracted so much attention. The review of the Ising problem describes the recent method developed by the authors for evaluation of the partition function for various two-dimensional Ising lattices from the properties of Pfaffians, as well as the older algebraic and combinatorial methods. Exact evaluations of correlations and magnetization are also reviewed.

The book naturally concludes with a chapter on the unsolved problems for the next generation of Ising enthusiasts. It is an excellent introduction for the ambitious young mathematician who hopes he will be the first to solve the three-dimensional Ising problem or even for the less ambitious ones who only wish to know what the others dream about.
G. Newell (Providence, R.I.)

Janeel, R.; Kahan, Th. 5571

★Électrodynamique des plasmas fondée sur la mécanique statistique. Tome I: Processus physiques et méthodes mathématiques.

Cours professé à la Sorbonne en troisième cycle.

Dunod, Paris, 1963. xx + 622 *pp.*

The renewal of interest in the physics of ionised gases, brought about by the stimulus of the attempt to achieve a controlled thermonuclear reaction and by developments in space physics, has led to a steadily increasing output of books on the subject, ranging from the introductory to monographs on special topics. The present book differs from most of the others in that the authors have adopted what might be called the encyclopaedic approach. Perhaps the strongest argument in favour of this is that, at best, it allows a uniform and comprehensive treatment of a very wide field; the pitfalls inherent in such an approach are apparent.

Volume I of the Electrodynamics of Plasmas deals with physical processes and mathematical methods, and runs to just over 600 pages in doing so (a list of topics to be included in the second volume suggests that it will not be less lengthy). While one understands such a work being bulky, one inevitably asks if it need run to as many pages as it does. In this case the reviewer feels that the answer must be "no", and the reason is to be found, in the first instance, in Chapter 1 which discusses the general properties of plasmas and fundamental atomic processes. No doubt it is proper to introduce and define the basic lengths and frequencies (Debye length, plasma frequency, etc.), although one might question to what extent this is necessary in a book unlikely to be used by complete beginners; what does not seem proper to this reviewer is to include a discussion of atomic collision phenomena, in itself a subject with a vast literature. The twenty-odd pages taken up in this way amount to nothing more than an incomplete catalogue of collision processes which could well have been left out.

Chapter 2 deals with the principles of statistical mechanics, while Chapters 3 and 4 are devoted to the motion of individual particles. Chapters 5–8 contain the main part of the work, the discussion of the Boltzmann equation, in this the treatment is fairly standard. Chapter 6, which devotes some 100 pages to general methods of handling the Boltzmann equation, is perhaps the most successful in the book; more recent developments in kinetic theory due to Grad, Ikenberry and Truesdell, amongst others, are touched on and the bulk of the chapter is taken up with the classical Chapman-Enskog theory. Other subjects which are mentioned rather than treated include the Vlasov equation and the phenomenological collision term of Bhatnagar, Gross and Krook. Unfortunately, the book does not end with the final chapter. There follow some 100 pages of appendices and one cannot avoid asking why, in this age, and in a book at this level, the authors feel obliged to take space to define vector operators grad and div, and to discuss Legendre functions, Hermite and Laguerre polynomials. The book ends with a short bibliography and an adequate index.

In attempting this difficult task the authors, in the first part of their work, have been something less than successful. It remains to say that the print is clear and the book attractively produced.

T. J. M. Boyd (Culham)

Ekstein, H. 5572

Time reversal and superselection. (Italian summary)

Nuovo Cimento (10) 23 (1962), 606–615.

Author's summary: "The physical requirements on the time-reversal operator are reformulated in terms of observables and proper states. These requirements entail superselection rules which are consistent with experiment."

Rau, Jayaseetha 5573

Relaxation phenomena in spin and harmonic oscillator systems.

Phys. Rev. (2) 129 (1963), 1880–1888.

Author's summary: "A method is developed for generating relaxation by introducing a fundamental interval τ and a stirring hypothesis. The application to spin and harmonic oscillator systems is discussed in some detail. All the results are obtained by exact calculations without applying perturbation theory, as the systems considered are simple and completely soluble. Equations similar to phenomenological Bloch equations are derived in the case of spin systems. The relaxation times obtained by the application of the theory are not only proportional to the strength of interaction, but also to the fundamental interval τ which plays an important role in the theory. It is shown that in the case of a harmonic oscillator system, an initial Boltzmann distribution relaxes to a final equilibrium Boltzmann distribution through a sequence of transient Boltzmann distributions."

Stephenson, John 5574

Ising-model spin correlations on the triangular lattice.

J. Mathematical Phys. 5 (1964), 1009–1024.

Author's summary: "A Pfaffian representation of the partition function of the triangular lattice is used to derive expressions for various two, four, and six spin correlations in terms of Pfaffians. The pair correlations along a diagonal are expressed as a Toeplitz determinant whose limiting form yields the spontaneous magnetization. At the ferromagnetic critical point the correlations decay as $1/r^{1/4}$ with approximately radial symmetry. At the antiferromagnetic zero point the ground state is highly degenerate—it has finite entropy—and on a given sublattice the pair correlations along a row decay as $\varepsilon/r^{1/2}$, where $\varepsilon = +\varepsilon_0$ on the sublattice containing the origin spin and $\varepsilon \simeq -\frac{1}{2}\varepsilon_0$ on the other two sublattices. Finally, the perpendicular susceptibility, $\chi_\perp$, which depends on a finite number of correlations, is calculated; its ferromagnetic behavior is similar to that of the perpendicular susceptibilities of the quadratic and honeycomb lattices, but for an antiferromagnet $\chi_\perp$ diverges as $1/T$ at low temperatures."

G. Newell (Providence, R.I.)

Newman, David S. 5575

Equation of state for a gas with a weak, long-range positive potential.

J. Mathematical Phys. 5 (1964), 1153–1157.

Author's summary: "A one-dimensional fluid model in which the pair interaction potential is exponential and repulsive is considered, and the equation of state in the 'long-range' limit is determined exactly. This model is complementary to one studied by Kac, in which the pair

potential consists of a hard core and an exponential, attractive tail. In that model a phase transition occurs, but in the current model there is no phase transition. This fact lends support to a conjecture of Ruelle that no phase transition occurs if the potential is bounded. The possibility of applying the method of Kac to a wider class of potentials is suggested, and some of the mathematical difficulties yet to be overcome are outlined."

N. L. Balazs (Stony Brook, N.Y.)

Oser, Hansjörg; Shuler, Kurt E.; Weiss, George H. 5576

Relaxation of a Lorentz gas with a repulsive r^{-s} force law.

J. Chem. Phys. 41 (1964), 2661–2666.

Authors' summary: "The relaxation of a Lorentz gas (dilute light particle subsystem in a heavy particle heat bath) has been studied for a repulsive r^{-s} central force law interaction between the light and heavy particles. The Fokker-Planck equation for the velocity distribution functions of the light particle subsystem has been solved numerically for $s=5$, 9, 13, 21, and ∞ for various initial conditions. The dependence of the rate of relaxation and of the form of the time-dependent velocity distribution function on the force law parameter s is discussed in light of the above results."

Oswatitsch, Klaus; Schwarzenberger, Rudolf 5577

★Übungen zur Gasdynamik.

255 Aufgaben nebst Lösungen mit einer Sammlung von Formeln und Tabellen.

Springer-Verlag, Vienna, 1963. x + 180 *pp*. $8.30.

A useful and interesting collection of problems and answers, with attention paid to numerical aspects.

Chu, C. K. 5578

Dynamics of ionizing shock waves: Shocks in transverse magnetic fields.

Phys. Fluids 7 (1964), 1349–1357.

Shock waves propagate through a non-ionized neutral gas and ionize it into a perfect conducting gas. The magnetic field is transverse. With the additional assumption of no ionization energy, it is found that the direction of the magnetic field is unchanged in both sides of the shock waves and that the imposed electric field ahead of the shock may not be arbitrarily imposed, but is determined by the shock structure. Then the piston problem is solved for these shocks. It is found that for a given ionization temperature these shocks are pure gasdynamic shocks at low piston speeds and approach classical hydromagnetic perpendicular shocks at high piston speeds.

L. N. Tao (Chicago, Ill.)

Guernsey, Ralph L. 5579

Kinetic equation for a completely ionized gas.

Phys. Fluids 5 (1962), 322–328.

This paper considers the problem of the time development of the velocity distribution function for a gas of particles with Coulomb interactions. The assumptions made are: (1) only the first two equations of the BBGKY hierarchy need to be considered, and (2) the deviations from thermal equilibrium are small and linear ratios of the equations are permissible. No assumption is made, however, that the time scale of relaxation of the two-particle function is short compared to the kinetic time scale governing the change of the one-particle functions. A complicated set of coupled integro-differential equations results, which are solved formally by a technique utilizing the theory of singular integral equations. *A. Lenard* (Princeton, N.J.)

Guernsey, Ralph L. 5580

Kinetic theory of the classical electron gas in a positive background. I. Equilibrium theory.

Phys. Fluids 7 (1964), 792–802.

An approximate method for calculating equilibrium properties of systems where the interaction consists of two parts, short range and weak interaction, is developed. By introducing formal parameters into the BBGKY hierarchy, a new set of equations is obtained which can be solved recursively. The solutions are then continued analytically in these parameters to obtain approximate solutions to the BBGKY hierarchy. The equation of state for an electron gas which is found agrees through second order in the plasma parameter with previous calculations by diagrammatic methods. *J. K. Thurber* (New York)

Macke, W.; Pegel, B. 5581

Anonyme Beschreibung und hydrodynamische Näherung für ein klassisches Plasma.

Ann. Physik (7) 12 (1963/64), 398–408.

The authors show that a description of a many-body system (e.g., a plasma) can be made in terms of the microscopic mass-density $\mu(\mathbf{r})=\sum_i m_i\delta(\mathbf{r}-\mathbf{r}_i)$ and a canonically conjugate field $\Phi(\mathbf{r})$. They show that a canonical transformation can be found, leading from the usual positions and momenta $\mathbf{r}_i$, $\mathbf{p}_i$ to the continuous canonical variables $\mu(\mathbf{r})$ and $\Phi(\mathbf{r})$. It is also shown that $\Phi(\mathbf{r})$ has the simple physical meaning of a velocity potential, such that $\mathbf{v}(\mathbf{r})=\operatorname{grad}\Phi(\mathbf{r})$. The canonical equations of motion are derived, and from these, by suitable averaging, a system of hydrodynamical equations is obtained. Whereas this first part is very elegant, the approximations necessary in order to close the hierarchy of hydrodynamical equations are less transparent in the reviewer's opinion.

R. Balescu (Brussels)

McCune, James E. 5582

Master equation for plasmas.

Phys. Fluids 7 (1964), 1306–1320.

Author's summary: "A straightforward derivation of the general kinetic equation describing the irreversible evolution of the N-momenta distribution function for a classical spatially homogeneous stable plasma ($nr_D^3>1$) is given. Collective effects are included. This plasma master equation reduces to the weak-coupling master equation when collective effects are neglected. The results are given first in terms of a coupled pair of equations analogous to those obtained by Bogoliubov for the kinetic description of a plasma. The equation for the generalized correlations is then solved for stable plasmas in terms of the N-momenta distribution function; the master equation then follows by substitution, and is shown to possess an H-theorem. The conditions under which these results

reduce to the kinetic equation of Balescu and Lenard are discussed. The origins of irreversibility in plasmas are discussed in the light of this and other recent work, and the properties of the stationary solutions of the general kinetic equations are analyzed." *R. Balescu* (Brussels)

Mermin, N. David; Canel, Eric 5583

Long wavelength oscillations of a quantum plasma in a uniform magnetic field.

Ann. Physics **26** (1964), 247–273.

An interesting discussion of the longitudinal plasma oscillation of an electron gas in a uniform magnetic field is given. The Coulomb interaction between the electron is treated in the random phase approximation. Most of the results are for the case of the degenerate gas and are thus of interest in metals. The spectrum of plasma oscillation in the presence of the field shows interesting resonances at the Larmor frequency and higher integral multiples of this frequency. Some of these effects may be observable in the spectrum of metals or semiconductors.

M. J. Stephen (New Haven, Conn.)

Minardi, E. 5584

Energy principle for electrostatic stability of a plasma.

Nuovo Cimento (10) **31** (1964), 674–678.

Engelmann, Feix, and the author [Nuovo Cimento (10) **30** (1963), 830–836] have formulated the equilibrium conditions for a non-uniform one-dimensional collisionless plasma in zero magnetic field. In the present paper their result is re-formulated as a variational principle. For a stable plasma the variational principle is to minimize the electrostatic energy subject to a certain constraint. On the other hand, if the electrostatic energy is maximum, then the plasma is unstable. *O. Penrose* (London)

Ron, Amiram 5585

Scattering of electromagnetic waves by an electron-phonon system.

Phys. Rev. (2) **132** (1963), 978–985.

A theory is developed to describe the scattering of electromagnetic waves by the electron density fluctuations in an electron-phonon system. Most attention is directed to frequencies of the electromagnetic radiation in the neighborhood of the plasma frequency. At these energies the total amount of scattered radiation is very small and the shifts due to the phonon resonances are small but may be observable with sensitive detectors.

M. J. Stephen (New Haven, Conn.)

Velibekov, É. R. [Velibekov, È. R.] 5586

A generalized self-consistent field method and collective excitations in the superconductivity theory.

Dokl. Akad. Nauk SSSR **150** (1963), 1015–1018 (*Russian*); *translated as Soviet Physics Dokl.* **8** (1963), 576–579.

The longitudinal collective excitations of a superconductor are investigated when the attractive pair interaction takes place in higher angular momentum states than the s state. The collection excitations have the same angular momentum as the attractive interaction. If the Coulomb interaction is taken into account, these longitudinal waves become ordinary plasma waves as in the case of an s state interaction. *M. J. Stephen* (New Haven, Conn.)

Levine, Arnold D. 5587

Lagrangian formulation of the phonon field equations.

J. Mathematical Phys. **5** (1964), 1615–1618.

An action principle is given for one space variable, x, and one time variable, t, which yields the equations of lattice vibrations as equations of motion. These are differential equations with respect to t, but difference equations with respect to x. The theory is therefore non-local in x. The resulting conservation laws of energy and momentum are also difference equations with respect to x. The boundary conditions are necessarily specified over one lattice spacing. The theory is then quantized in the canonical fashion. The commutator becomes a sum of periodically spaced delta-function singularities. The generalization to the three-dimensional case is discussed.

F. Rohrlich (Syracuse, N.Y.)

Verlet, Loup 5588

On the theory of classical fluids. III.

Physica **30** (1964), 95–104.

Part II (by the author and Levesque) appeared in Physica **28** (1962), 1124–1142 [MR **29** #830]. The author considers one way of systematically improving the Percus-Yevick (PY) and hypernetted chain (HNC) approximations to the pair distribution function of a classical fluid. Percus has shown how to obtain each of these approximations by truncating an appropriate functional Taylor expansion [Phys. Rev. Lett. 8 (1962), 462–463]. The author investigates the equations that result from retaining one more term in both of these expansions. In each case a second equation must be introduced to determine the approximate distribution function. The author obtains his second equation by introducing another functional equation. He assesses the accuracy of the resulting approximations by applying them to a system of hard cubes (in one, two, and three dimensions) and comparing the resulting fourth and fifth virial coefficients with the known exact coefficients for this system. (The author has subsequently [preprint] investigated a somewhat different second equation for each approximation and has also corrected an error in his tabulation of the fifth virial coefficient according to the improved HNC theory.) *G. R. Stell* (New York)

Falk, Harold 5589

Inequalities relating the nearest-neighbor spin correlation and the magnetization for the Heisenberg Hamiltonian.

J. Mathematical Phys. **5** (1964), 1478–1480.

Convexity properties of the free energy with respect to the interaction and temperature parameters are used to obtain inequalities for the short-range order and the magnetization of Heisenberg models of ferro- and anti-ferro-magnets. *R. Kubo* (Tokyo)

Mărgulescu, G. 5590

Les équations tensorielles de courbure dans la théorie de la fusion. (Romanian. Russian and French summaries)

Com. Acad. R. P. Romîne **12** (1962), 1111–1116.

RELATIVITY

See also 4652, 5193, 5205, 5206, 5207, 5208, 5421, 5503, 5531, 5563.

Nodvik, John S. 5591

A covariant formulation of classical electrodynamics for charges of finite extension.

Ann. Physics 28 (1964), 225–319.

This is a detailed and careful treatment of Lorentz covariant charge distributions and their dynamics. It includes much of the previous work on this subject as special cases. The charge distributions are assumed to be rigid in the sense that they remain invariant for the inertial observer who is instantaneously comoving with the charge center; the latter point is also assumed to be the center of mass. The charge distribution is in general characterized by two invariant shape functions, one for translatory and one for relativistic rotatory motion. The latter motion is studied in detail, a subject not easily found elsewhere. The equations of motion are derived from a variational principle, and the conservation laws are given. Radiation reaction terms are computed in perturbation expansion. In this approximation gyromagnetic ratios are obtained for various distributions. The paper is restricted to sufficiently small accelerations, so that differential pair production does not occur. This means that an inertial observer will see no world line of an element of charge change its (time-like) tangent vector from the future to the past light cone. *F. Rohrlich* (Syracuse, N.Y.)

DeWitt, C.; DeWitt, B. (Editors) 5592

★Relativité, groupes et topologie [Relativity, groups and topology].

Lectures delivered at Les Houches during the 1963 session of the Summer School of Theoretical Physics, University of Grenoble.

Gordon and Breach, Science Publishers, New York-London, 1964. xvi + 929 *pp*. $19.50.

This book is a collection of many outstanding contributions, mostly pedagogic in nature, which will be of great use, not only to students of general relativity and gravitation, but also to people interested in elementary particle theory. The authors and the editors, as well as the publisher, cooperated magnificently in producing from the lectures a real book which does not carry at all the characteristics of lecture notes.

The contents of this collection are as follows: (I) J. L. Synge, "Introduction to General Relativity"; (II) F. Gürsey, "Introduction to Group Theory"; (III) R. H. Dicke, "Experimental Relativity"; (IV) J. A. Wheeler, "Geometrodynamics and the Issue of the Final State"; (V) R. K. Sachs, "Gravitational Radiation"; (VI) R. Penrose, "Conformal Treatment of Infinity"; (VII) B. S. DeWitt, "Dynamical Theory of Groups and Fields"; (VIII) A. Lichnerowicz, "Propagateurs, Commutateurs et Anticommutateurs en Relativité Générale"; (IX) J. Weber, "Gravitational Radiation Experiments"; (X) C. W. Misner, "Differential Geometry and Differential Topology".

These papers will be reviewed individually.

P. Roman (Boston, Mass.)

Synge, J. L. 5593

Introduction to general relativity.

Relativité, Groupes et Topologie (*Lectures, Les Houches, 1963 Summer School of Theoret. Phys., Univ. Grenoble*), *pp*. 1–88. *Gordon and Breach, New York*, 1964.

This is a condensed, very readable introduction to classical general relativity. Occasionally, attention is drawn to some subtleties which are often neglected in the current textbook literature. A large selection of exercises makes the text even more valuable for the student.

P. Roman (Boston, Mass.)

Gürsey, Feza 5594

Introduction to group theory.

Relativité, Groupes et Topologie (*Lectures, Les Houches, 1963 Summer School of Theoret. Phys., Univ. Grenoble*), *pp*. 89–161. *Gordon and Breach, New York*, 1964.

This article serves as a good, but somewhat too concise, introduction for students who will have to use group-theoretical methods in further work. Standard topics about abstract groups, representations, and Lie groups are covered in a very clear and pedagogic manner, and the survey ends with a more specific chapter on semi-simple groups which are connected with general relativity and elementary particle theory. *P. Roman* (Boston, Mass.)

Dicke, R. H. 5595

Experimental relativity.

Relativité, Groupes et Topologie (*Lectures, Les Houches, 1963 Summer School of Theoret. Phys., Univ. Grenoble*), *pp*. 163–313. *Gordon and Breach, New York*, 1964.

In these clearly written and very stimulating lectures, primary emphasis is given to null experiments. In this chapter the Eötvös experiment, the Dicke experiment, the experiments on the isotropy of space, and an interesting Princeton ether drift experiment are discussed. Next, the so-called fundamental tests of general relativity (red shift, deflection of light, perihelion rotation) are reviewed. Finally, some cosmic experiments are mentioned. The main text is followed by 12 reprints of previously published papers of the author (and co-workers), concerning mainly the possibility of an additional scalar field in the theory of gravitation and the exploring of its consequences. Generally, a somewhat wider framework than the conventional one is used in discussing the experiments, so as to show the limitations imposed by experimental findings on the possible theories.

P. Roman (Boston, Mass.)

Wheeler, John Archibald 5596

Geometrodynamics and the issue of the final state.

Relativité, Groupes et Topologie (*Lectures, Les Houches, 1963 Summer School of Theoret. Phys., Univ. Grenoble*), *pp*. 315–520. *Gordon and Breach, New York*, 1964.

The underlying motive of this long and rich treatise is the following: What is the reason, meaning, interpretation, and what are the implications of situations where a space-like 3-geometry, followed forward in time, develops an infinite curvature? Many examples of such phenomena are discussed in detail and are classified. Furthermore, preliminary considerations on the quantum level are offered which may show a way out of these difficulties. An interesting feature of the text is that it gives a large number of problems, some of them being exercises for the

student and some of them being yet unsolved, and even fundamental, questions.

The abbreviated table of contents is given below. (1) Introduction and Outline, The Falling Star; (2) The Nature of Dynamics; (3) The Variational Principle of General Relativity; (4) The "Thin-Sandwich" Variational Problem, The Initial Value Problem, and Mach's Principle; (5) The Expanding and Recontracting Universe: Is the Shift Unique?; (6) The Time Symmetric Initial Value Problem and Closure as an Eigenvalue Problem; (7) Time Symmetry and the Thin Sandwich; (8) Regge Calculus and Schwarzschild Geometry; (9) Schwarzschild Geometry, Space-Like Slices, and the Nature of Time; (10) The Reissner-Nordstrom Geometry: Again a Singularity after a Finite Proper Time; (11) The Topological Description of Electric Charge; (12) Universes Activated by the Longest Wave, and a New Type of Singularity; (13) Geons; (14) The Collapsing Cloud of Matter Reconsidered; (15) Gravitational Collapse and the Creation and Annihilation of Matter. *P. Roman* (Boston, Mass.)

Sachs, R. K. 5597a

Gravitational radiation.

Relativité, Groupes et Topologie (*Lectures, Les Houches, 1963 Summer School of Theoret. Phys., Univ. Grenoble*), *pp.* 521–562. *Gordon and Breach, New York*, 1964.

Penrose, Roger 5597b

Conformal treatment of infinity.

Relativité, Groupes et Topologie (*Lectures, Les Houches, 1963 Summer School of Theoret. Phys., Univ. Grenoble*), *pp.* 563–584. *Gordon and Breach, New York*, 1964.

The first paper discusses how the gravitational field equations determine the evolution of waves; then a geometrical analysis of the far field is given; finally, some comments are made on approximation methods.

In the second paper, the standard initial-value problem, the discussion of the geometry of a congruence of null-geodesics, and the characteristic initial-value problem are followed by an introduction to the Bondi-Metzner-Sachs group and a discussion of asymptotic symmetries. Then, after a section on the algebra of two-component spinors, the algebra of the Weyl tensor is presented, followed by the peeling theorem and comments on algebraically special fields. Finally, a brief discussion of the linearized theory is given. The article ends with a fine selection of problems for the student, some unsolved as yet. A substantial list of references is given.

P. Roman (Boston, Mass.)

DeWitt, Bryce S. 5598

Dynamical theory of groups and fields.

Relativité, Groupes et Topologie (*Lectures, Les Houches, 1963 Summer School of Theoret. Phys., Univ. Grenoble*), *pp.* 585–820. *Gordon and Breach, New York*, 1964.

The unusual title of this treatise is meant to indicate the development of a framework within which the quantization of fields possessing infinite-dimensional invariance groups may be carried out in a manifestly covariant fashion. In line with this purpose, an overall space-time view is adopted from the outset, and canonical formulations are ignored. The quantum outlook is stressed from the beginning, even though quite a lot is done at the classical level. The fundamental tool employed is that of functional derivatives. Concerning the notion of infinite-dimensional invariance groups, it is stressed that they lead primarily to identities on the dynamical equations, in contradistinction to the more familiar finite-dimensional groups, which lead to conservation laws whenever the dynamical equations are satisfied.

In order to gain a glimpse of the richness of this article, we believe the best approach is to quote the table of contents: (1) Fundamentals; (2) Small disturbances; (3) Invariance groups; (4) Measurement; (5) Measurement of two observables, The uncertainty principle, Commutators; (6) Green's functions; (7) Supplementary conditions, Cauchy data, Reciprocity relations; (8) Poisson-Jacobi identity, Invariance of the Poisson bracket, Canonical theory; (9) Asymptotic invariants; (10) Field quantization, Creation and annihilation operators, Positive-definiteness of Hilbert space, Electromagnetic and gravitational fields, Helicity; (11) Theory of continuous groups; (12) The Yang-Mills group; (13) The general coordinate transformation group; (14) Metric and spin; (15) Anticommuting fields; (16) Specific Lagrangians; (17) Bi-tensors, bi-spinors, and Green's functions; (18) Conservation laws; (19) The canonical transformation group, Time reversal; (20) The Schwinger variational principle; (21) The S-matrix; (22) The quantum action functional, The "tree" theorem; (23) External field problems, Lowest-order radiative corrections; (24) Renormalization.

This article is not for the beginner, and even for the one who is seasoned in field theory, it will prove to be difficult, but most rewarding, reading. By way of only criticism we may mention that the difficulties of comprehension are compounded by the fact that topics are often not given full discussion at the points where they are raised, but are completed only in several later sections.

P. Roman (Boston, Mass.)

Lichnerowicz, A. 5599

Propagateurs, commutateurs et anticommutateurs en relativité générale.

Relativité, Groupes et Topologie (*Lectures, Les Houches, 1963 Summer School of Theoret. Phys., Univ. Grenoble*), *pp.* 821–861. *Gordon and Breach, New York*, 1964.

This mathematically elegant and very valuable review article is divided into four major sections. First, the general mathematical theory of propagators is given, followed by a discussion of tensorial fields. Then, spinorial fields of various order are discussed, and finally, comments on creation-annihilation operators are made.

The whole discussion is restricted to free fields defined over a curved space-time manifold.

P. Roman (Boston, Mass.)

Weber, Joseph 5600

Gravitational radiation experiments.

Relativité, Groupes et Topologie (*Lectures, Les Houches, 1963 Summer School of Theoret. Phys., Univ. Grenoble*), *pp.* 863–880. *Gordon and Breach, New York*, 1964.

This article consists of two main parts. In the first part, the feasibility of gravitational radiation experiments is illuminated, together with a discussion of the possibility

of generating gravitational waves, and a brief description of current experiments at the University of Maryland. In the second part, using the Fock-Papapetrou method, equations of motion involving both gravitational and non-gravitational forces are deduced in the framework of general relativity. The purpose of this discussion is that the ensuing equations can be used to discuss means of measurement of the Riemann tensor.

P. Roman (Boston, Mass.)

Hlavatý, V.; Mishra, R. S. 5601

Classification of space-time curvature tensor: General theory of λ-classification.

Rend. Circ. Mat. Palermo (2) **11** (1962), 319–350.

Let V_4 be a (4-dimensional) Lorentzian manifold. A tensor on V_4 having the algebraic symmetry properties of the Riemann curvature tensor (but not its differential properties) is called a curvature tensor. Using the concepts and notation of another paper in press, the authors develop a classification of curvature tensors by Segre characteristics. They do not decompose the curvature tensors with respect to the Lorentz group (into Weyl, trace-free Ricci and scalar parts) before classification. More details are promised in a sequel.

F. A. E. Pirani (Waltham, Mass.)

Kühnel, Adolf 5602

Equations of motion in the theory of gravitational perturbations.

Ann. Physics **28** (1964), 116–133.

Following Fok [*Theory of space, time and gravitation* (Russian), GITTL, Moscow, 1955; MR **18**, 445] the author derives the equations of motion of a system of point particles as a suitable limit of the equations of motion of spherical drops of a perfect compressible fluid. The essential difference from Fok's work is the use of the "fast approximation" method [B. Bertotti and J. Plebański, Ann. Physics **11** (1960), 169–200; MR **22** #5440]. The task is formidable, and several equations require a whole page to be written.

In the first-order equations, the singular terms arising in the limit of zero size (and infinite density) are removed by a mass renormalization. Radiation "anti-damping' terms occur like those in electrodynamics, but with the opposite sign. In the second approximation, two subtractions are needed: one for the mass and one for the coupling constant. The radiative terms in the second-order equations are symmetric under time reversal. *A. Peres* (Haifa)

Misra, R. M. 5603

Geometry of the electromagnetic field. (Italian summary)

Nuovo Cimento (10) **32** (1964), 1561–1583.

The author discusses canonical forms of the curvature tensor $R_\mu{}^\nu$ in the presence of matter (μ), pressure (p) and an electromagnetic energy momentum tensor $F_\mu{}^\nu$, permissible under the assumption that the electromagnetic field is non-null. It is found that only certain multiplicities of the eigenvalues of $R_\mu{}^\nu$ are allowed under these conditions. In every case, the electromagnetic invariant F, matter density μ, and pressure p are expressed as concomitants of the curvature tensor.

A. H. Klotz (Liverpool)

Pichon, Guy 5604

Diffusion du rotationnel dans les fluides visqueux incompressibles.

C. R. Acad. Sci. Paris **259** (1964), 2363–2365.

The author points out that the equations for the rotation of an incompressible viscous fluid contain but first derivatives with respect to the time coordinate, and that to this extent we are dealing here with a Cauchy problem resembling that of the diffusion equation.

{Reviewer's comment: Both the diffusion equation and Helmholtz equations deal with quantities that (more or less) conserve, in one case the number of diffusing particles, in the other the angular momentum. Presumably, for some classes of fluids (those free of dissipative processes) the dynamics of the fluid obeys time-reversible laws, whereas an intrinsically dissipative process (such as diffusion) never does. The Cauchy problems in these circumstances must be fundamentally inequivalent.} *P. G. Bergmann* (New York)

Radhakrishna, L. 5605

Some exact non-static cylindrically symmetric electrovac universes.

Proc. Nat. Inst. Sci. India Part A **29** (1963), 588–595.

Author's summary: "The non-static cylindrically symmetric metric of L. Marder [Proc. Roy. Soc. London Ser. A **244** (1958), 524–537; MR **21** #1885] is considered, and some exact non-static solutions of the Einstein-Maxwell field equations in vacuo are obtained. These solutions include as particular cases uniform electromagnetic fields, which are transforms of Levi-Civita's homogeneous electric or magnetic fields [Atti Accad. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (5) **26** (1917), 1° semestre, 519–531]."

B. Bertotti (Frascati)

Caricato, Gaetano 5606

Principio dell'azione stazionaria e principio di Fermat in relatività generale.

Rend. Mat. e Appl. (5) **22** (1963), 416–426.

The author discusses Fermat-type variational principles governing the behavior of a test particle in a Riemannian manifold. *P. G. Bergmann* (New York)

Bonazzola, Silvano 5607

Un vecteur de Poynting pour la radiation gravitationnelle en relativité générale.

C. R. Acad. Sci. Paris **259** (1964), 1011–1014.

The author assumes that in a general coordinate system in space-time it is possible to associate with the metric tensor $g_{\mu\nu}$ another metric tensor, the metric tensor of a Minkowski space in the same coordinate system. In terms of these two tensors he defines a four-vector, the energy-momentum four-vector of the gravitational field whose divergence with respect to the first metric vanishes. He observes that his vector is not "gauge-invariant" and obtains the condition that infinitesimal coordinate transformations have to obey in order that the energy-momentum four-vector be invariant under such infinitesimal coordinate transformations.

A. H. Taub (Berkeley, Calif.)

Schmutzer, Ernst 5608

Einige Bemerkungen zur Theorie der Spinoren und Bispinoren in der gekrümmten Raum-Zeit. (Russian summary)

Acta Phys. Acad. Sci. Hungar. 17 (1964), 57–65.

Author's summary: "Das Postulat der Lorentzinvarianz der Diracschen Gleichung führt zu zwei Transformationsversionen: (1) Konstanz der Dirac-Matrizen und daher das wohlbekannte Transformationsgesetz für Spinoren; (2) Auffassung der Dirac-Matrizen als Tensoren und Interpretation der Wellenfunktion als einer Art von Invarianten. In der vorliegenden Arbeit wird die Sachlage für die gekrümmte Raum-Zeit untersucht. Zu diesem Zweck wird die Theorie der Spinoren und der Bispinoren in der gekrümmten Raum-Zeit kurz dargestellt."

The author re-concludes that version (2) cannot be generalized to curved space-time, making use of the Lie-Cartan integrability conditions. "Bispinoren" are what are often called 4-component spinors.

W. Rindler (Dallas, Tex.)

Schöpf, Hans-Georg 5609

Clebsch-Transformationen in der allgemeinen Relativitätstheorie. (Russian summary)

Acta Phys. Acad. Sci. Hungar. 17 (1964), 41–55.

The author derives Clebsch transformations for relativistic perfect fluids, and then generalizes his results to an electrically charged fluid interacting with an electromagnetic field. A variational principle for the general case is given. *W. Rindler* (Dallas, Tex.)

Staruszkiewicz, A. 5610

On propagation of the Riemann tensor in the theory of gravitation.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 12 (1964), 271–273.

The author shows how to calculate, under the assumption of Einstein's field equations, all the components of the Riemann tensor and all its covariant derivatives on a spacelike hypersurface S ($x^0=0$) from the values of the metric and of the Christoffel symbols on S. From his formulae he deduces that if $R_{\lambda 0ik}$ $(i,k=1,2,3)$ vanishes on S, then $R_{\lambda\mu\nu\sigma}$ vanishes throughout space-time if it is real-variable analytic. It is then proved that $R_{\lambda 0ik}=0$ on S implies, without field equations, that S is embeddable in a four-dimensional Euclidean space.

W. Rindler (Dallas, Tex.)

Gregory, C. 5611

Search for extra-dimensional effects. (Italian summary)

Nuovo Cimento (10) **32** (1964), 1085–1091.

The author suggests that the presence of a fifth dimension, which is proposed in some unitary field theories [e.g., H. Weyl, Ann. Physik (4) **59** (1919), 101–133], might lead to extra statistical fluctuations, which could, for instance, become evident in Mössbauer effect experiments. A discussion of experimental data in the light of this conjecture is inconclusive.

{Reviewer's comments: The reference should have been to Kaluza [S.-B. Preuss. Akad. Wiss. **1921**, 966–972] and to projective theories (Veblen, Pauli, etc.) rather than to Weyl. Only some of these theories admit the fifth dimension as an additional degree of freedom, and the extent of the fluctuation associated with that degree of freedom is controlled.} *P. G. Bergmann* (New York)

Lal, K. B. 5612

Einstein's connections. III. Degenerate cases of the first class.

Proc. Nat. Inst. Sci. India Part A **29** (1963), 535–551.

This is substantially the summary of the author: In two recent papers [Lal and Mishra, Tensor (N.S.) **10** (1960), 218–237; MR **25** #1934; Proc. Nat. Acad. Sci. India Sect. A **32** (1962), 1–24; MR **28** #553], the solution of Einstein's first set of equations of the unified field theory has been studied in full detail in holonomic and non-holonomic forms for the degenerate cases $g=0$ and $a^2-4kb^2=0$ of the first class when $D\neq 0$, for the indices of inertia 0, 2 and 4, and conditions for the existence of at least one solution in each case have been stated. The reviewer [*Geometry of Einstein's unified field theory*, pp. 75–78, Noordhoff, Groningen, 1957; MR **20** #5067] gave a solution of the same in holonomic form for the index of inertia 2 and $D\neq 0$, which will remain similar in form for the indices of inertia 0 and 4 also. But this solution contains D as a factor in the denominator of some of its terms, and hence cannot be used for $D=0$. The object of this paper is to give solutions of the said equation in holonomic form for the degenerate cases $K+1=0$ and $3K-1=0$ of the first class when $D=0$, for the index of inertia 0 of $h_{\lambda\kappa}$ (in which case only $K+1$ can be zero) and for the index of inertia 0 and 4 of $h_{\lambda\kappa}$ (in which case only $3K-1$ can be zero) and to state conditions for the existence of at least one solution in each case. *V. Hlavatý* (Bloomington, Ind.)

Nguyen, Phong-Chau [Nguyen Phong Chau] 5613

Contribution à l'étude des théories du champ unifié du type théorie d'Einstein-Schrödinger. (English summary)

Ann. Inst. H. Poincaré 18, 303–357 (1964).

The author generalizes to some extent the Einstein-Schrödinger theory and proposes also some new physical interpretation of the variables occurring in the asymmetric theories. The main chapter headings of this rather comprehensive treatment are: (1) The difficulties encountered in the Einstein-Schrödinger theory; (2) The second-order approximate equations, and the conservation identities; (3) The Cauchy problem of the field equations; (4) Stationary unitary fields that are everywhere regular; (5) Physical interpretation, and the problem of (ponderomotive) motion; and (6) The static and spherically symmetric field of a charged particle.

P. G. Bergmann (New York)

Pimenov, R. I. 5614

An application of semi-Riemannian geometry to unified field theory. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 795–797.

The author shows that Einstein's gravitational field equations and both sets of Maxwell's equations are contained in the geometrical structure of a semi-riemannian manifold ${}^3V_5{}^4$ in which $g_i(5;\nu)=0$, $F_{ik}=2A_{[i;k]}$ and $A_i=g_{5i}/(g_{55})^{1/2}$ $(i,k=1,\cdots,4;\nu\leq 5)$.

A. H. Klotz (Liverpool)

Treder, Hans-Jürgen 5615

Eine Verallgemeinerung des Theorems von Einstein und Pauli.

Math. Nachr. **26** (1963/64), 353–359.

The theorem referred to in the title states that under certain boundary and regularity conditions there is no everywhere regular stationary solution of the Einstein vacuum field equations corresponding to a particle of non-vanishing mass. The author applies the technique of Einstein and Pauli to the case where the static space-time contains a world-line on which the metric has a vanishing determinant. He shows that this singular world-line may in some cases be identified with that of a monopole particle and gives an expression for the mass of the particle. *A. H. Taub* (Berkeley, Calif.)

ASTRONOMY

See also 5421.

Blitzer, Leon 5616

Synchronous and resonant satellite orbits associated with equatorial ellipticity. (Russian summary)

The Use of Artificial Satellites for Geodesy (Proc. First Internat. Sympos., Washington, D.C., 1962), *pp.* 52–61. *North-Holland, Amsterdam,* 1963.

The author considers orbits of such small eccentricity and inclination that the equations for the quantities expressing departures from circular motion and for the latitude can be considered to be linear in these quantities. Tesseral harmonics of the second order are considered. It is shown that, in general, the oscillations of the small quantities remain small, but that when the period of the satellite is 12 or 36 hours, the amplitudes of the induced oscillations increase linearly with time, due to resonance. The 24-hour satellite is another exception, and properties of the positions of stable and unstable equilibrium are discussed. *J. M. A. Danby* (New Haven, Conn.)

Garfinkel, B. 5617

Close satellite orbits of an oblate planet (historical survey). (Russian summary)

The Use of Artificial Satellites for Geodesy (Proc. First Internat. Sympos., Washington, D.C., 1962), *pp.* 3–5. *North-Holland, Amsterdam,* 1963.

A brief comparison is made of the accuracy and simplicity of the satellite theories of Brouwer, the author, and Vinti. *J. M. A. Danby* (New Haven, Conn.)

Morando, B. 5618

Recherches sur les orbites de résonance. (English and Russian summaries)

The Use of Artificial Satellites for Geodesy (Proc. First Internat. Sympos., Washington, D.C., 1962), *pp.* 42–51. *North-Holland, Amsterdam,* 1963.

The author considers satellite orbits with periods $(q/p)24$ hours, where q and p are integers. Different tesseral harmonics are considered independently, and the method of von Zeipel, as used by Hori, is used to isolate the terms that would lead to small divisors. Principal attention is confined to the motion of the 24-hour satellite. The author emphasizes that effects of eight harmonics should be taken into account. *J. M. A. Danby* (New Haven, Conn.)

Vinti, John P. 5619

Zonal harmonic perturbations of an accurate reference orbit of an artificial satellite.

J. Res. Nat. Bur. Standards Sect. B **67B** (1963), 191–222.

The author's well-known reference orbit, the construction of which is based on the separability of a Hamilton-Jacobi equation in oblate spheroidal coordinates, accounts for the conventional $O(J_2)$ and $O(J_2^2)$ perturbations of an artificial satellite. In the present paper the author derives the $O(J_2^2+J_4)$ and $O(J_3)$ perturbations in Delaunay variables that affect his reference orbit. Essentially, the results are continued in those of Kozai [Astronom. J. **67** (1962), 446–461]. *I. G. Izsak* (Cambridge, Mass.)

Alekseev, V. M. 5620

Remarks on the criteria for hyperbolic and hyperbolic-elliptic motion. (Russian)

Vestnik Moskov. Univ. Ser. III Fiz. Astronom. **1961**, no. 1, 67–75.

Alekseev, V. M. 5621

New examples of capture in the three-body problem.

Astronom. Ž. **39** (1962), 724–735 (*Russian*); *translated as Soviet Astronom. AJ* **6** (1963), 565–572.

Author's summary: "New examples are given of 'capture' phenomena in the problem of three mass points moving under mutual Newtonian attraction. Purely qualitative methods are used, and numerical integration is not employed. The examples are more general than in Sitnikov [Mat. Sb. (N.S.) **32** (74) (1953), 693–705; MR **15**, 356] and contain the maximum number of free parameters."

Traugott, S. C. 5622

A differential approximation for radiative transfer with application to normal shock structure.

Proc. 1963 Heat Transfer and Fluid Mech. Inst. (Calif. Inst. Tech., Pasadena, Calif., 1963), *pp.* 1–13. *Stanford Univ. Press, Stanford, Calif.,* 1963.

A differential approximation to one-dimensional gray radiative heat flux has been obtained which avoids the necessity of treating integro-differential equations. Various degrees of approximation lead from a moment method to various differential equations for the flux. The first two approximations are stated. The first approximation, equivalent to the Milne-Eddington approximation, is applied to the structure of a gray radiating normal shock wave with both viscosity and heat conductivity. It is found that two allowable starting slopes occur for the temperature-velocity profile, where in the non-radiating case there is only one. Linearization shows each slope to be related to a characteristic length for the shock, one associated with the collision and the other to the photon mean free path. The significance of these slopes in determining the shock profile is discussed as a function of shock strength, including the limits of no radiation, transparency, and opaqueness.

In an introductory discussion the author states that the conventional formulation of gas dynamics problems with

radiative transfer in terms of integro-differential equations "is not associated with some intrinsic action-at-distance of the radiation". This reviewer disagrees. The differential formulation arises from the assumption he uses: That the radiation field is nearly isotropic, so that an expansion in Legendre polynomials may be truncated after a few terms. In general, a hierarchy of differential equations is found, each one coupled with the next higher one. No rational truncation scheme (closure) seems evident unless anisotropy is small. This reviewer believes the expansion to be an asymptotic one in a sense analogous to that of the Chapman-Enskog expansion yielding partial differential equations asymptotic to the Boltzmann integro-differential equation for small Knudsen number. More physically, the integro-differential equation is an expression of the fact that radiative flux at a point comes from all over space, having been emitted elsewhere and, perhaps, absorbed and re-emitted along the way. The assumption of near isotropy, leading to the differential formulation rather than the conventional one, is formally equivalent to the substitution of an exponential function for an exponential integral as the kernel in the radiative heat flux integral. The relative error in this substitution grows linearly with the argument and hence becomes arbitrarily large for large argument. However, the integrated result for heat flux seems to be quite well approximated by this procedure. Thus, as an ad hoc approximation, the substitute kernel or differential approximation is a very useful and powerful tool, even when the radiation field is strongly anisotropic, as in shock wave. *I. M. Cohen* (Providence, R.I.)

★Astronomie. 5623

Volume publié sous la direction d'Evry Schatzman. Encyclopédie de la Pléiade, Vol. XIII.

Librairie Gallimard, Paris, 1962. xxiii + 1834 *pp*.

The strength of this book lies in its truly encyclopaedic coverage of astronomy; but though very many topics are discussed, the discussions are, on the whole, superficial. Formulas are often given which would bewilder the uninitiated, but also fail to satisfy the specialist. Nevertheless, there are some excellent descriptive articles, and the book is not unrewarding for the browser.

J. M. A. Danby (New Haven, Conn.)

Trumpler, Robert J.; Weaver, Harold F. 5624

★Statistical astronomy.

Dover Publications, Inc., New York, 1962. xx + 644 *pp*. $3.00.

A welcome appearance in paperback of a book [Univ. California Press, Berkeley, Calif.] that originally appeared in 1953. Contents: Elements of statistical theory. Statistical description of the galactic system: General analysis. Stellar motions in the vicinity of the sun. Luminosity-spectral type distribution. Space distribution of stars. Galactic rotation.

The final part makes use of stellar motions only.

J. M. A. Danby (New Haven, Conn.)

Lynden-Bell, D. 5625

On large-scale instabilities during gravitational collapse and the evolution of shrinking Maclaurin spheroids.

Astrophys. J. **139** (1964), 1195–1216.

Author's summary: "The collapse of a rotating, pressureless, self-gravitating spheroid is computed and shown to be unstable against small second harmonic deformation throughout the period of collapse into a disk. The disk actually formed may thus be considerably elongated along a diameter. This may be significant in the formation of galaxies. The instability due to friction that occurs in the classical sequence of Maclaurin spheroids at their point of bifurcation with the Jacobi ellipsoids only develops if the spheroid is shrinking more slowly than the rate at which viscosity can dissipate the energy. Small tides raised by another body have little effect except when they resonate at a natural frequency of the spheroid. Chandrasekhar's virial tensor method can be used to discuss time-varying ellipsoids provided the velocities are linear functions of position."

J. M. A. Danby (New Haven, Conn.)

GEOPHYSICS

Bullen, K. E. 5626

★An introduction to the theory of seismology.

Third edition.

Cambridge University Press, New York, 1963. viii + 381 *pp*. $9.50.

This is the third edition of Bullen's famous book on seismology; the second edition (1953) was reviewed in MR 15, 373. The first eight chapters are concerned, for the most part, with the classical linearized theory of wave propagation in elastic solids. The next ten chapters discuss the following subjects: the seismograph, the construction of travel-time tables, the seismological observatory, the earth's upper layers and deep interior, long-period oscillations of the earth, earthquakes, nuclear explosion seismology, and planetary seismology.

M. E. Gurtin (Providence, R.I.)

ECONOMICS, OPERATIONS RESEARCH, GAMES

See also 5356, 5357, 5667.

Batchelor, James H.; Athans, Chris N. 5627

★Operations research: An annotated bibliography. Vol. 4.

Saint Louis Academy Press, Saint Louis, Mo., 1964. xii + 477 *pp*. $16.00.

This volume continues the bibliography of publications in operations research and follows the format of earlier volumes [Batchelor, second edition, 1959; MR **21** #4058; Vol. 2, 1962; MR **28** #1000a; Vol. 3, 1963; MR **28** #1000b].

★Anwendung mathematischer Methoden in der Ökonomie. 5628

Herausgegeben von W. S. Nemtschinow [V. S. Nemčinov].

B. G. Teubner Verlagsgesellschaft, Leipzig, 1963. 437 *pp*. (1 insert) *DM* 28.50.

The original Russian [Izdat. Social'no-Èkon. Lit., Moscow, 1959] was reviewed earlier [MR **24** #B1655]. Some of the

papers of the original Russian volume were translations of English papers into Russian; they have now been translated into German.

Allen, W. R. 5629

Simple inventory models with bunched inputs.

Naval Res. Logist. Quart. **9** (1962), 265–273.

If N is a Poisson variable and $X_1, \cdots, X_N$ are independent random variables from the same population, then $S = X_1 + \cdots + X_N$ is a compound distribution whose d.f. is easily obtained. This note considers orders (output) and stock (input) arriving in a time period of length t to be "bunched" variables of this nature, with $\mathscr{E}(N) = \lambda t$ for some λ. If mean output and input are equal, the variance of the difference between output and input is considered as the parameter of interest and is listed for some simple compound distributions. It is noted that if transhipments in and out are considered, as further independent random inputs and outputs, their difference may be added to the stock-orders difference. There is no mention of any actual inventory system to which these elementary models apply: it is remarked that "this examination of an open loop system should aid in directing our attention to reasonable approaches in improving an inventory system by servo-control approaches", but we are not told how.

D. E. Barton (London)

Bartholomew, D. J. 5630

A multi-stage renewal process.

J. Roy. Statist. Soc. Ser. B **25** (1963), 150–168.

Author's summary: "This paper derives the theory of a multi-stage renewal process and discusses two applications. The process consists of several self-renewing aggregates arranged in series. When a failure occurs in the ith stage it is replaced by an item already operating in the $(i-1)$st stage; the vacancy thus created is filled by a member of stage $(i-2)$ and so on. The main object of the analysis is to determine the failure rates in each stage and the transfer rates between stages. Two rules for selecting the item to be transferred are considered; in one case the selection is made at random, in the other the oldest member is selected. General methods for obtaining transient and equilibrium solutions are given for both rules. Simple explicit solutions are obtained when the life distribution is negative exponential and for certain other distributions. The application of the theory to the study of wastage and promotion in hierarchical organizations and to the determination of optimum industrial replacement schemes is briefly discussed."

Ben-Israel, Adi 5631

Notes on linear inequalities. I. The intersection of the nonnegative orthant with complementary orthogonal subspaces.

J. Math. Anal. Appl. **9** (1964), 303–314.

Author's summary: "The intersections of the non-negative orthant in E^n with pairs of complementary orthogonal subspaces are investigated. Applications to linear inequalities and linear programming are then made by using Fredholm's alternative theorem."

W. F. Tyndall (Lancaster, Pa.)

Efron, Bradley 5632

Optimum evasion versus systematic search.

J. Soc. Indust. Appl. Math. **12** (1964), 450–457.

The author considers the following search-versus-evasion problem in the context of game theory. For his ith move Player I (the evader) chooses an integer m_i from $R = \{1, 2, \cdots, N\}$, and on his ith move Player II (the searcher) chooses a set S_i of M_i integers drawn from R. Each integer represents one of N separate rooms. Play terminates after n moves for each player or when II "finds" I (i.e., $m_i \in S_i$, for some i, $1 \leq i \leq n$), whichever occurs first.

Let P be a nondecreasing function on R. If Player II finds Player I on the j_0th move, II pays I $P(j_0)$ units; if there is no such j, then I receives $P(n+1)$. It is assumed that Player II's search is systematic, that is, the sets S_i are mutually disjoint. In this case, require $n \leq N$. By symmetry, II's optimal strategy is to choose with equal probability from among the permutations of $M_1 + M_2 + \cdots + M_n$ $(\leq N)$ distinct elements of R.

The main result of the paper establishes that Player I's optimal strategy is to choose with equal probability from among the permutations of n distinct elements of R. Thus Player I need never employ his option to return to a room which he has previously occupied.

W. F. Tyndall (Lancaster, Pa.)

Giammo, T. P. 5633

On the probability of success in a sudden death search with intermittent moves confined to a finite area.

SIAM Rev. **5** (1963), 41–51.

Red and Blue are two guerrillas searching for each other according to the following rules: Blue remains at a fixed point in his operating region for a time t_m and then jumps instantaneously to any other point of the region (with equal probability) where he stays for a further time t_m, then moves again, and so on. His first move is equally likely to be at any point in $(0, t)$. Similarly for Red. Independently and simultaneously, each searches the other's region randomly at a uniform rate so that, apart from the initial period, each scans a fixed proportion of the other's region before the latter jumps and then the search effectively starts again from scratch. This leads directly to the p.d.f. of the time before Red discovers Blue, and conversely: These are of course approximately exponential when the chance of discovery in any period is low. The usual actuarial formula is used then to give an expression for the chance that Blue discovers Red before Red discovers Blue, and conversely.

This model for guerrilla behaviour is attributed to Chaiken et al.; no justification is attempted in this paper.

D. E. Barton (London)

Feigin, L. I. 5634

On the realizability of a process of treatment of objects by a set of machines. (Russian)

Izv. Akad. Nauk SSSR Tehn. Kibernet. **1964**, *no.* 2, 24–29.

In connection with the Akers-Friedman scheduling problem, the author describes a network of finite automata which simulates a process of treatment of n objects by m machines and shows that a process is realizable if and only if a certain corresponding homogeneous system of strict linear inequalities is compatible.

G. N. Raney (Storrs, Conn.)

Garg, R. C. 5635

Analytical study of a complex system having two types of components.

Naval Res. Logist. Quart. **10** (1963), 263–269.

A complex system has components of two classes, S_1 and S_2. A failure in S_1 leads to reduced efficiency of the system, that of a component in S_2 leads to complete breakdown. The times to failure and to repair in each case are assumed exponential. The probabilities of each state as functions of time are found, and the steady-state probabilities evaluated. *K. J. Arrow* (Stanford, Calif.)

Geary, R. C.; McCarthy, M. D. 5636

★Elements of linear programming. With economic applications.

Griffin's Statistical Monographs & Courses, No. 15.

Hafner Publishing Co., New York, 1964. 126 *pp.* \$4.95.

This short, inexact, and often misleading book is intended to be an introduction to linear programming for students of economics. The book fails in presenting both "theory" and "applications".

Part I, "Theory", begins by considering a simple example which is solved graphically and by a labored use of a simplex method. There follows a general discussion of the linear programming problem and a simplex method for its solution. However, this discussion is misleading (e.g., there is no explicit recognition of the fact that a problem may not have a solution), and contains false statements (e.g., "the maximizing- or minimizing-point ... must be a vertex" is "Proposition I"), together with erroneous proofs. Duality is treated as a mystical gift (e.g., "the remarkable phenomenon").

Part II, "Applications", considers five simple linear programs arising in various applied contexts. Unfortunately, the potential uses and interpretations of the dual problem are not adequately discussed—in fact, not discussed at all in the first two examples dealing with a transportation problem and an activity analysis problem where interpretations are easily given.

Appendices, one pointing out the need for integer solutions in certain formulations and another dealing with convex programming, complete the book. Only three references to published work in the field of mathematical programming are made. *M. L. Balinski* (Princeton, N.J.)

Hoáng Tuy [Hoàng, Tuy] 5637

Sur une classe des programmes non linéaires.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 213–215.

This brief note is a French version of the author's earlier work in Vietnamese [Publ. Sect. Phys. Math. Comité État Sci., No. 1 (1963)]. It is suggested that the simplex method is applicable to the following bounded-variable programming problem. Minimize $f(x)$ subject to

$$\sum_{j=1}^{n} A_j x_j = B, \qquad 0 \le x_j \le \alpha_j \ (j = 1, \cdots, n),$$

where A_j, $B \in R^m$, $\alpha_j > 0$ $(j = 1, \cdots, n)$ and the numerical function f, defined on R^n, satisfies the property:

$$\text{(M)} \qquad \varphi(\lambda) = f(\lambda x + (1-\lambda)y)$$

is monotone in the interval $0 \le \lambda \le 1$ for each pair of points x, y belonging to the constraint set P of the problem. From the obvious property

$$(\mathrm{M}_1) \qquad f(\lambda x + (1-\lambda)y) \ge \min\{f(x), f(y)\}$$

for all $x, y \in P$ and $\lambda \in [0, 1]$, it follows that the minimum of $f(x)$ over P is attained at an extreme point of P. However, certain assertions depend on the claim that (M_2) if $x, y \in P$ and $f(x) \le f(y)$, then for all $\lambda \le 0$, $f(x) \le f(\lambda x + (1-\lambda)y)$. This inequality need not be true if $f(x) = f(y)$.

The article contains allusions to problems such as homogeneous linear fractional programming to which the hypotheses apply. However, no attempt is made to relate the present work to others in the field [see, for example, P. C. Gomory and R. E. Gilmore, Operations Res. **11** (1963), 863–888; in particular, note page 882]. *R. W. Cottle* (Holmdel, N.J.)

Kall, Peter 5638

Über eine Anwendung endlicher Markov-Ketten in der linearen und nichtlinearen Programmierung.

Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **3**, 89–109 (1964).

First, a Markov chain P is defined, and the necessary and sufficient condition is given for the existence of $P^* = \lim_{m\to\infty} P^m$. This is followed by a description of a program that has been developed to meet the demands of a certain material in a number, r, of successive years at minimum total cost. The demand in year i is to a certain extent determined by that year's political situation. These political situations are the external states of the system which are outside the control of the policymaker. p_{ij} is the probability that this year's political situation, E_i, will be E_j next year (it is, of course, possible that $E_j = E_i$); the p_{ij}'s are elements of a transfer matrix P which is square. To meet the demand in each successive year the material can be imported from foreign countries or produced at home. In the latter case, the combined capacity of the domestic plants is the determining factor; this capacity can be expanded or contracted (by not replacing what is obsolete or worn out). What is imported or produced domestically in each year is added to an existing stockpile, and what is needed to meet the demand is taken out of it. The various possible levels of the stockpile and the combined capacity of the domestic plants are the internal states of the system; they are under control of the policymaker. The objective is to minimize the total cost of the program given by the following recursive system of functional equations:

$$F_{E_i,k,r'} = \min_{k'}\Big(C_{E_i,k,k'} + v \sum_{j=1}^{K} p_{ij} F_{E_j,k',r'-1}\Big),$$

$$r' = 1, 2, \cdots, r,$$

where $F_{E_i,k,r'}$ is the total cost of a program starting with an external state E_i and an internal state k covering a period of r' years; $C_{E_i,k,k'}$ is the cost of that program in year r' (which is its first year since the years are counted backwards in time); k' is the internal state of the system at the end of year r' = the beginning of year $(r'-1)$; v is the discount factor; and p_{ij} is the probability that political situation E_i in year r' will be E_j in year $(r'-1)$; the probability of no change in political climate is then given by p_{ii}.

In the paper it is first proved that, due to the fact that the discount factor $v<1$, the total cost of the program $F_{E_i,k,r}$ will tend to a limit $F_{E_i,k}$ when the number of years, r, covered by it increases beyond all bounds. Then it is shown that if all elements of the transfer matrix P are strictly positive, the system will tend to a particular internal state for each possible external state. It is also shown that this "convergence in policy-space" will take place if the transfer matrix P is completely ergodic and at least one of the $p_{ii}>0$. Finally, it is shown that the same can be said of situations in which the restrictions, defining the requirements, and/or the cost functions are nonlinear. *H. F. Karreman* (Madison, Wis.)

Lur'e, A. L. 5639

An algorithm for solving a transportation problem by approximating by conditionally optimal plans. (Russian)

Vyčisl. Mat. 7 (1961), 151–160.

For the so-called transportation problem (viz., to find x_{ik}, $i=1,\cdots,n$, $k=1,\cdots,m$, such that $\sum_k x_{ik}=a_i$, $\sum_i x_{ik}=b_k$, $x_{ik}\geq 0$, $\sum_{i,k} c_{ik}x_{ik}=\min$, a_i, b_k, c_{ik} are given real numbers, a_i and b_k positive, $\sum_i a_i=\sum_k b_k$) the author adapts ideas from L. V. Kantorovič's linear-programming algorithm called "approximation by conditionally optimal plans" [*Mathematical methods of organization and planning of production* (Russian), Izdat. Leningrad. Goz. Univ., Leningrad, 1939; Dokl. Akad. Nauk SSSR **115** (1957), 441–444; MR **20** #769], expounds them in a fully self-contained fashion, and points out the resemblance to the (American-developed) so-called Hungarian method.

Consider a set of values x_{ik} (the "conditionally optimal plans") such that, if $c_{i_rk_p}>\min_i c_{ik_p}$, then $k_p=0$ in $x_{i_rk_p}$, $\sum_k x_{ik}\leq a_i$, $\sum_i x_{ik}\leq b_k$, $\sum_{i,k} x_{ik}=\max$. Find $N_k=b_k-\sum_i x_{ik}$. If $N_k=0$, end. If $N_k>0$, partition the index set $\{i\}$ into $\{i'\}$ and $\{i''\}$: If there is a k such that i_r satisfies $c_{i_rk}=\min_i c_{ik}$ and $N_k>0$, then $i_r\in\{i''\}$. Likewise, if there are i'' and k such that $c_{i_rk}=\min_i c_{ik}$, and $x_{i''k}>0$, then $i_r\in\{i''\}$. Other i belong to $\{i'\}$. Find $d=\min_{i',k'}(c_{i'k'}-\min_i c_{ik'})$, where k' is a k for which the number in the parentheses is positive for any i'. Every c_{ik} is augmented by d. With these improved c_{ik}, the routine begins anew.

The author says the method is most simple for small n and m. For larger n and m there are complications, arising from a fast increasing number of necessary iterations, which require certain complicated modifications of the routine. Practical applications of the method were shown by A. Aleksandrov, A. Lur'e, and Ju. Oleĭnik.
E. M. Fels (Pittsburgh, Pa.)

Weiss, George H. 5640

Optimal periodic inspection programs for randomly failing equipment.

J. Res. Nat. Bur. Standards Sect. B **67B** (1963), 223–228.

A system is assumed to fail in accordance with a known but arbitrary distribution of failure time. Both inspection and replacement take time; in addition, there is a probability, possibly positive, that any given inspection will fail to detect the failure. The problem posed is to choose that periodic inspection policy which minimizes expected down time of the system. (As the author notes, the optimal policy need not, in general, be periodic, but he seeks the optimum only among this restricted class because of its operational convenience.) It is assumed that replacement occurs only after failure is detected on an inspection. The optimal solution is found for the cases where inspection and replacement times are deterministic and where they are exponentially distributed. Some brief comparisons are made between this policy and the no-inspection (replacement at fixed time) policy.
K. J. Arrow (Stanford, Calif.)

Gomory, R. E.; Hu, T. C. 5641

Synthesis of a communication network.

J. Soc. Indust. Appl. Math. **12** (1964), 348–369.

From the authors' summary: "A communication network is a set of nodes connected by arcs. Every arc has associated with it a non-negative number called the branch capacity which indicates the maximum amount of flow that can pass through the arc. A communication network must have large enough branch capacities such that all message requirements (which can be regarded as flows of different commodities) can reach their destinations simultaneously. In general, these requirements vary with time. The present paper gives algorithms for min-cost synthesis of a communication network which is able to handle simultaneous flows of all time periods."

The time-varying synthesis problem for an m-arc network is first formulated as a linear program with a reasonable number of rows and an enormous number of columns. The authors exploit the special structure of this large linear program using decomposition methods to show that the problem can be reduced to a linear programming calculation involving only one $m\times m$ matrix, together with auxiliary $m\times m$ linear programming calculations of the type treated by L. R. Ford, Jr. and D. R. Fulkerson [Management Sci. **5** (1958), 97–101; MR **20** #4342].

A modest example is solved using both the primal and dual algorithms developed in the paper. It is stated that a synthesis problem for a ten-node twenty-arc network with two time periods was solved in ten minutes on an IBM 7094 using the dual method.

A useful survey of recent results in this area together with their references is included in the paper.
W. F. Tyndall (Lancaster, Pa.)

Mudrov, V. I. 5642

A remark on S. I. Zuhovickiĭ's paper, "On a problem of piecewise linear programming". (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 968–969.

It is shown that the minimization problem discussed in a paper by Zuhovickiĭ [same Ž. **3** (1963), 599–605; MR **28** #1741] is easily reduced to a standard linear programming problem. This was pointed out in the cited review of Zuhovickiĭ's paper. *L. Weiss* (Ithaca, N.Y.)

Rosen, J. B. 5643

Primal partition programming for block diagonal matrices.

Numer. Math. **6** (1964), 250–260.

Under consideration here are the l-block dual linear program (1) maximize $b_0'y+\sum_{i=1}^{l} b_i'x_i$ subject to $D_i y'+A_i'x_i\leq c_i$, $i=1,\cdots,l$, and the corresponding primal problem (2) minimize $\sum_{i=1}^{l} c_i'u_i$ subject to $\sum_{i=1}^{l} D_iu_i=b_0$, $A_iu_i=b_0$, $u_i\geq 0$, $i=1,\cdots,l$. (Lower case letters denote

column vectors and capital letters denote matrices. Transposition is indicated by a prime.) The following two paragraphs—excerpted (mutatis mutandis) from the author's introduction—provide an ample brief description of the algorithm.

Since (1) and (2) are dual linear problems, an optimal solution to either one also gives an optimal solution to the other. The method of solution described here gives a sequence of feasible solutions to (1) which are obtained by solving subproblems of (2). At each cycle a test is made for a complete problem optimum solution. When this test is satisfied, the optimum solution to (1) and (2) is given. If the test is not satisfied, a basis change is made in every non-optimal block and a new feasible solution is obtained with a non-decreased function value. If a finite solution exists, it is obtained in a finite number of cycles.

During the first cycle, l linear programming subproblems with m_i equations in the ith subproblem, where m_i is the number of equations in the ith block of (2), are solved. In every subsequent cycle, only a single linear programming problem with s equations, where s is the number of coupling equations in (2), is solved. The size of all subproblems remains unchanged throughout the solution procedure.

In addition to a demonstration of the validity of algorithm, experience with an IBM 7090/7094 computer program based on the algorithm is reported. Results are summarized on six test problems, the largest of which is a fifteen-block problem with 930 equations and 1200 variables. *R. W. Cottle* (Holmdel, N.J.)

Woo, Lin 5644

Remark on a paper by F. W. Sinden.

J. Math. Anal. Appl. **9** (1964), 162–163.

This note clarifies a point concerning the treatment of linear programming as a special case of quadratic programming given in Section VI of Sinden [same J. **5** (1962), 378–402; MR **26** #3523]. *E. M. L. Beale* (London)

Charnes, A.; Cooper, W. W.; Kortanek, K. 5645

Duality in semi-infinite programs and some works of Haar and Carathéodory.

Management Sci. **9** (1962/63), 209–228.

Authors' summary: "By constructing a new infinite-dimensional space for which the extreme point-linear independence and opposite sign theorems of Charnes and Cooper continue to hold, and, building on a little-known work of Haar (herein presented), an extended dual theorem comparable in precision and exhaustiveness to the finite space theorem is developed. Building further on this, a dual theorem is developed for arbitrary convex programs with convex constraints which subsumes in principle all characterizations of optimality or duality in convex programming. No differentiability or constraint qualifications are involved, and the theorem lends itself to new computational procedures."

Mangasarian, O. L.; Rosen, J. B. 5646

Inequalities for stochastic nonlinear programming problems.

Operations Res. **12** (1964), 143–154.

This paper extends Madansky's inequalities for stochastic linear programming problems [Management Sci. **6** (1959/60), 197–204; MR **22** #1457] to nonlinear programming problems under appropriate convexity and continuity assumptions. The authors point out that, since their problems are nonlinear anyway, the calculation of one upper and one lower bound may involve only slightly more effort than two solutions of a deterministic problem of the same size. *E. M. L. Beale* (London)

Haight, Frank A. 5647

★Mathematical theories of traffic flow.

Academic Press, New York-London, 1963. xi + 242 *pp.* $9.00.

This book, by setting out in a leisurely way the basic problems encountered by those concerned with a mathematical formulation of traffic flow, should persuade many readers to participate in the further development of this rather difficult subject. It is Volume 7 of the series of monographs on "Mathematics in Science and Engineering" and is written in a style which should make its contents quite accessible to the engineer-reader. The probabilist should not expect sophistication, which would be out of place in a book of this kind, and should not be deterred by the shortcomings of the mathematical presentation (e.g., an elliptic definition of the Poisson process on page 38, and a mysterious reference to Bayes on page 43). Chapter 1 summarises "Probability and Statistics" and Chapter 2 summarises "Theory of Queues"; most readers will know all this already. The book really begins with Chapter 3, "Fundamental Characteristics of Road Traffic", which deals with the flow-concentration diagram and the theory of car-following. Chapter 4, "Arrangement of Cars on a Road", is a study of the point-processes which arise in this context. Chapter 5, "The Simple Delay Problem", discusses delays to pedestrians trying to cross a road, and delays to vehicles at light-controlled intersections, etc. The "Miscellaneous Problems" dealt with in Chapter 6 include "left turning" (="right turning" in some parts of the world), "multi-laned highways", "sequences of traffic controls", "parking", "merging", "bottlenecks", and "the rush hour". The book concludes with a final chapter on "A Two-lane Road" and a discussion of "overtaking". There is a large bibliography.

D. G. Kendall (Cambridge, England)

Miyasawa, Koichi 5648

The n-person bargaining game.

Advances in game theory, *pp.* 547–575. *Princeton Univ. Press, Princeton, N.J.*, 1964.

For a class of games in which strategic problems do not complicate the bargaining features, the author simplifies Harsanyi's first general bargaining theory [*Contributions to the theory of games*, Vol. IV, pp. 325–355, Princeton Univ. Press, Princeton, N.J., 1959; MR **21** #4062]. It should be noted that Harsanyi has substantially revised his approach [Internat. Economic Rev. 4 (1963), 194–220], avoiding the reviewer's example [Bull. Amer. Math. Soc. **66** (1960), 70–73; MR **22** #13315], where one player can do better fighting all the rest than sitting down to bargain with them in the manner proposed. The author's theory gives the same paradoxical solution (15, 15, 0) to the same example. *J. R. Isbell* (New Orleans, La.)

Vilkas, È. I. 5649

Transformation of the matrix of a game and the value of the game. (Russian. Lithuanian and English summaries)

Litovsk. Mat. Sb. 4 (1964), 25–29.

There are no pairs of functions f, g on R to R such that the values of game matrices (a_{ij}) always satisfy $\mathrm{val}(f(a_{ij})) = g(\mathrm{val}(a_{ij}))$ except the order-preserving linear $f = g$.

J. R. Isbell (New Orleans, La.)

Vilkas, È. I. 5650

Decision regions of a parametric matrix game. (Russian. Lithuanian and English summaries)

Litovsk. Mat. Sb. 4 (1964), 31–35.

The author discusses a game matrix $A(X)$ depending on an s-dimensional vector X and the regions in which A has a given kernel, particularly for the linear case ($s = 1$) and the 2-by-2 case. *J. R. Isbell* (New Orleans, La.)

BIOLOGY AND BEHAVIORAL SCIENCES

See 5665.

INFORMATION, COMMUNICATION, CONTROL

See also 4678, 4690, 4691, 4692, 4704, 4951, 4952, 4953, 5240, 5289, 5290, 5297, 5337, 5414.

Cirlin, A. M. 5651

Determination of the spectral density of random processes as a problem of approximating a function with respect to its estimate. (Russian. English summary)

Avtomat. i Telemeh. 25 (1964), 1191–1197.

The determination of the spectral density of random processes by means of a decomposition of its estimate from a finite realization in a system of orthogonal functions is subject to the following conclusion. The accuracy of determination of coefficients of the decomposition depends on the scalar product of the dispersion of this estimate and the weight function of the decomposition. In particular, the accuracy increases when this scalar product decreases.

Additional conclusions are: (1) calculation of spectral density can be simplified by determining the coefficients of the decomposition in the selected system of orthogonal functions (instead of some selected ordinates); (2) by a choice of an adequate weight function one can diminish the effect of the lack of accuracy in the determination of spectral density; and (3) the method of moments suggested in a number of publications is not adequate for the determination of the random processes based on short realizations and a variable mathematical expectation.

N. Minorsky (Firenze)

Hackett, C. M., Jr. 5652

Word error rate for group codes detected by correlation and other means.

IEEE Trans. Information Theory IT-9 (1963), 24–33.

In this paper, the author derives general expressions for the probability of word error for several encoding/detection schemes using group codes. Correlation detection, digital decoding, Wagner codes, Wagnerised codes and direct transmission are the schemes treated, and in each case expressions are derived for the case where the additive channel noise is white and Gaussian. A good indication of the relative amounts of power required by two schemes operating at the same error rate is available from an asymptotic relationship as the signal to noise ratio approaches infinity. Numerical results are presented for three easily implemented codes.

I. M. Chakravarti (Chapel Hill, N.C.)

Jelinek, Frederick 5653

Loss in information transmission through two-way channels.

Information and Control 6 (1963), 337–371.

A discrete two-way channel, related to the pioneering work of Shannon in 1961 [Proc. Fourth Berkeley Sympos. Math. Statist. and Prob., Vol. I, pp. 611–644, Univ. California Press, Berkeley, Calif., 1961; MR 24 #B1250] and which has been further developed by Fano and the present author [Fano, *Transmission of information*, M.I.T. Press, Cambridge, Mass., 1961; MR 24 #B442; Jelinek, IEEE Trans. Information Theory IT-10 (1964), 5–17; MR 29 #4615] is considered. Certain restrictions are put on the probability. The terminals are connected to finite-memory stationary signal sources. These signal sources generate channel inputs depending on sequences of past outputs and inputs. A number of interesting results are deduced which appear in seven theorems and some of their corollaries given in the paper. The following extract from the author's summary will give some idea of the results obtained.

"Expressions for average information transmission rates in the left-to-right and right-to-left directions can be developed and their sum will be a simple information measure. When mutually independent messages are to be transmitted in opposite directions through the channel, it is desirable that they be encoded into sequences of strategy functions which, together with the received signals, constitute inputs to a transducer whose outputs are the channel input signals. The message source-encoder-transducer combinations are stochastically equivalent to signal sources whose outputs are governed by appropriate probabilities. We can interpret the transducer-channel combination as a derived two-way channel whose inputs are the strategy functions and whose outputs are the outputs of the underlying channel. Expressions for the information transmission rate through the two directions of the derived channel are developed and are compared to the expressions for the average information about outputs of the equivalent signal sources, transmitted through the underlying two-way channel. The values of the former expressions are found to be less than or equal to the values of the latter, the difference constituting a 'coding information loss'. A condition on the transmission probabilities enables us to define a class of lossless channels. Similarly, another class is defined having the property that, regardless of the strategy code used, the information transmitted through the derived channel will be strictly less than the information transmitted through the underlying channel. The consequences of the above results on the random selection of message codes are

discussed. It is shown that one can obtain the number of variables to be optimized when best random codes for lossy channels are desired, by using the number of variables for lossless channels as an exponent to the product of the size of the input and output signal alphabets. For the lossy channel class a simplified encoding procedure must in practice be applied, but as can be demonstrated, it will not yield optimal codes."

G. Bandyopadhyay (Kharagpur)

Karmazin, M. A. 5654

Solution of a problem of Shannon. (Russian)

Problemy Kibernet. No. 11 (1964), 263–266.

Let A and B be discrete memoryless channels with input [output] alphabets of the same size, and let P and Q be the matrices of their respective channel probability functions. Let $\lambda(n, D|A)$ and $\lambda(n, D|B)$ be, respectively, the minimum average probabilities of error for codes of length D and word length n. Shannon [Information and Control **1** (1958), 390–397; MR **20** #6952] proved that if

$$A = \sum_{\alpha,\beta} g_{\alpha\beta} F_\alpha B G_\beta, \qquad g_{\alpha\beta} \geq 0, \qquad \sum_{\alpha,\beta} g_{\alpha\beta} = 1,$$

where F_α and G_β are stochastic matrices, then, for all n and D, $\lambda(n, D|B) \leq \lambda(n, D|A)$. The author proves that the converse is not true.

J. Wolfowitz (Ithaca, N.Y.)

Rozen'blat-Rot, M. [Rosenblatt-Roth, M.] 5655

The concept of entropy in probability theory and its applications in the theory of transmission of information in communication channels. (Russian. French summary)

Teor. Verojatnost. i Primenen. **9** (1964), 238–261.

Let (S_i, Σ_i, μ_i) be σ-finite measure spaces, $i \in I$ the set of all integers. Let $(S, \Sigma, \mu) = \times_{i\in I} (S_i, \Sigma_i, \mu_i)$ and if $\alpha \subset I$, let $(S^\alpha, \Sigma^\alpha, \mu^\alpha) = \times_{i\in\alpha} (S_i, \Sigma_i, \mu_i)$. A stochastic process (not necessarily stationary) on (S, Σ) may be considered as a system of probability measures P^α on different Σ^α, $\alpha \subset I$. It is assumed that each P^α is absolutely continuous with respect to the restriction of μ to Σ^α, and let $\pi^\alpha = dP^\alpha/d\mu$ be the corresponding Radon-Nikodým derivative. The differential entropy of the process at time t is defined to be, if it exists, the limit as $n \to \infty$ of the expression $-n^{-1} \int_S \log \pi^{\alpha_n} d\mu$, where $\alpha_n = [t, t+n-1]$. This notion is used to obtain extensions of coding theorems of Shannon, McMillan and Feinstein.

L. Sucheston (Columbus, Ohio)

Cohen, Edward L. 5656

A note on perfect double error-correcting codes on q symbols.

Information and Control 7 (1964), 381–384.

The existence of perfect double error-correcting codes on q symbols ($q \geq 2$) is discussed. (i) References are given for $q = 2, 3, 5$. (ii) It is shown that there are no perfect double error-correcting codes for $q = 4, 6$. (iii) Some remarks are made concerning arbitrary q.

R. W. Hamming (Murray Hill, N.J.)

Golomb, S. W.; Posner, E. C. 5657

Rook domains, Latin squares, affine planes, and error-distributing codes.

IEEE Trans. Information Theory **IT-10** (1964), 196–208.

Authors' summary: "A problem originally suggested in the context of genetic coding leads naturally to the concept of rook packing and error-distributing codes. It is shown how various concepts in the theory of Latin squares, and also in coding theory, are best expressed in the form of questions about the placing of rooks on k-dimensional hyperchessboards of side n. A new species of combinatorial design suggested by this is the concept of optimal coloring. It is shown that the optimal colorings in certain cases correspond to duals of desarguian projective planes. Light is thereby shed on the problems of the existence of both finite projective planes and close-packed single-error-correcting codes. In particular, the existence of a certain close-packed nonbinary single-error-correcting code, listed by Golay as the first unknown case, has been ruled out by a well-known result concerning Latin squares."

Kislicyn, S. S. 5658

Average length of a binary code with minimum redundancy in the case of two groups of symbols having equal probabilities. (Russian)

Problemy Kibernet. No. 11 (1964), 267–270.

Continuing his investigation related to Huffman's minimum-redundancy binary codes [Teor. Verojatnost. i Primenen. 7 (1962), 342–343; MR **29** #1090], the author derives exact expressions for the average length of such codes in the case of the source alphabet consisting of $m+n$ independent symbols of which the first m symbols occur with equal probabilities p/m, and the remaining n symbols with equal probabilities q/n, where $p+q=1$.

S. Kotz (Toronto, Ont.)

Levenštein, V. I. 5659

Some properties of coding and self-adjusting automata for decoding messages. (Russian)

Problemy Kibernet. No. 11 (1964), 63–121.

A general survey of problems stated in the title and a detailed exposition of the results previously presented by the author in two of his other papers [Dokl. Akad. Nauk SSSR **140** (1961), 1274–1277; MR **25** #1967; ibid. **141** (1961), 1320–1323; MR **25** #1964].

S. Kotz (Toronto, Ont.)

Massey, James L. 5660

Reversible codes.

Information and Control 7 (1964), 369–380.

A code is defined to be reversible if its code-word set is invariant under a reversal of the digits in each code word. Such codes may have application in certain data storage and retrieval systems. It is shown that cyclic codes and convolutional codes are reversible when and only when their code-generating polynomials are self-reciprocal. Reversible codes are quite rare therefore, but it is shown that an important subclass of the Bose-Chaudhuri codes consists entirely of reversible codes. Techniques are developed by which any nonreversible cyclic code can be converted into a reversible cyclic code with at least as much error-correcting power, but at the cost of increased code redundancy. The redundancy of the derived code is at most twice that of the original code, and the derived code can be decoded by a decoder constructed for the original code.

Similarly, it is shown how any nonreversible convolutional code can be converted into a reversible convolutional code with at least as much error-correcting power, but at the cost of increased code constraint length. Again the derived code can be decoded by substantially the same decoder as for the original code.

R. W. Hamming (Murray Hill, N.J.)

Vasil'ev, Ju. L. 5661

On ungrouped, close-packed codes. (Russian)

Problemy Kibernet. No. 8 (1962), 337–339.

In this note an alternative method for constructing close-packed $(n, 3)$-codes is presented. A lower bound on the ratio of the number of different strongly ungrouped close-packed codes to the number of all grouped $(n, 3)$-codes is derived; strongly ungrouped codes mean those which are not grouped or are not shifts of grouped codes.

In particular, the conjecture of Shapiro and Slotnick [IBM J. Res. Develop. **3** (1959), 25–34; MR **20** #5092] on the existence of essentially one close-packed (n, d)-code for a given n and d is disproved.

This last result was also announced by the author in another paper [Dokl. Akad. Nauk SSSR **142** (1962), 263–265] in connection with the theory of disjunctive normal forms.

S. Kotz (Toronto, Ont.)

Amar, V.; Putzolu, G. 5662

On a family of linear grammars.

Information and Control 7 (1964), 283–291.

An even linear grammar (ELG) is a linear grammar in which the productions are all of the form

$$\delta \to \varphi,$$
$$\delta \to \varphi\delta'\varphi' \qquad \text{with } |\varphi| = |\varphi'|,$$

where δ, δ' are auxiliary letters, φ, φ' are terminal words and $|\varphi|$ denotes the length of φ. It is easily seen that the subsets of the free monoid Σ^* (where Σ is finite) produced by ELG's properly include the regular sets. Nonetheless, these special linear languages have a number of properties similar to regular sets.

Define a quasi-congruence relation on Σ^* as an equivalence relation R on Σ^* with the property

$$(\varphi, \psi) \in R \Rightarrow (\forall\varphi', \varphi'')_{\Sigma^*}[|\varphi'| = |\varphi''| \Rightarrow (\varphi'\varphi\varphi'', \varphi'\psi\varphi'') \in R].$$

Every even linear language is the union of some equivalence classes of a finite index quasi-congruence relation on Σ^*, and conversely. It is shown that the even linear languages are exactly those sets recognized by finite automata which scan a tape starting from the center and proceeding outward in both directions at the same time.

M. A. Harrison (Berkeley, Calif.)

Maslov, E. P. 5663

A statistical self-adjusting model. I. (Russian. English summary)

Avtomat. i Telemeh. 25 (1964), 1442–1450.

Author's summary: "The theory of statistical decision functions is applied to the design of self-adjusting models. To deduce basic correlations the notions of the dual control theory are used."

Bellman, Richard; Bucy, Richard 5664

Asymptotic control theory.

J. Soc. Indust. Appl. Math. Ser. A Control **2** (1964), 11–18.

The authors initiate the discussion of the problem of minimizing the functional

$$(*) \qquad J(u) = \frac{1}{2}\int_0^T (u^2 + L(x))\,dt$$

over all functions u, where $\dot{x} = f(x) + u$, $x(0) = c$. Let $V(c, t) = \min_u J(u)$. Then, $V(c, T)$ is increasing in T and uniformly bounded if $f(x) = ax$ and $L(x)$ satisfies certain mild restrictions. Since the existence of a sufficiently smooth solution to the partial differential equation

$$(**) \qquad V_T = \tfrac{1}{2}L(c) + acV_c - \tfrac{1}{2}V_c^2$$

is a sufficient condition for the variational problem $(*)$ to have a solution, an explicit solution to $(**)$ subject to the boundary conditions

$$V(c, T)\,|_{T=0} = 0, \qquad a < 0,$$
$$V(c, T)\,|_{T=0} = ac^2, \qquad a > 0,$$

is obtained. Further problems are suggested.

V. Lakshmikantham (Aurangabad)

Bellman, Richard; Gluss, Brian; Roth, Robert 5665

On the identification of systems and the unscrambling of data: Some problems suggested by neurophysiology.

Proc. Nat. Acad. Sci. U.S.A. **52** (1964), 1239–1240.

Let $t_0, t_1, \cdots, t_i, t_{i+1}, \cdots, t_n$ be a partition of the real interval $[t_0, t_n]$. Over each subinterval $[t_i, t_{i+1}]$, $i = 0, \cdots, n-1$, consider the kth-order differential equation $v^{(k)} = g(v^{(k-1)}, \cdots, v, t; a^i)$ with initial data $v^{(j)}(t_i) = C_j^i$, $j = 0, \cdots, k-1$; where the functional form of g is assumed known while a^i is a parameter vector to be chosen. Let $u(t)$ be a given function; the problem is to choose the vectors a^i, c^i and the partition points t_i so as to minimize a functional of the form

$$J(a^i, c^i, t_i) = \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} [u(t) - v(t, a^i, c^i)]^2\,dG(t).$$

A very brief description of a method of attack via dynamic programming is discussed while detailed results and motivation of the problem are deferred to subsequent papers.

H. Hermes (Providence, R.I.)

Bolčuk, O. P. 5666

Stability and autovibrations of a non-linear stabilization system. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 1294–1298.

Author's summary: "A method is developed which is convenient for investigating nonlinear gyroscopic systems. The method is based on a combined use of the methods of harmonic linearization and energy balance. Periodic regimes in the system are found from linearized equations of motion of the device, while the stability conditions are recorded by means of energy relations. The problem with a gap is solved in general form."

Fan, Liang-tseng; Wang, Chiu-sen 5667

On the optimization of multistage feedback processes.

J. Soc. Indust. Appl. Math. **12** (1964), 226–232.

The authors examine an iterative method proposed by D. F. Rudd [same J. 10 (1962), 448–453; MR 26 #5989] for the determination of optimal policy in multistage feedback processes based on dynamic programming formulations for multistage processes without feedback. Rudd's procedure is shown not to lead to the true maximum by explicit examination of a particular 3-stage process with feedback. A discrete version formulation of Pontryagin's maximum principle is proposed for feedback processes.

D. Rosenblatt (Washington, D.C.)

Harvey, C. A. 5668

Modes of finite response time control.

J. Soc. Indust. Appl. Math. Ser. A Control **2** (1964), 60–65.

Author's summary: "A linear autonomous system with a single control variable is considered. There are, in general, several modes of finite response time control for such a system. The concepts of single component regulation and multiple component regulation are defined. It is then shown that a multiple component regulation problem can be transformed into a single component regulation problem. Thus it is possible to express any of the modes of control considered as control of a single input, single output system."

T. F. Bridgland, Jr. (Columbia, S.C.)

Krivenkov, Ju. P. 5669

Sufficiency of the maximum principle for a linear problem of dynamic programming. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 1277–1280.

Das Problem optimaler Prozesse bei beschränkten Phasenkoordinaten wird im Buch von L. S. Pontrjagin, V. G. Boltjanskiĭ, R. V. Gamkrelidze und E. F. Miščenko [*Mathematische Theorie optimaler Prozesse* (russisch), Fizmatgiz, Moskau, 1961; MR **29** #3316a; englische Übersetzung, Interscience, New York, 1962; MR **29** #3316b; deutsche Übersetzung, Oldenbourg, München, 1964; MR **29** #3316c] behandelt. Die Arbeit gibt hinreichende Bedingungen dafür, damit die Linearizierung der Aufgabe eine Lösung hat. Diese Linearizierung fällt mit der linearen Aufgabe der dynamischen Programmierung zusammen [R. Bellman, *Dynamic programming*, Princeton Univ. Press, Princeton, N.J., 1957; MR **19**, 820].

Z. Daróczy (Debrecen)

Kudrevič, Ja. [Kudrewicz, J.] 5670

Stability of non-linear systems with feedback. (Russian. English summary)

Avtomat. i Telemeh. **25** (1964), 1145–1155.

The author defines a system by the input-output functional relation $y=\Gamma(x)$, where x is the input and y is the output of the system. He then defines the pseudo-norm

$$\|x\| = \lim_{T\to\infty}\left[\frac{1}{T}\int_0^T |x(t)|^2\,dt\right]^{1/2},$$

which may be regarded as the average power of the signal x. He considers a system to be stable "in the energy sense" if $\|\Gamma(x)\|=0$ for all inputs x such that $\|x\|=0$. He introduces the gain

$$|\Gamma| = \sup_{x\in E}\frac{\|\Gamma(x)\|}{\|x\|},\qquad E=\{x:\|x\|<\infty\}.$$

The rest of the paper is based on the theorem: "If the gain $|\Gamma|$ of the open-loop system Γ is less than one, then the closed-loop system with input z described by the equation $x=\Gamma(x)+z$ is stable in the energy sense". After proving this theorem he proceeds to calculate the gain $|\Gamma|$ for a number of nonlinear systems obtaining, among other results, the well-known circle criterion for a linear feedback system with a time-varying gain element and a particular case of the Popov stability criterion. The informed reader will recognize a substantial similarity between this work and the work of I. Sandberg and G. Zames (see, e.g., G. Zames [Proc. Nat. Electronics Conf., Vol. 20, pp. 725–730, Nat. Electronics Conf., Inc., Chicago, Ill., 1964] and I. Sandberg [ibid., pp. 737–741; Bell System Tech. J. **43** (1964), 1581–1599; MR **30** #1416; ibid. **43** (1964), 1601–1608; 1815–1817]).

E. Polak (Cambridge, Mass.)

Langenhop, C. E. 5671

On the stabilization of linear systems.

Proc. Amer. Math. Soc. **15** (1964), 735–742.

The results of Romanenko [Dopovidi Akad. Nauk Ukraïn. RSR **1962**, 863–867; MR **26** #432] and Berezanski on stabilizable systems of differential equations are generalized. Define a square matrix of complex elements by $G=\begin{pmatrix}A & B\\ P & Q\end{pmatrix}$, where A is n-by-n, Q is m-by-m, and B and P are correspondingly sized. Let $S_r=\{\mu_k: k=1,2,\cdots,r\}$, where the (complex) μ_k need not be distinct. Take h to be the rank of the matrix $H=(B, AB,\cdots, A^{n-1}B)$. The principal result is that $h\geqq r-m$ is a necessary and sufficient condition that, for each S_r, there exist P and Q such that the spectrum of G contains S_r.

The theorem is applied in specialized form to systems of differential equations $dx/dt=Ax+bu$, $du/dt=px+qu$, where $G=\begin{pmatrix}A & b\\ p & q\end{pmatrix}$, q a complex scalar. The system (A,b) is "stabilizable" if, given a non-empty set S of $n+1$ or fewer numbers, p and q can be chosen so that G has S as its spectrum. Then (A,b) is stabilizable if and only if $b, Ab,\cdots, A^{n-1}b$ are linearly independent. In particular, if (A,b) is a control system with state x and control u, the spectrum of G can be made to lie in the left half-plane, so that $x(t)\to 0$ and $u(t)\to 0$ as $t\to\infty$.

F. J. Beutler (Ann Arbor, Mich.)

Pyškin, I. V. 5672

An exact method for calculating transient processes and stability of sampled-data systems with finite sampling time. (Russian. English summary)

Avtomat. i Telemeh. **24** (1963), 1303–1316.

Author's summary: "Sampled-data systems with finite sampling time are analysed. With the help of the method of 'solution sewing' used for the solution of the canonical equations of the system corresponding to closed and open states of the key, there are obtained difference equations describing the system, and then the process at the system output. The process image is obtained as a ratio of two determinants whose elements are simple root functions of closed and open systems. The stability of such systems is considered."

Rapoport, I. M. 5673

On the stability of control processes. (Russian)

Dokl. Akad. Nauk SSSR **159** (1964), 288–291.

The author wishes to examine the stability of the linear time-varying feedback system described by the equations

$$(1)\qquad \begin{aligned} dx/dt &= A(t)x+a(t)u, \qquad v = (b(t), x), \\ u &= \int_{-\infty}^{t} v(t')r(t-t', t')\,dt'. \end{aligned}$$

In order to do this, he makes use of the "slow time" concept introduced by N. M. Bogoliubov and N. M. Krylov and writes his equations in the form

$$(2)\qquad \begin{aligned} dx/dt &= A(\tau)x+a(\tau)u, \qquad v = (b(\tau), x), \\ u(t) &= \int_{-\infty}^{t} v(t')r(t-t', \tau')\,dt', \end{aligned}$$

where $\tau=\varepsilon t$ is the slow time. He then seeks a solution of (2) in the form $x(t)=y(\tau)\exp\theta(t)$, $\theta'(t)=\lambda(\tau)$, $\|y(\tau)\|=\sqrt{(y(\tau), \overline{y(\tau)})}=1$. For $\varepsilon=1$ the norm of such a solution satisfies the differential equation $d\|x\|/dt=\operatorname{Re}\lambda(t)\|x(t)\|$, and hence the author uses as a criterion of local stability the following inequality: $\operatorname{Re}\lambda(t)<0$. The author computes the solution of (2) by an iterative method which results in a series representation for $\lambda(t)$, and then considers the first few terms of this series only.

E. Polak (Cambridge, Mass.)

Dreyfus, Stuart E. 5674

Some types of optimal control of stochastic systems.

J. Soc. Indust. Appl. Math. Ser. A Control **2** (1964), 120–134.

Although the results are not new, this paper has pedagogical value for its illuminating discussion of the effects of various control philosophies on three simple stochastic control situations. The continuous parameter examples are

$$\text{(I)}\qquad dx = (ax+bu)\,dt+dz, \quad \text{Error} = E\int_0^T u^2\,dt+Ex^2(T),$$

$$\text{(II)}\qquad dx = u\,dz, \qquad\qquad \text{Error} = Ex^2(T),$$

where z is a Wiener process and T is fixed. The author computes and compares the controls and errors for the (i) optimal "open-loop" control, (ii) optimum "open-loop" control re-initialized at each instant of time, and (iii) the feedback control derived from the principle of optimality.

H. J. Kushner (Providence, R.I.)

Egorov, A. I. 5675

Optimal control of processes in certain systems with distributed parameters. (Russian. English summary)

Avtomat. i Telemeh. **25** (1964), 613–623.

Author's summary: "A case of the optimum control problem is considered when the process is described by differential equations with second-order partial derivatives. Necessary criteria for optimality are found. In the case when the differential equations are linear, the criteria found are locally sufficient."

W. H. Fleming (Providence, R.I.)

Fadden, Edward J.; Gilbert, Elmer G. 5676

Computational aspects of the time-optimal control problem.

Computing Methods in Optimization Problems (*Proc. Conf., Univ. California, Los Angeles, Calif.*, 1964), pp. 167–192. *Academic Press, New York*, 1964.

Given the system $\dot{x}=F(t)x+b(t)u$, $x(t_0)=x_0$, it was shown by Neustadt [J. Math. Anal. Appl. 1 (1960), 484–493; MR 23 #B1612] and Eaton [ibid. 5 (1962), 329–344; MR 25 #4214] that the minimal time regulator problem can be solved via the maximization of $T(\eta)$, where

$$f(T(\eta), \eta) = \left(\eta, x_0-\int_{t_0}^{t} X^{-1}(\xi)b(\xi)\operatorname{sgn}(b^T(\xi)X^{-1}(\xi)^T\eta)\,d\xi\right) = 0$$

and $X^{-1}(t)$ is the adjoint fundamental matrix solution of $F(t)$. The maximization of T is usually carried out iteratively by a steepest descent method on η. This paper represents a careful theoretical and experimental exploration of the geometrical aspects of the steepest descent method on η. It concludes that convergence can be improved by carrying out the descent in a transformed ρ-space which is related to η by a nonsingular linear transformation P. Methods for computing P are discussed.

Y.-C. Ho (Cambridge, Mass.)

Petrov, V. A.; Skvorcov, G. V. 5677

A problem on the analytic construction of controllers. (Russian. English summary)

Avtomat. i Telemeh. **25** (1964), 1399–1403.

Authors' summary: "Based on Pontrjagin's maximum principle, a problem of synthesis is solved for a relay controller providing a second-order system optimality according to energy consumption and control rate."

Sirazetdinov, T. K. 5678

On the theory of optimal processes with distributed parameters. (Russian. English summary)

Avtomat. i Telemeh. **25** (1964), 463–472.

Author's summary: "A maximum principle is established as a necessary condition for optimality of control processes described by quasi-linear partial differential equations with several independent variables. In the case of a linear equation, the maximum principle is also a sufficient condition."

W. H. Fleming (Providence, R.I.)

Volin, Ju. M.; Ostrovskiĭ, G. M. 5679

On an optimal problem. (Russian. English summary)

Avtomat. i Telemeh. **25** (1964), 1414–1420.

Authors' summary: "Calculation of optimum conditions for a class of chemical-technological processes described by partial differential equations is considered."

Živogljadov, V. P. 5680

Application of statistical solution theory to control problems with indirect index. (Russian. English summary)

Avtomat. i Telemeh. **25** (1964), 1451–1461.

Author's summary: "The statistical solution theory is applied to the synthesis of optimal control systems whose parameters can be measured in definite time intervals so that the input control must be used. The obtained results are applied to Markov plants and delay systems. Some

examples of the synthesis of discrete and continuous control systems are described."

Bacon, Glenn C. 5681

The decomposition of stochastic automata.

Information and Control 7 (1964), 320–339.

Author's summary: "Finite-state stochastic sequential systems which are interconnections of two or more component systems exhibit a special structural property in their transition matrices as a consequence of the statistical independence of the next states of the component systems given the present states and the input to the interconnection. Along with the concept of a 'lumpable' stochastic matrix, this structural property is of central importance to the decomposition theory presented. These ideas allow a generalization to the stochastic case of the definitions used by Hartmanis [Information and Control **5** (1962), 25–43; MR **26** #2335] in his study of the decomposition of conventional switching circuits. With these definitions we are able to prove decomposition theorems for stochastic automata which reduce to those of Hartmanis in the deterministic case." *J. Hartmanis* (Schenectady, N.Y.)

Kim, Wan Hee; Chien, Robert Tien-Wen 5682

★Topological analysis and synthesis of communication networks.

Columbia University Press, New York-London, 1962. x+310 *pp.* $11.50.

Part I: Topological analysis of electrical networks. Fundamentals of the theory of linear graphs (Chapter 1); Analysis of the ordinary two-port networks (Chapter 2); Linear active multipoles. Part II: Flow-graph techniques for the solution of linear and sampled-data systems. Topological properties of signal-flow graphs (Chapter 1); Flow-graph analysis of sampled-data systems (Chapter 2); Flow-graph analysis of multirate sampled-data systems (Chapter 3). Part III: Analysis and synthesis of linear N-port networks. Formulation of N-port networks (Chapter 1); Analytic properties of Y- and Z-matrices (Chapter 2); Realization of a Y-matrix with $(n+1)$ nodes (Chapter 3); Generalized realization of a Y- and Z-matrix (Chapter 4). Part IV: Synthesis of single-contact switching networks and realization of loop and cut-set matrices. Fundamentals of the theory of contact networks (Chapter 1); Realization of loop and cut-set matrices. Part V: Analysis and synthesis of communication nets. Oriented communication nets (Chapter 1); Nonoriented communication nets (Chapter 2); Further discussion on communication nets (Chapter 3).

Koršunov, A. D. 5683

Lower bounds of complexity for contact networks which realize pairwise orthogonal functions of a Boolean algebra. (Russian)

Diskret. Analiz. No. 2 (1964), 42–47.

In this paper the number of contacts in a switching circuit containing transfer contacts only and realising a number k of pairwise orthogonal Boolean functions between a unique input terminal and k output terminals is found to be not less than $2k-2$. Application of this theorem shows that a tree circuit turns out to be one of such networks with the number of contacts being the possible minimum. *I. Cederbaum* (Tel-Aviv)

Kurki-Suonio, Reino 5684

On some sets of formal grammars.

Ann. Acad. Sci. Fenn. Ser. A I No. 349 (1964), 31 *pp.*

A presentation of some basic facts about regular sets and context-free languages is given. This includes characterization by finite-state automata and pushdown automata, respectively. The pushdown automaton discussed is a one-state device which accepts a word if the pushdown word at the end of the computation is an element of a pre-assigned finite set. No new results appear. *S. Ginsburg* (Van Nuys, Calif.)

Nečiporuk, É. I. 5685

On the complexity of networks in certain bases containing non-trivial elements with zero weights. (Russian)

Problemy Kibernet. No. 8 (1962), 123–160.

The author studies the problem of estimating the complexity, or cost, of circuits realizing Boolean functions of n variables. Various cases of this problem are distinguished according to the assumptions made concerning the class of admissible circuits, the basis from which the circuit elements are chosen, and the weights assigned to the basis elements. In the cases considered, the basis consists of a set E of elements of positive weight and a set Z of elements of weight zero. This corresponds to the practical problem of synthesizing circuits using minimal numbers of elements of certain types, while elements of other types may be used in unlimited number. Let $L(C)$ be the sum of the weights of the basis elements in the circuit C. If f is a Boolean function, let $L(f)=\inf L(C)$ for circuits C realizing f. The "Shannon function" $L(n)$ is defined to be $\max L(f)$ for Boolean functions f of n arguments. In the case where the admissible circuits are those called superpositions and the basis consists of elements of weight 0 realizing disjunction and conjunction and an element of weight 1 realizing inversion, exact expressions for $L(f)$ of the order of n are obtained. For certain other bases with non-empty set Z, asymptotic expressions for $L(n)$ are found. These have the form $A\cdot 2^n/n$ for superpositions, and the form $A\cdot 2^{(n/2)}$ for circuits of arbitrary topology. The author also obtains improvements in the lower and upper bounds for the minimal number of contacts in contact-rectifying circuits. *G. N. Raney* (Storrs, Conn.)

Ofman, Yu. [Ofman, Ju.] 5686

On the algorithmic complexity of discrete functions.

Dokl. Akad. Nauk SSSR **145** (1962), 48–51 (*Russian*); *translated as Soviet Physics Dokl.* 7 (1963), 589–591.

This paper studies by means of finite automata the complexity of discrete functions $d: D^k \to D^n$, where D^k is the k-dimensional vector space over the field of two elements. The complexity of d is defined by the minimal elements of the set of pairs (N, T), where N is the memory (or stages) required in an automaton which computes d, and T is the computation time of this automaton. (The author has assumed that the next state of a memory element can be only a function of the present states of two other memory elements.) A number of results are concerned with the rate of growth of N and T as the size of the problem grows. For example, it is shown that if $k=2m$, $n=m+1$ and d is the sum of two binary numbers of m digits, then d can be realized by an automaton with $N\sim m$ and $T\sim \log_2 m$. *J. Hartmanis* (Schenectady, N.Y.)

Pfeiffer, Paul E. 5687

★**Sets, events, and switching.**

McGraw-Hill Book Co., New York-Toronto-London, 1964. xiii+131 *pp.* $4.95.

This is an introductory book, written for undergraduate engineering students, although one would hope that mathematics majors would be exposed to it also. The author has very nicely combined material on sets, functions, Boolean algebra, and design of switching networks, so that a student may see the relationships between these areas without having to deduce it for himself. It is written in a very readable, yet not condescending, style. Each chapter is followed by a good selection of problems. On the slightly negative side, one notes that the definition of "minterm" appears on page 37, but this is not included in the index. And the inverse map f^{-1} of f is called the inverse function even if it is not a function. This book is certainly recommended for the audience for which it was written. *B. A. Galler* (Ann Arbor, Mich.)

Popkov, Ju. S. 5688

The influence of intense noise on periodic regimes in extremal sampled-data systems with independent search. (Russian. English summary)

Avtomat. i Telemeh. **25** (1964), 1462–1471.

Author's summary: "Error effect on periodic states in sampled-data extremal systems with independent search is analysed, based on the methods of statistic linearization and on the theory of Markov processes. Taking into account the frequency characteristic of the sampled-data extremal system with an error and correlation time less than the sampler repeating period, the author determines the error intensity which results in periodic state breakage. Data of theoretical and experimental study of a sampled-data extremal system with a second-order object and of a similar relay extremal system are described."

Weyh, Ulrich 5689

★**Elemente der Schaltungsalgebra.**

R. Oldenbourg, Munich, 1960. 116 *pp.* *DM* 13.80.

Table of contents: (1) Logische Funktionen; (2) Die Postulate der Schaltungsalgebra; (3) Die Rechenregeln der Schaltungsalgebra; (4) Analyse und Synthese von logischen Schaltungen; (5) Symbolische Darstellung von Schaltfunktionen; (6) Beispiele für die Realisierung von logischen Grundschaltungen in verschiedenen Techniken; (7) Beispiele für die Behandlung praktischer Probleme; (8) Anhang.

Aizerman, M. A. [Айзерман, М. А.]; Gusev, L. A. [Гусев, Л. А.]; Rozonoèr, L. I. [Розоноэр, Л. И.]; Smirnova, I. M. [Смирнова, И. М.]; Tal', A. A. [Таль, А. А.] 5690

★**Logic, automata and algorithms [Логика. Автоматы. Алгоритмы].**

Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1963. 556 *pp.* 1.45 *r.*

Bon manuel introductif (on y trouvera une présentation simple en général des notions de base dans les domaines suivants: automates finis (théorie et réalisation technique), réseaux neuroniques, algorithmes, fonctions récursives, machines de Turing), ce livre, selon les auteurs eux-mêmes, est destiné à exposer simplement aux ingénieurs la théorie, et aux mathématiciens la traduction technique de cette théorie. *F. Dejean* (Paris)

Blank, A. M. 5691

Expediency of automata. (Russian. English summary)

Avtomat. i Telemeh. **25** (1964), 1472–1483.

Author's summary: "A mathematical model is proposed for notions of 'expediency' and 'learning'. Peculiarities of finite automata possessing the expediency and the learning for an expediency partial criterion and peculiarities of functions producing expedient and learned finite automata are considered."

Černý, Ján 5692

A remark on homogeneous experiments with finite automata. (Slovak. English summary)

Mat.-Fyz. Časopis Sloven. Akad. Vied **14** (1964), 208–216.

Author's summary: "Some papers [e.g., E. F. Moore, *Automata studies*, pp. 129–153, Princeton Univ. Press, Princeton, N.J., 1956; MR **17**, 1140; S. Ginsburg, J. Assoc. Comput. Mach. **5** (1958), 266–280; MR **22** #10882] are concerned with the question whether there exists, for a given sequential machine, a homogeneous experiment which would bring it into a uniquely determined state, not depending on the initial state of the machine. The problem was solved for Moore's machines with distinguishable states. In the present paper the corresponding problem is treated for autonomous automata (i.e., without output). Necessary and sufficient conditions for the existence of such experiments are stated and estimates of their minimal length are established."

Dobrovidov, A. V.; Stratonovič, R. L. 5693

On the design of optimal automata functioning in random media. (Russian. English summary)

Avtomat. i Telemeh. **25** (1964), 1433–1441.

Authors' summary: "The synthesis of the automaton interacting in the optimal way with a random medium is considered, the sequence of the medium states being a Markov chain. Recurrent transformation of a posteriori probabilities is used. The quality of the optimum automaton operation is analysed and the automaton operation is compared with the linear policy [M. L. Cetlin, Dokl. Akad. Nauk SSSR **139** (1961), 830–833; MR **24** #B480]."

Dušskiĭ, V. A. 5694

Characteristic experiments with automata. (Russian)

Problemy Kibernet. No. 11 (1964), 257–262.

A simple experiment of length k for an automaton is an input word of k letters. A multiple experiment is an ordered n-tuple of simple experiments, and one such that consists of all simple experiments of length k is said to be complete of length k. An experiment $\bar{p}$ is characteristic for an automaton A if an automaton B is equivalent to A whenever $\bar{p}$ fails to distinguish them. It is shown that any automaton with n states that has a complete characteristic experiment has one of length n^2+n-1, and that for any

automaton A with more than one letter in its input and output languages and any simple experiment p, there is an automaton B, distinct from A, that p fails to distinguish from A. *E. J. Cogan* (Bronxville, N.Y.)

Even, Shimon 5695

Test for synchronizability of finite automata and variable length codes.

IEEE Trans. Information Theory IT-10 (1964), 185–189.

Author's summary: "A finite automaton is called synchronizable of Nth order if the knowledge of the last N outputs suffices to determine the state of the automaton at one time during the last N outputs (including the initial and the final states). In an analogous manner synchronizability of Nth order is defined for variable length codes. The paper describes a test for synchronizability on a more general model, the coding graphs, and shows that finite automata and variable length codes are special cases of it."

Schutzenberger, M. P. 5696

Certain elementary families of automata.

Proc. Sympos. Math. Theory of Automata (New York, 1962), pp. 139–153. *Polytechnic Press of Polytechnic Inst. of Brooklyn, Brooklyn, New York*, 1963.

Author's summary: "We attempt to relate the difficulty of the decision problem of certain algorithms (automata) with the underlying algebraic structure. In particular, we discuss the connection between 'push-down storage' and 'extension of a free group by a finite monoid'."

A. P. Ershov (Novosibirsk)

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Marcinkiewicz, Józef 5697

★**Collected papers.**

Edited by Antoni Zygmund. With the collaboration of Stanisław Łojasiewicz, Julian Musielak, Kazimierz Urbanik and Antoni Wiweger. Instytut Matematyczny Polskiej Akademii Nauk.

Państwowe Wydawnictwo Naukowe, Warsaw, 1964. viii + 673 *pp*. (3 *plates*)

This volume contains a biography and an analysis of Marcinkiewicz's work by Zygmund; a supplementary bibliography of papers related to Marcinkiewicz's work; and the text of Marcinkiewicz's 55 papers (the two that originally appeared in Polish have been translated into English). Marcinkiewicz contributed new ideas to several fields, and it seems that their potentialities have not even yet been fully explored. A number of open problems are mentioned in Zygmund's analysis.

R. P. Boas, Jr. (Evanston, Ill.)

Stoïlow, S. 5698

★**Œuvre mathématique.**

Éditions de l'Académie de la République Populaire Roumaine, Bucharest, 1964. xix + 416 *pp*. *Lei* 30.00.

This volume contains a reprinting of 65 papers of Stoïlow, ranging from his dissertation [Gauthier-Villars, Paris, 1916] to a brief abstract of his lecture at the Fifth Austrian Congress of Mathematicians at Innsbruck in 1960. The volume is particularly valuable for the inclusion of many relatively inaccessible papers published between 1940 and 1946. In addition to the Foreword by M. Nicolescu, there is a brief account of Stoïlow's work by C. Andreian Cazacu.

A. J. Lohwater (Providence, R.I.)

Minorski, W. P. [Minorskiĭ, V. P.] 5699

★**Aufgabensammlung der höheren Mathematik.**

Aus dem Russischen übersetzt von Eberhard Lacher und Gerhard Liebold; Bearbeitung der deutschen Ausgabe von Heinz Birnbaum. Lehrbücher der Mathematik.

VEB Fachbuchverlag, Leipzig, 1964. 302 *pp*. *MDN* 12.00.

This is a translation of the seventh Russian edition [Fizmatgiz, Moscow, 1962]; for a review of the fifth Russian edition [Fizmatgiz, Moscow, 1959], see MR 21 #6316.

★**American Mathematical Society Translations.** 5700
Series 2, Vol. 32: 17 papers on functions of complex variables.

American Mathematical Society, Providence, R.I., 1963. iv + 393 *pp*. $5.50.

This volume contains translations of four papers by L. K. Hua, two by L. I. Volkovyskiĭ, and one each by M. A. Lavrent'ev, V. I. Krylov, S. Ja. Havinson, A. A. Gol'dberg, B. N. Rahmanov, L. I. Ronkin, M. M. Džrbašjan, A. A. Kosmanova, B. Ja. Levin and I. V. Ostrovskiĭ, M. Š. Stavskiĭ, and N. N. Meĭman.

Bertolino, M.; Vasić, P. M.; Prešić, S. B. (Editors) 5701

★**Izabrana poglavlja iz matematike. II [Selected chapters from mathematics. II].**

Matematička Biblioteka 22. *Zavod za Izdavanje Udžbenika, Belgrade*, 1962. 192 *pp*. 600 *Dinara*.

This volume consists primarily of a collection of expository articles on various topics in mathematics. The articles and authors are as follows: Cyclic functional equations (pp. 5–23) by D. S. Mitrinović and D. Ž. Đoković; Application of functional analysis to the theory of partial differential equations (pp. 25–36) by S. Kurepa; Some applications of differential inequalities (pp. 37–45) by M. Bertolino; On some inequalities of orthogonal polynomials (pp. 47–53) by B. S. Popov; Three remarks on integration of differential equations by means of differentiation (pp. 55–59) by I. A. Šapkarev; Application of the calculus of residues to the calculation of some definite integrals (pp. 61–70) by D. Dimitrovski; The Staudt-Clausen Theorem (pp. 71–79) by K. Milošević-Rakočević; On nonsquare matrices (pp. 81–86) by M. Stojaković; On curves (pp. 87–96) by P. Papić; On the dimension of geometric constructions (pp. 97–104) by S. Mardešić; On polygons (pp. 105–128) by M. Jorgović; Convex bodies (pp. 129–141) by V. Devidé; On non-Euclidean Geometry (pp. 143–168) by D. Blanuša; On an analogue of Euler's rule (pp. 169–176) by S. Fempl; Several examples of the determination of the orbit of a point by means of vectors (pp. 177–180) by S. V. Pavlović; A direct proof of Steiner's theorem for an equilateral triangle (pp. 181–182) by V. Janeskoski. The papers are in Serbo-Croatian.

J. V. Wehausen (Berkeley, Calif.)

Mitrinović, D. S. (Editor) 5702

★**Zbornik matematičkih problema sa prilozima i numeričkim tablicama. I [Collection of mathematical problems with appendices and numerical tables. I].**

Third edition.

Zavod za Izdavanje Udžbenika, Belgrade, 1962. xvi + 503 *pp*. 1300 *Dinara*.

This collection of problems has been prepared by D. S. Mitrinović with the collaboration of D. D. Adamović, D. Ž. Đoković, Z. R. Popstojanović, and S. B. Prešić. It is divided into the following chapters, with the numbers in parenthesis giving the number of problems: Algebra (226); Theory of numbers (65); Analytic geometry and vectors (36); Differential calculus (128); Integral calculus (122); Sequences and series (97); Inequalities (256); Problems from various fields (248) (determinants, matrices, curvilinear and multiple integrals, differential and functional equations, special functions, real functions of a single variable, complex numbers and functions); Appendices and numerical tables. The problems have been gathered from many sources including journals such as the Amer. Math. Monthly, Math. Gazette, Mat. Tidsskr., etc., and various mathematical prize contests. They are well-selected, and the collection should prove to be very useful.

J. V. Wehausen (Berkeley, Calif.)

★Mathematics for physicists and engineers. 5703

Report of the O.E.E.C. seminar on "The Mathematical Knowledge Required by the Physicist and Engineer" (Project STP 17), Château de la Muette, Paris, February, 1961.

Organisation for Economic Co-operation and Development, Paris, 1961. 223 *pp.*

HISTORY AND BIOGRAPHY

Carruccio, Ettore 5704

Considerazioni sul significato di alcuni termini fondamentali degli Elementi di Euclide.

Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **98** (1963/64), 226–234.

The author suggests that in Euclid's *Elements* the term "*εὐθεῖα*" stands for "line segment" and not for "indefinite line", and similarly "*ἐπίπεδος ἐπιφάνεια*" for "plane region" (superficie piana limitata). The term "*ἴσος*" always means "equal in extension" (hence "equivalent") and not "superimposable by motion". There remains, however, a certain ambiguity in Euclid's use of "*κύκλος*" and "*περιφέρεια*", circle and circumference.

D. J. Struik (Belmont, Mass.)

Szabó, Árpád 5705

The transformation of mathematics into deductive science and the beginnings of its foundation on definitions and axioms. II.

Scripta Math. **27**, 113–139 (1964).

In diesem zweiten Teil [für den ersten Teil siehe dieselben Scripta **27** (1964), 27–48A; MR **29** #1120] geht der Verfasser auf Einzelheiten mathematischer Natur ein. Die Pythagoreer räumen der Geometrie als einer Art von Erfahrungswissenschaft geringere Bedeutung ein als der Arithmetik. Die Definition der Einheit als das Unteilbare verrät den Einfluss des Parmenides und findet ihr Gegenstück in der Geometrie bei der Definition des Punktes. Terminologie und Bedeutung des Teilungsproblems hat zunächst mit der Halbierung zu tun und steht in Beziehung zu den Begriffen gerade und ungerade. Das Axiom vom Ganzen und vom Teil hängt zusammen mit Zenos Paradoxien, die wohl im Sinne der modernen Mengenlehre gedeutet werden sollten und alsdann durchaus sinnvoll sind. Die Pythagoreische Zahlenlehre steht nicht im Gegensatz zur Auffassung der Eleaten, ist vielmehr als eine Weiterbildung anzusehen.

J. E. Hofmann (Ichenhausen)

Clagett, Marshall 5706

★Archimedes in the Middle Ages. Vol. I: The Arabo-Latin tradition.

The University of Wisconsin Publications in Medieval Science, No. 6.

The University of Wisconsin Press, Madison, Wis., 1964. xxix + 720 *pp.* $12.00.

Einzelheiten über das Bekanntwerden Archimedischer Schriften im lateinischen Mittelalter hat der Verfasser schon in zahlreichen früheren Veröffentlichungen mitgeteilt. Hier haben wir den ersten Band einer sehr verdienstvollen Gesamtdarstellung vor uns, deren Ziel es ist, die zum Teil noch unedierten Ms. mit bibliographischer Einleitung, im lateinischen Wortlaut mit englicher Übersetzung und sorgfältigen sachlichen Erläuterungen festzuhalten. Hauptthema ist die Kreismessung, die in zwei frühesten Übersetzungen nach der arabischen Fassung des Thâbit ibn Qurra (um 890) vorliegen, die eine wohl von Plato v. Tivoli (um 1140, Barcelona, unvollständig), die andere von Gerhard v. Cremona (um 1170, Toledo, Vergleich mit der arabischen Revision des aṭ-Ṭûsî, um 1250). Anschließend erscheinen drei stark von Gerhard beeinflußte Fassungen, dann drei weitere mit selbständigen Einschüben und Zusätzen.

Die Kreismessung ist auch in einer Paraphrase der Banû Mûsâ (um 850) enthalten, die ebenfalls in der Revision des aṭ-Ṭûsî vorliegt und wiederum von Gerhard übersetzt wurde. Sie enthält außerdem die "Heronische" Dreiecksformel, der genauere Ausführungen in einem Anhang gewidmet sind, und Stücke aus *De sphaera et cylindro* einschließlich der Kommentare des Eutokios (Einschiebung zweier geometrischer Mittel, Dreiteilung des Winkels, Näheres wieder im Anhang). Etwas anderen Aufbau zeigen die Übertragung des sogenannten Pseudo-Bradwardine (etwa 14. Jh.) und die *Quaestio* des Albert v. Sachsen (um 1360?).

Noch vor der Übersetzung Archimedischer Schriften durch Moerbeke (1269) findet sich zunächst die Zusammenstellung einiger Hauptsätze aus *De sphaera et cylindro* und dann ein umfangreiches Ms. des Joh. de Tinemue des Titels *De curvis superficiebus* mit wichtigen späteren Zusätzen. In sechs Anhängen erscheinen weitere Quadraturen, die nicht aus Archimedes stammen (Jordanus, Pseudo-Campanus), die "Möndchenquadratur" (vielleicht lateinisch aus dem Griechischen von Grosseteste, um 1240) und lateinische Archimedes-Zitate aus dem 12. bis 15. Jh.

Das hervorragend ausgestattete Werk ist mit einer eingehenden Bibliographie, mit einem umfangreichen und terminologisch besonders wichtigen Wortverzeichnis, mit einem Verzeichnis der zitierten Sätze aus Euklids Elementen, mit einer Übersicht über die angezogenen lateinischen Ms. und mit einem umfangreichen Namen- und Sachverzeichnis versehen. Schon aus dem bisherigen läßt sich Wichtiges über das Wiederaufleben der Archimedischen Betrachtungsweise erkennen. Weiteres erwarten wir von der Fortsetzung des Werkes, der wir mit größtem Interesse entgegensehen.

J. E. Hofmann (Ichenhausen)

Jayawardene, S. A. 5707
Documenti inediti negli archivi di Bologna intorno a Raffaele Bombelli e la sua famiglia.
Atti Accad. Sci. Ist. Bologna Cl. Sci. Fis. Rend. (11) **10** (1962/63), *no.* 2, 235–247.
Aus den Geburtsregistern von San Pietro, aus dem Stadtarchiv und aus den Einträgen in der Bibliothek des Archigimnasio di Bologna ergeben sich wichtige Aufschlüsse über die Familie der Bombelli-Mazoli, deren bedeutendster Vertreter der verdienstvolle Algebraiker und Zahlentheoretiker Raffaele (1526–1575/76) ist.
J. E. Hofmann (Ichenhausen)

Guseinov, A. I.; Agaev, G. N. 5708
On the history of the development of mathematical research in Azerbaijan. (Russian)
Izv. Akad. Nauk Azerbaĭdžan. SSR Ser. Fiz.-Tehn. Mat. Nauk **1964**, *no.* 3, 3–17.

Behnke, Heinrich; Köthe, Gottfried 5709
Otto Toeplitz zum Gedächtnis.
Jber. Deutsch. Math.-Verein. **66** (1963/64), *Abt.* 1, 1–16.
An extended account of the professional career of Toeplitz, together with a bibliography of 45 items.

LOGIC AND FOUNDATIONS

See also 5745, 5772, 5890, 5891, 6959, 6967, 6968, 6992, 6994, 6995, 7000.

Meigne, Maurice 5710
★Recherches sur une logique de la pensée créatrice en mathématiques.
Préface de Georges Bouligand.
Librairie Scientifique et Technique, Albert Blanchard, Paris, 1964. 132 *pp.* 18.00 *F.*
Der Verfasser entwickelt die Idee einer schöpferischen Logik der Mathematik. Gegenüber formalisierten Theorien wird der Einwand grundsätzlicher Unvollkommenheit erhoben, da der Sinn einer mathematischen Theorie nicht vollständig formal erfaßbar sei und die mathematische Forschung auf unformalisierbaren schöpferischen Intuitionen beruhe. Im Hinblick auf eine schöpferische Logik werden insbesondere Untersuchungen der Geometrie, Analysis und Gruppentheorie diskutiert. Die hierbei entwickelten Aspekte können nach Ansicht des Referenten zwar für heuristische Überlegungen und interpretative Vorstellungen von Bedeutung sein, betreffen aber nicht das besondere Anliegen der mathematischen Logik und Grundlagenforschung, mathematische Theorien in mathematisch exakter Weise zu untersuchen. Das Buch ist nicht frei von mathematischen Fehlern. So wird auf S. 29 die falsche Behauptung aufgestellt, daß die Menge der arithmetischen Ausdrücke (die etwa aus natürlichen Zahlen oder Zahlenvariablen und dem Additionszeichen gebildet werden können) nicht abzählbar sei.
K. Schütte (Kiel)

Devidé, Vladimir 5711
★Matematička logika. Prvi 1: Klasična logika sudova [Mathematical logic. Part 1: The classical propositional logic].
Posebna Izdanja Matematičkog Instituta, Kniga 3.
Matematički Institut u Beogradu, Belgrade, 1964. 228 *pp.*
In the Serbo-Croatian mathematical literature there is no book on mathematical logic; this is, perhaps, the reason why the Mathematical Institute in Belgrade incorporated this textbook in its series "Special Editions", which is supposed to cover only monographs, large original papers, and mathematical tables. The book is written in Serbo-Croatian, it is dedicated to the Japanese people, and it has an English summary.

This is a first introduction to mathematical logic; the present first volume is concerned only with propositional calculus, and in some 220 pages covers about the same material as the first 44 pages of the book of Mendelson; however, there are no problems and exercises, but there is an original part enumerating all possible bases of the propositional calculus: the beginning reader will know that there exist exactly two one-member bases, 34 two-member bases (all are quoted), and 10 three-member bases (all are quoted, too).

Detailed contents: Chapter 1 (Introduction) explains "the most serious, deepest, and most difficult, but also most brilliant, crisis of mathematics", i.e., the antinomies. Chapter 2 (Propositional algebra) contains usual material about functors, truth-values, bases, duality, and electrical switching networks. All bases with one, two and three members are enumerated. Chapter 3 (Propositional algebras as Boolean structures) describes the elements of lattice theory and Boolean algebra. Chapter 4 (Propositional logic) uses a slightly different axiom-system from that of Hilbert and Bernays; other axiom-systems are also exhibited, and the equivalence with the author's system is shown for many of them. Chapter 5 (Some properties of axiom-systems for propositional logic) discusses consistency, completeness, non-categoricity, and similar topics.

The exposition is lengthy, enthusiastic, and annoying. However, with some effort, the reader can learn from this book the basic facts of the propositional calculus.
V. Vučković (Notre Dame, Ind.)

Krnič, Luka [Krnić, Luka] 5712
A remark on the enumeration of bases of an algebra of logic formed by functions of one and two variables. (Russian. Serbo-Croatian summary)
Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hrvatske Ser. II **18** (1963), 13–16.
Es wird die von Devidé [Z. Math. Logik Grundlagen Math. 5 (1959), 265–279; MR **22** #A1512] angegebene Aufzählung sämtlicher aus ein- und zweistelligen Funktionen bestehenden Basen der Menge aller Funktionen der zweiwertigen Aussagenlogik mittels einer von Post gegebenen Charakterisierung der funktional-vollständigen Funktionenmengen der zweiwertigen Logik hergeleitet.
G. Asser (Greifswald)

Mo, Shao-kui [Mo, Shao-kuei] 5713
Enumerative quantifiers and predicate calculus.
Acta Math. Sinica **14** (1964), 218–230 (*Chinese*); *translated as Chinese Math.* **5** (1964), 239–253.

Singletary, Wilson E. 5714
Correction to "A complex of problems proposed by Post".
Bull. Amer. Math. Soc. **70** (1964), 826.

The author modifies the semi-Thue system σ and the propositional calculus $\mathfrak{P}_\sigma$ upon which the results in an earlier paper [same Bull. 70 (1964), 105–109; MR 28 #2040] depend. For σ, each word G_i and $\bar{G}_i$ is now required to have at least two letters. For $\mathfrak{P}_\sigma$, p_1 is replaced by $[p_1 \& p_3]$ and q_1 by $[q_1 \& q_2]$ throughout Axioms 3, 4 and 7; r_1 is replaced by $[r_1 \& r_3]$ in Axiom 7.

It is not known whether the original σ and $\mathfrak{P}_\sigma$ have the required properties. *G. F. Rose* (Pacific Palisades, Calif.)

Weinberger, Ota 5715

Théorie des propositions normatives. Quelques remarques au sujet de l'interprétation normative des systèmes K_1 et K_2 de M. Kalinowski. (Polish and Russian summaries)

Studia Logica 9 (1960), 7–25.

Skolem, Th. 5716

Studies on the axiom of comprehension.

Notre Dame J. Formal Logic 4 (1963), 162–170.

The author considers three forms of the axiom of comprehension.

(1) $(z_1)\cdots(z_n)(\mathsf{E}y)(x) \quad (x \in y \Leftrightarrow \varphi(x, z_1, \cdots, z_n)),$

(2) $(z_1)\cdots(z_m)(\mathsf{E}y)(x)$

$$\Bigl(x \in y \Leftrightarrow \prod_{u_1}\cdots\prod_{u_n} \varphi(x, z_1, \cdots, z_m, u_1, \cdots, u_n)\Bigr),$$

(3) $(\mathsf{E}y)(x) \quad \Bigl(x \in y \Leftrightarrow \prod_{u_1}\cdots\prod_{u_n} \varphi(x, u_1, \cdots, u_n)\Bigr).$

In (1), φ is either a propositional constant or is built up from atomic expressions $u \in v$ by negation, conjunction, and disjunction, and there are no further variables in φ other than $x, z_1, \cdots, z_n$. In (2), φ also contains the variables $u_1, \cdots, u_n$ and each $\prod_{u_i}$ is either a universal quantifier or existential quantifier with respect to u_i. In (3), φ contains no variables other than $x, u_1, \cdots, u_n$. In a previous paper [same J. 1 (1960), 13–22; MR 26 #4890], the author constructed a denumerable model M satisfying (1) for all φ without negation. Now he shows that M satisfies (2) for all φ without negation. In the cited paper it was indicated that there is no finite model for which (1) holds for negation-free φ. Here the author exhibits a five-element model satisfying (3) for all negation-free φ. Next, a denumerable model is given for which (1) holds for all φ in which no atomic formula $u \in u$ or $u \bar{\in} u$ occurs. Then the author gives a simpler proof of an earlier result of his, namely, the construction of a denumerable model satisfying (1) in 3-valued logic. All the models discussed also satisfy the axiom of extensionality. Finally, two number-theoretic models are exhibited which satisfy certain weak subtheories of Zermelo-Fraenkel-Skolem set theory. *E. Mendelson* (Flushing, N.Y.)

Vasil'ev, Ju. L. 5717

Comparison of the complexity of terminal and minimal disjunctive normal forms. (Russian)

Problemy Kibernet. No. 10 (1963), 5–61.

A disjunctive normal form (d.n.f.) is said to be terminal if removal of any disjunct or any conjunct within a disjunct leads to a non-equivalent d.n.f. If T is a d.n.f., let $Z(T)$ be the number of occurrences of letters in T, and let $I(T)$ be the number of disjuncts in T. If f is a given truth function, and $\mathfrak{T}(f)$ is the set of all terminal d.n.f.'s representing f, let $R(f) = \max Z(T_1)/Z(T_2)$ and $Y(f) = \max I(T_1)/I(T_2)$ for all T_1, T_2 in $\mathfrak{T}(f)$. If $p^{(n)}$ denotes the set of all truth functions of n variables, let $R(n) = \max R(f)$ and $Y(n) = \max Y(f)$ for all functions f in $p^{(n)}$. The author proves that $2^{n-8\sqrt{n}} \le Y(n) \le 2^{n-\log n}$ and $2^{n-8\sqrt{n}} \le R(n) \le 2^n$. Various additional results and applications are given. *E. Mendelson* (Flushing, N.Y.)

Rasiowa, H. 5718

On modal theories.

Acta Philos. Fenn. Fasc. 16 (1963), 201–214.

The author collects in this paper her results (and those of R. Sikorski) concerning the metatheory of pure and applied formal theories of modal logic. The language L of such modal theories is that of first-order logic with one additional unary propositional connective, necessity I. The basic semantical notions are introduced roughly as follows. Let J be a nonempty set, and let $\mathfrak{A}$ be a complete topological Boolean algebra. Let $\mathfrak{M}$ be a mapping associating with each m-place predicate of L a function from J^m into $\mathfrak{A}$. The mapping $\mathfrak{M}$ is extended to L in the obvious way by associating with the logical operations the corresponding Boolean operations in $\mathfrak{A}$; the interior operation of $\mathfrak{A}$ is associated with I, and g.l.b. and l.u.b. are associated with the quantifiers. $\mathfrak{M}$ is called a realization of L in I and $\mathfrak{A}$. An assignment v of elements of J to the free variables $x_1, \cdots, x_n$ occurring in a formula α of L is said to satisfy α in $\mathfrak{M}$ if $\mathfrak{M}$ maps α onto a function $\alpha_{\mathfrak{M}}(x_1, \cdots, x_n)$ for which $\alpha_{\mathfrak{M}}(vx_1, \cdots, vx_n)$ is the unit element of $\mathfrak{A}$. A wealth of interesting material is covered in survey fashion. The main results are generalizations of the fundamental notions and results concerning first-order theories and intuitionistic logic to modal theories. *E. Engeler* (Minneapolis, Minn.)

Büchi, J. Richard 5719

Turing-machines and the Entscheidungsproblem.

Math. Ann. 148 (1962), 201–213.

This paper provides (as the author remarks) the first really elegant proof of the unsolvability of the decision problem for q.t. (quantification theory—without identity). It also contains an extremely short proof of an improved version of Trachtenbrot's theorem (that the non-satisfiable sentences cannot be recursively separated from those satisfiable in a finite domain). The improvement obtained by the author is the addition "not even if only sentences of the form $E \& AEA$ are considered". This, of course, implies that the set of finitely satisfiable sentences is not recursive, again even if only sentences of the form $E \& AEA$ are considered.

The reviewer urges that this paper be regarded as the henceforth standard proof of Church's and Trachtenbrot's theorems. The problems Büchi mentions at the end of the paper (the "unrestricted domino problem" and the problem "T2") have been solved by students of Wang, but the solutions are not yet in print. *H. Putnam* (Cambridge, Mass.)

Pour-El, Marian Boykan; Howard, William A. 5720

A structural criterion for recursive enumeration without repetition.

Z. Math. Logik Grundlagen Math. 10 (1964), 105–114.

The authors generalize the result that the recursively enumerable sets are recursively enumerable without repetition (Friedberg) in the following way: any recursively enumerable class F with a partial recursive height function is recursively enumerable without repetition. By a height function is meant a function h defined on the finite subsets of members of F with the properties: (1) monotonicity: if $A \subseteq B$, then $h(A) \leq h(B)$; (2) convergence (the authors call this "the ascending chain condition"): if $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ is an ascending sequence of finite subsets of a fixed member of F, then the associated sequence of heights $h(A_1)$, $h(A_2)$, $\cdots$ eventually becomes constant; (3) unboundedness: for every finite subset A of a member of F, there exists a finite subset B, not necessarily of the same member of F, such that $A \subset B$, $h(A) < h(B)$.

The proof is by a complicated "priority" argument. In following the proof, the following remarks may be of help (the authors give little motivation). The first two conditions enable one to associate a "height" with each member of F, namely, if $S \in F$ is finite, "height" $(S) = h(S)$; if $S \in F$ is infinite, "height" $(S) = \max(h(S_i))$, S_i a finite subset of S. It can easily be proved from (1) and (2) that this maximum exists. Note that the "height" of $S \in F$ can be successively approximated from below by taking finite subsets, and that the approximations must eventually all be correct. The condition (3) implies that a finite subset of any member of F is also a finite subset of some member of F with "height" as great as one pleases. These facts allow one to modify Friedberg's original proof to obtain the theorem.

Several corollaries are obtained by the authors, including Friedberg's original result.

H. Putnam (Cambridge, Mass.)

Mostowski, A. 5721

On invariant, dual invariant and absolute formulas.
Rozprawy Mat. **29** (1962), 38 *pp.*

This paper is about the simple theory of types T_ω. A model $\mathscr{M}$ of T_ω consists of a pair $\langle R, S\rangle$, where $R = (R_0, R_1, \cdots)$ is a sequence of sets and $S = (S_0, S_1, \cdots)$ is a sequence of binary relations S_i between elements of R_i and R_{i+1}. The standard model $\mathrm{St}(R_0)$ is the model in which each R_{i+1} is the power set of R_i and each S_i is the ϵ relation between R_i and R_{i+1}. Let N be a ternary relation. We say that n represents N in a model $\mathscr{M}$ if (1) the set R_0 of individuals of $\mathscr{M}$ contains the field of N, (2) n is in R_5 and is a set of "ordered triples" of elements of R_0 in the sense of $\mathscr{M}$, and (3) $N(x, y, z)$ holds if and only if the "ordered triple" (x, y, z) belongs to n in the sense of $\mathscr{M}$. Let K be a class of models and let Φ be a formula of T_ω with a single free variable of order 5. We say that Φ is absolute with respect to K if for every $\mathscr{M}$ in K and every ternary relation N over R_0 which is represented in $\mathscr{M}$ by an element n, the conditions (a) n satisfies Φ in $\mathscr{M}$ and (b) N satisfies Φ in $\mathrm{St}(R_0)$ are equivalent. Φ is said to be invariant with respect to K if (a) implies (b), and dual invariant with respect to K if (b) implies (a). The main theorems are necessary conditions, in terms of first-order model theory, for a formula Φ of T_ω to be absolute, invariant, or dual invariant. We state two of the main results below. Let Z be a set of closed formulas of T_ω which are true in every standard model with infinitely many individuals. Let K be the set of all models $\mathscr{M}$ in which all formulas of Z are true. Theorem: If Φ is absolute with respect to K, then there is a first-order closed formula $S(M, N)$ with a unary predicate variable M and a ternary predicate variable N such that, for all infinite sets M and ternary relations N over M, N satisfies Φ in $\mathrm{St}(M)$ if and only if $S(M, N)$ holds. The author attributes to Scott the remark that the above theorem can also be proved using a result of the reviewer [Nederl. Akad. Wetensch. Proc. Ser. A **64** (1961), 477–495; MR **25** #3816] concerning ultraproducts. In the next theorem we assume further that Z is recursively enumerable and every formula in Z holds in all standard models. Theorem: If Φ is invariant with respect to K, then there is a first-order formula $F(M, N, Q_1, \cdots, Q_k)$ such that for all M and $N \subset M^3$, N satisfies Φ in $\mathrm{St}(M)$ if and only if there exists a model $\langle I, M, N, Q_1, \cdots, Q_k\rangle$ of F such that $I - M$ is infinite. There is also a similar result for dual invariant formulas Φ. The proofs are based upon formalizing in first-order logic the semantics for T_ω, and considering the models of the first-order theory so obtained. The author also states, proves, and uses a reformulation of a joint result of Ehrenfeucht and Mostowski [Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **9** (1961), 369–373; MR **26** #6043] concerning compact spaces of models.

H. J. Keisler (Madison, Wis.)

van Rootselaar, B. 5722

Algebraische Kennzeichnung freier Wortarithmetiken.
Compositio Math. **15**, 156–168 (1963).

An algebraic characterization is given of the free arithmetics of words studied by Vučković [Acad. Serbe Sci. Publ. Inst. Math. **14** (1960), 9–60; MR **22** #A3087]. It is used to establish some new properties of these arithmetics. A structure which is an additive semi-group with identity and right cancellation, is a multiplicative semi-group with identity, satisfies $0 \cdot x = 0$ and the left distributive law, and has no proper divisors of zero is called a near semi-ring ("Fasthalbring"). Let H be a near semi-ring with multiplicative center Z. Let ALE be the set of near semi-rings satisfying the following three conditions. Condition A: there is a mapping $\alpha: H \to Z$ satisfying (A1) $\alpha(x+y) = \alpha(y)$ if $y \neq 0$; (A2) $\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y)$; (A3) $\alpha(c) = c$ if $c \in Z$; (A4) $\alpha(x) = 0$ implies $x = 0$. Condition L: for all $x \in H$, $y + \alpha(x) = x$ is solvable in H. Since by additive right cancellation the solution y is unique, define $\beta: H \to H$ by $\beta(x) = y$. Condition E: for all $x \in H$ there is a natural number i such that $\beta^{(i)}(x) = 0$. In what follows let $H \in$ ALE and Z be its multiplicative center and let $V(n)$ $[V(\omega)]$ be the free arithmetic of words on n letters [on a countably infinite alphabet]. It is shown that for all H, if Z is a finite cyclic group of order n, $n > 1$, then H is isomorphic to $V(n)$ and that if Z is an infinite cyclic semi-group with identity, then H is isomorphic to $V(\omega)$. (Here, $+$ and $\cdot$ of the near semi-ring are identified with the first addition and multiplication of the arithmetic upon which the other additions and multiplications are dependent.) Noting that the additive semi-group generated by the elements of Z is just H, define

$$\lambda xy\, x - y : H \times H \to H$$

by (1) if $x, y \in Z$, then $x - y = 0$ whenever $x = y$, $x - y = x$ otherwise, (2) if $x = x_k + x_{k-1} + \cdots + x_1$ and $y = y_k + y_{k-1} + \cdots + y_1$ and $x, y \in H$, then $x - y = (x_k - y_k) + \cdots + (x_1 - y_1)$. For x, $y \in H$, define $x \leq y$ if and only if there exists

$t \in H$ such that $x+t=y$. Then under the assumption that the cancellation law holds in Z, the following are shown to hold in H: (a) the cancellation law, (b) $w\cdot(x-y)=w\cdot x-w\cdot y$, and (c) if $w\neq 0$ and $w\cdot x\leq w\cdot y$, then $x\leq y$. For both $V(n)$ and $V(\omega)$ the multiplicative centers consist of the empty word together with the letters of the alphabet, and in these centers cancellation holds. Thus (a), (b), and (c) hold for $V(n)$ and $V(\omega)$.

P. Axt (University Park, Pa.)

Nerode, A. 5723

A decision method for p-adic integral zeros of diophantine equations.

Bull. Amer. Math. Soc. **69** (1963), 513–517.

This paper gives a simple decision method for p-adic integral zeros of Diophantine equations. Since there is already a disproof procedure (show that f has no zeros mod p^n for some n), the problem reduces to giving a proof procedure. This is given in two theorems: (1) that a polynomial f has a zero in p-adic integers if and only if f has a zero in p-adic algebraic integers; and (2) that $ZA(p)$, the p-adic algebraic integers, can be construed naturally as a computable domain, i.e., the operations of addition and multiplication can be performed effectively in $ZA(p)$. Then the required proof procedure is to simply exhibit a zero of f in $ZA(p)$ and verify that it is a zero by computation. (1) is proved by the author via elimination, while the author's proof of (2) depends essentially on the Hensel-Rychlik theorem.

H. Putnam (Cambridge, Mass.)

Robinson, Julia 5724

The undecidability of exponential Diophantine equations.

Logic, Methodology and Philosophy of Science (*Proc.* 1960 *Internat. Congr.*), *pp.* 12–13. *Stanford Univ. Press, Stanford, Calif.*, 1962.

This paper announces the negative solution (obtained by the author jointly with Martin Davis and the reviewer) of the decision problem for exponential Diophantine equations (i.e., Diophantine equations in which exponentiation as well as addition and multiplication is permitted). The result is a consequence of the stronger result, obtained by the above authors, that every recursively enumerable set or relation is exponential Diophantine. Indeed, it has even been shown by Davis, the reviewer and the author that every recursively enumerable set is just the set of non-negative values taken on by an exponential polynomial, and analysis of the proof shows that a function $f(x_1,\cdots,x_n,2^y)$ will always suffice, f a polynomial. Whether or not every recursively enumerable set is Diophantine is still unknown; indeed, it is not even known whether or not there is a polynomial f (in n variables) whose non-negative values are just the prime numbers.

H. Putnam (Cambridge, Mass.)

Salomaa, Arto 5725

On the reducibility of events represented in automata.

Ann. Acad. Sci. Fenn. Ser. A I No. 353 (1964), 16 *pp.*

Let $\mathfrak{N}_k=\langle N_k,\lambda,r_1,\cdots,r_k\rangle$ be the free k-monadic algebra on one generator (N_k is the set of words on k symbols). A subset $\alpha\subseteq N_k$ is reducible if there exists a homomorphism $\rho:\mathfrak{N}_k\to\langle S,s_0,r_1',\cdots,r_k'\rangle$ and a subset $S'\subseteq S$ with $\mathrm{card}(S')<\mathrm{card}(\alpha)$ and $\rho^{-1}(S')=\alpha$. In the first section of the paper the author proves some properties of reducible sets. The second section is a study of "the representation theory of finite events".

J. W. Thatcher (Yorktown Heights, N.Y.)

Salomaa, Arto 5726

Axiom systems for regular expressions of finite automata.

Ann. Univ. Turku. Ser. A I No. 75 (1964), 29 *pp.*

For any regular expression α, $|\alpha|$ is the regular set denoted by α, and $\alpha=\beta$ is a valid equation if $|\alpha|=|\beta|$. The author considers the problem of axiomatizing the valid equations. Two deductive systems for equations are defined. Both $\vdash_1$ and $\vdash_2$ use the rules of equational deducibility and in addition: (1) If $\lambda\notin\beta$ and $\vdash_1\alpha=\beta\alpha\cup\gamma$ $[\vdash_1\alpha=\alpha\beta\cup\gamma]$, then $\vdash_1\alpha=\beta^*\gamma$ $[\vdash_1\alpha=\gamma\beta^*]$; (2) If $\vdash_2\alpha=\alpha\cup\gamma\beta^n\delta$ for every $n\geq 0$, then $\vdash_2\alpha=\alpha\cup\gamma\beta^*\delta$. In these rules, α,β,γ and δ are arbitrary regular expressions. Because the author deals with regular expressions (as opposed to terms in $\cup$, $\cdot$, $*$, and variables over regular sets), the condition of the first rule can be stated syntactically. The author exhibits a simple axiom set which is complete with respect to the infinitistic system ($\vdash_2$). A complete axiomatization is given for the valid equations between expressions on one symbol using $\vdash_1$. The problem of obtaining a complete axiomatization of all valid equations using either equational deducibility or $\vdash_1$ remains open.

J. W. Thatcher (Yorktown Heights, N.Y.)

SET THEORY

See also 5716, 5750, 5769, 5963, 5972, 6468.

Dumont, M. 5727

★Étude intuitive des ensembles.

Cours de Mathématiques pour l'Enseignement des Premier et Second Degrés.

Dunod, Paris, 1964. vii+87 *pp.* 7.80 *F.*

Andreoli, Giulio 5728

La ricostruzione del continuo lineare mediante insiemi ternari cantoriani (Semantica delle successioni binarie).

Ricerca (*Napoli*) (2) **14** (1963), *maggio–agosto*, 3–21.

Il problema trattato in questa memoria è quello inverso della formazione degli insiemi perfetti a partire dal continuo—formazione ottenuta eliminando i punti interni ad una infinità numerabile di intervalli disgiunti; quindi è formulato: "Data un insieme perfetto—nel nostro caso particolare quello ternario cantoriano—e date certe operazioni (in particolare) di dilatazione su questo eseguibili, ricostituire il continuo".

S. Marcus (Bucharest)

Charreau, André 5729

Sur les correspondances entre ensembles.

C. R. Acad. Sci. Paris **259** (1964), 2741–2743.

The notion of the kernel of a map: $E\to F$ is extended to apply to a correspondence between E and F, i.e., a subset of $E\times F$. Results concerning compositions of correspondences and inclusion relations between their kernels are established. *M. Morley* (Madison, Wis.)

Flor, Peter 5730

Über die Wertmengen fastperiodischer Folgen.

Monatsh. Math. **67** (1963), 12–17.

The author shows that for any bounded, countable set of real numbers which is dense in itself there exists a one-to-one function k from the integers onto the set with the property of being an "almost periodic sequence", i.e., for every ε there is an n such that $a \equiv b \pmod n$ implies $|k(a) - k(b)| < \varepsilon$. The proof is by an explicit construction. A second construction shows that for any bounded countable set of real numbers there exists an almost periodic sequence whose range is the set and which assumes each value in the set periodically. The author points out that the conditions given are (trivially) necessary as well as sufficient. *H. Putnam* (Cambridge, Mass.)

Katona, Gy. 5731

Intersection theorems for systems of finite sets.

Acta Math. Acad. Sci. Hungar. **15** (1964), 329–337.

Let $k \leq m < \omega$, and let $M = \{0, 1, \cdots, m-1\}$. Find a set A such that (1) each $a \in A$ is a subset of M; (2) for all $a, b \in A$, $a \cap b$ has power at least k; (3) A has the maximum power among all sets satisfying conditions (1) and (2). Erdős, Chao Ko, and Radó [Quart. J. Math. Oxford Ser. (2) **12** (1961), 313–320; MR **25** #3839] conjectured that if $m+k$ is even, then the set A of all $a \subseteq M$ of power $|a| \geq \frac{1}{2}(m+k)$ has the desired properties (1), (2), (3). The author proves their conjecture. For the case that $m+k$ is odd he also obtains a solution: A is the set of all $a \subseteq M$ of power $|a| > \frac{1}{2}(m+k)$ together with all $a \subseteq M - \{m-1\}$ of power $|a| = \frac{1}{2}(m+k-1)$. *H. J. Keisler* (Madison, Wis.)

Bose Majumder, N. C. 5732

On the construction of some SD-sets.

Bull. Calcutta Math. Soc. **54** (1962), 163–169.

The author constructs some sets, by omitting various collections of digits from the expansions of numbers on $[0, 1]$ in various bases, that have the property (δ) that their distance sets fill an interval of length 1. He constructs a set E with both property δ and the property that each point of $[0, 1]$ is halfway between a pair of points of E, but such that E is not symmetric.

R. P. Boas, Jr. (Evanston, Ill.)

Bose Majumder, N. C. 5733a

On the distance set of the Cantor set. II.

Bull. Calcutta Math. Soc. **54** (1962), 127–129.

Bose Majumder, N. C. 5733b

Category of some sets related to the distance set of the Cantor set.

Bull. Calcutta Math. Soc. **55** (1963), 91–95.

Part I of #5733a appeared in Proc. Nat. Inst. Sci. India Part A **27** (1961), 289–294 [MR **24** #A2537]. Let C be the Cantor set on $[0, 1]$. Then for almost all d the set E_1 of pairs (x, y) of points of C such that $y - x = d$ has the cardinal number of the continuum. The author deduces a new proof of this from results in his earlier paper [Bull. Calcutta Math. Soc. **52** (1960), 1–13; MR **24** #A3444]. In the second paper he shows that E_1 is a residual set, and that the set E_2 of numbers d such that there are only finitely many pairs (x, y) of points of C with $y - x = d$ is of first category. *R. P. Boas, Jr.* (Evanston, Ill.)

Myhill, John 5734

Variations on a theme of Bernays.

Notre Dame J. Formal Logic **4** (1963), 274–282.

Bernays' system of axioms for set theory (with classes) [*Essays on the foundations of mathematics*, pp. 3–49, Magnes Press, Hebrew Univ., Jerusalem, 1961; MR **29** #1034] is reformulated by the author, who first dispenses with the selector symbol and with class abstraction and then adds the definite article. *A. Lévy* (Jerusalem)

Rotman, B. 5735

A note on principal sequences.

Proc. Glasgow Math. Assoc. **6**, 133–135 (1964).

Theorem: For a given sequence of regressive functions assigning principal sequences to the limit numbers of the second number class, there exists, for any $i < \omega$, a non-denumerable subset of these limit numbers such that the principal sequences assigned to the members of this subset coincide in their first i places. Hence, "Denjoy's massive attempt [*L'énumération transfinie*, Livre II, Gauthier-Villars, Paris, 1952; MR 15, 408] to solve the problem of principal sequences must fail, according to his own description of that method (p. 584)".

L. Gillman (Rochester, N.Y.)

Slater, M. 5736

On a class of order-types generalizing ordinals.

Fund. Math. **54** (1964), 259–277.

The author investigates those order types for which the right half of every Dedekind cut has a first term. (Ordinals constitute the special case in which the type itself also has a first term.) He classifies these types in several ways, considers order-preserving maps between them, and presents an exhaustive discussion of their possible factorizations. *L. Gillman* (Rochester, N.Y.)

Sierpiński, W. 5737

Sur les ensembles raréfiés de nombres naturels.

Essays on the foundations of mathematics, pp. 300–303. *Magnes Press, Hebrew Univ., Jerusalem,* 1961.

Un ensemble E de nombres naturels est dit raréfié [muni de la propriété P] si, en le rangeant en ordre croissant $e_1 < e_2 < e_n$, l'on a

$$\lim_{n\to\infty} (e_{n+1} - e_n) = \infty \quad \left[\limsup_{n\to\infty} (e_{n+1} - e_n) = \infty\right].$$

La réunion d'un nombre fini d'ensembles raréfiés de nombres naturels est un ensemble raréfié (Théorème 1). La propriété P est plus faible que la propriété d'être raréfié. On ne sait pas si l'ensemble des nombres premiers est la réunion d'un nombre fini d'ensembles raréfiés. L'ensemble des nombres n^m ($n \in N$, $m \in N \setminus \{1\}$) possède la propriété P (Théorème 2) et d'après l'hypothèse de Pillai [Bull. Calcutta Math. Soc. **37** (1945), 15–20; MR 7, 145] est raréfié. *Đ. Kurepa* (Zagreb)

Hedrlín, Z. 5738

On a number of commuting transformations.

Comment. Math. Univ. Carolinae **4** (1963), 132–136.

Let X be an n-element set. A transformation is a mapping of X into itself. The author proves that any given transformation commutes with at least n different transformations. *G. Grätzer* (University Park, Pa.)

Dickmann, Maximo Alejandro 5739

König's theorem and the axiom of choice. (Spanish. English summary)

Rev. Un. Mat. Argentina **21**, 198–214 (1963).

L'auteur commence par établir un théorème en connexion directe avec le théorème connu de König de la théorie des nombres cardinaux; puis il démontre la proposition suivante: Si $\{A_\xi\}$, $\{B_\xi\}$, $\xi \in Z$, $Z \neq \varnothing$, sont deux familles d'ensembles satisfaisant aux conditions suivantes. (i) $\exists \alpha \in Z$, $\overline{\overline{B}}_\alpha \geqq \overline{\overline{Z-J}}+2$ avec $J=\{\xi \mid \xi \in Z,\ \overline{\overline{B}}_\xi \geqq 2\}$, (ii) $B_\xi \neq \varnothing$, $\forall \xi \in Z$, (iii) $\overline{\overline{J}} \geqq 2$, (iv) $\overline{\overline{A}}_\xi \leqq \overline{\overline{B}}_\xi$, $\forall \xi \in Z$, alors $\overline{\overline{\bigcup A_\xi}} \leqq \overline{\overline{\prod B_\xi}}$ $(\xi \in Z)$, et fait voir que ce théorème implique le précédent et aussi l'axiome du choix. Il prouve enfin que le théorème établi est le plus général relativement aux relations entre produits et sommes de familles arbitraires de nombres cardinaux. Toute une série d'exemples sont donnés, en vue de justifier l'indépendance et la nécessité des hypothèses. *A. Sade* (Marseille)

Gavrilov, G. P. 5740

On the cardinality of sets of closed classes of finite height in $P_{\aleph_0}$. (Russian)

Dokl. Akad. Nauk SSSR **158** (1964), 503–506.

Let P be the set of all functions whose arguments and values lie in a fixed denumerable set. A subclass of P is said to be closed if it is closed under composition of functions. P is defined to be the only class of height 0. If M is any closed subclass of P, M is said to be of height $n+1$ if, for any function f in $P-M$, the closure under composition M_f of the class $M \cup \{f\}$ has height $\leq n$, and, for at least one such f, M_f has height n. Denote by $K(n)$ the collection of all classes of height n. This paper proves that $K(1)$ has cardinality 2^c. The author also states that, for any n, $K(n)$ can be shown to have cardinality 2^c. *E. Mendelson* (Flushing, N.Y.)

Kurepa, Đuro 5741

On universal ramified sets. (Serbo-Croatian summary)

Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hrvatske Ser. II **18** (1963), 17–26.

Let α be a non-limit ordinal. Hausdorff [*Grundzüge der Mengenlehre*, Veit, Leipzig, 1914] constructed an $\aleph_\alpha$-universal totally ordered set of cardinal $\exp \aleph_{\alpha-1}$. J. B. Johnston [Proc. Amer. Math. Soc. 7 (1956), 507–514; MR **20** #3090] constructed an $\aleph_\alpha$-universal partially ordered set of cardinal $\exp \aleph_\alpha$. The present author constructs an $\aleph_\alpha$-universal ramified set of cardinal $\exp \aleph_{\alpha-1}$. (A ramified set is a partially ordered set in which the predecessors of each element form a chain.) *L. Gillman* (Rochester, N.Y.)

Vopěnka, Petr 5742

The independence of the continuum hypothesis. (Russian. English summary)

Comment. Math. Univ. Carolinae **5** (1964), *suppl. I*, 48 *pp.*

The following theorems are proved in this paper. (1) If the Gödel-Bernays set theory Σ^* is consistent, then it remains consistent after adjunction of any of the formulas (i) $2^{\aleph_0} \geq \aleph_n$; (ii) $2^{\aleph_0} \geq \aleph_\omega$; (iii) $2^{\aleph_0} \geq \aleph_{\omega_1}$, etc. (2) Let In be the axiom: "There is at least one regular aleph $\aleph$ with a positive limit index". If $\Sigma^* + \mathrm{In}$ is consistent, then it remains consistent after adjunction of the axiom: "There is a regular aleph with a positive limit index such that $\aleph \leq 2^{\aleph_0}$".

These theorems were proved earlier by Cohen [Proc. Nat. Acad. Sci. U.S.A. **50** (1963), 1143–1148; MR **28** #1118; ibid. **51** (1964), 105–110; MR **28** #2962] for Zermelo-Fraenkel set theory. The author's proof, while using essential ideas of Cohen's proof, differs considerably from the latter.

The first difference is that the author does not construct a semantical model for $\Sigma^* + $ (i) (or $\Sigma^* + \mathrm{In} + $ (ii)) but interprets these systems in Σ^*. In other words, he defines a class E of ordered pairs (by a normal formula) and proves that the axioms of $\Sigma^* + $ (i) [respectively, (ii)] are transformed in theorems of Σ^* if one substitutes E for $\in$. Since there are only finitely many axioms in Σ^* this proof can be carried out in Σ^* and does not require metamathematical arguments.

The basic tool is—as in Cohen's work—the notion of forcing. This notion is, however, for the author not a metamathematical one: He defines (again by a normal formula) a class Forc of triples $\langle p, \alpha, \beta \rangle$, where p is an open subset of a topological space c and α, β are ordinals. The formula $\langle p, \alpha, \beta \rangle \in$ Forc means that p forces the "limited statement $F_\alpha \in F_\beta$", where F_α and F_β are polynomials defined as in Cohen's paper.

The operation of adjunction of new sets to a given model is replaced in the author's proof by the ultraproduct operation. The field of his relation E consists of continuous functions φ defined on open dense subsets of c whose values are polynomials, discrete topology being used in the space of polynomials. A pair $\langle \varphi, \psi \rangle$ belongs to E if the union $\bigcup \{p : \langle p, \varphi(p), \psi(p) \rangle \in \text{Forc}\}$ belongs to a fixed ultrafilter $j \subset P(c)$ containing all open dense subsets of c. Functions φ which are not almost equal to a constant thus define "new sets".

The model ∇ obtained in this way satisfies the axioms of Σ; an additional construction is necessary to verify that the axiom of choice be true in the model.

In order to obtain the proofs of Theorems 1 and 2, the author has still to choose a suitable space c. He shows that formulas (i) and (ii) are satisfied in ∇ if c is the topological product of ν copies of the Cantor discontinuous space, where ν is $\aleph_n$, $\aleph_\omega$, $\aleph_{\omega_1}$ in case (i), and ν is the first regular aleph with a positive limit index in case (ii).

Finally, the author shows that there is an ordinal ν^* of the model ∇ (which can be made arbitrarily large by a suitable choice of ν) such that cardinals of ∇ which are $< \nu^*$ are absolute (i.e., coincide with cardinals of the model Δ constructed in ∇).

It is obvious that the author's method is not necessarily limited to the choice of c specified above but can be adapted to a multitude of other situations.

All proofs in the paper are given in full and with a

laudable precision; as it often happens, the precision is pushed so far that it sometimes makes the reading very hard. *A. Mostowski* (Warsaw)

Vopěnka, Petr; Bukovský, Leo 5743

The existence of a PCA-set of cardinal $\aleph_1$.

Comment. Math. Univ. Carolinae **5** (1964), 125–128.

The authors show that for a suitable ordinal ν the model V constructed by Vopěnka [cf. #5742 above] has the property that the formula $\mathrm{card}(L \cap N^N) = \aleph_1 < 2^{\aleph_0}$ is satisfied in V. Using a theorem of Addison [Fund. Math. **46** (1959), 337–357; MR **22** #A1523], who showed that $L \cap N^N \in \mathrm{PCA}$, the authors conclude that if the Gödel-Bernays set theory Σ is consistent, then the theorem "every non-denumerable PCA-set of irrationals is of power $2^{\aleph_0}$" is undecidable in Σ. *A. Mostowski* (Warsaw)

Erdős, P. 5744

An interpolation problem associated with the continuum hypothesis.

Michigan Math. J. **11** (1964), 9–10.

The note deals with families of analytic functions $f: K \to K$, K being the set of complex numbers. Such a family F is said to have the property P_0 provided that for every $z \in K$ the set $\{f(z); f \in F\}$ is countable. Answering a question of Wetzel, the author proves the following. If $c > \aleph_1$, every family F of analytic functions with the property P_0 is countable. If $c = \aleph_1$, some family of analytic functions with the property P_0 has cardinal c. The author asks whether for some family of distinct analytic functions f_α $(\alpha < \Omega_c)$ the set $\{f_\alpha(z)\}_\alpha$ has cardinal $< c$ for every complex number z. *Đ. Kurepa* (Zagreb)

Keisler, H. Jerome 5745

Ultraproducts and saturated models.

Nederl. Akad. Wetensch. Proc. Ser. A **67** = *Indag. Math.* **26** (1964), 178–186.

The author introduces the notion of an α-saturated structure, which is a refinement of the notion of a homogeneous universal model. A structure $\mathfrak{A}$ is said to be α-saturated, where α is a cardinal number, if for each sequence $\{a_\lambda\}_{\lambda<\xi}$, where $\xi < \alpha$, and every set Γ of formulas of the first-order language of $\mathfrak{A}$ with constants for the members of $\{a_\lambda\}_{\lambda<\xi}$ and with x as the only free variable if for every finite $\Lambda \subseteq \Gamma$ $(\exists x) \bigwedge_{\varphi \in \Lambda} \varphi(x)$ is true in $\mathfrak{A}$, then also $(\exists x)(\bigwedge_{\varphi \in \Gamma} \varphi(x))$ is true in $\mathfrak{A}$. The central theorem asserts that for appropriate sets I and ultrafilters D on I and for all structures $\mathfrak{A}_i$, $i \in I$ (of appropriate type), the ultraproduct $\prod_{i \in I} \mathfrak{A}_i/D$ is α-saturated. A consequence of this is that, assuming the generalized continuum hypothesis, two structures are elementary equivalent if and only if they have isomorphic ultrapowers. The paper contains a further discussion of the connections between the author's notion of good ideals [Ann. of Math. (2) **79** (1964), 338–359; MR **29** #3383] and α-saturated structures. The paper concludes with several (counter-)examples.

A. Lévy (Jerusalem)

Gaifman, Haim; Specker, E. P. 5746

Isomorphism types of trees.

Proc. Amer. Math. Soc. **15** (1964), 1–7.

A normal $\aleph_\alpha$ tree is any tree $(A, \leq)$ satisfying the following five conditions (N_1)–(N_5). (N_1) Every row is of power $\aleph_\alpha$; (N_2) Every node of the second [first] kind has $\aleph_\alpha$ points [a single point]; (N_3) Every point x of A has successors in every row of rank $> \rho(x)$ [$\rho(x)$ is the rank of x]; (N_4) Every path (i.e., every subchain that is an initial section) has $< \aleph_{\alpha+1}$ points; (N_5) To every path of cardinality $< \aleph_\alpha$ corresponds some point $x \in A$ which is a successor of every point of the path (for $\alpha = 0$ see the "suites distinguées" of the reviewer [Publ. Math. Univ. Belgrade **4** (1935), 1–138; Ph.D. Dissertation, Univ. Paris, Paris, 1935, in particular, p. 99]).

The authors prove the following. For every ordinal α such that $\aleph_\alpha^{\aleph_\xi} = \aleph_\alpha$ for every ordinal $\xi < \alpha$ there are at least $\aleph_{\alpha+2}$ isomorphism types of normal $\aleph_\alpha$ trees (the authors announce that the stronger wording also holds and is obtained from the foregoing by the substitution: "at least $\aleph_{\alpha+2}$" by "exactly $2^{\aleph_{\alpha+1}}$"). The proof is based on the existence of normal $\aleph_\alpha$ trees (cf. Aronszajn: $\alpha = 0$ in the reviewer's thesis [loc. cit.] and particularly in the reviewer's papers [Publ. Math. Univ. Belgrade **6/7** (1937), 129–160; Acta Math. **75** (1943), 139–150; MR **7**, 195]; see also Specker [Colloq. Math. **2** (1949), 9–12; MR **12**, 597]) and by the formation of new trees by the composition of those given. *Đ. Kurepa* (Zagreb)

COMBINATORIAL ANALYSIS

See also 5738, 5843, 6985.

Brown, William G. 5747

Enumeration of triangulations of the disk.

Proc. London Math. Soc. (3) **14** (1964), 746–768.

A triangulation of a disc is of type $[n, m]$ provided there are n nodes in the interior of the disc and $m+3$ nodes on its boundary. The author determines the number of rooted triangulations (up to root-isomorphisms) of type $[n, m]$ as

$$\frac{2(2m+3)!(4n+2m+1)!}{(m+2)!m!n!(3n+2m+3)!}.$$

He also enumerates the rooted triangulations of type $[n, m]$ having rotational or reflectional symmetry.

B. Grünbaum (Jerusalem)

Halberstam, H.; Laxton, R. R. 5748

Perfect difference sets.

Proc. Glasgow Math. Assoc. **6**, 177–184 (1964).

Let Π be the projective plane over $\mathrm{GF}(p^n)$, and C a collineation of Π of period $q = p^{2n} + p^n + 1$. If the points of Π are denoted by $0, 1, \cdots, q-1$, where $C(i) = i+1$, then the points of any line of Π form a perfect difference set mod q, called a Singer difference set (S.d.s.). The authors give a new proof of the theorem that the only multipliers of such a set are the powers of p (mod q). They then show that if A and B are any two S.d.s.'s (mod q), there is an integer t with $(t, q) = 1$ such that tA and B are equivalent (i.e., translates of each other mod q). Combining these results, it follows that the number of inequivalent S.d.s.'s (mod q) is $\varphi(q)/3n$. {Actually the paper deals only with the case where C is the collineation induced in $\Pi = \mathrm{GF}(p^{3n})^*/\mathrm{GF}(p^n)^*$ by the map $\alpha \to \zeta\alpha$ of $\mathrm{GF}(p^{3n})^*$

onto itself, where ζ is a generator of $GF(p^{2n})^*$. However, it is easily seen using canonical forms that every S.d.s. is equivalent to one obtained in this way.}

B. Gordon (Los Angeles, Calif.)

Hanani, H.; Ornstein, D.; Sós, Vera T. 5749

On the lottery problem. (Russian summary)

Magyar Tud. Akad. Mat. Kutató Int. Közl. **9** (1964), 155–158.

Let S be a system of combinations of k elements out of a set E of n elements such that every combination of l elements has at least d elements in common with a combination of S. Let N be the number of combinations of S and $N_0 = \min_S(N)$. The authors prove

(1) $$N_0 \geq \frac{n(n-l+1)}{k(l-1)^2};$$

(2) $$N_0 \sim \frac{n(n-l+1)}{k(l-1)^2};$$

(3) If $k \leq 5$ and $n/(l-1)$ is integral and $\equiv 1$ $(k-1)$, then equality holds in (1).

H. B. Mann (Madison, Wis.)

Lindström, Bernt 5750

On a combinatory detection problem. I. (Russian summary)

Magyar Tud. Akad. Mat. Kutató Int. Közl. **9** (1964), 195–207.

The present paper is an outgrowth of a paper by H. S. Shapiro and S. Söderberg on a coin-weighing problem [Amer. Math. Monthly **70** (1963), 1066–1070]. A family of subsets $T_1, T_2, \cdots, T_m$ of a given set S of $|S| = n$ elements is a detecting family for S if each subset M of S is uniquely determined by the m numbers $|M \cap T_i|$ $(i = 1, \cdots, m)$. The basic problem is to evaluate $f(n) = \min m$ over the class of all detecting families for S. It is easy to verify that $f(4) = 3$ and $f(5) = 4$, but the evaluation of $f(n)$ for general n appears hopelessly difficult. P. Erdős and A. Rényi, as well as B. Gordon, L. Moser, and the author, have shown that

(1) $$\liminf_{n\to\infty} \frac{f(n)\log n}{n} \geq \log 4.$$

The main result of this paper is that

(2) $$\limsup_{n\to\infty} \frac{f(n)\log n}{n} \leq \log 4,$$

whence

(3) $$\lim_{n\to\infty} \frac{f(n)\log n}{n} = \log 4.$$

This confirms a conjecture of Erdős and Rényi on the existence of the limit. A critical theorem in the derivation of (2) is the inequality

(4) $$f(k2^{k-1}) \leq 2^k - 1 \qquad (k = 2, 3, \cdots),$$

and this is derived by the construction of a certain "detecting matrix". The paper also contains another derivation of (1) based on information theory. The paper concludes by developing an analogous theory for a related detection problem of Erdős and Rényi.

H. J. Ryser (Syracuse, N.Y.)

Lindsey, John H., II 5751

Assignment of numbers to vertices.

Amer. Math. Monthly **71** (1964), 508–516.

The author proves the following theorem. The class of assignments of numbers to vertices of a rectangular parallelepiped of lattice points $(a_1, a_2, \cdots, a_n)$ in n-dimensions of sides l_1, $l_1 \geq l_2 \geq \cdots \geq l_n$, where $0 \leq a_i \leq l_i$, $1 \leq i \leq n$, which minimize the sum of the absolute values of the differences of the numbers assigned to neighboring vertices (two vertices agreeing in all but one coordinate being called neighbors) is exhausted by assignments of the following type: For any fixed i such that $l_i = l_n$, we number the points $(a_1, \cdots, a_n)$ such that $a_i = \pi(1)$; the order in which these points are numbered is governed by induction on n. Then we number the points $(a_1, \cdots, a_n)$ such that $a_i = \pi(2)$, etc., where π is a permutation of $0, 1, \cdots, l_i$.

N. S. Mendelsohn (Winnipeg, Man.)

Carlitz, L.; Riordan, J. 5752

Two element lattice permutation numbers and their q-generalization.

Duke Math. J. **31** (1964), 371–388.

A two-element lattice permutation can be described in a two-dimensional lattice as a path leading from $(0, 0)$ to a point (n, m) (where $0 \leq m \leq n$) with the conditions that the path has minimum length, viz., $m + n$, and that it does not contain points (a, b) with $a < b$. It can also be described as an election for two candidates A, B with final vote (n, m), which is such that none of the partial results gives a majority for B (in the paper it is less accurately said that all partial results correctly predict the winner). The number $a_{n,m}$ of such two-element lattice permutations was determined in 1887 by J. Bertrand [see P. A. MacMahon, *Combinatory analysis*, Vol. I, Section III, Chapter V. Cambridge Univ. Press, Cambridge, 1915] as

$$a_{n,m} = (n+1-m)(n+1)^{-1}\binom{n+m}{m}.$$

The authors consider $a_n(x) = \sum_{m=0}^{n} a_{n,m} x^m$ and show that this nth degree polynomial is characterized by the property that $(1-x)^{n+1}a_n(x)$ has the form $1 - xP_n(x(1-x))$, where P_n is again an nth degree polynomial; in fact, $P_n(u) = \sum_{m=0}^{n} a_{m,m} u^m$. A number of relations are derived for the generating function $a(x, y) = \sum_{n=0}^{\infty} x^n a_n(y)$.

The authors consider these formulas as the special cases $(q = 1)$ of a q-generalization. They define the nth degree polynomial $a_n(x, q)$ by the condition that there exist $c_0, \cdots, c_n$ such that

$$(x)_{n+1} a_n(x, q) = 1 - x \sum_{m=0}^{n} c_m (qx)^m (x)_m,$$

where $(x)_k$ denotes $(1-x)(1-qx)\cdots(1-q^{k-1}x)$. The coefficients $a_{n,m}(q)$ of this polynomial are studied in several ways.

{The reviewer remarks that $a_{n,m}(q)$ has the following combinatorial interpretation: It is the sum of all $q^{k_1+\cdots+k_m}$, where $k_1, \cdots, k_m$ are integers subject to the conditions $1 \leq k_1 \leq \cdots \leq k_m \leq n$, $k_1 \geq 1, \cdots, k_m \geq m$.}

N. G. de Bruijn (Eindhoven)

Carlitz, L. 5753

The generating function for $\max(n_1, \cdots, n_k)$.

Portugal. Math. **21** (1962), 201–207.

For $k \geq 1$ and $1 \leq r \leq k$, define M_r^k (briefly, M_r) to be the k-place function that selects the rth smallest member from a given set of k real numbers. Thus $M_1 = \min$, $M_k = \max$, while for $k = 3$, M_2 coincides with the function sometimes called "mid". The author evaluates the (formal) multiple power series

$$\mathfrak{M}_r^k = \sum_{n_1,\cdots,n_k=0}^{\infty} M_r(n_1, \cdots, n_k) x_1^{n_1} \cdots x_k^{n_k}$$

in closed form. To illustrate, we give the closed expressions for $(1-x_1)(1-x_2)(1-x_3)\mathfrak{M}_r^3$, $r = 1, 2, 3$. These are, respectively,

$$\frac{x_1x_2x_3}{1-x_1x_2x_3} \qquad (r = 1);$$

$$\frac{x_1x_2}{1-x_1x_2} + \frac{x_2x_3}{1-x_2x_3} + \frac{x_3x_1}{1-x_3x_1} - 2\frac{x_1x_2x_3}{1-x_1x_2x_3} \qquad (r = 2);$$

$$\frac{x_1}{1-x_1} + \frac{x_2}{1-x_2} + \frac{x_3}{1-x_3} - \frac{x_1x_2}{1-x_1x_2} - \frac{x_2x_3}{1-x_2x_3} - \frac{x_3x_1}{1-x_3x_1} + \frac{x_1x_2x_3}{1-x_1x_2x_3} \qquad (r = 3).$$

A. Sklar (Chicago, Ill.)

Gould, H. W. 5754

A binomial identity of Greenwood and Gleason.

Math. Student **29** (1961), 53–57.

Verfasser beweist mit einfachen kombinatorischen Methoden eine Verallgemeinerung der im Titel genannten Identität [Greenwood und Gleason, Canad. J. Math. **7** (1955), 1–7; MR **16**, 733], welche für beliebiges reelles x und für nichtnegative ganze Zahlen j, n die Gestalt

$$\sum_{k=0}^{j} \binom{x-n}{k}\binom{n}{j-k}\binom{x-k}{n} = \sum_{k=0}^{j} \binom{n}{k}\binom{x-n}{j-k}\binom{x-k}{n-k}$$

annimmt. *H. J. Kanold* (Zbl **103**, 249)

Chowla, S. 5755

On a restricted random walk.

Norske Vid. Selsk. Forh. (*Trondheim*) **37** (1964), 91–94.

The random walk is of unit steps, right or left, on a line, with no steps to the left of the start point. It is shown that g_{nk}, the number of such walks with n steps and final point k, is given by

$$g_{n,n-2k} = \binom{n}{k} - \binom{n}{k-1}, \qquad k = 0, 1, \cdots, m,$$

with m the integral part of $\frac{1}{2}n$ ($g_{n,n-2k-1} = 0$). As an immediate consequence, the total number with n steps, the sum of the g_{nk}, is $\binom{n}{m}$, a result also obtained (without specific identification in present terms) by Ora Engelberg [J. Appl. Probability **1** (1964), 168–172; MR **28** #4561]. It is interesting to notice that $g_{n,n-2k} = a_{n-k,k}$, where a_{nm} is the number of lattice permutations with n elements of one kind, m of a second, and $n \geq m$ (the numbers associated with the simplest ballot problem), a mapping which gives the main result at once. *J. Riordan* (Murray Hill, N.J.)

Erdős, P.; Moser, L. 5756

On the representation of directed graphs as unions of orderings. (**Russian summary**)

Magyar Tud. Akad. Mat. Kutató Int. Közl. **9** (1964), 125–132.

If m voters each rank n candidates in order of preference, an oriented graph may be formed in which an arc is oriented from vertex i to vertex j if and only if candidate i is ranked above candidate j by more than half the voters. Let $m(n)$ denote the smallest number of voters for which it is possible to obtain every oriented graph with n vertices in this way. Stearns [Amer. Math. Monthly **66** (1959), 761–763; MR **21** #7799] showed that $m(n) > c_1 n/\log n$, and here it is shown that $m(n) < c_2 n/\log n$, where the c_i denote constants.

It is also shown that if $f(n)$ denotes the largest integer t such that every tournament with n vertices contains a transitive subtournament with t vertices, then $[\log_2 n] + 1 \leq f(n) \leq 2[\log_2 n] + 1$.

J. W. Moon (Edmonton, Alta.)

ORDER, LATTICES

See also 5752, 5767, 5768, 5866, 5867, 5973, 6280, 6289, 6294, 6462, 6661.

Fillmore, Peter A. 5757

An Archimedean property of cardinal algebras.

Michigan Math. J. **11** (1964), 365–367.

The concept of cardinal algebra has been introduced and investigated by A. Tarski [*Cardinal algebras*, Oxford Univ. Press, New York, 1949; MR **10**, 686]. A cardinal algebra A can be partially ordered by defining $a \leq b$ to mean $a + x = b$ for some $x \in A$. The author proves that if $na \leq (n+1)b$ holds for all $n < \infty$ ($a, b \in A$), then $a \leq b$.

E. T. Schmidt (Budapest)

Perles, Micha A. 5758

A proof of Dilworth's decomposition theorem for partially ordered sets.

Israel J. Math. **1** (1963), 105–107.

A theorem of Dilworth's states that if P is a partially ordered set in which the maximal number of elements in an independent subset is k, then P is the union of k chains. The author gives a simple inductive proof of this theorem when $|P|$ is finite and indicates how the general case can be deduced from the finite case.

J. McLaughlin (Ann Arbor, Mich.)

Perles, Micha A. 5759

On Dilworth's theorem in the infinite case.

Israel J. Math. **1** (1963), 108–109.

In answer to a question attributed to P. Erdős, the author exhibits, for each infinite cardinal $\aleph_\alpha$, a partially ordered set T_α such that (1) $|T| = \aleph_\alpha$, (2) T_α contains no infinite independent subset, (3) T_α is not decomposable into less than $\aleph_\alpha$ chains. This shows the impossibility of extending Dilworth's theorem [#5758 above] to the case where the cardinals of the independent subsets are not bounded.

J. McLaughlin (Ann Arbor, Mich.)

Holland, Samuel S., Jr. 5760

Distributivity and perspectivity in orthomodular lattices.

Trans. Amer. Math. Soc. **112** (1964), 330–343.

A lattice L with 0 and 1 is "orthocomplemented" if L admits an order-reversing involution $a \to a^{\perp}$ such that $a^{\perp}$ is a complement of a for every a, and is "orthomodular" if $b = a \vee (b \wedge a^{\perp})$ for every $b \geqq a$. The lattice of projections in a von Neumann algebra is not modular, in general, but is always orthomodular, and the general question raised here is, roughly, whether it is possible to erect a structure theory of complete orthocomplemented lattices that will generalize projection lattices in the manner in which the theory of continuous geometries englobes the structure theory of the lattice of projections of a finite von Neumann algebra. By way of answer the author examines nine theorems that are valid in the lattice projections of an arbitrary von Neumann algebra (as well as in an arbitrary continuous geometry) and shows that two of them (when suitably reformulated) remain valid in a complete orthocomplemented lattice, but that the other seven do not. The more interesting of the two results that do generalize concerns the basic problem of adding perspectivities and goes as follows. Write $a \perp b$ if $b \leq a^{\perp}$ and $a \sim b$ if there exists an x such that $a \wedge x = b \wedge x = 0$ and $a \vee x = b \vee x = a \vee b$ ("strong perspectivity"). Let $\{a_\alpha\}$ and $\{b_\alpha\}$ be similarly indexed families in a complete orthocomplemented lattice L and suppose (1) $a_\alpha \vee b_\alpha \perp a_\beta \vee b_\beta$ for $\alpha \neq \beta$, and (2) $a_\alpha \sim b_\alpha$ for all α. Then $\bigvee_\alpha a_\alpha \sim \bigvee_\alpha b_\alpha$. *A. Brown* (Ann Arbor, Mich.)

Grätzer, G. 5761

Boolean functions on distributive lattices.

Acta Math. Acad. Sci. Hungar. **15** (1964), 195–201.

The author extends the notion of a Boolean function on a Boolean algebra, which was investigated in an earlier paper of the author [Rev. Math. Pures Appl. 7 (1962), 693–697], to a Boolean function on a distributive lattice L (with 0 and 1). A Boolean function on L is defined by the property that it enjoys the substitution property under any congruence relation on L. With every Boolean function $f(x_1, x_2, \cdots, x_n)$ there is associated a characteristic function φ of 2^n into L, and it is shown that φ determines f uniquely. Necessary and sufficient conditions are stated in order that a function $\varphi: 2^n \to L$ be a characteristic function. From this result several other results are derived. A Boolean function f is a lattice polynomial if and only if its characteristic function φ is monotone, and if and only if L contains no proper Boolean interval. It is also shown that a Boolean function f on a distributive lattice L with 0 and 1 can be uniquely extended to any Boolean algebra B that contains L as a sublattice. The author also investigates those Boolean functions which are automorphisms or dual-automorphisms. If $f(x)$ is a Boolean function on L, then the correspondence $x \to f(x)$ is an automorphism if and only if $f(x) = x$. *Ph. Dwinger* (Delft)

Karolinskaja, L. N. 5762

Congruence lattices on distributive lattices. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 1037–1054.

A class K of universal algebras is said to be abstract if $A \in K$ and $A \cong B$ imply $B \in K$. The author gives a characterization of congruence lattices $\Theta(L)$ for a given abstract class of distributive lattices L. These $\Theta(L)$ are characterized by the property of containing a sublattice C satisfying certain conditions. These conditions are satisfied by the sublattice C of all congruence relations $\Theta(I_a)$, defined as the smallest congruence relation under which the principal ideal I_a generated by $a \in L$ is a congruence class. Then L can be recognized as isomorphic to C. {Reviewer's remark: The condition (III) in Theorem 1 may be omitted.} *E. T. Schmidt* (Budapest)

van Albada, P. J. 5763

A self-dual system of axioms for Boolean algebra.

Nederl. Akad. Wetensch. Proc. Ser. A **67** = *Indag. Math.* **26** (1964), 377–381.

The author shows that a class B which (i) contains elements 0, 1 and a dual a' to any element a, (ii) is closed under addition and multiplication, and (iii) satisfies the (demonstrably independent) axioms

$$a(b+c) = ab+ac, \qquad (b+c)a = ba+ca,$$

$$a+bc = (a+b)(a+c), \qquad a+0 = 0+a = a,$$

$$aa' = 0, \qquad a1 = a, \qquad a+a' = 1$$

for all elements a, b, c, is a Boolean algebra. Thus, a Newman algebra is a Boolean algebra if and only if $a+bc = (a+b)(a+c)$ for all elements.

R. L. Goodstein (Leicester)

Fenstad, Jens Erik 5764

On representation of polyadic algebras.

Norske Vid. Selsk. Forh. (*Trondheim*) **37** (1964), 36–41.

A relatively short new proof of the well-known representation theorem for locally finite polyadic algebras is given. The paper is almost self-contained and includes a discussion of the equivalence of the representation theorem and the Gödel completeness theorem.

H. J. Keisler (Madison, Wis.)

Gaifman, H. 5765

Infinite Boolean polynomials. I.

Fund. Math. **54** (1964), 229–250.

The summary of results given in the Introduction to this important paper can hardly be improved by the reviewer. "This work treats Boolean polynomials (on a fixed set of variables) in which the operations of join, $\bigvee$, and meet, $\bigwedge$, may apply to sets of arbitrary high powers. Thus if X is any set of such polynomials, then $\bigvee X$ and $\bigwedge X$ are Boolean polynomials as well. If the variables of such a polynomial are given values in a complete Boolean algebra, then the polynomial has a value in this Boolean algebra, which is defined in a natural way, and two polynomials which always yield the same values for the same variables are identified. The main result of Part I is that the Boolean polynomials in $\aleph_0$ variables do not constitute a set."

Boolean polynomials are defined using Bernays set theory without the axiom of choice. The properties of the Boolean polynomials mentioned above are established, and the principal result that the Boolean polynomials in $\aleph_0$ variables do not form a set is proved. From this it follows directly that there is no completely free Boolean algebra with $\aleph_0$ generators. This corollary provides a negative answer to a question posed by L. S. Rieger [Fund. Math. 38 (1951), 35–52; MR 14, 347; errata,

MR 14, p. 1278]. The same result is proved more generally for the class of complete, (δ, ∞)-distributive Boolean algebras, generated by δ elements, where δ is any infinite regular cardinal. These results have been obtained independently using different methods by A. W. Hales [ibid. 54 (1964), 45–66; MR 29 #1162]. The results on Boolean polynomials are also used to prove theorems on the power of the free α-complete Boolean with β generators; for example, if B is such an algebra then the cardinality of B is at least $\max\{\alpha, \beta\}$. *R. S. Pierce* (Seattle, Wash.)

Kluvanek, Igor' [Kluvánek, Igor]; Riečan, Beloslav 5766

Some properties of Bernoulli schemata. (Russian. English summary)

Mat.-Fyz. Časopis Sloven. Akad. Vied **14** (1964), 83–88.

Die Verfasser definieren eine Invariante in einer Booleschen σ-Algebra mit einem gegebenen Automorphismus. Es sei $A(n)$ die Algebra von allen Teilmengen einer Menge mit n Elementen, in der ein Mass definiert ist, welches den Wert $1/n$ für alle Mengen, die nur einen Punkt besitzen, erreicht. In einem gewissen cartesischen Produkt $P(n)$ von abzählbar vielen Exemplaren der Algebra $A(n)$ wird ein Automorphismus auf eine natürliche Weise definiert. Mit Hilfe der eingeführten Invariante wird gezeigt, dass für $n \neq m$ die Algebren $P(n)$, $P(m)$ nicht isomorph sind. Ähnliche Resultate sind auch für gewisse Quotientenalgebren von $P(n)$ in Kraft, welche die Verfasser Bernoullische dynamische Systeme nennen.

M. Novotný (Brno)

GENERAL MATHEMATICAL SYSTEMS

See also 5757, 5764, 5876, 5887.

Andreoli, Giulio 5767

Catene booleane e markowiane di partizioni; tassinomie ed algoritmi sugli alberi.

Ricerca (Napoli) (2) **13** (1962), *settembre–dicembre*, 3–12.

A given set is partitioned into pairwise disjoint subsets, and this initial partitioning is iterated for the subsets, altogether an indexed number of times. The final class of subsets, together with the intermediate stages, can be regarded as a hierarchy of Boolean orthonormal algebras under the usual operations. But the process also provides what is known as a "tree" in the literature of finite mathematics. In the set-theoretic interpretation certain measure-theoretic concepts are applicable. If each subset within a partition is assigned a measure (say area), then the chain of partitions created can be typical of a Markov chain. A given sequence (chain) of successive partitions

$$\Pi^{(0)}, \Pi_r', \Pi_{rs}'', \Pi_{rst}''', \cdots, \Pi^{(\nu)}$$

is called by the author a "taxonomy" (tassinomia). For the tree analogy, an intermediate or final subset determines an oriented path through the taxonomy.

There is defined a "product" of two partitions of a set (the totality of intersection sets) and through this a product of two taxonomies over the same initial set. Two partitions are called independent if the measure of the Boolean (set) product is the numerical product of the subset measures; the paper deals only with independent partitions. Some algorithmic properties are developed, as well as a sum operation and an exploration of lattice properties under the suitable bound definitions. Taxonomies over two initial sets are handled via Cartesian product. The paper is exploratory and vaguely suggestive of eventual application to stochastic processes.

I. Sussman (Santa Clara, Calif.)

Andreoli, Giulio 5768

Algoritmi booleani subordinati ad algoritmi numerici conversi. Algebre di Frobenius-Boole.

Ricerca (Napoli) (2) **14** (1963), *gennaio–aprile*, 3–9.

A hypercomplex system in which the usual operator ring or field is replaced by a Boolean algebra, with the structure constants belonging to the minimal (zero-one) algebra, is identified as a "Frobenius algebra" if, when $A_rA_s = A_t$ for the underlying group, the constants $\gamma_{rs\rho}$ are unity for $\rho = t$ and zero otherwise. A generalization introduced is to allow $u_ru_s = \sum \gamma_{rs\rho} u_\rho$, where the ρ take on exclusively zero and unit values.

A primitive Frobenius-Boolean algebra is essentially an n-atomic Boolean algebra whose atoms are the basis elements, and any Frobenius algebra is a direct sum of such primitive algebras. An algebra is "convex" if in any basis the structure constants are non-negative. Those Frobenius algebras whose basis elements are precisely the elements of the underlying group are convex algebras. The Boolean algebra of n atoms is a particular case of a transition algebra; and in the opposite sense a transition algebra is a convex algebra in which the coefficient set is related to a Boolean algebra.

I. Sussman (Santa Clara, Calif.)

Robert, Pierre 5769

Théorie générale de la disjonction.

C. R. Acad. Sci. Paris **258** (1964), 34–37.

The author studies disjoint subsets of a φ-space [N. Bourbaki, *Les structures fondamentales de l'analyse*, Livre I, Théorie des ensembles, Chapitre III, § 4, no. 5; ibid. Chapitre III, § 3, no. 3; Actualités Sci. Indust., No. 1243, Hermann, Paris, 1956; MR 17, 1062]. He generalizes the notion of linear or algebraic disjointness of two sets. The paper is a statement of results, detailed proofs to be published at a later date. *S. Rubinstein* (Seattle, Wash.)

Dlab, V. 5770

A generalization of dependence relations.

Proc. Colloq. Abelian Groups (Tihany, 1963), pp. 49–50. *Akadémiai Kiadó, Budapest*, 1964.

Apart from an exposition of an earlier paper [Publ. Math. Debrecen **9** (1962), 324–355; MR 27 #81], the author gives without proof the following result. Let δ be a generalized algebraic dependence relation for S. Then there exist a set L and a one-to-one mapping ϕ of $\mathfrak{P}(S)$ onto $\mathfrak{P}(L)$ such that

$$X_1 \subseteq X_2 \quad \text{if and only if} \quad \phi(X_1) \subseteq \phi(X_2)$$

and

$$[x, X] \in \delta \quad \text{if and only if} \quad (\phi(x)) \cap \phi(X) \neq \varnothing.$$

O. Pretzel (Berlin)

Foster, Alfred L.; Pixley, Alden F. 5771
Semi-categorical algebras. II.
Math. Z. **85** (1964), 169–184.

Part I appeared earlier [same Z. **83** (1964), 147–169; MR **29** #1165]. As in Part I, the authors consider algebras of arbitrary but fixed species $S=(n_1, n_2, \cdots)$. At first, semi-primal algebras are characterized in terms of properties of the lattice of congruence relations. Notation: With each set $\mathfrak{U}$ of S-identities is associated the equational class $C(\mathfrak{U})$ of all algebras of species S satisfying the members of $\mathfrak{U}$. The members of $C(\mathfrak{U})$ are called $\mathfrak{U}$-algebras or, if $\mathfrak{U}$ consists of the identities $|\mathfrak{A}|$ satisfied by an algebra $\mathfrak{A}$, $\mathfrak{A}$-algebras. When all of the algebras of $C(\mathfrak{U})$ have permutable congruence relations [a distributive lattice of congruence relations], then $\mathfrak{U}$ is said to be a permutable [distributive] set of identities. Theorem 3.1: An algebra $\mathfrak{A}$ is semi-primal if and only if (a) $\mathfrak{A}$ is finite, having at least two elements; (b) The subalgebras of $\mathfrak{A}$ are simple and have no proper automorphisms and no two distinct subalgebras of more than one element are isomorphic; (c) The identities $|\mathfrak{A}|$ are both permutable and distributive.

While every subalgebra of a semi-primal algebra is simple, this is not generally true of semi-categorical algebras. This leads to the definition: $\mathfrak{A}$ is quasi-simple if all homomorphic images of subalgebras of $\mathfrak{A}$ are isomorphic with subdirect products of subalgebras of $\mathfrak{A}$. Theorem 5.4 (general representation theorem): If $\mathfrak{A}$ is a finite algebra having at least two elements and satisfying (a) $\mathfrak{A}$ is quasi-simple, and (b) $|\mathfrak{A}|$ are distributive identities, then $\mathfrak{A}$ is semi-categorical. Theorem 6.1 (on direct product representations): If $\mathfrak{A}$ is a semi-categorical algebra having permutable identities and having no simple subalgebras, then every finitely generated $\mathfrak{A}$-algebra is isomorphic with the direct product of subalgebras of $\mathfrak{A}$. There is also a partial converse of this theorem. Applications to semi-categorical lattices and to commutative rings.

H.-J. Hoehnke (Berlin)

Grätzer, G. 5772
On the class of subdirect powers of a finite algebra.
Acta Sci. Math. (*Szeged*) **25** (1964), 160–168.

Let $\mathfrak{A}$ be a finite algebra with a finite number of finitary operations, and let $\mathrm{Sp}(\mathfrak{A})$ denote the class of sub-direct products of copies of $\mathfrak{A}$. The author gives a sufficient condition for $\mathrm{Sp}(\mathfrak{A})$ to be an elementary class (defined by a single closed sentence). The condition is too complicated to give here in detail, but it may be stated roughly as follows: For some positive integer N, every finite system of equations with no solution in $\mathfrak{A}$ entails a similar system with at most N unknowns. The chain of length n, considered as a lattice, satisfies this condition with $N=n$, whence the class of n-bounded lattices is elementary, a result of Anderson and Blair [Math. Ann. **143** (1961), 187–211; MR **25** #2988]. The theorem also covers several other well-known examples and carries over to relational systems with minor changes. The author gives an example to show that $\mathrm{Sp}(\mathfrak{A})$ is not always elementary and concludes by listing a number of related problems.

P. J. Higgins (London)

Janov, Ju. I. 5773
Systems of identities for algebras. (Russian)
Problemy Kibernet. No. 8 (1962), 75–90.

Let $F(X)$ be the free algebra of some finitary species $(n_1, \cdots, n_m)$, $m<\aleph_0$, X being the set of free generators, card $X=\aleph_0$. For any binary relation Σ on $F(X)$ let its closure $R\Sigma$ be the least congruence such that $f\Sigma g$ ($f, g \in F(X)$) implies $\eta(f)R\Sigma\eta(g)$ for any endomorphism $\eta: F(X)\to F(X)$. For any algebra A of the given species define $\Sigma(A)$ on $F(X)$ by putting $f\Sigma(A)g$ if and only if $\vartheta(f)=\vartheta(g)$ for any homomorphism $\vartheta: F(X)\to A$. The existence of a finite A with $R\Delta\neq\Sigma(A)$ for all finite $\Delta\subset\Sigma(A)$ has been proved by R. C. Lyndon [Proc. Amer. Math. Soc. **5** (1954), 8–9; MR **15**, 676]. The closure operation R satisfying $\Sigma\subset R\Sigma=RR\Sigma$ and $R\Sigma(A)=\Sigma(A)$ is replaced by the author by S_n ($n=1, 2, \cdots$) defined in the following manner. Consider all mappings $\sigma: X\to X$ with card $\sigma(X)\leq n$ and their extensions to homomorphisms $\bar\sigma: F(X)\to F(X)$. Define $fS_n\Sigma g$ if and only if $\bar\sigma(f)R\Sigma\bar\sigma(g)$ for all $\bar\sigma$'s described above. Hence $R\Sigma\subset S_n\Sigma$, and since $S_nS_n\Sigma=S_n\Sigma$, we have $RS_n\Sigma=S_nR\Sigma=S_n\Sigma$ for any natural n. The author proves that for any finite A with card $A=n$ there exists a finite $\Delta\subset\Sigma(A)$ with $S_n\Delta=\Sigma(A)$.

The paper also concerns some other questions. Equationally complete Σ's are considered (i.e., those for which $R\Sigma$ is maximal) and for this property a necessary and sufficient condition is found. Next, some conditions are given for A under which $\Sigma(A)\subset\Sigma(A')\neq F(X)\times F(X)$ for some algebra A' with $\Sigma(A')=R\Delta$ for some finite Δ.

The author uses the language of formulae, identities, etc., rather than the language of free algebras and homomorphisms used here by the reviewer for brevity. For R and S_n intrinsic definitions are given.

K. Drbohlav (Prague)

THEORY OF NUMBERS

See also 5723, 5730, 5821, 5824, 5926, 6061, 6062, 6203, 6204, 6395, 6624.

Bojarincev, A. E. 5774
Some generalizations of Wilson's theorem. (Russian)
Ural. Gos. Univ. Mat. Zap. **3**, *tetrad* 3, 21–24 (1962).

It is well known that Wilson's congruence, $(p-1)!\equiv -1 \pmod p$, holds if and only if $p>1$ is prime. The author proves these generalizations. If $m>0$ is an integer, and $p\geq m$, then p is prime if and only if $(m-1)!(p-m)!\equiv(-1)^m \pmod p$. If also n is a positive integer, then p is prime if and only if

$$\frac{(m-1)!(np-m)!}{(n-1)!p^{n-1}}\equiv-(-1)^{m+n} \pmod p.$$

If $n>0$ is even, and $p\nmid n$, then p and $p+n$ are both prime if and only if $[(n!-1)p-n](p-1)!\equiv n \pmod{p(p+n)}$.

W. J. LeVeque (Ann Arbor, Mich.)

Brown, J. L., Jr. 5775
Generalized bases for the integers.
Amer. Math. Monthly **71** (1964), 973–980.

The author proves the following theorem: Let f_i, k_i, m_i ($i=1, 2, 3, \cdots$) be integers, the f_i being positive and the k_i and m_i being non-negative. Any integer n such that $-\sum_1^N m_if_i\leq n\leq\sum_1^N k_if_i$ can be expressed uniquely in the form $n=\sum_1^N \alpha_if_i$, where each α_i is an integer satisfying

$-m_i \leq a_i \leq f_i$ if and only if $f_{s+1} = 1 + \sum_1^s (k_i + m_i) f_i$. A number of interesting special cases are studied.

N. S. Mendelsohn (Winnipeg, Man.)

Carlitz, L. 5776

Note on a q-identity.

Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 78–83 (1962), 19–20.

By expanding each side into a power series and equating coefficients, the author shows that the identity

$$\sum_{r=1}^{\infty} q^{2r} \frac{1+q^{4r-2}}{(1-q^{4r-2})^2} = \left(\sum_{r=1}^{\infty} \frac{q^r}{1+q^{2r-1}}\right)^2$$

of S. Fempl [see #6069 below] leads to the known number-theoretic identity

$$\sigma(n) = \sum_{\substack{r+s=2n \\ r,s\geq 1}} \rho(2r-1)\rho(2s-1),$$

where $\sigma(n)$ is the sum of the divisors of n and $\rho(2n-1) = \sum_{d|2n-1} (-1)^{(d-1)/2}$.

A. Sklar (Chicago, Ill.)

Vorobyov, N. N. [Vorob'ev, N. N.] 5777

★The Fibonacci numbers.

Translated and adapted from the first Russian edition (1951) by Norman D. Whaland, Jr., and Olga A. Titelbaum. Survey of Recent East European Mathematical Literature, a project conducted by Alfred L. Putnam and Izaak Wirszup. Topics in Mathematics.

D. C. Heath and Co., Boston, Mass., 1963. v+47 pp. \$1.40.

This is one of a series of booklets "Topics in Mathematics" translated and adapted from the Russian series of Popular Lectures in Mathematics. As stated, "the purpose of these booklets is to introduce the reader to various aspects of mathematical thought and to engage him in mathematical activity of a kind that fosters habits leading to independent creative work. The series will make available to students of mathematics at various levels, as well as to other interested readers, valuable supplementary material to further their mathematical knowledge and development." "Each booklet is largely self-contained."

This particular booklet is elementary in nature, and can be interesting to people with two years of high-school algebra: students, as well as those who work mathematical problems for recreation. It deals with the elementary properties of Fibonacci numbers, their applications to geometry, and their connection with the theory of continued fractions.

Specifically, the contents are as follows: A historical introduction. Chapter 1, simplest properties of Fibonacci numbers—sums, sums of squares and other sums, connections with binomial coefficients, the Binet formula, sums of related series, computing Fibonacci numbers with large indices. Chapter 2, number-theoretical properties of the Fibonacci numbers—euclidean algorithm, commensurable and incommensurable line segments, properties of greatest common divisors, divisibility properties of Fibonacci numbers. Chapter 3, the Fibonacci numbers and continued fractions—definitions of continued fractions, convergents, particular continued fractions, Fibonacci numbers and the euclidean algorithm, continued fraction expansion of real numbers. Chapter 4, the Fibonacci numbers and geometry—mean proportional, the Golden Section, regular decagons, Golden Rectangles.

On the whole, the translators have rendered a valuable service in making the contents of this booklet accessible to English-speaking students.

E. Frank (Chicago, Ill.)

Ehrhart, Eugène 5778

Sur un problème de partition d'un nombre.

C. R. Acad. Sci. Paris **259** (1964), 2746–2747.

Let $a_1 = 2, \cdots, a_{16} = 53$ be the first 16 primes, and write C_n for the number of solutions of the equation $\sum_{i=1}^{16} a_i X_i = 0$ in integers $X_i \geq 0$. There is a polynomial $P(n)$ of degree 15 such that C_n is the nearest integer to $P(n)$.

B. J. Birch (Manchester)

Ehrhart, Eugène 5779

Sur la partition des nombres.

C. R. Acad. Sci. Paris **259** (1964), 3151–3153.

Author's summary: "The functions C_n, C_n' which count, respectively, the non-negative and the positive solutions of the diophantine equation $\sum_{i=1}^{k} a_i X_i = n$ are related by $C_n' = (-1)^{k-1} C_{-n}$. Application: If the a's are coprime by pairs, then $C_n = P(n) + \psi(n)$, where $\psi(n)$ is periodic; $P(n)$ can now be given a relatively concise form."

{The reviewer presumes that the a's are positive integers and that, for $-n<0$, C_{-n} is defined to be $P(-n)+\psi(-n)$; owing to the lack of a definition for C_{-n}, the proof appears incomplete, though the results are essentially correct.}

B. J. Birch (Manchester)

Inkeri, K. 5780

On Catalan's problem.

Acta Arith. **9** (1964), 285–290.

Catalan's equation $x^p - y^q = 1$ is discussed for the case p, q prime, $p>3$, $q>3$, $p\equiv 3 \pmod 4$, and q does not divide the class number $h(p)$ of the quadratic field $k(\sqrt{-p})$. If these conditions hold and if there is a non-zero integral solution x, y, then it is shown that $p^q \equiv p \pmod{q^2}$, $x \equiv 0 \pmod{q^2}$, and $y \equiv -1 \pmod{q^{2p-1}}$. If we also have $p>q$ and $q\equiv 3 \pmod 4$, then the roles of p and q can be reversed to obtain additional conditions. These results imply that Catalan's equation is unsolvable for a large number of pairs p, q.

W. H. Mills (Princeton, N.J.)

Pignataro, Salvatore 5781

Una osservazione sull'ultimo teorema di Fermat. (English summary)

Rend. Accad. Sci. Fis. Mat. Napoli (4) **30** (1963), 281–286.

On sait que le dernier (ou grand théorème) de Fermat s'énonce comme suit : L'équation diophantienne

$$(1) \qquad x^n = y^n + z^n,$$

n entier >2, est impossible en entiers rationnels tous $\neq 0$. On sait que ce théorème n'a pas encore été démontré pour tout nombre premier n.

Dans cette note, l'auteur établit un critère pour l'existence éventuelle de solutions de l'équation (1) en entiers positifs tous $\neq 0$. Il y expose en effet une méthode ramenant la recherche de solutions éventuelles de (1) à celle des solutions représentant un certain nombre impair

positif moyennant une forme canonique (quadratique binaire), m étant un nombre positif impair, il s'agit d'une représentation de la forme

$$(2) \qquad m = \xi x^2 - \eta y^2.$$

L'auteur note cette forme $(\xi, -\eta)$, où ξ et η sont supposés premiers entre eux, et où x et y sont des entiers positifs premiers entre eux et respectivement premiers avec η et ξ. Il démontre le théorème suivant : Si un nombre positif impair est représentable par (2), alors il est représentable, d'une infinité de façons, par des nombres positifs qui peuvent être rangés, à partir de deux solutions minimales, en deux suites de nombres croissants. Moyennant ce théorème, il arrive à un critère pour l'existence éventuelle d'une solution (ξ^k, η^k, ζ^k), qui représente ζ^{2k+1} par la forme $(\xi, -\eta)$, $2k+1 \geq 3$, ξ et η étant supposés de parité différente, donc ζ impair (ξ^k, η^k, ζ^k) étant une solution primitive en entiers positifs de l'équation (1) que nous écrirons sous la forme

$$(1') \qquad x^n - y^n = z^n,$$

$$(1'') \qquad \xi^{2k+1} - \eta^{2k+1} = \zeta^{2k+1}.$$

(1″) peut évidemment s'écrire sous la forme $\xi(\xi^k)^2 - \eta(\eta^k)^2 = \zeta^{2k+1}$ d'où il résulte que (ξ^k, η^k) est une solution représentant ζ^{2k+1} moyennant la forme canonique $\xi x^2 - \eta y^2$.

A propos de l'équation $\xi x^2 - \eta y^2 = \zeta^{2k+1}$, il est connu, d'après la théorie des formes quadratiques, que le produit $\xi\eta$ doit être un résidu quadratique de ζ^{2k+1}, et partant, de ζ. L'auteur démontre finalement que, s'il existe une solution ξ^k, η^k de l'équation $\xi x^2 - \eta y^2 = \zeta^{2k+1}$, (ξ^k, η^k) doit être la solution minimale absolue parmi l'infinité de solutions représentant ζ^{2k+1} par cette forme. Ainsi le critère de l'auteur réduit le problème de Fermat à la détermination d'une méthode donnant la solution minimale absolue pour tout nombre impair ζ^{2k+1} représentable par une certaine forme canonique, quadratique binaire $(\zeta, -\eta)$ pour laquelle le produit $\xi\eta$ est un résidu quadratique de ζ, où x et y désignent toujours deux entiers positifs premiers entre eux et premiers respectivement, avec η et ξ. *A. Gloden* (Luxembourg)

Postnikova, L. P. 5782

Distribution of solutions of the congruence

$$x^2 + y^2 \equiv 1 \pmod{p^n}.$$

(Russian)

Mat. Sb. (*N.S.*) **65 (107)** (1964), 228–238.

Let p be a prime >3, n any natural number ≥ 13, and let $A(T, T_1)$ denote the number of solutions of the congruence $x^2 + y^2 \equiv 1 \pmod{p^n}$ such that $y \not\equiv \pm 1 \pmod{p}$, $0 \leq x \leq T$, $0 \leq y \leq T_1$. It is proved that for $1 \leq T \leq p^n$ and $p^{(4n+12)/6} \leq T_1 \leq p^n$

$$A(T, T_1) - TT_1 p^{-n-1}\{p - 2 - (-1)^{(p-1)/2}\} \ll e^{7n \log^2 n} T_1^{1-1/(84n^2 \log 3n)}.$$

The proof is a combination of I. M. Vinogradov's method with an elementary form of p-adic analysis, no knowledge of the latter being presupposed.

{There are plenty of misprints and some minor inaccuracies such as $12(n-1)^2 \geq 15n^2$ on p. 236, but they can be eliminated. A reference to Vinogradov's theorem [*Selected works* (Russian), p. 369, Izdat. Akad. Nauk SSSR, Moscow, 1952; MR **14**, 610] (where there are puzzling misprints) should be replaced by one to his original paper [Izv. Akad. Nauk SSSR Ser. Mat. **14** (1950), 199–214, p. 209; MR **12**, 161].} *E. Fogels* (Riga)

Rumsey, Howard, Jr.; Posner, Edward C. 5783

On a class of exponential equations.

Proc. Amer. Math. Soc. **15** (1964), 974–978.

The purpose of the paper is the proof of the following theorem. Let a be an odd integer or $a=2$, and let q_s $(1 \leq s \leq j)$ and r_t $(1 \leq t \leq k)$ be $j+k$ distinct primes, all prime to a. Then the equation $a^x = B + C$ $(B = q_1^{y_1} \cdots q_j^{y_j}$, $C = r_1^{z_1} \cdots r_k^{z_k})$ has only a finite number of solutions in non-negative integers x; $y_1, \cdots, y_j$; $z_1, \cdots, z_k$. All such solutions may be found in a finite number of steps. The fact that the number of solutions is finite follows from a more general statement in Gel'fond [*Transcendental and algebraic numbers* (Russian), GITTL, Moscow, 1952; MR **15**, 292; English transl., p. 37, Dover, New York, 1960; MR **22** #2598]. The fact that all solutions may be found constructively seems new. The proof is based on the factorization of a^{2x} in the field $R((-BC)^{1/2})$ (R is the field of rationals) and follows readily from the following lemma. Let D be a positive integer, H an algebraic integer in $R((-D)^{1/2})$, not a unit, u a unit in $R((-D)^{1/2})$ and $(p_1, p_2, \cdots, p_m)$ a fixed set of rational primes, all relatively prime to H. Then there exist effectively computable constants $X_1, \cdots, X_m$ such that $x_i \geq X_i$ $(1 \leq i \leq m)$ implies that $uH^r - \bar{u}\bar{H}^r = p_1^{x_1} p_2^{x_2} \cdots p_m^{x_m}(-D)^{1/2}$ has no solutions in non-negative integers r; $x_1, \cdots, x_m$. The proof of the lemma is obtained by considering separately the three particular cases $u = \pm 1$; $u = \pm i$, $D = 1$; $u = \frac{1}{2}(\pm 1 \pm i3^{1/2})$, $D = 3$.

{Misprints: in (7) replace i by 1, and reference [3] should read reference [2].} *E. Grosswald* (Paris)

Sprindžuk, V. G. 5784

On the measure of the set of S-numbers in a p-adic field. (Russian)

Dokl. Akad. Nauk SSSR **151** (1963), 1292.

Aus der Dissertation des Verfassers werden hier folgende Ergebnisse (ohne Beweis) angegeben, die sich auf das Analogon zur Mahlerschen Vermutung im p-adischen Fall [siehe F. Kasch und den Referent, Math. Z. **72** (1959/60), 367–368; MR **22** #A3713] beziehen: Für fast alle $\xi \in K_p$ ist $\theta_n(\xi) \leq \frac{3}{2}(1 + 1/(2n))$ $(n = 3, 4, 5, 6, 7)$, $\theta_n(\xi) \leq \frac{3}{2}n$ $(n \geq 8)$, $\theta_3(\xi) = \frac{4}{3}$. *B. Volkmann* (Stuttgart)

Schmidt, Asmus L. 5785

Approximation theorems of Borel and Fujiwara.

Math. Scand. **14** (1964), 35–38.

Another short proof of the following three well-known results. Let ξ be real, with the incomplete denominators a_n and the convergents p_n/q_n, where $n \geq 0$. Of the three consecutive convergents p_{n-1}/q_{n-1}, p_n/q_n, p_{n+1}/q_{n+1}, at least one satisfies

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^2\sqrt{5}};$$

and if $a_{n+1} \geq 2$, at least one satisfies

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^2\sqrt{8}},$$

while either p_n/q_n, or both of p_{n-1}/q_{n-1} and p_{n+1}/q_{n+1} satisfy

$$\left|\xi-\frac{p}{q}\right|<\frac{2}{5q^2}.$$

K. Mahler (Canberra)

Bernstein, Leon 5786

Periodische Jacobische Algorithmen für eine unendliche Klasse algebraischer Irrationalzahlen vom Grade n und einige unendliche Klassen kubischer Irrationalzahlen.

J. Reine Angew. Math. **214/215** (1964), 76–83.

In three recent papers [same J. **212** (1963/64), 31–38; MR **27** #5727; ibid. **213** (1963/64), 137–146; MR **29** #2434; Math. Nachr. (to appear)] the author studied periodic continued fractions for special nth roots. The present paper contains the following new results. (1) Let D, d, n be integers such that $n\geq 3$, $d|D$, $n-1\leq d\leq D/(n-1)$, and let $\omega=\sqrt[n]{(D^n+dD)}$, $f_s=f_s(\omega, D)=\sum_{i=0}^{s}\binom{n-1+i-s}{i}\omega^{s-i}D^i$ $(s=1,\ 2,\ \cdots,\ n-1)$, $f_0=f_0(\omega, D)=1$. Then the system of $n-1$ numbers f_1/d, f_2/d^2, $\cdots, f_{n-1}/d^{n-1}$ produces a pure periodic Jacobi algorithm with a primitive period of length n if $D\neq d^{n-1}$ and of length 1 if $D=d^{n-1}$. Explicit expressions for the form of this period are given here. (2) Let d, D be integers, $d|D$. If $\omega=\sqrt[3]{(D^3+dD)}$, then the numbers $(\omega+2D)/d$, $(\omega^2+\omega D+D^2)/d^2$ form a pure periodic Jacobi algorithm with a primitive period of length 3 or 1 according as $D\neq d^2$ or $D=d^2$, respectively. The form of the primitive period is given in this paper. *E. Frank* (Chicago, Ill.)

Bernstein, Leon 5787

Periodische Kettenbrüche beliebiger Periodenlänge.

Math. Z. **86** (1964), 128–135.

The following theorems are proved concerning the explicit form of two periodic continued fractions of arbitrarily prescribed length. Each continued fraction represents a given binomial quadratic surd. (1) Let a and n be integers, $N=\frac{1}{2}(n-1)$, and the integer

$$x_n=2^{-n+1}\sum_{i=0}^{N}\binom{n}{2i+1}a^{n-2i-1}(a^2+4)^i,\qquad n=1,2,\cdots.$$

Let d be an integer such that (A) $D=\frac{1}{2}(a+d\cdot x_{n+1})$ is integral. Then $\sqrt{(D^2+1+d\cdot x_n)}=[D,\overline{a,a,\cdots,a,2D}]$, a continued fraction with period of length $n+1$. A number d of the type required by (A) does not exist if a is odd and simultaneously $n+1$ is divisible by 3. (2) Let a and b be integers, $a|b$, and the integer

$$f_n(ab)=2^{-2n-1}\sum_{i=0}^{n}\binom{2n+2}{2i+1}(ab)^{n-i}(ab+4)^i,$$

$$n=0,1,\cdots.$$

Let d be an integer such that (B) $D=\frac{1}{2}[b+adf_n(ab)]$ is integral. Then

$$\sqrt{(D^2+2D:a-df_{n-1}(ab))}=[D,\overline{a,b,a,b,\cdots,a,b,a,2D}],$$

a continued fraction with period of length $2n+2$. A number d of the type required by (B) does not exist if b is odd and simultaneously $n+1$ is divisible by 3.

E. Frank (Chicago, Ill.)

Davenport, H. 5788

A remark on continued fractions.

Michigan Math. J. **11** (1964), 343–344.

It is shown that, for any irrational number θ and for any prime P, at least one of the numbers

$$P^2\theta,\ \theta,\ \theta+1/P,\ \cdots,\ \theta+(P-1)/P$$

has a simple continued fraction whose partial denominators a_n satisfy $a_n>P-2$ for infinitely many indices n.

W. T. Scott (Tempe, Ariz.)

Šalát, Tibor 5789

Eine metrische Eigenschaft der Cantorschen Entwicklungen der reellen Zahlen und Irrationalitätskriterien. (Russian summary)

Czechoslovak Math. J. **14** (89) (1964), 254–266.

Let $\{q_n\}$ be an infinite sequence of integers larger than 1, and for each $x\in[0, 1)$ let

$$x=\sum_{n=1}^{\infty}\frac{\varepsilon_n(x)}{q_1\cdots q_n}$$

be the Cantor expansion of x relative to $\{q_n\}$, so that $\varepsilon_n(x)$ is an integer such that $0\leq\varepsilon_n(x)\leq q_n-1$ always, and $\varepsilon_n(x)<q_n-1$ for infinitely many n. It is shown that, for almost all x, $\liminf\varepsilon_n(x)/q_n=0$, and that if $\limsup q_n=\infty$, then for almost all x, the set of limit points of $\{\varepsilon_n(x)/q_n\}$ is the whole interval $[0, 1]$. The proofs use the notion of a homogeneous set and the following condition for homogeneity. Let $Q_n=q_1\cdots q_n$, and let i_n^k be the interval $[k/Q_n, (k+1)/Q_n)$ for $0\leq k<Q_n$. Let B be a measurable set in $[0, 1)$. Then B is homogeneous if $\operatorname{mes}(B\cap i_n^k)=\operatorname{mes}(B\cap i_n^{k'})$ for all k, $k'<Q_n$.

As an application, the author considers four sufficient conditions for the irrationality of x, in terms of the Cantor expansion of x, given by A. Oppenheim [Amer. Math. Monthly **61** (1954), 235–241; MR **15**, 781] and P. H. Diananda and A. Oppenheim [ibid. **62** (1955), 222–225; MR **16**, 908]. (For example, x is irrational if 1 is a limit point of $\{\varepsilon_n/q_n\}$.) He shows, for each of three of the four cases, that the condition is satisfied by almost all x if $\{q_n\}$ has one kind of behavior, and by almost no x otherwise. The remaining condition is shown to hold for almost all x under certain circumstances.

W. J. LeVeque (Ann Arbor, Mich.)

Freiman, G. A. 5790

Inverse problems in additive number theory. VIII. On a conjecture of P. Erdős. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, no. 3 (40), 156–169.

The present paper depends primarily on Part VII of the series [same Izv. **1962**, no. 6 (31), 131–144; MR **26** #2420]. It gives an affirmative response to a conjecture of P. Erdős contained in one of his problem collections [Monographies de L'Enseignement Mathématique, No. 6, pp. 81–135; L'Enseignement Mathématique, Université, Genève, 1963; MR **28** #2070]: Let A be a set of non-negative numbers arranged as an increasing sequence, $a_0=0$, $a_i<a_{i+1}$. Let $A(x)$ be the number of elements of A which are less than or equal to x; let $A_2(x)$ be the similar function for $A+A$. Suppose $\lim_{x\to\infty}A(x)/x=0$, then $\delta=\limsup_{x\to\infty}A_2(x)/A(x)\geq 3$. The proof is by the method of trigonometrical sums over parallelepipeds defined by vectors of partial sequences

from A. Several other theorems are obtained of which the following (Theorem 4) may be given without further definitions: If $\delta \leqq 4.1$ and $\lim_{x\to\infty} A(x)/x = 0$, there are infinitely many $a_i \in A$ such that $2a_i < a_{i+1}$.

J. D. Swift (Los Angeles, Calif.)

Freiman, G. A. 5791

On the addition of finite sets. (Russian)

Dokl. Akad. Nauk SSSR **158** (1964), 1038–1041.

This paper is primarily concerned with the structure of sets $K = \{a_0, a_1, \cdots, a_{k-1}\}$ of integers such that the sum set $K+K$ of numbers of the form $a_i + a_j$ (each representable number being counted once only) has comparatively few members. The author shows that such sets are obtained by a linear map from the integral points in a convex set in affine space of comparatively small dimension. More precisely, he proves the following (Theorem 1). Suppose that the number T of elements of $K+K$ satisfies $T < Ck$, where $C \geqq 2$, and suppose that k is sufficiently large. Then there exists an integer $n \leqq [C-1]$, a homomorphism φ of the additive group Z_n of integral points in n dimensions into the corresponding group Z_1 in one dimension and a convex subset D of n-dimensional space such that (1) $K \subset (D \cap Z_n)\varphi$; (2) the sets $D \cap Z_n$ and $(D \cap Z_n)\varphi$ are "isomorphic up to the first degree", i.e., φ is an isomorphism and induces an isomorphism in the sets of sums of pairs of elements; (3) the number of elements in $D \cap Z_n$ is at most $c_1 k$, where $c_1 = c_1(C)$.

The proof is rather complicated and depends essentially on earlier work of the author [Izv. Vysš. Učebn. Zaved. Matematika **1962**, no. 3 (28), 151–157; MR **27** #2464; ibid. **1964**, no. 6 (43), 168–178; see also #5790 above].

In conclusion, the author states a similar result about sets D in n-dimensional euclidean space which satisfy an inequality $\mu^*(2D) \leqq C\mu^*(D)$, where μ^* is the outer Lebesgue measure and C is a constant; it is mentioned that there are similar results for lower asymptotic density of sequences of integers.

J. W. S. Cassels (Cambridge, England)

Mullender, P. 5792

Some remarks on a method of Mordell in the Geometry of Numbers.

Acta Arith. **9** (1964), 301–304.

The "method" of the title leads to an inequality connecting the critical determinant of an n-dimensional star body with that of a related $(n-1)$-dimensional star body. Several authors [see, e.g., C. G. Lekkerkerker, J. Reine Angew. Math. **206** (1961), 20–25; MR **26** #2401] have attempted to analyse the underlying principles of the method; the present paper seeks to elucidate it further.

Let $F(x)$, $G(x)$ be the distance functions of two star bodies K_F, K_G of finite type in n-dimensional Euclidean space, X. Suppose that K_F possesses a group Ω of automorphs such that the adjoint of an automorph of K_F is an automorph of K_G; that is, there is a group of non-singular matrices, A, such that $F(Ax) = F(x)$, $G(\tilde{A}x) = G(x)$ for all $x \in X$, where $\tilde{A}$ is the transposed inverse of A. The main theorem reads as follows. Suppose that there are in X a k-dimensional linear sub-space R and an $(n-k)$-dimensional linear sub-space S, perpendicular to R, such that (a) the k-dimensional and $(n-k)$-dimensional star bodies $R \cap K_F$ and $S \cap K_G$ are of finite type in R, S, respectively; (b) to any k-dimensional linear sub-space R' of X such that the k-dimensional star body $R' \cap K_F$ is of finite type in R' there corresponds an automorph of K_F in Ω transforming R' into R. Then, for $k = n-1$,

$$(\Delta_{K_F})^k \geq \left(\frac{\Delta_{R\cap K_F}}{\Delta_{S\cap K_G}}\right)^n \cdot (\Delta_{K_G})^{n-k},$$

where Δ_{K_F} denotes the critical determinant of K_F, etc.

The integer k is introduced in order to indicate possible generalizations of the theorem. However, it is not easy to see what the right generalization should be, and it may be a difficult problem to solve, if indeed there is a solution at all. For an example of a particular case in which the method can be carried further, see a paper by the reviewer [Mathematika **2** (1955), 132–140; MR **17**, 1060].

J. V. Armitage (Durham)

Arkin, Joseph 5793

Congruences for the coefficients of the k-th power of a power series.

Amer. Math. Monthly **71** (1964), 899–900.

Put $(\sum_{n=0}^{\infty} C_n x^n)^k = \sum_{n=0}^{\infty} C_n^{(k)} x^n$, where the C_n and k are integers. It is proved that $C_n^{(k)} \equiv 0 \pmod{k/(n, k)}$ $(n = 1, 2, 3, \cdots)$. As an application let $\tau(n)$ denote the Ramanujan τ-function defined by

$$x \prod_{n=1}^{\infty} (1-x^n)^{24} = \sum_{n=1}^{\infty} \tau(n) x^n.$$

Then $\tau(3n) \equiv \tau(3n+2) \equiv 0 \pmod 3$, a congruence due to Lahiri [Bull. Calcutta Math. Soc. **38** (1946), 193–206; MR 8, 567]; also, $\tau(2n) \equiv 0 \pmod 8$.

L. Carlitz (Durham, N.C.)

Gupta, H.; Vaidya, A. M. 5794

The number of representations of a number as a sum of two squares.

Amer. Math. Monthly **70** (1963), 1081–1082.

The authors obtain what they call an apparently new expression for $r_2(n)$ (the number of representations of n as sum of two squares) in the form

$$r_2(n) = 4 \prod \left\{\left[\frac{\alpha+2}{2}\right] + (-1)^{[p/2]}\left[\frac{\alpha+1}{2}\right]\right\},$$

where $n = 2^\lambda \prod p^\alpha$ is the canonical factorization of n. {Reviewer's remark: This is the same as (16.9.5) in Hardy and Wright's *An introduction to the theory of numbers* [fourth edition, Clarendon, Oxford, 1960].}

M. V. Subbarao (Edmonton, Alta.)

Horadam, E. M. 5795

Ramanujan's sum for generalised integers.

Duke Math. J. **31** (1964), 697–702.

Let $\{p\}$ be a sequence of real numbers (generalized primes) such that $0 < 1 < p_1 < p_2 < \cdots$. Form the set $\{l\}$ of all p-products $p_1^{v_1} p_2^{v_2} \cdots$, where the v_i are integers $\geqq 0$, of which all but a finite number are 0. These numbers are called generalized integers; it is assumed that no two are equal if their v's are different. Also the l's are numbered so that $0 < 1 < l_2 < l_3 < \cdots$. Ramanujan's sum for $\{l\}$ is defined by means of

$$c(l_n, l_r) = \sum_{d \mid (l_n, l_r)} \mu(l_r/d) d.$$

More generally the author defines

$$c_k(l_n, l_r) = \sum_{\substack{d \mid l_r \\ d^k \mid l_n}} \mu(l_r/d)d^k,$$

thus generalizing the function

$$c_k(n, r) = \sum_{(x, r^k)_k = 1} e(nx, r^k)$$

introduced by E. Cohen [same J. **22** (1955), 543–550; MR **17**, 238]. The present paper generalizes a number of results concerning $c_k(n, r)$, and also some properties of the class E_k of k-even functions, that is, functions $f(l_n, l_r)$ such that

$$f(l_n, l_r) = f((l_n, l_r^k)_k, l_r).$$

The class E_k for functions of integers has been studied by P. J. McCarthy [J. Reine Angew. Math. **203** (1960), 55–63; MR **22** #2574]. For example, it is proved that if $f(l_n, l_r) \in E_k$, then

$$f(l_n, l_r) = \sum_{d \mid l_r} \alpha(d) c_k(l_n, d),$$

where

$$\alpha(d) = \frac{1}{l_r^k} \sum_{\delta \mid l_r} f\left[\left(\frac{l_r}{\delta}\right)^k, l_r\right] c_k\left[\left(\frac{l_r}{d}\right)^k, \delta\right].$$

L. Carlitz (Durham, N.C.)

Carlitz, L. 5796

Extended Bernoulli and Eulerian numbers.

Duke Math. J. **31** (1964), 667–689.

Extensions of Bernoulli and Eulerian numbers and polynomials are studied by the author. The extended Bernoulli numbers $\beta(n)$ and polynomials $\beta(n, z)$ are defined by

$$\frac{\log \zeta(s)}{\zeta(s) - 1} = \sum_{n=1}^{\infty} \beta(n)/n^s, \qquad \zeta(s) = \sum_{n=1}^{\infty} 1/n^s,$$

$$\frac{(\zeta(s))^z \log \zeta(s)}{\zeta(s) - 1} = \sum_{n=1}^{\infty} \frac{\beta(n, z)}{n^s},$$

respectively, while the extended Eulerian numbers $H(n, \lambda)$ and polynomials $H(n, \lambda, z)$ are defined by

$$\frac{1 - \lambda}{\zeta(s) - \lambda} = \sum_{n=1}^{\infty} \frac{H(n, \lambda)}{n^s},$$

$$\frac{(1 - \lambda)(\zeta(s))^z}{\zeta(s) - \lambda} = \sum_{n=1}^{\infty} \frac{H(n, \lambda, z)}{n^s}, \qquad \lambda \neq 1.$$

A number of algebraic properties of these numbers and polynomials are discussed, most of which are generalizations of the corresponding properties of the Bernoulli and Eulerian numbers and polynomials. Expressions for $\beta(n)$, $H(n, z)$ are obtained for special values of n, and multiplication theorems are proved, e.g.,

$$\sum_{j=0}^{m-1} \beta\left(n, mh, z + \frac{j}{m}\right) = m\beta(n, h, mz),$$

where

$$\frac{h \log \zeta(s)}{(\zeta(s))^h - 1} (\zeta(s))^{hz} = \sum_{n=1}^{\infty} \frac{\beta(n, h, z)}{n^s}, \qquad h \neq 0.$$

Ennola, Veikko 579

On a problem about the Epstein zeta-function.

Proc. Cambridge Philos. Soc. **60** (1964), 855–875.

Let $h(m, n) = \alpha m^2 + 2\delta mn + \beta n^2$, where $\alpha\beta - \delta^2 = 1$, an write $Q(m, n) = 2 \cdot 3^{-1/2}(m^2 + mn + n^2)$ and

$$Z_h(s) = \sum \sum \{h(m, n)\}^{-s},$$

the summation being over all integers m, n with th exception of $m = n = 0$. The series converges for Re $s > 1$ The reviewer [Proc. Glasgow Math. Assoc. **1** (1953) 149–158; MR **15**, 507], Cassels [ibid. **4** (1959), 73–80 MR **22** #7975], the author [#5798a below] and Dianand [#5798b below] proved that $Z_h(s) \geq Z_Q(s)$ for real $s >$ where the sign of equality is needed only if h is equivalen to Q. The author considers the three-dimensional analogu of this problem and makes the conjecture that, for $s >$ $Z_f(s) \geq Z_R(s)$, strict inequality holding except when f i equivalent to R. Here

$$f(m, n, p) = am^2 + bn^2 + cp^2 + 2rnp + 2spm + 2tmn,$$

the determinant being $\frac{1}{2}$, and

$$R(m, n, p) = m^2 + n^2 + p^2 + mp + np.$$

The zeta-functions $Z_f(s)$ and $Z_R(s)$ are defined similarl by infinite series that are absolutely convergent fo Re $s > \frac{3}{2}$. This conjecture he is unable to prove, but b very complicated analysis he shows that the form R give a local minimum for $Z_f(s)$. *R. A. Rankin* (Glasgow

Ennola, Veikko 5798

A lemma about the Epstein zeta-function.

Proc. Glasgow Math. Assoc. **6**, 198–201 (1964).

Diananda, P. H. 5798

Notes on two lemmas concerning the Epstein zeta function.

Proc. Glasgow Math. Assoc. **6**, 202–204 (1964).

The reviewer's paper [same Proc. **4** (1959), 73–80; MR **2** #7975] contained a silly blunder which apparentl vitiated the whole argument. Ennola notes the blunder an shows that it can be rectified by a more elaborate argu ment on the same general lines as the original on Diananda shows that the proof of the reviewer's resul can be considerably simplified.

J. W. S. Cassels (Cambridge, Englan

Grosswald, E. 579

Considerations concerning the complex roots of Rie mann's zeta-function.

Publ. Math. Debrecen **10** (1963), 157–170.

The results are too elaborate to quote in detail, but th general idea is to connect the upper bound θ of the re parts of the zeros of $\zeta(s)$ with the behaviour of functio such as

$$F(x) = \sum_{n=1}^{\infty} \frac{(-x)^n}{\zeta(2n)\varphi(n)}, \qquad \Psi(u) = \sum_{n=2}^{\infty} \frac{u^{n-2}}{\zeta(\frac{1}{2}n)\varphi(\frac{1}{2}n)}.$$

M. Riesz proved that, when $\varphi(s) = \Gamma(s)$, a necessary an sufficient condition for the Riemann hypothesis ($\theta = \frac{1}{2}$) that $F(x) = O(x^\alpha)$ as $x \to \infty$, for every fixed $\alpha > \frac{1}{4}$. Theorem extends results of this type to general classes Φ and of functions

into a different shape. Theorem 4 asserts that, if, for some φ in Φ, the integral function $\Psi(u)$ is of order $<\frac{1}{2}$ (or of order $\frac{1}{2}$ and minimal type), then $\theta=1$. Another type of result gives analogues, for various arithmetical functions $R_1(x)$, of the theorem that $R(x)=\psi(x)-x$ is $O(x^\alpha)$ if $\alpha>\theta$ but not if $\alpha<\theta$. (This seems to be the intention of Theorem 3 and the succeeding remarks, though the italicized statement of the theorem appears to be nothing more than a list of definitions.) [The basic assumptions about $\varphi(s)$ are stated in a form that does not seem to accord with what is assumed in proofs. Thus, on p. 158, the hypothesis (I) is stated only for fixed σ and varying t, and (II) for fixed t and varying σ; but any normal application of Cauchy's theorem, such as the one on p. 162, presumes uniform estimates of the integrand. There are other inaccuracies of presentation.]

A. E. Ingham (Cambridge, England)

Nevanlinna, Veikko 5800

Über die elementaren Beweise der Primzahlsätze und deren äquivalente Fassungen.

Ann. Acad. Sci. Fenn. Ser. A I No. 343 (1964), 52 *pp.*

The prime number theorem (PNT) asserts that

$$\text{(I)}\qquad \pi(x)\sim x/\log x,\qquad x\to\infty,$$

where $\pi(x)$ is the number of primes $p\leq x$. (I) is known to be equivalent to the following assertions:

$$\text{(II)}\qquad \vartheta(x)\sim x,\qquad x\to\infty,$$

where $\vartheta(x)$ denotes the sum $\vartheta(x)=\sum_{p\leq x}\log p$, and

$$\text{(III)}\qquad \psi(x)=\sum_{p^r\leq x}\log p=\sum_{n\leq x}\Lambda(n)\sim x,\qquad x\to\infty,$$

where the function $\Lambda(n)$ is defined as being $\log p$ when n is a prime p or one of its powers and 0 otherwise [e.g., A. E. Ingham, *The distribution of prime numbers*, p. 13, Cambridge Univ. Press, London, 1932].

All previous proofs up to 1948 have been by "transcendental" arguments involving some appeal to the theory of functions of a complex variable, but then A. Selberg and P. Erdős [see, e.g., J. G. van der Corput, Math. Centrum Amsterdam Scriptum No. 1 (1948); MR **10**, 597] succeeded in producing an elementary proof of the PNT, not depending on analytical ideas remote from the problem itself. In various versions elementary proofs were given by A. Selberg [Ann. of Math. (2) **50** (1949), 305–313; MR **10**, 595], P. Erdős [Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 374–384; MR **10**, 595], and others.

The purpose of this paper is essentially to give a simple elementary proof of the PNT and to deduce its equivalent formulas while exhibiting the parallelism running through the proofs. The starting point in the first chapter of this paper is Selberg's asymptotic formula:

$$\text{(a)}\qquad \rho(x)\log x+\int_{1-}^{x+}\rho(x/t)\,d\psi(t)=O(x)\qquad(\rho(x)=\psi(x)-x),$$

which is effected to this form from Selberg's original formula through application of the method of T. Tatuzawa and K. Iseki [Proc. Japan Acad. **27** (1951), 340–342; MR **13**, 725]. The PNT is then deduced first by eliminating the differential $d\psi$ to get Selberg's inequality:

$$\text{(b)}\qquad |\rho(x)|\log^2 x\leq 2\int_2^x|\rho(x/t)|\log t\,dt+O(x\log x).$$

After the substitution of $r(x)=e^{-x}\rho(e^x)=e^{-x}\psi(e^x)-1$ the author deduces from this inequality by elementary arguments $r(x)=o(1)$, which is equivalent to the PNT [Soc. Sci. Fenn. Comment. Phys.-Math. **27** (1962), no. 3; MR **26** #2416].

The second chapter starts with the following notations:

$$\varepsilon(x)=\int_{1-}^{x+}d\psi(t)/t-\log x+C,$$

$$M(x)=\sum_{n\leq x}\mu(n),$$

$$B(x)=\sum_{n\leq x}\mu(n)/n=\int_{1-}^{x+}dM(t)/t,$$

where C is the Euler constant, and μ denotes the Möbius function. Then it is shown that as $x\to\infty$

$$\text{(IV)}\qquad \varepsilon(x)=o(1),$$

$$\text{(V)}\qquad M(x)=o(x),$$

$$\text{(VI)}\qquad B(x)=o(1)$$

are each equivalent to the formula (III). Selberg's asymptotic formula is derived for the function $\varepsilon(x)$, and through the use of the ideas of Pitt [*Tauberian theorems*, Oxford Univ. Press, London, 1958; MR **21** #5109] and Selberg, a new variant of Selberg's proof is given. The equivalence of (III), (IV), (V), and (VI) is proved by showing the logical relations among them. It is shown that if $\rho(x)$ in (a) and (b) is replaced by $M(x)$ or $B(x)$, then an almost complete analogy for the asymptotic expression and inequality of Selberg holds.

In the third chapter the author deduces the PNT for arithmetical progressions corresponding to (I), (II), and (III), which was first proved in an elementary way by Selberg [Canad. J. Math. **2** (1950), 66–78; MR **11**, 419]. Here again, the lemma of Tatuzawa and Iseki is used to derive Selberg's asymptotic formula and inequality. Then the method in the chapter is carried out, resulting in the almost complete analogy with the first chapter. Finally, an elementary proof is given of Dirichlet's theorem about primes in arithmetic progressions [cf., e.g., A. Selberg, Ann. of Math. (2) **50** (1949), 297–304; MR **10**, 595].

S. Ikehara (Tokyo)

Breusch, Robert 5801

An asymptotic formula for primes of the form $4n+1$.

Michigan Math. J. **11** (1964), 311–315.

By an elementary method (based on factoring of the number $\prod(r^2+s^2)$, $1\leq r\leq n$, $1\leq s\leq n$) the author proves the asymptotic formula

$$\sum_{p\leq x}p^{-1}\log p=\tfrac{1}{2}\log x+O(1),$$

where p runs through the primes $4n+1$ (or $4n-1$). The author notes that a similar proof can still be given for progressions $6n\pm1$.

E. Fogels (Riga)

Fogels, E. 5802

On the abstract theory of primes. I.

Acta Arith. **10** (1964/65), 137–182.

The theory in question arises from replacing the positive integers by a countable multiplicative semigroup G of real numbers $a\geq1$ with some asymptotic distribution law.

The role of the primes is taken over by the generators of G. To get the equivalent of arithmetic progressions, G is split into classes H_i $(i=1,\cdots,h)$ forming a group K under the multiplication given by: If $a \in H_i$, $a' \in H_j$, then $aa' \in H_k$ with $k=k(i,j)$. The class H_i corresponds to an arithmetic progression.

The author's aim is to get, under suitable assumptions, an upper estimate for the smallest generator of G in H_i, valid simultaneously for all i. A parameter D is introduced corresponding to the common difference in an arithmetic progression, and it is supposed that $D>3$, $1 \leq h \leq D^{c_0}$, $c_0>0$. The asymptotic distribution law which is assumed is $\sum 1 = \alpha x + O(D^{c_1}x^{1-\theta})$ for large x, where the summation is over all $a \in H_i$ such that $a \leq x$, and $\alpha = D^l$; the constants l, c_1, θ do not depend on i.

For even h let K_j be any subgroup of K with index 2. Then it is shown that

$$\lim_{x\to\infty}\Big(\sum_{a\in K, a\leq x} a^{-1} - \sum_{a\in K_j, a\leq x} a^{-1}\Big) = C_j = C_j(D).$$

Now the main theorem is: (i) If $\theta > \frac{1}{2}$, there is a $c>0$ such that for any $x \geq 1$, H_i has a generator in the interval (x, xD^c). For odd h this holds also if $\theta \leq \frac{1}{2}$. (ii) For even h and $\theta \leq \frac{1}{2}$, the conclusion of (i) holds if there is a $c_2>0$ such that $C_j > D^{-c_2}$ for any j (in this case c also depends on c_2). (iii) Let $\pi(x, H_i)$ be the number of generators (generalized primes) in H_i not exceeding x. Then constants c_3, c_4 exist such that $\pi(x, H_i) > D^{-c_3}x/\log x$ for $x > D^{c_4}$.

The proof, running to over twenty-five lemmas, is long, involved, and not easy to follow. The method is analytical in nature and employs the following zeta-functions and L-series: $\zeta(s,H) = \sum_{a\in H} a^{-s}$, $\zeta(s,\chi) = \sum_a \chi(a)a^{-s}$ (χ is a character of the group K and $\chi(a)$ is $\chi(H)$ for all $a \in H$). One then has, as for the ordinary case, $\zeta(s,\chi) = \prod_b [1-\chi(b)b^{-s}]^{-1}$, where b runs over all generators of G.

R. D. James (Vancouver, B.C.)

Kelly, John B. 5803

Partitions with equal products.

Proc. Amer. Math. Soc. **15** (1964), 987–990.

The author proves the following theorem: If k is an integer ≥ 3, there exists an $N(k)$ such that every $n \geq N(k)$ can be partitioned into k parts in $(k-1)$ different ways so that the products of the integers in these $(k-1)$ partitions are equal. What is more, the integers which occur in all these partitions are pairwise different.

S. Knapowski (Marburg)

Linnik, Ju. V. 5804

★**The dispersion method in binary additive problems.**

Translated by S. Schuur.

American Mathematical Society, Providence, R.I., 1963. x + 186 pp. \$12.30.

The original Russian [Izdat. Leningrad. Univ., Leningrad, 1961] was reviewed earlier [MR **25** #3920].

Novoselov, E. V. 5805

A new method in probabilistic number theory. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 307–364.

This paper is a continuation of the author's previous investigations on topological methods in number theory [Učen. Zap. Elabužsk. Gos. Ped. Inst. **8** (1960), 3–[illegible]; Izv. Vysš. Učebn. Zaved. Matematika **1961**, no. 1 (2[illegible]), 119–129; MR **27** #6249; ibid. **1961**, no. 3 (22), 66–[illegible]; MR **28** #77; ibid. **1963**, no. 5 (36), 71–88; MR **28** #78].

The ring S of integer rational numbers may be to[pologized] by means of the topology for which the princi[pal] ideals constitute a neighbourhood base of 0. By co[m]pletion of S we get a compact topological ring $\mathfrak{S}$; t[he] elements of $\mathfrak{S}$ are called polyadic numbers. The set $\mathfrak{S}$ [is] also a bicompact topological group; accordingly, th[ere] exists a normed Haar measure μ on $\mathfrak{S}$. The completion [of] μ is a probabilistic measure P.

The author gives a number of applications of the theo[ry] of measure P and the corresponding integral calculus [to] the value distribution theory of the arithmetical functi[ons] and to the calculation of their means.

In particular, the author obtains some results on t[he] almost periodicity of the arithmetical functions due [to] Erdős, Hartman, Kac, van Kampen and Wintner [K[ac], van Kampen and Wintner, Amer. J. Math. **62** (194[0]), 107–114; MR **1**, 203; van Kampen and Wintner, ibid. [62] (1940), 613–626; MR **2**, 41; van Kampen, ibid. **62** (194[0]), 627–634; MR **2**, 41; Erdős and Wintner, ibid. **62** (194[0]), 635–645; MR **2**, 41; Hartman and Wintner, ibid. [62] (1940), 753–758; MR **2**, 41; Wintner, ibid. **67** (194[5]), 173–193; MR **6**, 260; Wintner, ibid. **67** (1945), 481–4[?]; MR **7**, 147]. From Kolmogorov's three series criterion [for] the convergence of the series of the independent rand[om] variables he deduces the Erdős theorem [J. London Ma[th.] Soc. **13** (1938), 119–127] on the sufficient condition [for] the existence of the asymptotic distribution law of addit[ive] number-theoretic functions. He also proves a theorem [of] Rényi [Acad. Serbe Sci. Publ. Inst. Math. 8 (1955), 1[?]–162; MR **17**, 944], a theorem of Hardy and Ramanu[jan] [Quart. J. Pure Appl. Math. 48 (1917), 76–92], and so[me] results of the reviewer [*Probabilistic methods in the the[ory] of numbers* (Russian), Gosudarstv. Izdat. Politič. i Nau[č.] Lit. Litovsk. SSR, Vilna, 1959; MR **22** #A134; sec[ond] enlarged edition, 1962; MR **26** #3691].

J. Kubilius (Vilni[us])

Roth, K. F. 5[806]

Remark concerning integer sequences.

Acta Arith. **9** (1964), 257–260.

Let N be a positive integer, and let $\mathcal{N}$ be a set of disti[nct] positive integers not exceeding N. For each integer [q,] $0 < m \leq N$, and each residue class $h \pmod q$, denote [by] $\Phi_{q,h}(\mathcal{N}; m)$ the number of elements of $\mathcal{N}$ not exceed[ing] m and in the residue class, and by $\Phi_{q,h}^*(\mathcal{N}; m)$ t[he] corresponding "expectation",

$$\Phi_{q,h}^*(\mathcal{N}; m) = \eta\Phi_{q,h}(\mathcal{J}; m),$$

where $\mathcal{J}$ is the set $\{1,\cdots,N\}$ and

$$\eta = N^{-1}\sum_{\substack{n\leq N\\ n\in\mathcal{N}}} 1.$$

For each m and q, define

$$V_q(m) = \sum_{h=1}^{q}\{\Phi_{q,h}(\mathcal{N}; m) - \Phi_{q,h}^*(\mathcal{N}; m)\}^2.$$

It is shown that for every positive integer Q,

$$\sum_{q=1}^{Q} q^{-1}\sum_{m=1}^{N} V_q(m) + Q\sum_{q=1}^{Q} V_q(N) > c\eta(1-\eta)Q^2N,$$

where c is an absolute constant. Qualitatively and crudely speaking, this theorem shows that unless a sequence of integers has density nearly 0 or 1, its elements cannot be too evenly distributed simultaneously among all residue classes. *W. J. LeVeque* (Ann Arbor, Mich.)

Kesten, H. 5807

The discrepancy of random sequences $\{kx\}$.

Acta Arith. **10** (1964/65), 183–213.

For $0 \leq a \leq b \leq 1$, let

$$f(\xi; a, b) = 1 \quad \text{if } a \leq \xi \leq b,$$
$$= 0 \quad \text{if } 0 \leq \xi < a \text{ or } b < \xi \leq 1,$$
$$f(\xi+1; a, b) = f(\xi; a, b),$$

and for $1 < b \leq 1+a$, put

$$f(\xi; a, b) = f(\xi; a, 1) + f(\xi; 0, b-1).$$

For each real x, let

$$D_N(x) = N^{-1} \sup_{0 \leq a \leq b \leq 1+a} \left| \sum_{k=1}^{N} f(kx; a, b) - N(b-a) \right|.$$

The principal result of this deep paper is that for every fixed $\varepsilon > 0$,

$$\lim_{N\to\infty} \operatorname{mes}\left\{ x \in [0;1] : \left| \frac{N D_N(x)}{\log N \log\log N} - \frac{2}{\pi^2} \right| > \varepsilon \right\} = 0.$$

The proof depends on a fairly explicit, but rather complicated, connection between $D_N(x)$ and the continued fraction expansion of x.

W. J. LeVeque (Ann Arbor, Mich.)

Philipp, Walter 5808

An n-dimensional analogue of a theorem of H. Weyl.

Compositio Math. **16**, 161–163 (1964).

The results of this lecture were published in Arch. Math. **12** (1961), 429–433 [MR **27** #3614].

Stevens, Harlan 5809

Kummer's congruences of a second kind.

Math. Z. **79** (1962), 180–192.

Mit $a_n \in R_m(X)$ (Polynombereich wie in einer vorangehenden Arbeit [Math. Nachr. **24** (1962), 219–227; MR **27** #1407]) werde die Summe

$$T(h; n, r) = \sum_{s=0}^{r} (-1)^s \binom{r}{s} a_h^{\,r-s} a_{n+sh}$$

gebildet. Dabei sei die feste Zahl $m = p_1^{e_1} \cdots p_k^{e_k}$. Bestehen für die untereinander verschiedenen Primzahlen $p_1, \cdots, p_k$ die Kummerschen Kongruenzen 2. Art $T(p_i; n, r) \equiv 0 \pmod{p_i^r}$, $1 \leq i \leq k$, für alle ganzen $n \geq 0$, $r \geq 0$, so beweist Verfasser daß $T(m; n, r) \equiv 0 \pmod{m^r}$ ist. Er erhält bei ungeradem m mit $r_i = [\frac{1}{2}(r+1)]$ die Kongruenz (1) $T(m; n, r) \equiv 0 \pmod{p_1^{(e_1-1)r+r_1} p_2^{(e_2-1)r+r_2} \cdots}$ aus der Gültigkeit von $T(p_i; n, r) \equiv 0 \pmod{p_i^{r_i}}$ für $1 \leq i \leq k$. Auch für gerade m gilt eine (1) entsprechende Kongruenz. Weiter werden Kummersche Kongruenzen für das Hurwitzsche Produkt von h Folgen $\{a_n^{(1)}\}, \{a_n^{(2)}\}, \cdots, \{a_n^{(h)}\}$ und für die Reziproke einer Folge $\{a_n\}$ untersucht. Anwendungen auf die Eulerschen Zahlen und die Hermiteschen Polynome verschärfen Ergebnisse von L. Carlitz [Acta Arith. **6** (1960), 149–158; MR **22** #7968]. *H. Salié* (Zbl **106**, 32)

Bertrandias, Françoise 5810

Sur une caractérisation de certains ensembles de nombres algébriques.

C. R. Acad. Sci. Paris **258** (1964), 1666–1668.

The author announces theorems which extend the Pisot theory of S-numbers over the complex or p-adic fields to a combined theory over the ring of adèles over the rational field and its completions.

K. Mahler (Canberra)

Ramanujam, C. P. 5811

Sums of m-th powers in p-adic rings.

Mathematika **10** (1963), 137–146.

Let K be an algebraic number field, and m a positive integer. Siegel has conjectured that there is a $G(m)$ such that every totally positive integer of large enough norm that is a sum of mth powers of integers of K is a sum of at most $G(m)$ such mth powers. The point is that $G(m)$ should be independent of the particular field K. The reviewer [Proc. Cambridge Philos. Soc. **57** (1961), 449–459; MR **26** #1306] has shown that to prove Siegel's conjecture it is enough to deal with the $\mathfrak{p}$-adic analogue.

Suppose then that A is the ring of integers of a $\mathfrak{p}$-adic field, whose residue class field has p^f elements, and let $J_m(A)$ be the subring generated by mth powers of elements of A. The author proves that each element of $J_m(A)$ is a sum of at most $8m^5$ mth powers.

The author first analyses what $J_m(A)$ looks like in the unramified case; the proof of his theorem in this case is then natural. Next, by a device, he deals quite simply with the case where the residue class field is large ($p^f \geq m$). Finally, he forces through the case $p^f < m$; the details of this part are (unavoidably) rather messy.

The author's final estimate $8m^5$ for the number of variables necessary is surprisingly good, though by no means best possible. The reviewer has proved a similar (less precise but slightly more general) theorem [Acta Arith. **9** (1964), 169–176; MR **29** #3462].

B. J. Birch (Manchester)

FIELDS AND POLYNOMIALS

See also 5824, 5847, 5869.

Northcott, D. G.; Reufel, M. 5812

Contributions to the specialization theory of polynomial modules.

Proc. Roy. Soc. Ser. A **281** (1964), 291–309.

Authors' summary: "This paper is concerned with the further development of ideas which appear in three other papers [D. G. Northcott, Proc. London Math. Soc. (3) **15** (1965), 1–25; MR **30** #135; D. G. Northcott and M. Reufel, Abh. Math. Sem. Univ. Hamburg **28** (1965), 16–49; and M. Reufel, Bonn. Math. Schr. No. 19 (1963); MR **28** #3031]. In particular, it analyzes the connexions between saturated and Z-complete modules and shows the precise role of these notions in the theory of reductions by means of an arbitrary valuation. This part of the paper opens the way for the development of a relative reduction theory

with the aid of which one can study the behavior of embedded components."

W. E. Deskins (E. Lansing, Mich.)

Schofield, R. 5813

Products of linear forms.

Proc. Cambridge Philos. Soc. **60** (1964), 1032–1033.

Let F be a square-free homogeneous polynomial in $K[x_1 \cdots x_n]$ and $F_i = \partial F/\partial x_i \neq 0$. The following interesting factorization criterion is proved: F is a product of linear forms in some algebraic extension if and only if for all j, k and some i

$$F_i^2 F_{jk} - F_i F_k F_{ij} - F_i F_j F_{ik} + F_j F_k F_{ii} = 0 \pmod F.$$

A. Evyatar (*Gutwirth*) (Haifa)

Jakovlev, A. V. 5814

The immersion problem for fields. (Russian)

Dokl. Akad. Nauk SSSR **150** (1963), 1009–1011.

The author considers the problem of constructing the extension field K of a normal extension field k of a field k_0, where the Galois group $\mathfrak{G}$ of K over k_0 and the Galois group $\mathfrak{F}$ of k over k_0 lie in an exact sequence

$$1 \to \mathfrak{A} \to \mathfrak{G} \xrightarrow{\pi} \mathfrak{F} \to 1;$$

here $\mathfrak{A}$ is periodic of period n, π is the restriction map of Galois groups and k contains a primitive nth root of unity.

This problem is reduced to a group extension problem $0 \to C \to X \to \hat{\mathfrak{A}} \to 0$, where $\hat{\mathfrak{A}} = \mathrm{Hom}_z(\mathfrak{A}, C)$ and C is the multiplicative group of k. *S. Rubinstein* (Seattle, Wash.)

Lur'e, B. B. 5815

On the problem of imbedding with a kernel without center. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 1135–1138.

We are given a finite group G and a homomorphism φ of G onto the Galois group F of a normal field extension k/k_0. The problem of imbedding is to find a Galois algebra A over k_0 with Galois group G such that the subalgebra fixed by $B = \ker \varphi$ is k_0-isomorphic to k, and such that the automorphism induced by $g \in G$ on this subalgebra is isomorphic to that induced by $\varphi(g)$ on k.

The author makes a simple reduction of this problem when B has no center, essentially factoring G by the centralizer of B. He then combines this technique with results of Delone and Faddeev [Mat. Sb. (N.S.) **15** (**57**) (1944), 243–284; MR **6**, 200] to reduce the problem in several special cases (e.g., when B is meta-cyclic without center) to compatibility of G with k.

E. C. Dade (Pasadena, Calif.)

Kolchin, E. R. 5816

The notion of dimension in the theory of algebraic differential equations.

Bull. Amer. Math. Soc. **70** (1964), 570–573.

The author introduces a highly important measure of the size of the solution set of a system of algebraic (ordinary or partial) differential equations and announces some of its properties whose proofs and details will appear in a forthcoming book. This measure is a certain numerical polynomial $\omega_\mathfrak{p}$ in one indeterminate (in the sense [of] Zariski and Samuel) defined for every prime different[ial] polynomial ideal $\mathfrak{p}$; it is called the differential dimensi[on] polynomial of $\mathfrak{p}$ and is somewhat related to Hilber[t's] "characteristic function" for homogeneous polynom[ial] ideals. This polynomial contains more information abo[ut] $\mathfrak{p}$ than the differential dimension of $\mathfrak{p}$ which can be deriv[ed] from it easily. In particular, two distinct prime different[ial] ideals $\mathfrak{p}$, $\mathfrak{p}'$ with $\mathfrak{p} \subset \mathfrak{p}'$ have distinct differential dimensi[on] polynomials $\omega_\mathfrak{p}$, $\omega_{\mathfrak{p}'}$ with $\omega_\mathfrak{p} > \omega_{\mathfrak{p}'}$ with respect to a suitab[le] total ordering of these numerical polynomials. While [it] is only a birational invariant, several differential [bi]rational invariants can be derived from it, for examp[le] the typical differential dimension and the differential ty[pe] of $\mathfrak{p}$ which, in classical terminology, describe the numb[er] of arbitrary functions and the number of their variab[les] on which the solution of the corresponding syst[em] depends. *A. Jaeger* (Cincinnati, Oh[io])

ABSTRACT ALGEBRAIC GEOMETRY

See also 6058, 6061, 6062.

Dantoni, Giovanni 58[17]

Ideali e varietà algebriche.

Univ. e Politec. Torino Rend. Sem. Mat. **20** (1960/6[1]) 149–156.

The author proposes some natural conditions on a latt[ice] W of ideals in $k[x_0, \cdots, x_r]$ such that one could say th[at] the ideals of W correspond in a one-to-one way with t[he] objects in a class of (generalized) algebraic varieties [in] projective r-space, S_r. Examples: the ideals coincidi[ng] with their radical and the ideals with no improper co[m]ponent. *A. Mattuck* (Cambridge, Ma[ss.])

Mallol, Rafael 581[8]

On the decomposition of an algebraic variety in exte[n]sions of the ground field. (Spanish)

Collect. Math. **14** (1962), 257–259.

Mallol, Rafael 581[9]

A remark on a paper of the author. (Spanish)

Collect. Math. **15** (1963), 3.

Let P be a generic point over k of the k-variety V, N t[he] smallest normal extension field of k containing $k(P) \cap$ [...] and Σ an extension field of k. Then the Σ-components [of] V are its $(\Sigma \cap N)$-components.

M. Rosenlicht (Berkeley, Cali[f.])

Matsumura, Hideyuki; Monsky, Paul 58[20]

On the automorphisms of hypersurfaces.

J. Math. Kyoto Univ. **3** (1963/64), 347–361.

The authors consider a hypersurface $H_{n,d}$ of degree d [in] the projective space P_{n+1} of dimension $n+1$ and the gro[up] $\mathrm{Aut}(H_{n,d})$ of all biregular transformations of $H_{n,d}$ o[nto] itself. The main results are as follows. (1) If $H_{n,d}$ is no[n]singular and $n \geq 2$, $d \geq 3$, then $\mathrm{Aut}(H_{n,d})$ is finite except [in] the case $n = 2$, $d = 4$. (2) If $H_{n,d}$ is generic over the pri[me] field and if $n \geq 2$, $d \geq 3$, then $\mathrm{Aut}(H_{n,d})$ is trivial except [in] the case that the ground field has characteristic $p > 0$ a[nd]

$n=2$, $d=4$. The proofs use a theorem of Severi and Lefschetz, according to which any positive divisor on a non-singular $H_{n,d}$ is cut out by a hypersurface in P_{n+1} provided that $n \geq 3$, and which reduces to the consideration of the group of all projectivities leaving $H_{n,d}$ invariant as a whole. From then on one needs a rather elementary but lengthy examination of polynomial algebra.

P. Cartier (Strasbourg)

Samuel, Pierre 5820

Le théorème de Hahn-Banach en géométrie algébrique.

Univ. e Politec. Torino Rend. Sem. Mat. **20** (1960/61), 127–134.

A discussion of a number of results proved elsewhere [e.g., J. Reine Angew. Math. **204** (1960), 1–10; MR **24** #A3165]. The main result concerns a morphism $f: V' \to V$ of normal varieties, the groups $\mathfrak{G}$ of divisors on V whose support does not contain $f(V')$ and $\mathfrak{G}'$ of divisors on V' with real coefficients, and affirms the existence of a homomorphism $\mathfrak{G} \to \mathfrak{G}'$ which is of local character, preserves positivity, and is the correct thing for divisors of functions.

M. Rosenlicht (Berkeley, Calif.)

Eichler, Martin 5821

★**Einführung in die Theorie der algebraischen Zahlen und Funktionen.**

Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften, Mathematische Reihe, Band 27.

Birkhäuser Verlag, Basel-Stuttgart, 1963. 338 *pp.* *sFr.* 59.00.

Three books peacefully coexist between the covers of this work: one on algebraic functions of one variable, an introduction to algebraic number theory, and an exposition of modular forms with some applications to analytic number theory. Their association is rather loose, and occasionally a turn of the page means not only a change of scene but a new cast of characters as well. However, the modular forms gain by being placed in the context of algebraic function theory and serve to illustrate it as well, while the elementary substratum common to algebraic numbers and functions is well known.

The book on algebraic functions has the usual material leading up to the Riemann-Roch theorem and the Abel-Jacobi theorem for Riemann surfaces. Most books stop there, but this one goes on to the higher differentials, reduction modulo p, and a long chapter on correspondences ending with the Riemann hypothesis for function fields. The approach is algebraic, studying a function field K of transcendence degree 1 over its constant field k, but it is not invariant: almost always the author proves his result first for a field $k(x)$, and then extends it by studying the layer $K/k(x)$. In other words, the corresponding Riemann surface is thought of as a finite covering of the sphere, rather than as an abstract complex manifold.

The ideal theory is done simultaneously for number and function fields, using both the Dedekind and the Kronecker approaches (the account of the latter is quite elegant). Valuations are not introduced explicitly, nor are the completions. For those with concrete tastes or analytic backgrounds, the same material is redone from scratch (for function fields) via power series and Puiseux expansions; then come the Riemann-Roch theorem and the differentials. After that is the chapter on Riemann surfaces—the basic facts of their construction and topology are assumed, but summarized rapidly—going as far as the Abel-Jacobi theorem, but not discussing uniformization. Finally, we have the chapter on correspondences, done both field-theoretically, as in Deuring's works, and in the classical case by Riemann surface theory. Roquette's proof of the Castelnuovo inequality is given.

The book on number theory is of more modest dimensions: beyond the aforementioned chapter on ideal theory, there is an account of the Hilbert ramification theory, a rapid treatment of class number and the unit theorem, the Minkowski estimates, and quadratic and cyclotomic fields.

The book on modular forms must be hunted for, but it is all there. An appendix to the first chapter gives in twenty pages the elementary facts about the general symplectic group of nth degree, gives the fundamental domain for $n=1$, proves the transformation law for theta functions, and discusses their connection with quadratic forms, including proofs of the reciprocity law and the sign of the Gauss sum. The modular forms appear in the chapter on Riemann surfaces. The account is classical, including Eisenstein series and the forms of level N, the theta functions being special cases. Finally, in the last chapter on correspondences, we get the ring of modular correspondences, its representation on the differentials and its connection with the Ramanujan conjecture—this last being proved at the end of the book for a special case by using the Riemann hypothesis for function fields.

Naturally, in all of this one of the author's problems has been that the pages of a book are linearly ordered, but their mathematical content cannot be. The sequence of topics he has opted for is as good as any, but will be particularly hard on anyone interested in the Riemann-Roch theorem, essential steps of the proof (which is modeled after Dedekind and Weber) being strewn over three chapters. Three or four different definitions of divisor are given and proved equivalent. One of these, the "linear divisor", is apparently a purely ad hoc invention to enable the author to give the linear core of the theorem in Chapter I; here it is shown to be formally somewhat like Minkowski's theorem on linear forms, and the two appear side by side as a sort of prologue to the rest.

The expository style is variable. In places it is leisurely; in others (particularly the number theory and theta functions) rapid. Theorems are not numbered or even labeled as theorems; they are distinguished only by italic script, and sometimes have to share the same sentence with discussion. The notations and running hypotheses are rarely included in the statements, but must usually be extracted from the context (which in at least one case means the preceding chapter). Little local problems (conflicting notation, interpolations in proofs without warning, discontinuities of argument) are also troublesome. In one place a starred—and thus presumably omittable—section is used essentially in the very next paragraph.

The above do not make the book unreadable, but they do add extrinsic difficulties to a hard enough subject. There are good historical notes, as well as brief summaries of the existing state of things. The book is really a text, not a monograph, despite the virtual absence of exercises. For those interested in algebraic functions, it is probably a better source than the comparable book of Chevalley

[*Introduction to the theory of algebraic functions of one variable*, Amer. Math. Soc., New York, 1951; MR 12, 64]; it is more discursive, attempts to illuminate the subject by using a variety of methods, and of course goes much further. The treatment of the other two subjects is more or less classical; it is the best source for the theory of modular correspondences, however. The general shortcomings of the field-theoretic approach for the study of higher-dimensional problems, and even for the deeper aspects of the one-dimensional case, have been noted elsewhere, and more passionately; no other method, however, will take one so far in such a short space. The book is recommended for those studying algebraic functions, as well as for those interested in the other two subjects in a fairly broad context.

A. Mattuck (Cambridge, Mass.)

Gröbner, Wolfgang 5822

Applicazioni delle serie di Lie nella geometria algebrica.

Univ. e Politec. Torino Rend. Sem. Mat. **20** (1960/61), 217–226.

A one-dimensional Lie series is defined as the expansion, in powers of t, of the function $\exp(tD)f(z)$, where

$$D = \vartheta_1(z)\frac{\partial}{\partial z_1} + \cdots + \vartheta_n\frac{\partial}{\partial z_n},$$

and where $\vartheta_1(z), \cdots, \vartheta_n(z), f(z)$ are holomorphic functions of $z_1, \cdots, z_n$ in a certain domain. Similarly, one may define a d-dimensional Lie series by means of the function $\exp(t_1D_1 + \cdots + t_dD_d)$, where $D_1, D_2, \cdots, D_d$ are differential operators analogous to D.

If V_d is an irreducible algebraic variety of a projective space S_n of homogeneous coordinates $(x_0, x_1, \cdots, x_n)$, all the forms containing V_d constitute a prime ideal $(f_1, f_2, \cdots, f_l)$, say. Writing $f_{ik} = \partial f_i/\partial x_k$, we can form $d+1$ operators

$$D_j = \vartheta_{j0}\frac{\partial}{\partial x_0} + \vartheta_{j1}\frac{\partial}{\partial x_1} + \cdots + \vartheta_{jn}\frac{\partial}{\partial x_n} \quad (j = 0, 1, \cdots, d)$$

associated with V_d. The author shows that the parameters $t_1, t_2, \cdots, t_d$ which appear in the corresponding Lie series are simple integrals (not in general of the first species) attached to V_d.

L. Roth (London)

Hoyt, William L. 5823

Embeddings of Picard varieties.

Proc. Amer. Math. Soc. **15** (1964), 26–31.

Let V be a subvariety of a projective space. Assume that V is an irreducible normal variety. Let H be a hypersurface of degree d in the ambient projective space. When we denote by $P(V)$, $P(V\cdot H)$ the Picard varieties of V, $V\cdot H$, respectively (assuming always that $V\cdot H$ is defined, irreducible and normal), there is a natural homomorphism λ of $P(V)$ into $P(V\cdot H)$. The reviewer proved that λ is a purely inseparable injection and that it is surjective when $\dim V > 2$ [Natur. Sci. Rep. Ochanomizu Univ. **4** (1954), 164–171; MR **16**, 163] when H is a generic hypersurface over a defining field of V. In this paper the author proves that λ is an injection (i.e., an everywhere biregular, birational isomorphism of $P(V)$ into $P(V\cdot H)$) for almost all H, provided that the degree of the hypersurface H is sufficiently large.

T. Matsusaka (Waltham, Mass.)

Katayama, Koji 58[…]

On the Hilbert-Siegel modular group and abeli[…] varieties. II.

J. Fac. Sci. Univ. Tokyo Sect. I **9**, 433–467 (1963).

Es sei F ein total reeller algebraischer Zahlkörper vo[…] Grade r über dem Körper der rationalen Zahlen. Es sei […] der Ring der ganzen Zahlen von F. Für eine natürlich[…] Zahl n sei 1_n die n-reihige Einheitsmatrix und $I = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$[…] Für jede Primstelle $\mathfrak{p}$ von F sei $F_\mathfrak{p}$ die $\mathfrak{p}$-Vervollständigun[…] von F. Für die endlichen Primstellen $\mathfrak{p}$ sei $\mathfrak{g}_\mathfrak{p}$ der Ring d[…] ganzen Zahlen von $F_\mathfrak{p}$ und $\mathfrak{u}_\mathfrak{p}$ die Gruppe der Einheit[…] von $F_\mathfrak{p}$. Der Verfasser betrachtet dann die Gruppen

$$G = \{\sigma \in \mathrm{GL}(2n, F) |\ \sigma I {}^t\sigma = m(\sigma)I,\ m(\sigma) \in F\},$$

$$G_\mathfrak{p} = \{\sigma \in \mathrm{GL}(2n, F_\mathfrak{p}) |\ \sigma I {}^t\sigma = m(\sigma)I,\ m(\sigma) \in F_\mathfrak{p}\},$$

$$\mathfrak{U}_\mathfrak{p} = \{\sigma \in \mathrm{GL}(2n, \mathfrak{g}_\mathfrak{p}) |\ \sigma I {}^t\sigma = m(\sigma)I,\ m(\sigma) \in \mathfrak{u}_\mathfrak{p}\}.$$

Zwei $\mathfrak{g}$-Gitter $\mathfrak{M}$, $\mathfrak{N}$ im $2n$-dimensionalen Zeilenvekto[…] raum $\mathfrak{B} = \mathfrak{B}(2n, F)$ über F heißen G-äquivalent, wenn […] ein $\sigma \in G$ mit $\mathfrak{M}\sigma = \mathfrak{N}$ gibt. Analog wird für endliches […] die $G_\mathfrak{p}$-Äquivalenz zweier $\mathfrak{g}_\mathfrak{p}$-Gitter $\mathfrak{M}_\mathfrak{p}$, $\mathfrak{N}_\mathfrak{p}$ von $\mathfrak{B}_\mathfrak{p}$ […] $\mathfrak{B}(2n, F_\mathfrak{p})$ definiert. Die beiden $\mathfrak{g}$-Gitter $\mathfrak{M}$, $\mathfrak{N}$ heißen […] demselben Geschlecht gehörig, wenn für jedes endliche […] die $\mathfrak{p}$-Vervollständigungen $\mathfrak{M}_\mathfrak{p}$, $\mathfrak{N}_\mathfrak{p}$ von $\mathfrak{M}$, $\mathfrak{N}$ $G_\mathfrak{p}$-äquivale[…] sind. Das erste Resultat der vorliegenden Arbeit ist dan[…] Jedes Geschlecht von $\mathfrak{g}$-Gittern besteht aus genau […] G-Äquivalenzklassen, wobei h die Klassenzahl von F i[…] (Theorem 1). Es seien $\mathfrak{p}_1, \cdots, \mathfrak{p}_r$ die unendlichen Pri[…] stellen von F. Es werden dann die direkten Produk[…] $J_{G,\infty} = G_{\mathfrak{p}_1} \times \cdots \times G_{\mathfrak{p}_r}$ und

$$J_{G,0} = \{s \in \prod_{\mathfrak{p}<\infty} G_\mathfrak{p} |\ s_\mathfrak{p} \in \mathfrak{U}_\mathfrak{p} \text{ für fast alle } \mathfrak{p} < \infty\}$$

betrachtet, wobei $s_\mathfrak{p}$ jeweils die $\mathfrak{p}$-Koordinate von s i[…] Das direkte Produkt $J_G = J_{G,\infty} \times J_{G,0}$ heißt die Idelgrup[…] von G. Unter Benutzung früherer Ergebnisse des Ve[…] fassers [dasselbe J. **9** (1962), 261–291; MR **26** #6158] ko[…] struiert der Verfasser zu jedem Element von J_G ei[…] polarisierte Abelsche Mannigfaltigkeit vom Typ $\mathfrak{g}$ u[…] zeigt, daß auf diese Weise auch alle polarisierten Ab[…] schen Mannigfaltigkeiten vom Typ $\mathfrak{g}$ erhalten werd[…] (Theorem 5). Die Elemente $D_\infty \in J_{G,\infty}$ können in der Fo[…]

$$D_\infty = \left(\begin{pmatrix} A^{(1)} & B^{(1)} \\ C^{(1)} & D^{(1)} \end{pmatrix}, \cdots, \begin{pmatrix} A^{(r)} & B^{(r)} \\ C^{(r)} & D^{(r)} \end{pmatrix}\right)$$

geschrieben werden, wobei $A^{(i)}$, $B^{(i)}$, $C^{(i)}$, $D^{(i)}$ n-reihi[…] quadratische Matrizen mit reellen Elementen sind. Es […] $\mathfrak{R}_\infty = \{D_\infty \in J_{G,\infty} | A^{(i)} = D^{(i)}, C^{(i)} = -B^{(i)}$ für $i = 1, \cdots,$[…] und $\mathfrak{R} = \mathfrak{R}_\infty \times \mathfrak{U}_0$ mit $\mathfrak{U}_0 = \prod_{\mathfrak{p}<\infty} \mathfrak{U}_\mathfrak{p}$. Zu zwei Element[…] $D, D' \in J_G$ seien gemäß Theorem 5 polarisierte Abels[…] Mannigfaltigkeiten $\wp$, $\wp'$ vom Typ $\mathfrak{g}$ konstruiert. D[…] Verfasser zeigt, daß $\wp$ und $\wp'$ genau dann isomorph si[…] wenn $GD\mathfrak{R} = GD'\mathfrak{R}$ gilt. Der zu G gehörige Hecke[…] Ring $\mathfrak{R}$ wird mit Hilfe von $J_{G,0}$ und $\mathfrak{U}_0$ in Anlehnung […] Shimura [Ann. of Math. (2) **76** (1962), 237–294] definie[…] Der Verfasser zeigt, daß $\mathfrak{R}$ kommutativ ist. Aufgrund v[…] Theorem 1 gilt eine Zerlegung $J_{G,0} = \bigcup_{\lambda=1}^{h} Gs_\lambda\mathfrak{U}_0$ […] gewissen $s_\lambda \in J_{G,0}$. Mit einem geeigneten $D_\infty \in J_{G,\infty}$ w[…] $D_\lambda = D_\infty \times s_\lambda \in J_G$ gesetzt und gemäß Theorem 5 eine […] D_λ gehörige polarisierte Abelsche Mannigfaltigkeit […] vom Typ $\mathfrak{g}$ konstruiert. Man setze $A = A_1 \times \cdots \times A_h$. D[…] Verfasser zeigt dann, daß jedes Element von $\mathfrak{R}$ e[…] Isogenie von A definiert.

E. Gottschling (Berl[…]

Baily, Walter L., Jr. 5825

A correction to "On the moduli of Abelian varieties with multiplications".

J. Math. Soc. Japan **16** (1964), 182.

The author remarks that G. Shimura pointed out that Lemma 1, p. 370, of the author's paper [same J. **15** (1963), 367–386; MR **28** #3995] does not hold when k is an imaginary extension. He indicates the modifications in subsequent results. *R. J. Crittenden* (Providence, R.I.)

Barsotti, Iacopo 5826

Metodi analitici per varietà abeliane in caratteristica positiva. Capitoli I, II.

Ann. Scuola Norm. Sup. Pisa (3) **18** (1964), 1–25.

This paper is devoted to an extension of the theory of Witt vectors. The new objects which are introduced arise in connection with points of order p^n on an abelian variety in characteristic p and will be used to study these varieties —the endomorphisms, the local analytic group, etc.

Let R be a ring of characteristic p in which every element has a unique pth root, which contains a perfect subfield k, and which is assumed also to be complete under a reasonable topology. Vect R denotes then the (topological) ring of infinite Witt vectors over R. The new objects bear approximately the same relation to the Witt vectors as Laurent series do to ordinary power series.

First Cov R (the Witt covectors) is introduced as the set of sequences $(\cdots, x_{-2}, x_{-1})$ from R satisfying a suitable convergence condition. Like the Witt vectors, they are added by certain universal formulas for the components, and there are natural operators t, π, and p on them. Cov R is a topological (Vect k)-module. The bivectors $(\cdots, x_{-1}, x_0, x_1, \cdots)$ are then defined as those sequences for which $(\cdots, x_n)$ is a covector for all n. They form a topological group which is completable to Biv R. This last is a complete topological ring, as well as a topological (Vect R)-module. Finally, the topology in R permits the introduction of analysis—the exponential, logarithmic and Artin-Hasse exponential series are introduced and briefly studied.

A. Mattuck (Cambridge, Mass.)

Lubin, Jonathan 5827

One-parameter formal Lie groups over $\mathfrak{p}$-adic integer rings.

Ann. of Math. (2) **80** (1964), 464–484.

Soit $\mathfrak{o}$ un anneau de valuation discrète complet, de caractéristique 0, d'idéal maximal $\mathfrak{p}$ tel que le corps résiduel $k=\mathfrak{o}/\mathfrak{p}$ soit de caractéristique $p>0$. L'auteur étudie les lois de groupe formel à un paramètre F à coefficients dans $\mathfrak{o}$, et spécialement l'anneau des endomorphismes formels $\mathrm{End}_{\mathfrak{o}}(F)$ d'une telle loi de groupe. En associant à un tel endomorphisme (qui est une série formelle à coefficients dans $\mathfrak{o}$) le coefficient de son terme du premier degré, il montre qu'on obtient un isomorphisme de $\mathrm{End}_{\mathfrak{o}}(F)$ sur un sous-anneau fermé de $\mathfrak{o}$. D'autre part, par réduction mod $\mathfrak{p}$, la loi de groupe F donne une loi de groupe formel F^* sur k; si la hauteur h de F^* est finie, on montre que l'homomorphisme $\mathrm{End}_{\mathfrak{o}}(F) \to \mathrm{End}_k(F^*)$ est injectif, ce qui a pour conséquence que $\mathrm{End}_{\mathfrak{o}}(F)$ a un corps des fractions qui est une extension de $\mathbf{Q}_p$ dont le degré est un diviseur de h. On en déduit que lorsqu'on considère des anneaux de valuation discrète complets assez grands $\mathfrak{o}'$ contenant $\mathfrak{o}$, $\mathrm{End}_{\mathfrak{o}'}(F)$ devient indépendant de $\mathfrak{o}'$ (anneau absolu des endomorphismes de F). L'auteur montre ensuite que si les coefficients des termes de degré total $\le p^h$ dans F appartiennent à un sous-anneau de $\mathfrak{o}$ dont le corps des fractions est une extension non ramifiée de $\mathbf{Q}_p$, alors $\mathrm{End}_{\mathfrak{o}}(F)$ est intégralement clos, et son corps des fractions une extension non ramifiée de $\mathbf{Q}_p$. Ceci l'amène à considérer une classe spéciale de lois de groupe F, dites "pleines", pour lesquelles l'anneau absolu d'endomorphismes $\mathrm{End}(F)$ est intégralement clos et de rang h (égal à la hauteur de F^*) sur $\mathbf{Z}_p$; l'auteur montre que si en outre $\mathfrak{o}/\mathfrak{p}$ est algébriquement clos, ces lois de groupe sont déterminées à isomorphisme près par $\mathrm{End}(F)$. En outre il construit des lois de groupe "pleines" pour lesquelles l'anneau $\mathrm{End}(F)$ est isomorphe à l'anneau des entiers dans une extension finie quelconque de $\mathbf{Q}_p$. Il montre d'autre part qu'il y a des lois de groupe non "pleines".

J. Dieudonné (Paris)

LINEAR ALGEBRA

See also 5857, 5924, 5926, 6285, 6318, 6368, 6376, 6614, 6620, 6821.

Wedderburn, J. H. M. 5828

★Lectures on matrices.

Dover Publications, Inc., New York, 1964. vii + 200 pp. $1.65.

This edition is an unabridged and unaltered republication of the work first published in 1934 by the American Mathematical Society [New York, 1934] as Volume XVII in their Colloquium Publications.

Table of Contents: I. Matrices and vectors; II. Algebraic operations with matrices. The characteristic equation; III. Invariant factors and elementary divisors; IV. Vector polynomials. Singular matric polynomials; V. Compound matrices; VI. Symmetric, skew, and Hermitian matrices; VII. Commutative matrices; VIII. Functions of matrices; IX. The automorphic transformation of a bilinear form; X. Linear associative algebras; Appendix I. Notes (historical); Appendix II. Bibliography. Index to bibliography.

Badaljan, A. E. 5829

On a variational problem. (Russian. Armenian summary)

Izv. Akad. Nauk Armjan. SSR Ser. Fiz.-Mat. Nauk **17** (1964), no. 3, 3–6.

In dealing with large velocities one meets the following problem: Given the quadratic matrix $A=[a_{ij}]$, $i,j=1,2,\cdots,n$, and let Φ be the set of all admissible mutual uniform self-mappings of the set $\{1,2,\cdots,n\}$, determine

$$\min_{\varphi\in\Phi} \max_i \{a_{i\varphi(i)}\}, \qquad i=1,2,\cdots,n,$$

and the mapping φ for which the above-mentioned extremum is attained. One theorem regarding necessary and sufficient conditions for this extremum is investigated. The proof is very complicated, hence the author does not give the general formula. At the end of the paper it is shown that the same special problem can also be treated and solved. *D. P. Rašković* (Belgrade)

Hakuhara, Masuo 5830

Sur la dépendance linéaire de trois applications linéaires.

Funkcial. Ekvac. **6** (1964), 47–54.

Let H_1, H_2, H_3 be linear transformations from a vector space V to a vector space V' such that $H_i(X) \subset H_j(X) + H_k(X)$ for any subspace X of V and any permutation (i, j, k) of $(1, 2, 3)$. The author asserts that under these conditions H_3 is a linear combination of H_1 and H_2, and he generalizes this to $n+1$ linear transformations, with an additional condition. The proofs are based on a lemma for which an incorrect demonstration is given. Thus the asserted results remain doubtful.

The author does not even mention the field of scalars underlying V and V', but he assumes tacitly that this field is algebraically closed. Without this assumption, his results are false. *O. Wyler* (Albuquerque, N.M.)

Stoer, Josef 5831

On the characterization of least upper bound norms in matrix space.

Numer. Math. **6** (1964), 302–314.

For any norm on n-space, the set $\{x : \|x\| \leq 1\}$ is compact, convex and contains 0; conversely, such a set defines a vector norm. It is harder to describe all (sub-multiplicative) matrix norms, although those induced by vector norms, $\|A\| = \sup \|Ax\|/\|x\|$, form a simple subclass. The author proves that they are exactly the minimal matrix norms. He also defines the dual matrix norm of A by $\sup \mathrm{Re}(\mathrm{trace}\, AB/\|B\|)$, and investigates some of its properties. *W. G. Strang* (Cambridge, Mass.)

Černikov, S. N. 5832

Fundamental theorems in the theory of linear inequalities. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 1181–1190.

The author points out and proves several logical relations among some well-known basic theorems on finite systems of linear inequalities. *Ky Fan* (Evanston, Ill.)

Ehlich, Hartmut 5833

Determinantenabschätzung für binäre Matrizen mit $n \equiv 3 \bmod 4$.

Math. Z. **84** (1964), 438–447.

Let $A = [a_{ij}]$ be a matrix of order n with entries $a_{ij} = \pm 1$ and let α_n denote the maximum of the absolute values of the determinants of all such $(+1, -1)$-matrices of order n. It has been previously shown that

$$\begin{aligned} \alpha_n^2 &\leq n^n = h_n^2 && \text{for } n \equiv 0 \pmod 4, \\ &\leq (n-1)^{n-1}(2n-1) && \text{for } n \equiv 1 \pmod 2, \\ &\leq 4(n-2)^{n-2}(n-1)^2 && \text{for } n \equiv 2 \pmod 4, \end{aligned}$$

and the present paper concentrates on the case $n \equiv 3 \pmod 4$. Define the sets of matrices of order m $(1 \leq m \leq n)$

$$\mathfrak{a}_m = \{C_m \mid C_m = [c_{ij}],\ c_{ii} = n,\ c_{ij} \equiv 3 \pmod 4,\ i, j = 1, \cdots, m\},$$

and let C_m^* denote those matrices in $\mathfrak{a}_m$ with $\det C_m^*$ maximal. One knows that $\alpha_n^2 \leq \det C_n^*$ for $n \equiv 3 \pmod 4$, and the main portion of the present paper deals with the evaluation of $\det C_n^*$. This evaluation takes the form $\det C_n^* = \max\{D_n(s) \mid 1 \leq s \leq n\}$, where the function $D_n(s)$ is evaluated explicitly. One may deduce, for example, that

$$\alpha_n^2 \leq \frac{4 \cdot 11^6}{7^7} (n-3)^{n-7} n^7, \qquad n \equiv 3 \pmod 4,\ n \geq 63.$$

H. J. Ryser (Syracuse, N.Y.)

Wojtas, M. 5834

On Hadamard's inequality for the determinants of order non-divisible by 4.

Colloq. Math. **12** (1964), 73–83.

If A_n is a real $n \times n$ matrix with $-1 \leq a_{ij} \leq 1$, then $|A_n|^2 \leq n^n$, with equality possible only for $n = 1, 2$, or $4t$ $(t = 1, 2, 3, \cdots)$. For odd n this bound was improved to $|A_n|^2 \leq (2n-1)(n-1)^{n-1}$ by G. Barba [Giorn. Mat. Battaglini (3) 71 (1933), 70–86]. The present author considers the case $n \equiv 2 \pmod 4$ and shows that $|A_n|^2 \leq 4(n-1)^2(n-2)^{n-2}$. This result has also appeared recently (apparently independently) in H. Ehlich [Math. Z. **83** (1964), 123–132; MR **28** #4003]. The interested reader should also see H. Ehlich [#5833 above].

L. D. Baumert (Pasadena, Calif.)

Brauer, Alfred 5835

On the characteristic roots of non-negative matrices.

Recent Advances in Matrix Theory (Proc. Advanced Seminar, Math. Res. Center, U.S. Army, Univ. Wisconsin, Madison, Wis., 1963), pp. 3–38. Univ. Wisconsin Press, Madison, Wis., 1964.

This is a general discourse on the subject of characteristic roots of real matrices with non-negative elements, principally concerned with the author's contributions over the past few years. The topics discussed include the following: theorems concerning the location of characteristic roots in ovals of Cassini; characteristic roots of stochastic matrices; the Perron-Frobenius theorems on positive matrices; characteristic roots of power-positive matrices; the theorems of Frobenius on non-negative matrices; stochastic matrices with real characteristic roots α, β such that $|\alpha| > |\beta| \geq |\gamma|$ for any other root γ.

The results given have for the most part been published previously, but some new work is included. Nearly all proofs of theorems are given.

B. N. Moyls (Vancouver, B.C.)

Carranza, César 5836

The decomposition of a vector space by normal endomorphisms. (Spanish)

Univ. Nac. Ingen. Inst. Mat. Puras Apl. Notas Mat. **1** (1962/63), 89–95.

The two theorems of this paper are particular cases of Theorem 2 and Proposition 4, respectively, of Bourbaki [*Les structures fondamentales de l'analyse*, Livre II, Algèbre, Chapitre 9, Actualités Sci. Indust., No. 1272, Hermann, Paris, 1959; MR **21** #6384; ibid., Chapitre 7, no. 3, Actualités Sci. Indust., No. 1179, Hermann, Paris, 1952; MR **14**, 237]. *P. Abellanas* (Madrid)

Dionísio, J. J. [Dionísio, José Joaquim] 5837

An extension of a result of Schur. (Portuguese. English summary)

Univ. Lisboa Revista Fac. Ci. A (2) **9** (1961/62), 307–310.

The theorem stated in this paper says that if E^n is a unitary space and $\lambda \neq 0$ is a characteristic root of a linear transformation represented by the matrix A, then, in order that no non-zero vector in N_λ, the null space of $A-\lambda I$, is orthogonal to a non-zero vector in the null space of the adjoint transformation $A^*-\bar{\lambda}I$, the dimension of N_λ has to be equal to the multiplicity of the root λ in the characteristic polynomial of A. The author proves the theorem by showing that λ must be a simple root of the minimal polynomial, but he overlooks the fact that his condition also implies that dim $N_\lambda = 1$.

M. J. Wonenburger (Toronto, Ont.)

Dionísio, José Joaquim 5838

Non-negative matrices. (Portuguese)

Univ. Lisboa Revista Fac. Ci. A (2) **10** (1963), 5–35.

This is an expository paper describing the spectral properties of non-negative matrices, i.e., square matrices whose entries are non-negative real numbers. Proofs are included. Some of the results are proved for Perron matrices, defined by the author as square real matrices for which there exists a real polynomial $f(x)$ such that the coefficients of the matrix $f(A)$ are all positive.

M. J. Wonenburger (Toronto, Ont.)

Pruitt, William E. 5839

Eigenvalues of non-negative matrices.

Ann. Math. Statist. **35** (1964), 1797–1800.

Let $P=(p_{ij})$, $i,j=0,1,2,\cdots$, be a non-negative infinite matrix which is irreducible in the sense that the transitive closure of the relation $\{(i,j); p_{ij}>0\}$ contains all pairs (i,j), and let λ be a positive number. The author gives a necessary and sufficient condition (too complicated to reproduce here) for the existence of a non-negative, non-zero vector x satisfying $Px=\lambda x$. The method adopted is to use some results of Vere-Jones [Quart. J. Math. Oxford Ser. (2) **13** (1962), 7–28; MR **25** #4571] to reduce the problem to the case in which $\sum_i p_{ij} \leq 1$ for all j, and then to apply a theorem of Harris [Proc. Amer. Math. Soc. **8** (1957), 937–942; MR **19**, 989] and Veech [Proc. Amer. Math. Soc. **14** (1963), 856–860; MR **27** #6302] to the transpose of P. A by-product of the argument is a description of the set of non-negative solutions x of the inequality $Px \leq \lambda x$.

J. F. C. Kingman (Cambridge, England)

Bogdanov, Ju. S. 5840

Transforming a variable matrix to canonical form. (Russian)

Dokl. Akad. Nauk BSSR **7** (1963), 152–154.

The work reported on was presented in a seminar at Leningrad University in 1947; applications of the work have appeared, making it desirable to present proofs. Theorem 1 is as follows. Let $\Delta=(a,b)$. Let A be a matrix of elements, each of which is a continuous function of t, $t\in\Delta$. If $\tau\in\Delta$ is such that in some neighborhood of τ, multiplicities (and degrees of the elementary divisors) of the proper values of A do not change, τ is called a c-point; the set of c-points Δ_A is clearly everywhere dense. There is a matrix V, continuous on Δ_A, that transforms A into Jordan form. Nothing more is true; all such matrices V may be discontinuous at every point of $\Delta\setminus\Delta_A$. Theorem 2 states a corresponding result for a matrix A of elements, each of which is holomorphic in a region of the z-plane.

J. L. Brenner (Palo Alto, Calif.)

Pless, Vera 5841

On Witt's theorem for nonalternating symmetric bilinear forms over a field of characteristic 2.

Proc. Amer. Math. Soc. **15** (1964), 979–983.

Let k be a field of characteristic 2, let f be a bilinear form on a finite-dimensional vector space V over k, which is non-alternating ($f(x,x)\neq 0$ for some $x\in V$) and non-degenerate. Moreover, it is assumed that the values $f(x,x)$ lie in a subfield of k which is contained in k^2. Under these assumptions there exists a unique h in V such that $f(h,x)^2=f(x,x)$ for all x in V. The author proves a theorem of Witt type for this situation. The theorem can be stated as follows. Let σ be a linear transformation defined on a subspace U of V such that $f(\sigma(x),\sigma(y))=f(x,y)$ for x, y in V. Then σ can be extended to a linear transformation defined on all of V with the same property if and only if the following holds: (i) if $h\in U$, then $\sigma(h)=h$; (ii) if $h\notin U$, then $h\notin\sigma(U)$. From this, one gets as corollaries results on maximal isotropic subspaces of V and conditions for two subspaces of V to be transformable into each other by a transformation in the orthogonal group of f.

T. A. Springer (Utrecht)

Virier, Marcel 5842

Sur certaines équivalences de matrices.

C. R. Acad. Sci. Paris **259** (1964), 497–500.

Let $M\in(p\times p)_K$ be a $p\times p$ matrix over a commutative field K. If $x^k\Phi(x)$ is the minimal polynomial of M, where $\Phi(0)\neq 0$, the rank of M^k is called the height of M. The note is concerned with $mp\times mp$ matrices T that are partitioned into $p\times p$ matrices M_{ij} $(i,j=1,2,\cdots,m)$ which commute in pairs. The set of matrices $\{M_{ij}\}$ generates a commutative K-ring H. The nil ideal of H is denoted by N. An equivalence transformation on T is a sequence of elementary transformations acting on the rows or columns of T, where T is regarded as a matrix over H. The following theorem is proved: T is equivalent to a diagonal matrix over H/N in which the heights of the diagonal terms do not increase.

W. Ledermann (Brighton)

Fulkerson, D. R. 5843

The maximum number of disjoint permutations contained in a matrix of zeros and ones.

Canad. J. Math. **16** (1964), 729–735.

Let A be an arbitrary m-by-n (0, 1)-matrix with $m\leq n$. Let

$$(1)\qquad A=P_1+P_2+\cdots+P_p+R,$$

where R is a (0, 1)-matrix and each P_i $(i=1,\cdots,p)$ is a permutation matrix. Let $\pi=\pi(A)$ denote the maximum value of the integer p among decompositions of A of the

form (1). The main theorem in this paper gives an explicit formula for $\pi(A)$, namely,

(2) $$\pi(A) = \min_{A'}\left[\frac{N_1(A')}{s(A')}\right],$$

where A' is an e-by-f minor of A, $N_1(A')$ denotes the number of 1's in A', $s(A') = e+f-n$, brackets denote the largest integer, and the minimum is over all minors A' of A with $s(A') > 0$. The proof is constructive and singles out a critical minor A' of A that yields the minimum in (2). In case A is n-by-n with precisely p 1's in each row and column, then the formula reduces to the familiar $\pi(A) = p$.

Using the formula and a result due to Haber, one may evaluate

(3) $$\tilde{\pi} = \min_{A \in \mathfrak{A}} \pi(A),$$

where $\mathfrak{A}$ is the class of all m-by-n (0, 1)-matrices that have the same row and column sums as A. The determination of the minimum value of p in the decomposition (1) for a fixed A under the assumption that R contains no permutation and the determination of the maximum value of $\pi(A)$ over the class $\mathfrak{A}$ generated by A remain open.

H. J. Ryser (Syracuse, N.Y.)

Gáspár, Gyula 5844

Die Charakterisierung der Determinanten über einem unendlichen Integritätsbereich mittels Funktionalgleichungen.

Publ. Math. Debrecen **10** (1963), 244–253.

In a previous paper [same Publ. **3** (1954), 257–260; MR **17**, 338] the author gave an axiomatic characterization of the determinant function on matrices over an integral domain (commutative and with 1) of characteristic 0. In the present paper this characterization is extended to matrices over any infinite integral domain R. Let R_n be the ring of n-square matrices with entries in R and let φ be a function on R_n into R, not identically zero, satisfying the following axioms: (a) $\varphi(A+B) = \sum \varphi(C)$, where the summation extends over all 2^n matrices C whose jth column is either the jth column of A or the jth column of B, $j = 1, \cdots, n$; (b) $\varphi(AB) = \varphi(A)\varphi(B)$; (c) $\varphi(aA) = a^n\varphi(A)$, for all A, B in R_n and all a in R. Then $\varphi(A)$ is the determinant of A.

The paper concludes with counter-examples showing that the axioms (a), (b) and (c) are independent and that the condition that R be infinite cannot be dropped.

H. Minc (Santa Barbara, Calif.)

Marcus, Marvin 5845

The Hadamard theorem for permanents.

Proc. Amer. Math. Soc. **15** (1964), 967–973.

Let per A denote the permanent of the $n \times n$ matrix $A = (a_{ij})$. The author and M. Newman [Ann. of Math. (2) **75** (1962), 47–62; MR **25** #96] conjectured that the inequality

(*) $$\operatorname{per} A \geq a_{11}a_{22}\cdots a_{nn}$$

is valid for every non-negative hermitian matrix A, and a result somewhat weaker than (*) was subsequently established by the author and H. Minc [Trans. Amer. Math. Soc. **104** (1962), 510–515; MR **25** #3059]. In the present paper it is shown that if A is a non-negative hermitian $n \times n$ matrix and $1 \leq i \leq n$, then

(**) $$na_{ii} \operatorname{per} A_i \geq \operatorname{per} A \geq a_{ii} \operatorname{per} A_i,$$

where A_i denotes the matrix obtained when the ith row and the ith column are deleted in A. From (**), the inequality (*) follows by a simple process of induction. The cases of equality in both (*) and (**) are discussed fully.

L. Mirsky (Sheffield)

Slater, Morton L.; Thompson, Robert J. 5846

A permanent inequality for positive functions on the unit square.

Pacific J. Math. **14** (1964), 1069–1078.

The van der Waerden conjecture states that the permanent $P = \sum_{\sigma \in \mathfrak{S}} \prod_{i=1}^n a_{i\sigma(i)}$ of a doubly stochastic matrix (a_{ij}), where $\mathfrak{S}$ is the group of all permutations of $1, \cdots, n$, satisfies $P \geq n!n^{-n}$. It follows from this conjecture that $\prod_{i=1}^n a_{i\sigma(i)} \geq n^{-n}$ for some $\sigma \in \mathfrak{S}$. Marcus and Minc [Proc. Amer. Math. Soc. **13** (1962), 571–579; MR **25** #3057] proved the following related result. If $\prod a_{ii}$ is not exceeded by any other term of the permanent, then

(1) $$\sum \log a_{ii} \geq \sum\sum a_{ij} \log a_{ij} \geq -n \log n.$$

The authors give a very nice proof of (1), and they generalize (1) to functions on $I \times I$, where $I = [0, 1)$ is the half-open unit interval, as follows. Let f be a measurable function on $I \times I$ satisfying: (1°) $f(x, y) > 0$, and

$$\int_0^1 f(x, y)\,dx = 1 = \int_0^1 f(x, y)\,dy$$

for all $x, y \in I$; (2°) for all measure-preserving bijective transformations T on I, $f(x, Tx)$ is measurable, and

$$-\infty < \int_0^1 \log \frac{f(x, Tx)}{f(x, x)}\,dx \leq 0;$$

(3°) $$\int_0^1 \left|\log \frac{f(x, x+\delta)}{f(x, x)}\right| dx \to 0 \text{ as } \delta \to 0$$

(put $f(x, y+1) = f(x, y)$).

Then $f \log f$ is integrable on $I \times I$, and

$$\infty > \int_0^1 \log f(x, x)\,dx \geq \int_0^1 \int_0^1 f(x, y) \log f(x, y)\,dx\,dy \geq 0.$$

O. Wyler (Albuquerque, N.M.)

ASSOCIATIVE RINGS AND ALGEBRAS

See also 5817, 5826, 5842, 5871, 5877, 5879, 5880a–c, 5883, 5884, 5921, 6338.

Salmon, Paolo 5847

Serie convergenti su un corpo non archimedeo con applicazione ai fasci analitici.

Ann. Mat. Pura Appl. (4) **65** (1964), 113–125.

Soient K un corps valué complet pour une valeur absolue valuative, et A l'anneau de la valuation correspondante. Une série formelle restreinte est un élément $\sum_i c_i X^i \in K[[X_1, \cdots, X_n]]$ tel que c_i tende vers 0 quand $|i|$ tend vers l'infini (i désignant un multi-indice); la condition que c_i tende vers 0 signifie que, pour tout point x du

polydisque unité U de K^n, la série $f(x) = \sum_i c_i x^i$ est absolument convergente; les fonctions f ainsi obtenues s'appellent les fonctions strictement holomorphes sur U. Les séries formelles restreintes forment un sous-anneau $K\{X_1, \cdots, X_n\}$ de $K[[X_1, \cdots, X_n]]$, dans lequel le théorème de préparation de Weierstrass est vrai, et qui est noethérien et factoriel. Soient $\mathfrak{O}$ le faisceau des germes de fonctions holomorphes sur le polydisque U, et $s_1, \cdots, s_p$ des sections strictement holomorphes de $\mathfrak{O}$; alors le faisceau des relations entre les s_i est de type fini, et est engendré par un nombre fini de relations à coefficients strictement holomorphes. *P. Samuel* (Paris)

Fried, Erwin [Fried, Ervin] 5848
Beiträge zur Theorie der Frobenius-Algebren.
Math. Ann. **155** (1964), 265–269.

Let A be a Frobenius algebra with radical N and let I be the annihilator of N. An idempotent e is called principal if the residue class of e in A/N is in the center. A is called reduced if no principal idempotent except 0 and 1 has the property $eI = Ie$. A reduced algebra is indecomposable (in fact, indecomposability is equivalent to $eI_2 \neq I_2 e$, where I_2 is the annihilator of N^2). Let e be principal; then $eI = Ie$ if and only if eAe is Frobenius and the annihilator of its radical is eIe. If e is a minimal principal idempotent, then eAe is reduced.

If A is reduced, then I is a power of N, A/N is a direct sum of isomorphic simple algebras, and the minimal, orthogonal, primitive idempotents $e_1, \cdots, e_n$ can be chosen so that $e_i I = I e_{i+1}$. *D. Zelinsky* (Evanston, Ill.)

Jacobson, N. 5849
Clifford algebras for algebras with involution of type D.
J. Algebra **1** (1964), 288–300.

Let k be a field of characteristic $\neq 2$. An algebra with involution over k is a pair $(\mathfrak{A}, J)$, where $\mathfrak{A}$ is a finite-dimensional associative algebra over k and J an involution in $\mathfrak{A}$. This paper deals mainly with algebras with involution which are simple of type D_l. These are defined as follows: There is an algebraic extension K of k and an isomorphism of the K-algebra $\mathfrak{A} \otimes_k K$ onto the algebra of matrices over K with $2l$ rows and columns, which transforms $J \otimes 1$ into the adjoint mapping relative to a nondegenerate symmetric matrix, which has maximal Witt index. One may assume K to be a finite Galois extension of k.

In this paper a construction is given of the even Clifford algebra $\mathfrak{C}^+(\mathfrak{A}, J)$ of any algebra with involution of type D_l. This generalizes the even Clifford algebra of a quadratic form of even rank. The construction is by a Galois descent argument, which is more or less standard. The author then gives some results concerning the structure of $\mathfrak{C}^+(\mathfrak{A}, J)$. It is an algebra with involution ι over k. In the case where $\mathfrak{A}$ is the algebra of matrices with $2l$ rows and columns over k and J the adjoint mapping relative to a nondegenerate symmetric matrix S, $\mathfrak{C}^+(A, J)$ is isomorphic to the even Clifford algebra of the quadratic form associated with S, and ι is the main involution in that algebra. $\mathfrak{C}^+(A, J)$ is either the direct sum of two simple ideals $\mathfrak{C}_1$ and $\mathfrak{C}_2$ (case D_{II}) or is simple (case D_{III}). The most interesting case is D_{II}, l even. Then ι maps the simple ideals $\mathfrak{C}_1$ and $\mathfrak{C}_2$ into themselves. It is proved that in this case the sum of the elements of the Brauer group of k, defined by $\mathfrak{A}$, $\mathfrak{C}_1$ and $\mathfrak{C}_2$, is zero. The proof of this fact involves an interesting generalization of the classical principle of triality. *T. A. Springer* (Utrecht)

Mauldon, J. G. 5850
Nonorthogonal idempotents whose sum is idempotent.
Amer. Math. Monthly **71** (1964), 963–973.

The main result of the paper is contained in Theorem 1: For every integer $n > 2$ there exists a ring $\mathfrak{R}$ (with unit) whose additive group is torsion-free, containing a set $\{a_1, a_2, \cdots, a_n\}$ of n idempotents whose sum is idempotent, such that $a_i a_j \neq 0$ for all $i, j = 1, 2, \cdots, n$. The author first constructs a ring $\mathfrak{R}_1$ containing four mutually non-orthogonal idempotents whose sum is the unit e. The tensor product $\mathfrak{R}_k$ of a suitable number of copies of $\mathfrak{R}_1$ (regarded as an algebra over the integers) satisfies the conditions of the theorem if $n \not\equiv 2 \pmod 3$. The author also constructs a ring $\mathfrak{R}_0$ containing four mutually non-orthogonal idempotents whose sum is zero. The proof of the theorem is then completed by considering rings of the form $\mathfrak{R}_0 \otimes \mathfrak{R}_k$. *R. A. Melter* (Amherst, Mass.)

Onodera, Takesi 5851
On semi-linear normal basis.
J. Fac. Sci. Hokkaido Univ. Ser. I **18** (1964), 23–33.

In this paper a result of Nakayama concerning semi-linear normal basis for fields, and for division rings under certain assumptions [Amer. J. Math. **71** (1949), 241–248; MR **10**, 425], is extended to what has been called strictly Galois extensions of division rings. In particular, the following theorem is proved: Let Δ be a division ring and let Φ be a division subring of Δ. Suppose that $[\Delta:\Phi]_r = n$, that $\mathfrak{G}_0$ is a group of automorphisms of Δ whose fixed subring is Φ, and that the order of $\mathfrak{G}_0$ is n. Let Λ be a division subring of Δ which is invariant under $\mathfrak{G}_0$. If $[\Delta:\Lambda]_r = nq + r$ $(0 \leq r < n)$, then there exist $q+1$ elements $\xi_1, \xi_2, \cdots, \xi_q, \rho$ in Δ such that $\{\xi_i^\sigma \mid \sigma \in \mathfrak{G}_0, i = 1, 2, \cdots, q\}$, together with suitable r-elements in $\{\rho^\sigma \mid \sigma \in \mathfrak{G}_0\}$, is a right basis for Δ over Λ. If $[\Delta:\Lambda]_r$ is infinite, then there exist elements ξ_α, $\alpha \in A$, in Δ such that $\{\xi_\alpha^\sigma \mid \sigma \in \mathfrak{G}_0, \alpha \in A\}$ is a right basis for Δ over Λ, where A is a set of cardinality $[\Delta:\Lambda]_r$.

A result of Kasch on normal bases for Galois extensions of division rings [Math. Ann. **126** (1953), 447–463; MR **15**, 597] is generalized to semi-linear normal basis [see also #5852 below]. *J. K. Goldhaber* (College Park, Md.)

Kishimoto, Kazuo; Onodera, Takesi; Tominaga, Hisao 5852
On the normal basis theorems and the extension dimension.
J. Fac. Sci. Hokkaido Univ. Ser. I **18** (1964), 81–88.

Let Δ be a division ring, Φ a division subring of Δ, and $\mathfrak{G}$ the group of automorphisms of Δ which fix Φ. Kasch [Math. Ann. **126** (1953), 447–463; MR **15**, 597] has proved that Δ is $\mathfrak{G}\Delta_r$-isomorphic to $\mathfrak{G}\Delta$, if and only if the centralizer of Φ in Δ is the center of Δ or is contained in Φ. In the paper this result is extended in several ways to simple rings. The following is an example of the type of theorem which the authors prove. Let A be an Artinian ring which is Galois and finite over B, let $\mathfrak{G}$ be the group of automorphisms of A over B, and let N be a $\mathfrak{G}$-invariant simple subring of A. Suppose that the centralizer of B in A is different from the two-by-two matrix ring over GF(2). Then

$[\mathfrak{G}N_r : N_r]_r = [A : B]$ if and only if the centralizer of B in A is the center of A or is contained in N.

The paper also includes an improvement of a theorem of Takazawa and Tominaga [J. Fac. Sci. Hokkaido Univ. Ser. I 15 (1961), 198–201; MR 25 #2099]. In particular, the following theorem is proved. Let A be a simple ring with center C, $\mathfrak{H}$ an F-group of A of order p^e (p a prime), and B the subring of A which is fixed by $\mathfrak{H}$. Let Z denote the center of B. If Z does not contain a primitive pth root of unity, then the centralizer of B in A is $C[Z]$ and $[A : B]$ is a divisor of p^e (and a multiple of $p^f = [\mathfrak{H} : \mathfrak{H}_0]$ times the exponent of $\mathfrak{H}_0$, where $\mathfrak{H}_0$ is the subgroup of $\mathfrak{H}$ consisting of all the inner automorphisms contained in $\mathfrak{H}$). Furthermore, if A is not of characteristic p, then $[A : B] = p^e$.

J. K. Goldhaber (College Park, Md.)

Elizarov, V. P.; Pilatovskaja, A. I. 5853

Sufficient conditions for the existence of a quotient ring. (Russian)

Sibirsk. Mat. Ž. 5 (1964), 1191–1194.

Let R be an associative (not necessarily commutative) ring and S a subset of R which is closed under multiplication but which does not contain zero. The definition of generalized right quotient ring $R(S)$ for R with respect to S was given by V. P. Elizarov [Izv. Akad. Nauk SSSR Ser. Mat. 24 (1960), 153–170; MR 22 #9513]. In the present paper there are given various conditions, each of which is sufficient for the existence of $R(S)$. For example, the existence is guaranteed when the minimal condition is satisfied for right ideals in R. Further, we have the theorem that if R is a ring without zero-divisors and if every direct sum of principal right ideals admits only a finite number of summands, then R can be imbedded in a right quotient division ring. This is a generalization of A. W. Goldie's theorem [Proc. London Math. Soc. (3) 8 (1958), 589–608; MR 21 #1988].

E. Inaba (Tokyo)

Balakrishnan, R. 5854

On general rings with descendant chain condition.

Math. Ann. 156 (1964), 337–339.

Soit R un anneau unitaire (non commutatif). Pour $a \in R$, on note $\mathfrak{b}(a)$ l'idéal bilatère engendré par a. Comparaison des ensembles suivants: (a) ensemble A des $a \subset R$ tels que tout élément de $1 + \mathfrak{b}(a)$ soit inversible; (b) ensemble B des $a \in R$ tels que l'idéal bilatère engendré par $1 + \mathfrak{b}(a)$ soit égal à R; (c) intersection I des idéaux bilatères maximaux de R; (d) ensemble Z des $a \in R$ tels que $\mathfrak{b}(a)$ soit nilpotent. Alors $I = B$. Si R est artinien à gauche, on a $Z = A$. Un anneau artinien à gauche R tel que $A = B = I = Z$ n'a qu'un nombre fini d'idéaux premiers, tous maximaux.

P. Samuel (Paris)

Nobusawa, Nobuo 5855

On a generalization of the ring theory.

Osaka J. Math. 1 (1964), 81–89.

Let A, B be left R-modules for a ring R. Let M be a subgroup of $\mathrm{Hom}_R(A, B)$ and N a subgroup of $\mathrm{Hom}_R(B, A)$. If f, f^1 are in M and g, g^1 in N, then the products fgf^1 and gfg^1 are defined and may lie in M and N, respectively. The author abstracts this situation by considering two abelian groups M, N and postulating the existence of products mnm^1 as an element of M, and nmn^1 as an element of N. These products are subject to several axioms, based on the above concrete situation. The author defines simple and semi-simple with minimum condition in this situation, and proves analogues of the Wedderburn theorems. Thus, a simple pair M, N is shown to consist of the $m \times n$ and $n \times m$ matrices over a division ring with the product the matrix product, and a semi-simple pair is a direct sum of such pairs.

A. Rosenberg (Ithaca, N.Y.)

Samuel, Pierre 5856

Un exemple d'anneau factoriel.

Bol. Soc. Mat. São Paulo 15 (1960), 1–4 (1964).

The object of this paper is to prove the following result. Let K be a noetherian unique factorization domain, and let $x_1, \cdots, x_n$ be indeterminates over K with weights $q(1), \cdots, q(n) > 0$, respectively. Let $F(x)$ be an irreducible element of $K[x]$ which is an isobar of weight q. We put $A = [x, z]$, where $z^c = f(x)$ and c is prime to q. Then A is a unique factorization domain if one of the following conditions holds: (1) $c \equiv 1 \pmod q$; (2) Every projective K-module of finite type is free.

Y. Nakai (Hiroshima)

Ohm, Jack; Schneider, Hans 5857

Matrices similar on a Zariski-open set.

Math. Z. 85 (1964), 373–381.

Let D be an integral domain; for every $\mathfrak{p} \in \mathrm{Spec}(D)$, denote by $D(\mathfrak{p})$ the quotient field of $D/\mathfrak{p}$. Let A be an n-by-n matrix over D; denote by $A(\mathfrak{p})$ the matrix over $D(\mathfrak{p})$ obtained from A by reduction mod $\mathfrak{p}$, and by $\rho_i(\mathfrak{p})$ $(i = 1, \cdots, n)$ the degrees of the invariant factors of $A(\mathfrak{p})$ (in descending order). If B' is a matrix similar to $A(\mathfrak{p})$, then the solution space of $A(\mathfrak{p})X - XB' = 0$ has dimension

$$\tau(\mathfrak{p}) = \sum (2i - 1)\rho_i(\mathfrak{p}).$$

For $\mathfrak{p} \in \mathrm{Spec}(D)$ the following are equivalent: (1) $\rho_i(\mathfrak{p}) = \rho_i(0)$ for every i; (2) there exists a dense set V_i containing $\mathfrak{p}$ such that ρ_i is constant on V_i; (3) there exists a dense set V containing $\mathfrak{p}$ on which τ is constant; (4) τ is minimal at $\mathfrak{p}$. Furthermore, the set U of all $\mathfrak{p} \in \mathrm{Spec}(D)$ for which the preceding properties hold is open. Now let B be another n-by-n matrix over D, and let V be a dense subset of $\mathrm{Spec}(D)$ on which τ (for A) is constant; if $A(\mathfrak{p})$ is similar to $B(\mathfrak{p})$ for every $\mathfrak{p}$ in V, and if $\mathfrak{q} \in V$, then there exists a neighbourhood U of $\mathfrak{q}$ such that A is similar to B on U (i.e., in the corresponding ring of quotients of D).

Now let X be a topological space, K a field, and H a ring of functions $f: X \to K$ such that, for every $f \neq 0$ in H, the zero-set of f is discrete. If no point of X is open, this implies that A is a domain. We associate to every x in X the ideal $\mathfrak{p}_x$ of all f in H such that $f(x) = 0$, thus getting a continuous mapping j of X into $\mathrm{Spec}(H)$; for every non-discrete subset V of X, $j(V)$ is dense in $\mathrm{Spec}(H)$. Then the above result about similarity of matrices remains true with D replaced by H and $\mathrm{Spec}(D)$ by X. This gives a result of Wasow [J. Math. Anal. Appl. 4 (1962), 202–206] about holomorphic matrices.

P. Samuel (Paris)

Sands, A. D. 5858

Primitive rings of infinite matrices.

Proc. Edinburgh Math. Soc. (2) 14 (1964/65), 47–53.

Let R be a ring. Let $M(R)$ denote either the ring of row-finite matrices over R or the ring of infinite matrices over

R in which each matrix has only a finite number of nonzero entries. It is shown that R is primitive if and only if $M(R)$ is, and that an ideal P of R is primitive if and only if $M(P)$ is a primitive ideal of $M(R)$. An example of a primitive ideal in $M(R)$, for a certain R, which is not of the form $M(P)$, is given. The question of the Jacobson radical of $M(R)$ is also touched upon.

A. Rosenberg (Ithaca, N.Y.)

Besserre, Annie 5859

Quelques propriétés d'un couple de modules.

C. R. Acad. Sci. Paris **259** (1964), 22–23.

If A is a commutative ring, $E' \subset E$ are modules and $\mathfrak{a}$ is an ideal, then $\mathfrak{a}E \cap E' = \mathfrak{a}E'$ if and only if $\mathfrak{a}_\mathfrak{m} \cap E_\mathfrak{m} = E_\mathfrak{m}'$ for all maximal ideals $\mathfrak{m}$. It follows that whenever E, E' are finitely generated projective modules, the set P of prime ideals such that $\mathfrak{p}E \cap E' = \mathfrak{p}E'$ is open in the Zariski topology of the spectrum. If the hypothesis on E and E' is dropped, P need not be open.

A. Heller (Urbana, Ill.)

Kertész, Andor 5860

Eine kennzeichnende Eigenschaft der injektiven Moduln.

Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe **11** (1962), 737–739.

Let G be an R-module, $\{g_\beta\}$ a subset of G, $\{f_\beta\}$ a set of linear forms in indeterminates $\{x_\alpha\}$. The author considers three conditions on the system of linear equations $\{f_\beta = g_\beta\}$: "compatible" means $\sum s_\beta f_\beta = 0$ (finite sum, $s_\beta \in R$) implies $\sum s_\beta g_\beta = 0$; "solvable" means there are substitutions $x_\alpha \to h_\alpha$ ($h_\alpha \in G$) sending every f_β to g_β; "translatable (übertragbar)" means there are substitutions for a nonempty set of x_α's such that the resulting system of equations is compatible. Theorem: Solvable implies translatable implies compatible; G is injective if and only if compatible implies translatable, and then all three concepts coincide. A short proof can be given as follows: Let F be the free module on $\{x_\alpha\}$ and H the submodule generated by $\{f_\beta\}$. "Compatible" means there is a homomorphism from H to G carrying every f_β to g_β; "solvable" means there is a homomorphism from F to G carrying f_β to g_β; "translatable" means $F = F_1 \oplus F_2$, with F_1 free and nonzero, and there is a homomorphism from $F_1 + H$ to G carrying f_β to g_β. It is then clear that the conditions are ordered as above and that if G is injective, they all coincide. That compatible implies translatable only if G is injective follows by taking $F = R$ (or rather, R with a unit adjoined if, with the author, we do not postulate a unit in R).

D. Zelinsky (Evanston, Ill.)

Fuchs, L. 5861

Ranks of modules.

Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **6** (1963), 71–78.

A subset $\{a_i\}$ of a left R-module M is called independent if $\sum \lambda_i a_i = 0$ ($\lambda_i \in R$) implies $\lambda_i a_i = 0$. A module has rank 1 if every set of two or more elements is dependent. Consider maximal independent subsets U of M whose elements (1) have prime order [equivalently, generate cyclic modules all of whose nonzero elements have the same order] and (2) generate cyclic modules of rank 1. Under suitable hypotheses on R, such sets U exist in every module, and for every prime ideal P, the cardinal number of elements of order P in such a set U is an invariant of the module, independent of U, called the P-rank of M. The hypotheses on R are that every module contains a module of rank 1, and every module of rank 1 contains an element of prime order [equivalently, R/P is an Ore domain for every prime P, and, for every left ideal L, there is a λ in R such that $L:\lambda$ is prime]. These hypotheses are satisfied by commutative Noetherian rings.

D. Zelinsky (Evanston, Ill.)

Kertész, A. 5862

On ranks of modules. A remark to the preceding paper of L. Fuchs.

Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **6** (1963), 79–82.

The author proves that the definition of P-rank in the preceding paper [#5861] works in modules over arbitrary rings (a module is said to have rank 0 if it has no elements satisfying (1) and (2) in #5861 above). The rings satisfying the hypotheses of the paper above then appear as the rings for which only the zero module has rank 0. If H is a submodule of G, then the 0-rank of H plus the 0-rank of G/H is the 0-rank of G.

D. Zelinsky (Evanston, Ill.)

Northcott, D. G. 5863

The Hilbert function of the tensor product of two multigraded modules.

Mathematika **10** (1963), 43–57.

For K a field, consider the polynomial ring R:

$$K[X_{11}, \cdots, X_{1m_1}, X_{21}, \cdots, X_{2m_2}, \cdots, X_{p1}, \cdots, X_{pm_p}]$$

in p sets of variables. Let D be the direct sum of p copies of the non-negative integers. For $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_p) \in D$ let $R^{(\alpha)}$ be the set of those polynomials which are homogeneous of degree α_1 in the first set of variables, homogeneous of degree α_2 in the second set of variables, and so on. In this way R is a D-graded ring. Let E and L be finitely generated D-graded R-modules. Then the expressions of the Hilbert functions $H(\alpha, E)$ (here $H(\alpha, E) = \dim_K E^{(\alpha)}$) and $H(\alpha, L)$ as polynomials (for large α) are related to the expression of $H(\alpha, E \otimes_R L)$ as a polynomial (for large α). This reduces to the standard theory when $p = 1$. The computations involved are lengthy and are of particular interest to the specialist.

D. K. Harrison (Eugene, Ore.)

Wong, Edward T. 5864

Atomic quasi-injective modules.

J. Math. Kyoto Univ. **3** (1963/64), 295–303.

If R is a ring with unity, then a unital module ${}_RM$ is called quasi-injective if for every submodule N of M and every $f \in \operatorname{Hom}({}_RN, {}_RM)$, f can be extended to an element of $\operatorname{Hom}({}_RM, {}_RM)$. This paper extends some of the results of the reviewer and the author [J. London Math. Soc. **36** (1961), 260–268; MR **24** #A1295] to rings having an atomic lattice of closed right ideals.

R. E. Johnson (Rochester, N.Y.)

Kanzaki, Teruo 5865

On commutator rings and Galois theory of separable algebras.

Osaka J. Math. **1** (1964), 103–115; *correction, ibid.* **1** (1964), 253.

The author generalizes the definition of the word "Galois" due to Auslander and Goldman [Trans. Amer. Math. Soc. 97 (1960), 367–409; MR 22 #12130] to the non-commutative case. Let S be a ring, G a finite group of automorphisms of S, and $R = S^G$. Consider S as a right R-module. Then there is a natural homomorphism, j, of the trivial crossed product of S and G to $\mathrm{Hom}_R(S, S)$. S is called a Galois extension of R if j is an isomorphism and if S is a finitely generated projective right R-module. Before doing Galois theory proper, the author studies the centralizers of Azumaya subalgebras of central Azumaya algebras. He generalizes the relevant result of Auslander and Goldman [loc. cit.] to yield a theorem as satisfactory in this case as the one for simple algebras over fields. He then goes on to study the basic properties of Galois extensions along the lines of the cited paper by Auslander and Goldman, and proves, inter alia, that if H is a subgroup of G and $T = S^H$, then S is a Galois extension of T. For the rest of the paper the author assumes that S is a central Azumaya C-algebra, that no element of G induces the identity on C, and that C is a Galois extension of C^G. He then shows that in order that S be Galois over R, it suffices to suppose that the trivial crossed product of S with G is an Azumaya algebra. He ends up by proving a "fundamental theorem of Galois theory", but, unfortunately has to assume that C is an integral domain for the harder half. Since the author's assumption excludes inner automorphisms, this theory cannot, as yet, be regarded as a true generalization of the Galois theory of simple algebras as presented, e.g., in Jacobson [*Structure of rings*, Amer. Math. Soc., Providence, R.I., 1956; MR 18, 373].

A. Rosenberg (Ithaca, N.Y.)

Bleicher, Michael N.; Schneider, Hans 5866

The decomposition of cones in modules over ordered rings.

J. Algebra 1 (1964), 233–258.

On considère un A-module unitaire à gauche sur un anneau totalement ordonné A. On dit qu'un sous-ensemble P de E est un cône si on a $0 \in P$, $P + P \subseteq P$ et $aP \subseteq P$ $(a \geqq 0)$. On voit alors qu'un cône est convexe. Par définition un cône est propre s'il n'a pas de sous-modules non nuls et un cône est premier s'il n'est pas somme directe de deux cônes propres. L'ensemble des sous-cônes d'un cône propre S qui sont facteurs directs de S forme une algèbre de Boole. Dans le cas où A satisfait à la condition de chaîne ascendante et E à la même condition pour ses facteurs directs, tout sous-cône propre de E admet une décomposition première finie. On introduit la notion de PF-module: module sur un anneau ordonné tel que chaque cône propre contienne un élément libre. On établit alors que le cône des matrices hermitiennes semi-définies est premier. *J. Guérindon* (Rennes)

Hayes, Allan 5867

A representation theory for a class of partially ordered rings.

Pacific J. Math. 14 (1964), 957–968.

This paper presents three theorems on commutative partially ordered rings (po-rings), two of which generalize standard results on lattice-ordered rings. The first is a characterization of those po-rings A which are subdirect unions of ordered rings without non-zero nilpotent elements. The condition for A to be such a ring is that the positive cone A^+ contain all squares and have the property that if $a \notin A^+$, then $pa - a^{2n} \notin A^+$ for all $p \in A^+$ and $n > 0$. A ring with this property is called an f^*-ring. The second main theorem involves the following definition. A po-ring A is called r-Archimedean if, whenever $na + pa \leq b$ holds for all integers $n \geqq 0$ and $p \in A^+$, then $a \leqq 0$. It is shown that an r-Archimedean f^*-ring is a subdirect union of r-Archimedean ordered rings without non-zero nilpotent elements, and conversely. The last main theorem is the analogue of a result of Johnson's [Proc. London Math. Soc. (3) 12 (1962), 207–225; MR 25 #5082]: an Archimedean ordered f^*-ring is isomorphic to a ring of extended real-valued ($R \cup \{\pm\infty\}$) functions which are finite on dense open sets of a locally compact Hausdorff space. *R. S. Pierce* (Seattle, Wash.)

Almeida Costa, A. 5868

Sur la théorie générale des demi-anneaux.

Publ. Math. Debrecen 10 (1963), 14–29.

An algebraic system with two binary compositions (sum and product) is a semi-ring ("demi-anneau") if it is an additive semi-group and a multplicative semi-group with left- and right-distributive laws. Ideals, m-systems, and p-systems are defined, and properties analogous to those concerning the similarly named concepts in rings are demonstrated for general semi-rings. Some sharper results are obtained for ordered semi-rings (with lattice structure such that for all x, y, $x + y = x \vee y$ and $xy \leqq x \wedge y$). The notion of μ-system of ideals is introduced, and the fact that for any semi-ring the semi-ring of its ideals (with operations suitably defined) is ordered becomes a useful tool in investigating the ideal structure of semi-rings. This is particularly true for "μ-semi-rings", a concept analogous to the concept "μ-ring" investigated in a previous paper of the author [Univ. Lisboa Revista Fac. Ci. A (2) 8 (1960), 131–144; MR 26 #2459]. For this class of semi-rings, a body of theorems resembling those of the earlier paper is derived. *W. R. Ballard* (Missoula, Mont.)

Koch, H. 5869

Über Halbkörper, die in algebraischen Zahlkörpern enthalten sind.

Acta Math. Acad. Sci. Hungar. 15 (1964), 439–444.

Let P $[R]$ be the field of all rational [real] numbers, K a finite extension of P. Theorems: Every semi-field $H \subset K$ is simple; If K is minimal with respect to $K \supset H$ and if $0 \in H$, then there exists a set Φ of isomorphisms $\varphi : K \to R$ such that H consists exactly of all $b \in K$ with $\varphi(b) \geqq 0$ for all $\varphi \in \Phi$. Let a be some generator of $K \mid P$, $b \in K$. Let every isomorphism $\varphi : K \to R$ satisfy $\varphi(b) > 0$ or $\varphi(a) < 0$. Then there exists a rational function $r(x)$ with non-negative rational coefficients such that $b = r(a)$.

K. Drbohlav (Prague)

NON-ASSOCIATIVE ALGEBRA

Chawla, L. M. 5870

An interpretation of Rees' exchange algebras in relation to the nuclei of a non-associative linear algebra.

J. Natur. Sci. and Math. 3 (1963), 147–162.

Let $M(x_1, \cdots, x_r)$ be a multilinear form on the vector spaces $V_1, \cdots, V_r$ over the field F. Let $E_{ij}(M)$ be the set of all linear transformations U on V_i such that there is a linear transformation U^1 on V_j with

$$M(x_1, \cdots, x_iU, \cdots, x_r) = M(x_1, \cdots, x_jU^1, \cdots, x_r).$$

Each $E_{ij}(M)$ is an algebra known as an exchange algebra of M. These algebras were first studied by Rees [same J. 1 (1961), no. 1, 57–70; MR 27 #2517].

Each non-associative algebra A has various multilinear forms associated with it. In particular, a trilinear form M defined by Bruck [Trans. Amer. Math. Soc. 56 (1944), 141–199; MR 6, 116] in terms of the multiplicative constants of A is associated with A. If N_μ, N_λ and N_ρ are the middle nucleus, left nucleus and right nucleus of A, respectively, then it is shown that t is an element of N_μ if and only if R_t belongs to $E_{12}(M)$; t is an element of N_λ if and only if L_t belongs to $E_{13}(M)$ and t is an element of N_ρ if and only if R_t is an element of $E_{23}(M)$. Similar relationships are demonstrated between the various intersections of the nuclei of A and the exchange algebras of a quartic form and a pentic form on A. Various other forms on A are examined. *R. H. Oehmke* (Princeton, N.J.)

Herstein, I. N. 5871

Sugli anelli semplici alternativi.

Rend. Mat. e Appl. (5) **23** (1964), 9–13.

A (not necessarily associative) algebra A over a field F is called a composition algebra if it admits a non-degenerate quadratic form g such that $g(xy) = g(x)g(y)$. It is known that such algebras have $[A:F] = 1, 2, 4$, or 8; a simple proof may be found in Jacobson [Rend. Circ. Mat. Palermo (2) 7 (1958), 55–80; MR 21 #66]. It has also been shown by Kaplansky [Proc. Amer. Math. Soc. 4 (1953), 956–960; MR 15, 596] that a composition algebra is, with some trivial exceptions, a simple alternative ring. In this note, which is an announcement of results of which detailed proofs will appear later, the author shows how the theory of simple non-associative alternative rings may be subsumed under that of composition algebras. This yields an alternative proof of a theorem of Kleinfeld [Ann. of Math. (2) 58 (1953), 544–547; MR 15, 392] that such a ring is a Cayley-Dickson algebra. More precisely, the author sketches a proof of the fact that a simple non-associative alternative ring of characteristic $\neq 2$ is an algebra over a field F, and that every element satisfies a quadratic equation with coefficients in F. The constant term then yields the desired quadratic form. *A. Rosenberg* (Ithaca, N.Y.)

Kleinfeld, Erwin 5872

Classification theorems of simple non-associative rings with some applications to projective planes.

Rend. Mat. e Appl. (5) **23** (1964), 4–8.

The author discusses the relationship between the existence of projective planes and the existence of non-associative division rings. Some of the recent developments in the determination of the simple rings in certain classes of non-associative rings are outlined. It is indicated how these results determine the existence of classes of projective planes and the importance, from the point of view of projective planes, of obtaining further results on certain classes of simple rings. *R. H. Oehmke* (Princeton, N.J.)

HOMOLOGICAL ALGEBRA

See also 5814, 5856, 5859, 5865, 5905, 5906, 5949, 5950, 5952, 5960.

Dedecker, Paul 5873

Les foncteurs $\mathscr{E}xt_\Pi$, $\mathbf{H}_\Pi{}^2$ et $H_\Pi{}^2$ non abéliens.

C. R. Acad. Sci. Paris **258** (1964), 4891–4894.

In the usual theory of extensions of groups one classifies short exact sequences

$$0 \to H \to E \to G \to 0,$$

where G and H are given, and which induce a specific action of G on H, i.e., a specific homomorphism φ of G into the quotient group of automorphisms of G by inner automorphisms. In the present paper the group of automorphisms is replaced by an arbitrary group Π, together with a crossed Π-module structure on H. The study of Π-extensions generalizes the study of ordinary extensions of G by H for all possible actions. The resulting "global" classification theorems are in terms of groupoids and principal sets over groups (octopi) and probably take account of all the structure that is to be found in the extension problem. *J. W. Gray* (Urbana, Ill.)

Deheuvels, René 5874

Homologie des ensembles ordonnés et des espaces topologiques.

Bull. Soc. Math. France **90** (1962), 261–321.

Importante contribution à l'algèbre homologique classique permettant, entre autres, de définir une homologie d'un espace à valeurs dans un préfaisceau et une cohomologie à valeurs dans un cofaisceau. Ce mémoire développe une note précédente [C. R. Acad. Sci. Paris **250** (1960), 2492–2494; MR **26** #1884]. Les questions traitées pourraient actuellement être généralisées en tenant compte des résultats récents de Grothendieck et ses élèves sur les topologies et les faisceaux. Voir, par exemple, Michael Artin [*Grothendieck topologies* (mimeographed notes), Harvard, 1962] et Michel Demazure [Topologies et faisceaux, Séminaire de Géométrie Algébrique, Inst. Hautes Études Sci. Publ. Math., 1963].

L'auteur part du fait que la théorie de Čech repose sur la structure d'ordre de l'ensemble des ouverts d'un espace topologique. Sur un ensemble ordonné $\mathscr{E}$, $\preccurlyeq$ considéré comme catégorie (les morphismes $a \to b$, $a, b \in \mathscr{E}$, étant les relations $a \preccurlyeq b$), un système covariant [contravariant] A de coefficients est un foncteur de même variance $A: \mathscr{E} \to \mathscr{C}$ où $\mathscr{C}$ est une catégorie abélienne. Soit $\mathscr{C}(\mathscr{E})[\mathscr{C}^*(\mathscr{E})]$ la catégorie dont les objets sont de tels systèmes covariants [contravariants]. Si $\mathscr{C}$ est suffisamment raisonnable on définit des foncteurs sections $\Gamma: \mathscr{C}(\mathscr{E}) \to \mathscr{C}$, et co-sections $L: \mathscr{C}(\mathscr{E}) \to \mathscr{C}$, solutions de problèmes universels duaux. Les dérivés à droite $r^n\Gamma$ et à gauche l^nL fournissent alors la cohomologie $H^n(\mathscr{E}, -): \mathscr{C}(\mathscr{E}) \to \mathscr{C}$, et l'homologie $H_n(\mathscr{E}, -): \mathscr{C}(\mathscr{E}) \to \mathscr{C}$ à valeurs dans un système de coefficients $A \in \mathscr{C}(\mathscr{E})$. Dualement pour $A \in \mathscr{C}^*(\mathscr{E})$.

L'une des principales applications consiste à associer à tout ensemble ordonné $\mathscr{E}$ un schéma simplicial $K = K(\mathscr{E})$ et à tout schéma simplicial K un ensemble ordonné $\mathscr{E} = \mathscr{E}(K)$. Cela est fait de telle manière que coïncident l'homologie des $\mathscr{E}$ et K à valeurs dans un système contravariant de coefficients ainsi que la cohomologie à valeurs dans un système covariant, les seules qui soient

définies pour un schéma simplicial K. Il devient donc possible de définir l'homologie et la cohomologie de K à valeurs dans un système A, quelle que soit sa variance; d'où par exemple l'homologie d'un espace à valeurs dans un préfaisceau.

Parmi les notions fondamentales mentionnons celle d'"ordre" ρ d'un ensemble ordonné $\mathscr{E}$ dans un autre $\mathscr{E}'$: Il s'agit essentiellement d'un ordre sur la somme disjointe $\mathscr{E} \cup \mathscr{E}'$ tel qu'aucun élément de $\mathscr{E}'$ ne précède un élément de $\mathscr{E}$. Toute application croissante $\lambda: \mathscr{E} \to \mathscr{E}'$ définit un tel ordre ρ_λ. Un "morphisme au-dessus de ρ" entre système $A \in \mathscr{C}(\mathscr{E})$ et $A' \in \mathscr{C}(\mathscr{E}')$ est essentiellement un système B sur l'ordre ρ de $\mathscr{E} \cup \mathscr{E}'$. Un morphisme au-dessus de ρ_λ correspond à une transformation naturelle du foncteur A dans le foncteur $A' \circ \lambda$. Ces notions jouent un rôle essentiel dans les théorèmes d'unicité. Fondamentale aussi est la notion d'"hyperdérivé" d'un foncteur composé

$$G \circ F: \mathscr{C} \to \mathscr{C}' \to \mathscr{C}''.$$

Pour $A \in \mathscr{C}(A)$ on considère des résolutions projective $\mathscr{P}A$ et injective $\mathscr{J}A$ d'où des $\mathscr{C}'$-complexes $F\mathscr{P}A$, $F\mathscr{J}A$. En en prenant à nouveau des résolutions on obtient alors quatre $\mathscr{C}''$-complexes $G\mathscr{P}F\mathscr{P}A$, $G\mathscr{P}F\mathscr{J}A$, $G\mathscr{J}F\mathscr{P}A$, $G\mathscr{J}F\mathscr{J}A$ dont l'homologie fournit par définition les hyperdérivés du foncteur GF. Par exemple $(lr)_nGF(A) = H^n(G\mathscr{P}F\mathscr{J}A)$. Exemple parmi les théorèmes: Sur un espace X on note $\mathscr{E}$ l'ensemble ordonné de ses ouverts et $\mathscr{F}$ une sous-famille de $\mathscr{E}$ suffisamment riche. D'où un ordre canonique π de $\mathscr{E}$ dans $\mathscr{F}$. Soit d'autre part $A \in \mathscr{C}(\mathscr{E})$ un préfaisceau sur X, $\mathscr{C}$ étant une catégorie abélienne AB5 au sens de A. Grothendieck [Tôhoku Math. J. (2) **9** (1957), 119–221; MR **21** #1328]. Alors l'homologie de X à valeurs dans A se calcule au moyen des cochaînes de $\mathscr{F}$ à valeurs dans πA.

Un mémoire complémentaire est annoncé devant étudier la dualité homologie-cohomologie et les relations avec la théorie de Borel-Moore: Homologie à coefficients dans un faisceau sur un espace localement compact.

P. Dedecker (Lille)

Ehresmann, Charles 5875

Groupoïdes sous-inductifs.

Ann. Inst. Fourier (Grenoble) **13** (1963), *fasc.* 2, 1–60.

From the introduction: "Les principaux résultats de ce mémoire sont: l'existence et les propriétés des groupoïdes sous-préinductifs des atlas faibles complets et des atlas complets attachés à un groupoïde sous-préinductif; les théorèmes de complétion d'un groupoïde prélocal; les théorèmes d'existence du groupoïde sous-inductif des filtres sur un groupoïde sous-préinductif. Cet article est une généralisation, et une étude plus approfondie, de la théorie des groupoïdes inductifs, qui interviennent dans de nombreuses questions; il peut donc être considéré comme la suite d'un article antérieur [mêmes Ann. **10** (1960), 307–332; MR **22** #8525]."

There are 24 definitions, 45 propositions, 41 corollaries, 20 theorems and no examples. *J. W. Gray* (Urbana, Ill.)

Ehresmann, Ch. [Ehresmann, Charles] 5876

Sous-structures et catégories ordonnées.

Fund. Math. **54** (1964), 211–228.

From the introduction: "Nous avons défini [Ann. Sci. École Norm. Sup. (3) **80** (1963), 349–426] la notion de sous-structure d'une structure dans une catégorie d'homomorphismes $(\mathscr{C}, p, \mathscr{H}, \mathscr{S})$ lorsque $\mathscr{C}$ est mun d'une structure de catégorie inductive. Ici, nous allo généraliser cette notion, en définissant les $(\mathscr{C}', p)$-injectio où $(\mathscr{C}, p, \mathscr{H})$ est un foncteur de $\mathscr{H}$ vers $\mathscr{C}$, et $\mathscr{C}'$ u sous-catégorie de $\mathscr{C}$. En particulier, nous verrons que l résultats sont valables en remplaçant la catégorie inducti $(\mathscr{C}, <)$ par une catégorie ordonnée. Ensuite, nous définisso les catégories sous-préinductives et sous-inductives étudions les propriétés du pseudo-produit."

J. W. Gray (Urbana, Il

Enochs, Edgar 58

Torsion free covering modules.

Proc. Amer. Math. Soc. **14** (1963), 884–889.

For R an integral domain, let E be any R-module. By torsion-free cover of E is meant a torsion-free R-modu $T(E)$, together with a homomorphism t_E from $T(E)$ to such that (1) every homomorphism from every torsic free R-module F to E factors through $T(E)$, and (2) t kernel of t_E contains no non-zero pure submodules of $T($ It is proved that there exists one and, up to ismorphis only one torsion-free cover of E. If K is the kernel of then it is shown that $0 \to \mathrm{Ext}_R{}^i(F, K) \to \mathrm{Ext}_R{}^i(F, T(E))$ $\mathrm{Ext}_R{}^i(F, E) \to 0$ is exact for every integer i and torsion-fr R-module F. If M is a maximal ideal of R and T $T(R/M)$ is a submodule of a torsion-free R-module F, wi $T \cap (M \cdot F) = M \cdot T$, then T is a direct summand of Hence T is a sort of M-adic integer. In particular, for the integers, the projective cover of a cyclic group of or p is the p-adic integers. The elegance of the meth remind this reviewer of the construction of inject envelopes. *D. K. Harrison* (Eugene, O

Guerra, Juan 58

Algebraic theory of projective and inductive lim (Spanish)

Univ. Nac. Ingen. Inst. Mat. Puras Apl. Notas Mat (1962/63), 156–169.

Harada, Manabu 58

Structure of hereditary orders over local rings.

J. Math. Osaka City Univ. **14** (1963), 1–22.

Let R be a discrete rank one valuation ring of a field and let Σ be a central simple K-algebra. A subring Γ o is called an h-order if Γ is an order of R in Σ (i.e., $\Gamma \cdot K = \Sigma$, $\Gamma \supset R$, and Γ is finitely generated as an module) and if Γ is hereditary (every left ideal of Γ is p jective). These are of interest since Auslander and Go man proved that an order Γ is a maximal order (the ord ordered by inclusion; maximal orders always exist and unique up to isomorphism) if and only if Γ is an h-or with $\Gamma/\mathrm{rad}(\Gamma)$ a simple ring. In the present note a s prisingly complete theory is given for all h-orders. It proved that all maximal chains of h-orders have fi length, and in fact the same length; call it n. Let Γ be h-order. Let $N = \mathrm{rad}(\Gamma)$, the radical of Γ. Then Γ/N i direct sum of full matrix rings over division rings, these division rings are isomorphic to a fixed division r Δ, so we may write $\Gamma/N \simeq \Delta_{m_1} \oplus \cdots \oplus \Delta_{m_r}$ (where Δ_m is ring of m-by-m matrices over Δ). For $i = 1, \cdots, r$ let M_i the maximal ideal of Γ which consists of all $x \in \Gamma$ such t the ith component of $x + N \in \Gamma/N$ is zero in the ab

breakdown of Γ/N. Let N^{-1} be the set of $y\in\Sigma$ with $NyN\subset\Gamma$. Then $N^{-1}M_1N, N^{-2}M_1N^2, \cdots, N^{-s-1}M_1N^{s+1}$ is proven to be the set of all maximal ideals of Γ, so, after re-indexing if necessary, we may assume that $M_2=N^{-1}M_1N$, $M_3=N^{-2}M_1N^2$, etc. We agree to distinguish r-tuples only up to cyclic permutations (so

$$(m_1, m_2, \cdots, m_r) = (m_2, \cdots, m_r, m_1),$$

for instance). With this agreement, $(m_1, \cdots, m_r)$ is independent of the indexing and so is an invariant of Γ; it is called the form of Γ. $m_1, m_2, \cdots, m_r$ are positive integers and it is proven that $m_1+m_2+\cdots+m_r=n$. Conversely, if $k_1, k_2, \cdots, k_s$ are positive integers with $k_1+k_2+\cdots+k_s=n$, then it is proven that there is an h-order which has $(k_1, k_2, \cdots, k_s)$ for form. Most important of all, two h-orders are isomorphic if and only if they have the same form. The maximal orders are the ones with form (n). The minimal h-orders are the ones with form $(1, 1, \cdots, 1)$. Hence, minimal h-orders always exist, and they are all isomorphic. An h-order Γ has a constant form $(t, t, \cdots, t)$ if and only if $\mathrm{rad}(\Gamma)$ is a principal ideal. Other ramifications are given. *D. K. Harrison* (Eugene, Ore.)

Harada, Manabu 5880a

Hereditary orders in generalized quaternions D_γ.

J. Math. Osaka City Univ. **14** (1963), 71–81.

Harada, Manabu 5880b

Multiplicative ideal theory in hereditary orders.

J. Math. Osaka City Univ. **14** (1963), 83–106.

Harada, Manabu 5880c

Hereditary orders which are dual.

J. Math. Osaka City Univ. **14** (1963), 107–115.

Hereditary orders, introduced by Auslander and Goldman [Trans. Amer. Math. Soc. **97** (1960), 1–24; MR **22** #8034], were first studied by the author in his thesis [ibid. **107** (1963), 273–290; MR **27** #1474], and their structures were completely determined (modulo knowledge of maximal orders) by the reviewer in his thesis [Ph.D. Dissertation, Princeton Univ., Princeton, N.J., 1963; Bull. Amer. Math. Soc. **69** (1963), 721–724; MR **27** #2543].

In the first paper, the author studies hereditary orders in quaternion algebras over the rationals, using the known structure of maximal orders in this case. In the second, the author generalizes successfully all of Deuring's definitions and results on maximal orders (of non-arithmetic nature): normal ideals, characteristic products, ideal class group, discriminant [Deuring, *Algebren*, Springer, Berlin, 1935]. The starting point of the third is the reviewer's observation (in his thesis) that the endomorphism ring of a projective module over an hereditary order is also hereditary; the consequences are fully investigated.

A. Brumer (Ann Arbor, Mich.)

Ito, Yoshihiko 5881

On relative cohomology group of groups. (Japanese. English summary)

Sci. Rep. Fac. Lib. Arts Ed. Gifu Univ. Natur. Sci. **3** (1962/63), 155–160.

The author gives the cup-product of Adamson's relative cohomolgy groups. Let G be an abelian group, H_i $(i=1, 2)$ subgroups, $H_3=H_1\cap H_2$, $\bar G=G(Z)$ the group ring of G over the rational integer ring Z, and A and $B\bar G$-modules, respectively.

He shows existence of a $\bar G$-homomorphism $\varphi_{r,s}$ of the $(r+s)$-relative chain group $C_{r+s}(G:H_3)$ into $C_r(G:H_1)\otimes C_s(G:H_2)$ such that $\varphi_{r,s}\partial^3_{r+s-1}=\partial'_{r+1}\varphi_{r+1,s}+\partial''_{s+1}\varphi_{r,s+1}$ and $(\varepsilon_1\otimes\varepsilon_2)\varphi_{0,0}=\varepsilon_3$, where ε_i denotes the usual mapping of $C_0(G:H)$ into Z, $\partial_r'=\partial_r^1\otimes 1_s$, and $\partial_s''=(-1)^r(1_r\otimes\partial_s^2)$. Then he obtains the cup-product $\cup$ of $H^r([G:H_1], A)$ and $H^s([G:H_2], B)$ into $H^{r+s}([G:H_3], A\otimes B)$ naturally.

K. Masuda (Okayama)

Kleisli, H.; Wu, Y. C. 5882

On injective sheaves.

Canad. Math. Bull. **7** (1964), 415–423.

A short proof is given of the existence of sufficiently many injections in the category of sheaves of abelian groups. It is based on the Eckmann-Schopf technique [Arch. Math. **4** (1953), 75–78; MR **15**, 5] and uses the terminology of Maranda [Trans. Amer. Math. Soc. **110** (1964), 98–135; MR **29** #1236]. *J. W. Gray* (Urbana, Ill.)

Lazard, Daniel 5883

Sur les modules plats.

C. R. Acad. Sci. Paris **258** (1964), 6313–6316.

It is proven that in the category of left unitary modules over a unitary ring, a module M is flat if and only if it is the inductive limit of a directed system of free modules of finite type. A number of consequences are mentioned for tensor, symmetric, and exterior algebras.

J. W. Gray (Urbana, Ill.)

Nakayama, T. 5884

Class group of cohomologically trivial modules and cyclotomic ideals.

Acta Arith. **9** (1964), 245–256.

If Λ is a left Noetherian ring, $\mathfrak{P}$ the category of finitely generated left Λ-modules and $\mathfrak{F}$ that of finitely generated left Λ-modules of finite projective dimension, then the inclusion $\mathfrak{P}\subset\mathfrak{F}$ induces $\mathfrak{G}\mathfrak{P}\approx\mathfrak{G}\mathfrak{F}$, where $\mathfrak{G}$ is the Grothendieck group. In the special case that Λ is the integer group ring of a finite group, then $\mathfrak{F}$ just contains the cohomologically trivial modules.

But further, Swan [Ann. of Math. (2) **71** (1960), 552–578; MR **25** #2131] showed that if $P\in\mathfrak{P}$, then $Q\otimes P$ is free. Thus the inclusion $\mathfrak{R}\subset\mathfrak{F}$, where $\mathfrak{R}$ is the category of finite cohomologically trivial Λ-modules, induces a homomorphism $\mathfrak{G}\mathfrak{R}\to\mathfrak{G}\mathfrak{P}$ whose image consists precisely of the projective class group $\mathfrak{C}\mathfrak{P}$, regarded as the subgroup of $\mathfrak{G}\mathfrak{P}$ of elements of virtual rank 0.

These observations are used to make explicit computations in case the group is of prime order, the elements of $\mathfrak{C}\mathfrak{P}$ then being represented by ideals in the cyclotomic field [Rim, ibid. (2) **69** (1959), 700–712; MR **21** #3474].

A. Heller (Urbana, Ill.)

Pupier, René 5885

Sur les décompositions de morphismes dans les catégories à sommes ou à produits fibrés.

C. R. Acad. Sci. Paris **258** (1964), 6317–6319.

This paper should include a reference to Grothendieck

[*Séminaire Bourbaki*, 1959/60, Exp. 190, Secrétariat mathématique, Paris, 1960; MR 23 #A2273], where one can find most of the remarks concerning normal monomorphisms (essentially strict monomorphisms), universal monomorphisms, their duals, and the relations between these notions and fibre products and sums. In this context, the weak image of a morphism is defined to be the fibre product of the sub-objects of the range through which the morphism factors. An epimorphism is called strong if its image is its range. This gives a unique decomposition of an arbitrary morphism into the composition of a strong epimorphism and a monomorphism. Under suitable conditions this decomposition has the expected desirable properties. *J. W. Gray* (Urbana, Ill.)

Sonner, Johann 5886

Universal solutions and adjoint homomorphisms.

Math. Z. **86** (1964), 14–20.

The author discusses the relation between the solution of left universal (= co-universal) and right universal problems, and adjoint homomorphisms. Thus if $\mathscr{A}$ and $\mathscr{B}$ are categories and F is a functor from $\mathscr{A}^0 \times \mathscr{B}$ to the category $\mathfrak{St}$ of sets and mappings, then there is a universal map for each $\mathscr{A}$-object a if and only if the identity mapping of $\mathscr{B}$ admits a left adjoint. Examples (which, it is claimed, cannot be treated by previous methods) include symmetric and exterior algebras and tensor products of pairs of modules (over a commutative coefficient ring).

P. M. Cohn (Chicago, Ill.)

Švarc, A. S. 5887

Duality of functors. (Russian)

Dokl. Akad. Nauk SSSR **148** (1963), 288–291.

The author constructs a notion of duality of functors in a wide class of categories. This class includes categories of abelian groups, topological spaces, Banach spaces, sets, partially ordered sets, and lattices. The proofs are supplied. *S. Rubinstein* (Seattle, Wash.)

GROUP THEORY AND GENERALIZATIONS

See also 5722, 5748, 5802, 5849, 5881, 5960, 6472, 6492, 6985.

★Topics in Abelian groups. 5888

Proceedings of the Symposium on Abelian Groups, held at New Mexico State University, June 4–8, 1962. Edited by J. M. Irwin and E. A. Walker.

Scott, Foresman and Co., Chicago, Ill., 1963. 368 pp. $6.00.

The papers contained in this volume will be reviewed individually. There is a list of 30 unsolved problems presented at the symposium, together with an account (pp. 357–364) of the present status of the 86 unsolved problems stated by L. Fuchs in his book [*Abelian groups*, Publ. House Hungar. Acad. Sci., Budapest, 1958; MR 21 #5672].

Hall, Marshall, Jr.; Senior, James K. 5889

★The groups of order 2^n ($n \le 6$).

The Macmillan Co., New York; Collier-Macmillan, Ltd., London, 1964. 225 pp. $15.00.

The ambitious task the authors set themselves in this b is the determination of all groups of order 2^n, where n: They not only succeed, but also supply us with a great of information about the groups on their list. For $n =$ and 3 these groups are, of course, well known. It turns that there are 14 groups of order 16, 51 of order 32, 267 of order 64.

The classification begins with the notion of "fam which was introduced by Philip Hall [J. Reine Ang Math. **182** (1940), 130–141; MR **2**, 211]. For a group G $Z(G)$ and $[G, G]$ denote the center of G and the c mutator subgroup of G, respectively. The natural from $G \times G$ to $[G, G]$ lifts to a map from $G/Z(G) \times G/Z$ to $[G, G]$. Call this map φ. For another group G' call corresponding map φ'. Then G and G' are said to bel to the same family if there exist isomorphisms ρ of G/Z with $G'/Z(G')$ and λ of $[G, G]$ with $[G', G']$ such $\varphi' \circ \rho \times \rho = \lambda \circ \varphi$. "Belonging to the same family" is cle an equivalence relation. The groups under considera fall into 27 families. The procedure is first to determine families which can occur, and then to classify the gro which occur within each family. Of these two tasks first is apparently the more difficult. It is accomplis with the use of Schur's theory of the multiplier of a gr [I. Schur, ibid. **127** (1904), 20–50].

A careful reading of the first chapter, which deals definitions and notation, is sufficient to enable the rea to use the tables. The remaining chapters deal with theory which leads to the classification.

The first tables deal with family invariants, i.e., th invariants of a group which depend only on the famil which the group belongs. The length of the lower cen series is an example. Following these are tables which the groups of order 2^n, $n \le 6$, in each of the 27 fami The groups are given by generators and relations. For group we are supplied, among other things, with the n ber of elements of each order, and the order of the a morphism group.

Finally, there are 130 pages of lattice diagrams gi abundant information about the lattice of normal groups of all the various groups under consideration.

The format of this book is rather unusual. The dim sions are 14 inches high, 17 inches wide, and ½ inch th It is interesting to conjecture about the eventual loca of the book in a typical library. Moreover, the first pages have three pages per page. More precisely, the ten sides have three columns, each column being giv page number. Thus the eleventh side turns out to be 31.

Undoubtedly this book will serve as a useful tes ground for group-theoretical conjectures. It is perhaps fortunate that the testing ground has such extrem awkward dimensions. *Michael Rosen* (Providence,

Fridman, A. A.

Degrees of unsolvability of the word problem for fin presented groups. (Russian)

Dokl. Akad. Nauk SSSR **147** (1962), 805–808.

The main theorem of this important paper is that, g any recursively enumerable degree of unsolvability α, can effectively find a finitely presented group whose problem is of degree α. The proof, outlined in the pa proceeds more or less as follows.

A Turing machine with a halting-type problem

degree α is constructed. The author then constructs a semi-group for which the problem of equality to a fixed generator q is of degree α. Finally, following W. W. Boone [Ann. of Math. (2) **70** (1959), 207–265], the author constructs the required group B. The difficulty is in showing that the word problem for B reduces to the above-mentioned problem for the semi-group. This is accomplished by (1) solving the word problem for the group G obtained from B by removing the defining relation $q^{-1}tqk = kq^{-1}tq$, and (2) considering a variation of the extended word problem in B relative to the subgroup generated by all generators other than k. *M. Greendlinger* (Ivanovo)

Britton, John L. 5891

The word problem.

Ann. of Math. (2) **77** (1963), 16–32.

The proof given in this paper for P. S. Novikoff's theorem on the unsolvability of the word problem for groups is the simplest so far published, certainly for those familiar with free products with amalgamated subgroups. This is the main tool in the proof of the following principal lemma.

Let G be obtained from H by adding the generators p_i and the defining relations $p_i^{-1}A_{i,j}p_i = B_{i,j}$, where for every fixed i the mapping in H determined by $A_{i,j} \to B_{i,j}$ is an isomorphism. If $w = 1$ in G and contains at least one of the p_i, then w contains a subword $p_a^{-1}Cp_a$ [or $p_aCp_a^{-1}$], where C belongs to the subgroup of H generated by the $A_{a,j}$ [or the $B_{a,j}$].

The group proven to have an unsolvable word problem is virtually the same as in W. W. Boone [same Ann. (2) **70** (1959), 207–265]. However, here the author builds it up from a free group in five stages along the lines suggested by the principal lemma. Repeated application of the latter, together with some intricate uses of induction, yields the desired result.

{E should read E^* in line 7 from the bottom on p. 20. E should read F in (4) on p. 25.}

M. Greendlinger (Ivanovo)

Piccard, Sophie 5892

Les groupes pseudo-libres et les groupes fondamentaux.

C. R. Acad. Sci. Paris **259** (1964), 24–26.

A group is called pseudo-free if an infinite factor group is obtained by equating all the elements of some generating set. A group generated by a set of elements such that no finite subset can be replaced by a smaller set is called fundamental. Several results and examples dealing with these concepts are given. *M. Greendlinger* (Ivanovo)

Pschaar, Günter 5893

Über ein spezielles dreifaches schiefes Produkt.

Acta Sci. Math. (*Szeged*) **25** (1964), 143–148.

Das dreifache schiefe Produkt $\mathfrak{G} \circ \Gamma \circ G$ der Gruppen $\mathfrak{G} = (a, b, \cdots)$, $\Gamma = (\alpha, \beta, \cdots)$, $G = (A, B, \cdots)$ hat die Multiplicationsvorschrift

$$(a, \alpha, A)(b, \beta, B) = (ab^{\alpha}, \alpha\beta^{A}, AB^{a}),$$

wobei b^{α}, β^{A}, B^{a} drei beliebige Funktionen mit Werten in $\mathfrak{G}$, Γ, G bezeichnen. Mit den Anfangsbedingungen

$$a^{\varepsilon} = a; \qquad \alpha^{E} = \alpha; \qquad A^{e} = A;$$
$$e^{\alpha} = e; \qquad \varepsilon^{A} = \varepsilon; \qquad E^{a} = E,$$

für beliebige $a \in \mathfrak{G}$, $\alpha \in \Gamma$, $A \in G$ und für $e \in \mathfrak{G}$, $\varepsilon \in \Gamma$, $E \in G$ beweist der Verfasser dass $\mathfrak{G} \circ \Gamma \circ G$ eine Gruppe bildet dann, und nur dann wenn die folgenden Relationen erfüllt sind: für alle $a, b, c, \alpha, \beta, \gamma, A, B, C$

$$(1) \quad \begin{array}{lll} (bc)^{\alpha} = b^{\alpha}c^{\alpha}; & (\beta\gamma)^{A} = \beta^{A}\gamma^{A}; & (BC)^{a} = B^{a}C^{a}; \\ (c^{\beta})^{\alpha} = c^{\alpha\beta}; & (\gamma^{B})^{A} = \gamma^{AB}; & (C^{b})^{a} = C^{ab}; \\ c^{\beta^{A}} = c^{\beta}; & \gamma^{B^{a}} = \gamma^{B}; & C^{b^{\alpha}} = C^{b}. \end{array}$$

Definiert man die zulässigen Normalteiler $\mathfrak{G}_1$, Γ_1, G_1 von $\mathfrak{G}$, Γ, G durch $\mathfrak{G}_1 = (a \mid a \in \mathfrak{G},\ A^{a} = A$ für alle $A \in G)$, usw., ergibt dann den zweiten Satz so:

$$\mathfrak{G} \circ \Gamma \circ G/(\mathfrak{G}_1, \Gamma_1, G_1) \cong \mathfrak{G}/\mathfrak{G}_1 \times \Gamma/\Gamma_1 \times G/G_1$$

mit $(\mathfrak{G}_1, \Gamma_1, G_1) \cong \mathfrak{G}_1 \times \Gamma_1 \times G_1$ d.h. $\mathfrak{G} \circ \Gamma \circ G$ ist isomorph zu einer Schreierschen Erweiterung von $\mathfrak{G}_1 \times \Gamma_1 \times G_1$ durch $\mathfrak{G}/\mathfrak{G}_1 \times \Gamma/\Gamma_1 \times G/G_1$. *V. S. Krishnan* (Madras)

Hancock, V. Ray 5894

Commutative Schreier semigroup extensions of a group.

Acta Sci. Math. (*Szeged*) **25** (1964), 129–134.

The paper continues the study of complete structures [Wiegandt, Acta Sci. Math. (Szeged) **19** (1958), 93–97; MR **20** #2387; ibid. **19** (1958), 219–223, MR **20** #6476; the author, Proc. Amer. Math. Soc. **11** (1960), 71–76; MR **22** #4787]. The main theorem is that if S is a group, Q a commutative semigroup with identity element, Q' the maximal cancellative homomorphic image of Q, and Q^* the difference-group of Q', then the group of equivalence classes of commutative Schreier extensions of S by Q is isomorphic to, and can be obtained from, the group of commutative group extensions of S by Q^*. An immediate consequence of this theorem is that a commutative semigroup is complete if and only if it is a divisible group.

R. Artzy (Princeton, N.J.)

Azleckiĭ, S. P. 5895

On the factorization of finite groups. (Russian)

Ural. Gos. Univ. Mat. Zap. **3**, *tetrad* 3, 3–17 (1962).

Expository article. First factorizations into cyclic, Abelian, nilpotent and solvable factors are considered. Then the orders of the factors are subject to certain conditions or one of the factors is assumed to be a normal subgroup. *L. Fuchs* (Budapest)

Azleckiĭ, S. P. 5896

On the theory of π-special groups. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 969–975.

A finite group G is π-special, where π is a set of primes, if G has a normal, nilpotent Hall π-subgroup. Define the π-commutator subgroup $K_\pi(G)$ of G as the subgroup of G generated by all commutators of π-elements of G, the π-Frattini subgroup $\Phi_\pi(G)$ as the intersection of all maximal π-subgroups of G. The author proves the following analogue of a result due to Wielandt: G is π-special if and only if $K_\pi(G) \leq \Phi_\pi(G)$ and $\Phi_\pi(G)$ is nilpotent. The remainder of the article consists of necessary and/or sufficient conditions for a group to be solvable in terms of properties of $H_\pi{}^0$-subgroups, these being refinements of H^0-subgroups introduced by the author in an earlier paper [Ukrain. Mat. Ž. **16** (1964), 220–225; MR **28** #5107].

P. Fong (Berkeley, Calif.)

Morris, A. O. 5897

A note on the multiplication of Hall functions.

J. London Math. Soc. **39** (1964), 481–488.

Pursuing his efforts to make explicit the coefficient in the expansion of a product $Q_{(\lambda)}Q_{(\mu)}$ of two Hall functions, the author succeeds in the special case $Q_{(\lambda)}q_r$ in improving on his previous theorem [Proc. London Math. Soc. (3) **13** (1963), 733–742; MR **27** #3712; cf. also D. E. Littlewood, ibid. (3) **11** (1961), 485–498; MR **24** #A173]. The improvement consists in recognizing that the coefficient f in the expansion

$$Q_{(\lambda)}q_r = \sum_{(\nu)} f^{(\nu)}_{(\lambda)r}(t)Q_{(\nu)}$$

can be written as a product of factors

$$f^{(\nu)}_{(\lambda)r}(t) = \prod_{i=1}^{s} (1-t^{r_i}),$$

with certain further limitations which are natural in the context. In the second part of the paper this theorem is used to simplify the proof of I. J. Davies [Proc. Edinburgh Math. Soc. (2) **13** (1962), 1–4; MR **26** #3780] in the case $(\mu)=(p^r)$. *G. de B. Robinson* (Toronto, Ont.)

Beaumont, R. A.; Pierce, R. S. 5898

Isomorphic direct summands of abelian groups.

Math. Ann. **153** (1964), 21–37.

In a previous paper the authors proved that "most large" abelian groups are isomorphic to a proper direct summand of themselves. In particular, this is true for every torsion group of power greater than the continuum. In this paper they point out that a group G has an isomorphic proper direct summand if and only if there exist φ, θ in the endomorphism ring of G such that $\theta\circ\varphi=1$ and $\varphi\circ\theta\neq 1$. For this reason they make a systematic study of systems $\langle G;\varphi,\theta\rangle$, where G is an abelian group and φ and θ are endomorphisms of G with $\theta\circ\varphi=1$. Such systems are equivalent to Δ-modules, where $\Delta=Z[X,Y]/(XY-1)$. Here $Z[X,Y]$ is the polynomial ring in two non-commuting indeterminates and $(XY-1)$ is the ideal generated by $XY-1$. They show that every Δ-module is naturally an extension of an automorphic module by a shift module, where automorphic and shift modules are of a certain explicit type which may be considered known. For this reason they study $\mathrm{Ext}_\Delta{}^1$. For K an automorphic module and T any module (so possibly a shift module) they construct a seven-term exact sequence which, among other things, pins $\mathrm{Ext}_\Delta{}^1(T,K)$ between $\mathrm{Hom}_Z(T,K)$ and $\mathrm{Ext}_Z(T,K)$, where Z denotes the integers. The maps in this exact sequence are natural, and the condition that they be one-one, onto, etc., is equivalent to certain natural situations in the original problem. The authors now interpret this general theory in special cases and harvest a number of interesting special results.

D. K. Harrison (Eugene, Ore.)

Fuchs, L. 5899

On algebraically compact abelian groups.

J. Natur. Sci. and Math. **3** (1963), 73–82.

The known properties of algebraically compact abelian groups are presented systematically. The many useful equivalent definitions of this fundamental concept, the Fleischer-Kaplansky characterization by number invariants, and the known relation to cotorsion groups, are all given in the author's usual perspicuous style.

D. K. Harrison (Eugene, Ore.)

Rangaswamy, K. M. 5900

A note on algebraic compact groups.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 369–371.

It is proved that every algebraically compact group is the completion of a direct sum of cyclic groups. Also, a number of conditions on a cotorsion group are given which are equivalent to algebraic compactness. The proofs are homological. *D. K. Harrison* (Eugene, Ore.)

Golema, K.; Hulanicki, A. 5901

The structure of the factor group of the unrestricted sum by the restricted sum of Abelian groups. II.

Fund. Math. **53** (1963/64), 177–185.

Part I (by A. Hulanicki) appeared in Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **10** (1962), 77–80 [MR **26** #5060]. Let $G_1, G_2, G_3, \cdots$ be a sequence of abelian groups whose cardinalities are not greater than that of the continuum. Let S^* be the direct product of these groups, and let S be the direct sum. A well-known theorem of Hulanicki states that S^*/S is algebraically compact. This result is refined in this paper to give necessary and sufficient conditions on a group A that it be isomorphic to S^*/S for some sequence of groups. These conditions are explicit and, although technical, they are not complicated.

D. K. Harrison (Eugene, Ore.)

Irwin, J. M.; Khabbaz, S. A. 5902

On generating subgroups of Abelian groups.

Proc. Colloq. Abelian Groups (Tihany, 1963), pp. 87–97. *Akadémiai Kiadó, Budapest*, 1964.

The authors call an Abelian group G Σ^n-cyclic when G is generated by n subgroups, each of which is a direct sum of cyclic groups. One of the authors, Khabbaz, submitted a question on the conditions under which Abelian groups are Σ^2-cyclic. J. D. Reid [Proc. Amer. Math. Soc. **13** (1962), 222–225; MR **24** #A3190] gave a solution in the torsion-free case.

In this paper, the authors give a solution in the general case. The main results are as follows: (1) Every reduced p-group is Σ^2-cyclic. (2) A p-group G is Σ^2-cyclic if and only if G is not a finite direct sum of $Z(p^\infty)$'s directly summed with a group of bounded order. (3) A group G is Σ^2-cyclic if and only if (a) G/T (T a maximal torsion subgroup of G) is free of finite rank and T is Σ^2-cyclic or (b) the torsion-free rank of G is infinite. (4) A group G is Σ^n-cyclic for some n if and only if G is Σ^2-cyclic. (5) A p-group G is generated by n of its basic subgroups for some integer n if and only if G is generated by two of its basic subgroups.

K. Honda (Tokyo)

Loiko, N. 5903

The S-isomorphism of torsion-free Abelian groups (Russian)

Ural. Gos. Univ. Mat. Zap. **3**, *tetrad'* 1, 67–71 (1961).

For a group G, let $S(G)$ denote the lattice of all cosets of all

subgroups of G together with the empty set, where the intersection is the set-theoretical intersection and the union of the cosets aH, bF ($a, b \in G$; H, F subgroups of G) is aK; here K is the subgroup generated by H, F and $a^{-1}b$. An S-isomorphism φ of G and G_1 is a lattice-isomorphism of $S(G)$ and $S(G_1)$; φ induces a one-to-one mapping between G and G_1, since the elements are cosets of the identities e, e_1. If $\varphi(e) = e_1$ and if G, G_1 are torsion-free Abelian, then this induced mapping is an isomorphism between G and G_1.

L. Fuchs (Budapest)

Loiko, N. V. 5904

S-isomorphisms of mixed Abelian groups of rank $r \geq 2$. (Russian)

Ural. Gos. Univ. Mat. Zap. 4, *tetrad* 1, 57–59 (1963).

The main result of the preceding paper [#5903] is extended to mixed Abelian groups of torsion-free rank ≥ 2.

L. Fuchs (Budapest)

Loonstra, F. 5905

A-ordering of the group $\mathrm{Ext}(B, A)$.

Proc. Colloq. Abelian Groups (*Tihany*, 1963), *pp.* 105–114. *Akadémiai Kiadó, Budapest*, 1964.

Let A be an abelian group. By an extension of A is meant a pair (G, α), where G is an abelian group and α is a monomorphism of A into G. Write $(G_1, \alpha_1) < (G_2, \alpha_2)$ for two extensions of A if there exists a homomorphism f from G_1 to G_2 with $\alpha_2 \circ f = \alpha_1$. An extension a is called equivalent to an extension b (of A) if $a < b$ and $b < a$. Then the class $V(A)$ of all equivalence classes of extensions of A is a partially ordered class and, indeed, has all the properties of a lattice except that of being a set. For an abelian group B one can consider all pairs (β, G), where G is an abelian group and β is an epimorphism from G to B, and analogously to the above, develop a lattice-like class $P(B)$. There are obvious natural maps from $\mathrm{Ext}(B, A)$ into $V(A)$ and into $P(B)$. This situation is considered in detail in the special case when A and B are finite.

D. K. Harrison (Eugene, Ore.)

Viljoen, G. 5906

A contribution to the extensions of abelian groups.

Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 6 (1963), 125–132.

The Baer and Whitehead problems are considered. A new proof is given of a theorem of Chase on Whitehead groups. If $\mathrm{Ext}(L, C(\infty)) = 0$, $|L| = \alpha > \aleph_0$, $\alpha^{\aleph_0} = \alpha$, then the rank of $L/(pL)$ is α for every prime p. If $\mathrm{Ext}(L, T) = 0$ for every torsion abelian group T, $|L| = \alpha > \aleph_0$, $\alpha^{\aleph_0} = \alpha$, then the rank of $L/(pL)$ is α for every prime p. Other results of particular interest to the specialist are proved.

D. K. Harrison (Eugene, Ore.)

Levin, Frank 5907

On some varieties of soluble groups. I.

Math. Z. 85 (1964), 369–372.

Let $S \subseteq G$ be a subset of a group, S^G its normal closure. It is proved that—with the usual notation for left-normed commutators—$[s, x, y] = [s, y, x]$ for all $s \in S$, $x, y \in G$, implies the same relation for all $s \in S^G$ and $x, y \in G$. Moreover, the latter is equivalent to $[s, [x, y]] = 1$ for all $s \in S^G$ and $x, y \in G$. It follows that the law

$$[x_1, \cdots, x_n, x_{n+1}, x_{n+2}] = [x_1, \cdots, x_n, x_{n+2}, x_{n+1}]$$

defines the same variety as the law $[x_1, \cdots, x_n, [x_{n+1}, x_{n+2}]] = 1$. The proof uses and extends results by I. D. Macdonald [same Z. 76 (1961), 270–282; MR 24 #A166].

Hanna Neumann (Canberra)

Macdonald, I. D. 5908

Generalisations of a classical theorem about nilpotent groups.

Illinois J. Math. 8 (1964), 556–570.

With skillful manipulation of commutator calculation the author obtains several interesting theorems on finite p-groups. Some of his main results are as follows. (1) Let G be a metabelian p-group of class precisely $2n$ with every proper subgroup of class n. Then $n = 1$. (2) Let G be a group of order p^{r+m} in which every subgroup of order p^r has class n, while G itself is not of class n. Then (i) G has a set of at most $m+n$ generators; (ii) the class of G is bounded by $f(m, n)$, where f is a function of m and n but not of r. (3) Let G be a group of order p^{r+4} in which every subgroup of order p^r has class 2. Then the class of G is 4; and G is metabelian if $p > 2$. Here the author requires some results of Blackburn [Proc. London Math. Soc. (3) 11 (1961), 1–22; MR 23 #A208] and Heineken [Illinois J. Math. 5 (1961), 681–707; MR 24 #A1319]. (4) Let a prime $p \geq 5$ be given. Then there is a p-group of order p^{11} and of class precisely 6 with every proper subgroup having class 3.

N. Itô (Nagoya)

Neumann, B. H. 5909

"Subsemigroups of nilpotent groups": An acknowledgement.

Proc. Roy. Soc. Ser. A 281 (1964), 436.

The author acknowledges that the results of his paper with T. Taylor [same Proc. 274 (1963), 1–4; MR 28 #3100] were anticipated by A. I. Mal'cev [Ivanov. Gos. Ped. Inst. Učen. Zap. Fiz.-Mat. Nauki 4 (1953), 107–111; MR 17, 825].

Weichsel, Paul M. 5910

A decomposition theory for finite groups with applications to p-groups.

Trans. Amer. Math. Soc. 102 (1962), 218–226.

The author introduces a notion of decomposition for finite groups which includes direct products and subdirect products as special cases. A group G is called an "in-direct product" if G belongs to $\{A_\alpha\}^*$, where $\{A_\alpha\}$ is the set of all proper subgroups and factor groups of G and $\{A_\alpha\}^*$ is the "closure" of the set $\{A_\alpha\}$ which is the smallest set of groups containing $\{A_\alpha\}$ and closed under the operations of taking finite direct products, subgroups, and factor groups. It is shown that an in-direct product which is not a subdirect product or a factor group of a direct product contains a normal subgroup which is an abelian p-group.

This notion is applied to p-groups and the following theorems obtained. Theorem: Let G be a p-group, not of class two and let $Z(G)$ denote the center of G. If $G/Z(G)$ is a product of elementwise commuting proper subgroups,

then G is an in-direct product. Theorem: Let G be a p-group of class two. G is not an in-direct product if and only if $Z(G)$ is cyclic and G may be generated by two elements.
J. E. Adney (E. Lansing, Mich.)

Thompson, John G. 5911
Fixed points of p-groups acting on p-groups.
Math. Z. **86** (1964), 12–13.

The author proves two theorems. (1) Let p be an odd prime. Let $\mathfrak{B}$ be a p-group such that its Frattini subgroup $D(\mathfrak{B})$ is elementary abelian and central in $\mathfrak{B}$. Let $\mathfrak{A}$ be a p-group of operators acting on $\mathfrak{B}$ such that $V = \mathfrak{B}/D(\mathfrak{B})$ is a free $Z_p(\mathfrak{A})$-module, where $Z_p(\mathfrak{A})$ denotes the group ring of $\mathfrak{A}$ over the field of p elements. Let $C_{\mathfrak{B}}(\mathfrak{A})$ and $C_V(\mathfrak{A})$ be centralizers of $\mathfrak{A}$ in $\mathfrak{B}$ and V, respectively. Then $C_{\mathfrak{B}}(\mathfrak{A})D(\mathfrak{B})/D(\mathfrak{B}) = C_V(\mathfrak{A})$. The proof rests on a theorem of Blackburn on p-groups of maximal class [Acta Math. **100** (1958), 45–92; MR **21** #1349]. (2) Let p be an odd prime. Let $\mathfrak{G}$ be a p-solvable group. Let $\mathfrak{A}$ be a p-subgroup of $\mathfrak{G}$ such that its centralizer in $\mathfrak{G}$ has the form: $C_{\mathfrak{G}}(\mathfrak{A}) = Z(\mathfrak{A}) \times \mathfrak{T}$, where $Z(\mathfrak{A})$ is the center of $\mathfrak{A}$ and $\mathfrak{T}$ has an order prime to p. Let $\mathfrak{K}$ be a subgroup of $\mathfrak{G}$ such that $\mathfrak{K}$ has an order prime to p and the normalizer of $\mathfrak{K}$ contains $\mathfrak{A}$. Then $\mathfrak{K}$ is contained in the largest normal subgroup of $\mathfrak{G}$ which has order prime to p.
N. Ito (Nagoya)

Held, Dieter 5912
Closure properties and partial Engel conditions in groups.
Illinois J. Math. **8** (1964), 706–712.

Let $\mathfrak{p}$ be a set of primes and σ a partial ordering of $\mathfrak{p}$ such that $p\sigma p$ is false for every prime p in $\mathfrak{p}$. A σ-segment is a nonempty subset $\mathfrak{s}$ of $\mathfrak{p}$ defined by the following property: If q belongs to $\mathfrak{s}$ and $p\sigma q$, then p belongs to $\mathfrak{s}$, too. The group G is called $\mathfrak{a}$-closed for $\mathfrak{a}$ a set of primes if products of $\mathfrak{a}$-elements of G are $\mathfrak{a}$-elements. A torsion group is called $(\mathfrak{p}, \sigma)$-dispersed if it is $\mathfrak{s}$-closed for every σ-segment $\mathfrak{s}$ of $\mathfrak{p}$. The author investigates sufficient conditions in order that the set of all $\mathfrak{p}$-elements of a torsion group G constitute a $(\mathfrak{p}, \sigma)$-dispersed subgroup of G. One of the conditions given by the author is as follows. (0) Every finitely generated subgroup of G satisfies the maximum and minimum conditions for subgroups. (1) Epimorphic images of subgroups of G are called factors of G. If $p \in \mathfrak{p}$, then p-factors of G are locally finite. (2) If p and q are different primes such that $p \in \mathfrak{p}$ and $q\sigma p$ is false, and if a is a p-element and b a q-element of G, then for almost every positive integer i the order of an iterated commutator $a^{(i)} \circ b$ is divisible by primes r with $r\sigma p$ only. (3) Any infinite factor F of G which can be generated by finitely many $\mathfrak{p}$-elements contains an element of prime order p such that there is no element of F of prime order q with $q\sigma p$.
N. Ito (Nagoya)

Kemhadze, Š. S. 5913
Some properties of factorizable groups. (Russian. Georgian summary)
Soobšč. Akad. Nauk Gruzin. SSR **35** (1964), 257–262.

Die nicht notwendig endliche Gruppe G sei faktorisierbar in der Form $G = AB$. Ist $A^{(n)} = B^{(n)} = 1$ und gibt es eine Normalreihe der Länge s von B nach G, so ist G auflösbar von einer Länge $\leq s + (n-1)s$. Ist $A' = B' = 1$, so hat G ein Zentrum $\neq 1$ oder $G'' = 1$. {Bemerkung des Referenten: Daß die Kommutativität von A und B die von G' nach sich zieht, ist wohlbekannt, siehe N. Itô [Math. Z. **62** (1955), 400–401; MR **17**, 125]. Der Beweis des Verfassers für seine schwächere Aussage ist fehlerhaft.}
H. Salzmann (Frankfurt a.M.)

Kemhadze, Š. S. 5914
Groups generated by nilpotent and ZA-subgroups. (Russian)
Sibirsk. Mat. Ž. **5** (1964), 827–837.

A group is an N-group if every proper subgroup of it is properly contained in its normalizer. The paper starts out with the remark that the N-group G has a non-trivial center provided there is a normal subgroup A in G with non-trivial center and a ZA-group (i.e., the ascending central sequence terminates at the "top" only) B such that $G = AB$. In a similar way the author obtains the following Theorem 5: If A, B are ZA-groups, subnormal in the group G such that $G = AB$, then G is ZA. The corresponding result holds with "locally nilpotent" replacing ZA (Theorem 6).
O. H. Kegel (Frankfurt a.M.)

Kemhadze, Š. S. 5915
The factorization of groups by accessible subgroups. (Russian)
Sibirsk. Mat. Ž. **5** (1964), 838–843.

In the same vein as in the preceding paper [#5914], the author proves that if A and B are permutable nilpotent subnormal subgroups of the group G, then the product AB is a nilpotent subnormal subgroup of G.
O. H. Kegel (Frankfurt a.M.)

Polovickiĭ, Ja. D. 5916
Groups with extremal classes of conjugate elements. (Russian)
Sibirsk. Mat. Ž. **5** (1964), 891–895.

G is a group with extremal classes of conjugate elements if G induces in each class a^G an extremal group of permutations, i.e., if $G/\mathfrak{C}(a^G)$ is a finite extension of a (divisible) commutative group with minimal condition, where $\mathfrak{C}(A)$ denotes the centralizer of A in G. For a discussion of extremal groups cf. R. Baer [Trans. Amer. Math. Soc. **79** (1955), 521–540; MR **17**, 125]. If G has extremal classes, then G' is a torsion group. Theorem: G is a group with extremal classes if and only if (a) each class a^G generates a subgroup which is either extremal or an infinite cyclic extension of an extremal group, and (b) $G/\mathfrak{C}(a^G)$ is a torsion group for each $a \in G$. Corollary. G has finite classes of conjugate elements if and only if each class generates a cyclic extension of a finite group.
H. Salzmann (Frankfurt a.M.)

Maass, Hans 5917
Die Multiplikatorsysteme zur Siegelschen Modulgruppe.
Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II **1964**, 125–135.

Let Γ_n be the modular group of degree $n > 1$. A multiplier system on Γ_n of integral dimension (the only dimensions of

interest when $n>1$) is a character v of Γ_n of absolute value 1. The author proves that

$$(\Gamma_n:\Gamma_n') = 2, \quad n=2,$$
$$= 1, \quad n>2.$$

Since v is identically 1 on Γ_n', there is a nontrivial multiplier system only for Γ_2. The author characterizes Γ_2' by means of congruences and exhibits the unique character v on Γ_2 by its action on the generators.

In a note added in proof the author remarks that, unknown to him, the results of this paper had already been obtained by I. Reiner [Proc. Amer. Math. Soc. **6** (1955), 987–990; MR **17**, 710]. The reader should note that the author writes Γ_n for the modular group of degree n whereas Reiner writes Γ_{2n}. *J. Lehner* (College Park, Md.)

Götzky, Martin 5918

Eine Kennzeichnung der orthogonalen Gruppen unter den unitären Gruppen.

Arch. Math. **15** (1964), 261–265.

Let K be a sfield of characteristic $\neq 2$, $U_n(K,f)$ a unitary group with $n\geqq 2$. The author gives a proof of a theorem of F. Bachmann giving a condition on $U_n(K,f)$ which implies that K is commutative, and the involution in K is the identity (hence $U_n(K,f)$ is then an orthogonal group). The condition involves symmetries with respect to non-isotropic hyperplanes, and reads as follows: If three hyperplanes H_i are orthogonal to non-isotropic linearly dependent vectors, and s_i is the symmetry with respect to H_i ($i=1,2,3$), then $s_1s_2s_3$ is again a symmetry with respect to a non-isotropic hyperplane.

J. Dieudonné (Paris)

Tsuzuku, Tosiro 5919

A characterization of finite projective linear groups.

Proc. Japan Acad. **40** (1964), 155–156.

Let G be a finite group with a Bruhat decomposition satisfying the axioms of Steinberg [Canad. J. Math. **9** (1957), 347–351; MR **19**, 387]. Suppose the Weyl group is the symmetric group of degree $n\geqq 4$ and that the canonical set of generators is $(1,2),\cdots,(n-1,n)$. The author sketches a proof of the theorem that under these hypotheses G has a homomorphic image isomorphic with $\mathrm{PSL}_n(q)$ or A_7. The proof is made by constructing the appropriate projective space and showing that G has a flag-transitive action on this space. The conclusion then follows from a theorem of D. G. Higman [Illinois J. Math. **6** (1962), 434–446; MR **26** #663]. {Reviewer's note: It is easy to see that the case A_7 cannot, in fact, occur.}

J. McLaughlin (Ann Arbor, Mich.)

Brauer, Richard 5920

Some applications of the theory of blocks of characters of finite groups. I.

J. Algebra **1** (1964), 152–167.

This paper consists primarily of the deduction of some consequences and refinements of the author's main theorems on p-blocks [Math. Z. **63** (1956), 406–444; MR **17**, 824; ibid. **72** (1959/60) 25–46; MR **21** #7258]. There are several results which bound the order of $G/\mathfrak{O}_p'(G)$ in terms of various properties of G. In connection with this the author introduces the concept of "deficiency class", and some results are obtained concerning groups of deficiency class 0. In case $p=2$ analogous results are also obtained about groups of deficiency class 1. The final part of the paper is a discussion of how the main theorem in the second paper mentioned above may be modified if the modular characters in a block of $\mathfrak{C}_G(\pi)$ are replaced by an arbitrary basis of the Z-module spanned by these modular characters. Here π is an arbitrary p-element of G.

W. Feit (New Haven, Conn.)

Conlon, S. B. 5921

Twisted group algebras and their representations.

J. Austral. Math. Soc. **4** (1964), 152–173.

Let G be a finite group and F a field. A twisted group algebra $A(G)$ of G over F is an associative algebra over F with an F-basis $\{u(g)\}$ in one-to-one correspondence with the elements $g\in G$, and such that $u(g)u(g')=f(g,g')u(gg')$, $f(g,g')\in F$, $f(g,g')\neq 0$. The author first shows that, by extending the field F and replacing the $u(g)$ by suitable scalar multiples of themselves, a twisted group algebra can be normalized so that the structure constants $f(g,g')$ are manageable (the precise conditions are too involved to describe here). Let $A(G)$ be a normalized twisted group algebra of G over F, let H be a subgroup of G, and let $A(H)$ be the subalgebra of $A(G)$ determined by H. Let L be a left $A(H)$-module. The induced module L^G is defined to be the left $A(G)$ module $A(G)\otimes_{A(H)}L$. The first main result is a generalization of P. A. Tucker's result on ordinary induced representations [Amer. J. Math. **84** (1962), 400–420; MR **26** #1353]. Let $A(H)$ be the restriction of a normalized twisted group algebra $A(G)$ over an algebraically closed field F to a normal subgroup H of G. Let L be an indecomposable $A(H)$-module, and S the subgroup of G consisting of all $s\in G$ such that $u(s)\otimes L\simeq L$. The theorem states that the decomposition of L^G into indecomposable submodules is entirely determined by the decomposition of a certain twisted group algebra $A(S/H)$ into left ideals, there being a one-to-one correspondence between left ideal components of $A(S/H)$ and components of L^G such that two left ideal components are isomorphic if and only if the corresponding components of L^G are isomorphic. The dimensions over F of the components of L^G are given in terms of the dimensions of the corresponding left ideals.

The author next adapts A. Rosenberg's version [Math. Z. **76** (1961), 209–216; MR **24** #A158] of R. Brauer's main theorem on blocks to the case of twisted group algebras. Rather complete generalizations are also given, for twisted group algebras, of D. G. Higman's theory of relatively projective modules over group algebras [Duke Math. J. **21** (1954), 377–381; MR **16**, 794] and J. A. Green's work on the vertex and source of an indecomposable module [Math. Z. **70** (1958/59), 430–445; MR **24** #A1304].

C. W. Curtis (Eugene, Ore.)

Fischer, Bernd 5922

Die Brauersche Charakterisierung der Charaktere endlicher Gruppen.

Math. Ann. **149** (1962/63), 226–231.

Let G be a finite group and $C(G)$ its character ring. If U is a subgroup of G and φ an element of $C(U)$, then φ^G denotes

the induced generalized character of G. For a set of subgroups $\mathfrak{M}$ of G let $V(\mathfrak{M})$ be the set of functions $\sum_{U \in \mathfrak{M}} a_U \varphi_U{}^G$, where a_U is an integer and φ_U is an element of $C(U)$. Finally, let $U(\mathfrak{M})$ denote the complex-valued class functions on G whose restriction to a subgroup W of $\mathfrak{M}$ is an element of $C(W)$. Brauer and Tate [Ann. of Math. (2) **62** (1955), 1–7; MR **16**, 1087] showed $V(\mathfrak{M})$ is an ideal of the ring $U(\mathfrak{M})$ and $U(\mathfrak{M}) \supseteq C(G) \supseteq V(\mathfrak{M})$. The main theorem of this paper is that the following properties of a set of subgroups $\mathfrak{M}$ of a finite group G are equivalent: (a) $U(\mathfrak{M}) = C(G)$; (b) $V(\mathfrak{M}) = C(G)$; (c) If E is an elementary nilpotent subgroup of G, then there is a subgroup of G contained in a member of $\mathfrak{M}$ and conjugate to E. (A finite group is called elementary nilpotent if it is the direct product of a cyclic group and a p-group.)

J. E. Adney (E. Lansing, Mich.)

Reiner, Irving 5923

On the number of irreducible modular representations of a finite group.

Proc. Amer. Math. Soc. **15** (1964), 810–812.

Let K be a field of characteristic p, and G a finite group. Let n be the L.C.M. of the orders of the p-regular elements of G, and let δ be a primitive nth root of 1 over K. Each K-automorphism of $K(\delta)$ is given by a mapping $\delta \to \delta^t$ for some integer t representing a unit in $Z/(n)$. The multiplicative group T of $Z/(n)$ is isomorphic to the Galois group of $K(\delta)$ over K. Two p-regular elements $a, b \in G$ are called K-conjugate if $b^t = x^{-1}ax$ for some $x \in G$ and $t \in T$. Berman proved [Dokl. Akad. Nauk SSSR **106** (1956), 767–769; MR **17**, 1181] that the number of irreducible K-representations of G equals the number of p-regular K-conjugacy classes of G. The author gives a new proof of Berman's theorem using the theory of Brauer characters of G [see C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, Chapter XII, Interscience, New York, 1962; MR **26** #2519].

C. W. Curtis (Eugene, Ore.)

Srinivasan, Bhama 5924

The modular representation ring of a cyclic p-group.

Proc. London Math. Soc. (3) **14** (1964), 677–688.

Let G be a cyclic group of order p^a, and let K be a field of characteristic p, for some prime p. The set of all isomorphism classes of finitely generated KG-modules forms a commutative ring R with operations $\{M\} + \{M'\} = \{M \oplus M'\}$, $\{M\} \cdot \{M'\} = \{M \otimes M'\}$. The ring R is a free Z-module with a basis consisting of the p^a distinct isomorphism classes $\{M_i\}$, $i = (M_i : K)$, $1 \leq i \leq p^a$, of indecomposable ZG-modules. For each pair of basis elements $\{M_n\}$ and $\{M_m\}$, there exist integers a_{nml} such that $\{M_n\}\{M_m\} = \sum_l a_{nml}\{M_l\}$. The author determines the integers a_{nml} explicitly. In case $n + m - 1 < p$, the formula reduces to $\{M_n\}\{M_m\} = \sum_{j=1}^{n} \{M_{n+m-2j+1}\}$, which agrees with a classical formula for tensor products of cyclic nilpotent matrices at characteristic zero [D. E. Littlewood, *The theory of group characters and matrix representations of groups*, p. 196, Oxford Univ. Press, New York, 1940; MR **2**, 3]. J. A. Green proved that if C is the complex field, then $R \otimes_Z C$ is a semi-simple algebra, without actually calculating the integers a_{nml} [Illinois J. Math. **6** (1962), 607–619; MR **26** #5106].

C. W. Curtis (Eugene, Ore.)

Farahat, H. K. 5925

Note on a paper of Tsuzuku.

Proc. Glasgow Math. Assoc. **6**, 196–197 (1964).

The author corrects an argument used by T. Tsuzuku [Nagoya Math. J. **22** (1963), 79–82; MR **27** #4853] in proving the following theorem. Let G be a doubly transitive permutation group of degree n and ρ its representation by $n \times n$ permutation matrices. If K is any commutative ring with unity, then ρ is decomposable over K if and only if n is an invertible element of K.

G. de B. Robinson (Toronto, Ont.)

Levine, Jack; Korfhage, Robert R. 5926

Automorphisms of abelian groups induced by involutory matrices, general modulus.

Duke Math. J. **31** (1964), 631–653.

Let Z_q denote the ring of rational integers modulo q, and let n be a fixed positive integer. The authors study the collection of all $n \times n$ matrices X with entries in Z_q, satisfying $X^2 = I$, the identity matrix. It is convenient to view such involutory matrices as representing involutions on the additive group G which is the direct sum of n copies of Z_q.

Hereafter, let $q = 2^m$, $m > 1$, which is the case of main interest in the present paper. (Previous articles by various authors have dealt with the cases where $m = 1$, or where q is a power of an odd prime.) Given an involution μ on G, its fixed group F is defined by $F = \{x \in G : \mu(x) = x\}$. The authors show easily that the additive structure of F cannot be arbitrary, but rather there exist non-negative integers α, β and γ such that F is a direct sum of α copies of Z_2, β copies of $Z_{2^{m-1}}$, and γ copies of Z_{2^m}. We shall say that such a direct sum has type $\langle \alpha, \beta, \gamma \rangle$. Conversely, the more difficult result is established: Given non-negative integers α, β and γ such that

$$(*) \qquad n - 2\gamma \leq \alpha + \beta \leq n - \gamma,$$

there exist involutions on G whose fixed group has type $\langle \alpha, \beta, \gamma \rangle$.

Fix some canonical basis for G; then each involutory matrix X defines an involution μ on G, and hence also defines a fixed group F. It is not hard to show that the number of involutory matrices X with a given fixed group F_0 of type $\langle \alpha, \beta, \gamma \rangle$ depends only upon α, β and γ, rather than upon the group F_0. Let F_0 be a subgroup of G of type $\langle \alpha, \beta, \gamma \rangle$, chosen in some canonical way. Let $r(\alpha, \beta, \gamma)$ be the number of involutory matrices X with fixed group F_0, and let $s(\alpha, \beta, \gamma)$ be the number of subgroups of G of type $\langle \alpha, \beta, \gamma \rangle$. Then the total number $N(q, n)$ of involutory $n \times n$ matrices with entries in Z_q is given by

$$N(q, n) = \sum r(\alpha, \beta, \gamma) s(\alpha, \beta, \gamma),$$

where the summation extends over all triples (α, β, γ) of non-negative integers satisfying (*).

The factor $s(\alpha, \beta, \gamma)$ can be computed directly from the Yeh-Delsarte formula. The determination of $r(\alpha, \beta, \gamma)$ is more difficult, and depends upon exhibiting all involutory matrices X which fix F_0, and then deciding which of these have F_0 as fixed group. The authors eventually obtain an explicit formula for $N(q, n)$.

The same methods yield a formula for $N(p^a, n)$, where p is an odd prime. The arguments are considerably simpler in this case, however. Since $N(\prod p_i^{a_i}, n) = \prod N(p_i^{a_i}, n)$, the authors are thus able to evaluate $N(t, n)$ for each positive integer t.

I. Reiner (Urbana, Ill.)

Cilang, H. [Zieschang, H.] 5927

On automorphisms of plane groups. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 57–60.

Let G be a plane group with a compact fundamental region R, and let N be the closed orientable surface obtained by identifying equivalent boundary points of R. An autohomeomorphism of N is called admissible if it maps every branch point onto a branch point of the same order. G admits the following presentation:

$$G = \langle p_1, \cdots, p_m, a_1, b_1, \cdots, a_g, b_g;$$

$$p_1^{k_1} = 1, \cdots, p_m^{k_m} = 1, \prod_{j=1}^{m} p_j \cdot \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = 1 \rangle.$$

Let $F = \langle p_1, \cdots, p_m, a_1, b_1, \cdots, a_g, b_g \rangle$. Theorem 1: Every automorphism of G is induced by an automorphism α of F satisfying the following equations:

$$\alpha(p_i) = T_i p_{n_i} T_i^{-1}, \qquad i = 1, \cdots, m,$$

$$\alpha\left(\prod_{j=1}^{m} p_j \cdot \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1}\right) = T \cdot \prod_{j=1}^{m} p_j \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} \cdot T^{-1},$$

where $\begin{pmatrix} 1 \cdots m \\ n_1 \cdots n_m \end{pmatrix}$ is a permutation with $K_{n_i} = K_i$ and $T_1, \cdots, T_m, T$ are arbitrary words in F.

The author sketches a proof of the equivalence of the group-theoretical Theorem 1 and the following generalization of Nielsen's theorem (Theorem 2): Every automorphism of G is induced by an admissible autohomeomorphism of N. *M. Greendlinger* (Ivanovo)

Fried, E. 5928

On the subgroups of an Abelian group that are ideals in every ring.

Proc. Colloq. Abelian Groups (*Tihany*, 1963) *pp.* 51–55. *Akadémiai Kiadó, Budapest*, 1964.

A ring over a given (abelian) group G is a ring with additive group isomorphic to G. The author obtains characterizations for the subgroups of an arbitrary group G which are ideals in every ring over G. Let $\mathfrak{J}(G)$ be that ideal of the endomorphism ring $\mathfrak{E}(G)$ of G which is generated by all homomorphic images of G into the additive group of $\mathfrak{E}(G)$. Subgroups H^* and $\bar{H}$ of G corresponding to the subgroup H are defined thus: H^* is generated by all g such that $g\varphi \in H$ whenever $\varphi \in \mathfrak{J}(G)$, and $\bar{H}$ is generated by all $h\varphi$ with $h \in H$ and $\varphi \in \mathfrak{J}(G)$. Then Theorem 2 states that each of the following four conditions is necessary and sufficient for H to be an ideal in every ring over G: (1) $\bar{H} \subseteq H$, (2) there exists a fully invariant subgroup T of G such that $\bar{T} \subseteq H \subseteq T^*$, (3) there exists a fully invariant subgroup S of G such that $\bar{S} \subseteq H \subseteq S$, (4) $H \subseteq H^*$. The proofs are elementary. The author also considers groups G in which only the fully invariant subgroups are ideals in all rings over G. Finally, two problems are posed. *I. D. Macdonald* (Newcastle, N.S.W.)

Leptin, H. 5929

Einige Bemerkungen über die Automorphismen Abelscher p-Gruppen.

Proc. Colloq. Abelian Groups (*Tihany*, 1963), *pp.* 99–104. *Akadémiai Kiadó, Budapest*, 1964.

Let G be a reduced Abelian p-group, and let S be the socle of G. For every ordinal μ, put $S^\mu = S \cap (p^\mu G)$. Then any endomorphism η of G induces in the factor group $S^\mu/S^{\mu+1}$ an endomorphism η_μ. H. Freedman [Proc. London Math. Soc. (3) **12** (1962), 77–99; MR **24** #A2215] proved the following theorem. Let $p \geq 5$, and let G be of finite Ulm type, i.e., $p^{n\omega}G = 0$ for a natural number n. If an automorphism α of G satisfies $\alpha_\mu = 1$ for every μ and $\alpha^p = 1$, then $\alpha = 1$ on pG.

In the present paper, the author shows that the assumption of finiteness of Ulm type is superfluous by proving the following theorem. Let $p \geq 5$, and let G be a reduced p-group. If an automorphism α of G satisfies $\alpha_\mu = 1$ for every $\mu < \omega$, then $\alpha = 1$ on pG if and only if $\alpha^p = 1$. *K. Honda* (Tokyo)

Banaschewski, B. 5930

On lattice-ordered groups.

Fund. Math. **55** (1964), 113–122.

Lorenzen [Math. Z. **45** (1939), 533–553; MR **1**, 101] has shown that a lattice-ordered Abelian group has a realization as a subgroup of a direct product of totally ordered groups, and Ribenboim [Pacific J. Math. **10** (1960), 305–308; MR **22** #1624] has shown that this realization can be made completely regular. The problem in the case of a general lattice-ordered group G is approached here by studying the epimorphisms of G to totally ordered groups by using the concept of a zero-dimensional subgroup of a product of totally ordered groups. A full subgroup H of $\prod T_\alpha$, $\alpha \in I$, is zero-dimensional if, in the topology on I defined by the basis $s(u) = \{\alpha \mid u(\alpha) \neq 0\}$, $u \in H$, of open sets, the sets $s(u)$ are also closed. (Zero-dimensionality can be shown to be equivalent to complete regularity.) Various results concerning these concepts are expressed in terms of properties of the prime filters in the lattice P of positive elements of G, and these are used to prove the equivalence of the following statements: (i) G has a zero-dimensional realization, (ii) G has a realization, (iii) no strictly positive element of G is disjoint from any of its conjugates, (iv) each ultrafilter in P is normal.

The author compares his method of proof with that of Ribenboim and is able to generalize another of the latter's results concerning the lattice of filets of P. Some further results are found in the special case of a complete lattice-ordered group. *J. C. Ault* (Leicester)

Ciampa, Salvatore 5931

Osservazioni sull'ordinabilità dei gruppi.

Ann. Scuola Norm. Sup. Pisa (3) **18** (1964), 111–136.

Let G be a group with identity e. For $H \subseteq G$, let H^+ be the subsemigroup generated by H. Let $H^{-1} = \{x^{-1} : x \in H\}$, and $H^* = \bigcap \{[K \cup (H - K)^{-1}]^+ : K \subseteq H\}$. Call H weakly stable (w.s.) if $H^* \subseteq H$. The author obtains a number of properties of H^*. For example, $H^* = \bigcup \{F^* : F \subseteq H, F$ finite$\}$. If $H \cap H^* = \varnothing$, then $H^* = \varnothing$. If $e \notin H$ and H is w.s., then every subset of H is w.s. If $e \notin H^*$, H is w.s.

These results are applied to obtain several equivalent conditions for G to be right orderable (i.e., for the existence of a total order $\leq$ on G for which $a \leq b$ always implies $ac \leq bc$). For example, G is right orderable if and only if G^0 $(= \{x \in G : x \neq e\})$ is w.s., or equivalently if and only if $H^* = \{e\}$ whenever $H \neq \varnothing$ is finite. As the author points out, some of these conditions were found by Conrad [Michigan Math. J. **6** (1959), 267–275; MR **21** #5684], and another is

somewhat similar to a result of the reviewer [Proc. Amer. Math. Soc. 13 (1962), 217–219; MR 24 #A3216]. Corollary: A simple group G is right orderable if and only if, for some $H \subseteq G$, $H \cup H^{-1} = G^0$ and $H^+ \neq G$.

In the last section, the extendibility (to a right total order) of a right partial order is discussed. An example is given of a right orderable group having a right partial order which is not extendible. Theorem: A given right partial order is extendible if the set of elements not comparable with e is the union of two subsemigroups. This leads to some sufficient conditions for the extendibility of every right partial order. *E. J. Tully, Jr.* (Davis, Calif.)

Curzio, Mario 5932

Elementi $\bigcup$-quasi-distributivi in alcuni reticoli di gruppi.

Ricerche Mat. **13** (1964), 70–79.

Secondo una definizione dello stesso autore [stessi Ricerche **6** (1957), 96–110; MR **20** #905] un elemento a di un reticolo $\mathscr{R}$ si dice $\cup$-quasi distributivo ($\cup$-q.d.) in $\mathscr{R}$ se

$$x, y \in \mathscr{R}: x \cup a = y \cup a \Rightarrow x \cup a = (x \cap y) \cup a.$$

In questa nota si caratterizzano gli elementi $\cup$-q.d. del reticolo $\mathscr{L}(G)$ dei sottogruppi di un gruppo G finito o abeliano, dimostrando che, se G è un tale gruppo, $A \in \mathscr{L}(G)$ è $\cup$-q.d. se e solo se l'applicazione

$$X \in \mathscr{L}(G) \to A \cup X$$

è un endomorfismo di $\mathscr{L}(G)$ e riconducendone la determinazione a noti risultati di Higman, Sato e Zappa.

Si prova inoltre che se G è un gruppo qualunque (un gruppo finito risolubile) la classe degli elementi $\cup$-q.d. nel reticolo dei sottogruppi normali di G (nel reticolo dei sottogruppi di composizione di G) coincide con quella degli elementi neutri. *L. A. Rosati* (Firenze)

Dubreil-Jacotin, Marie-Louise 5933

Sur les images homomorphes d'un demi-groupe ordonné.

Bull. Soc. Math. France **92** (1964), 101–115.

The author investigates o-homomorphic images of partially ordered semigroups. The results are similar to, but more general and more extended than, recent results of L. Fuchs [Acta Sci. Math. (Szeged) **25** (1964), 139–142; MR **29** #3554]. Among other results, the author proves the following (we use notation defined in the review cited above). If S is a partially ordered semigroup containing an element e such that for each $a \in S$, $\langle e^{\cdot} . a\rangle = \langle e. {}^{\cdot}a\rangle$, and $\langle e^{\cdot} . e\rangle = \{x \in S \mid x \geq e\}$, then there exists a unique o-homomorphism β of S onto a partially ordered semigroup $\bar{S}$ with neutral element $e\beta$, such that $a\beta \leq e\beta$ implies $a \leq e$. Then $a\beta = b\beta$ if and only if $\langle e^{\cdot} . a\rangle = \langle e^{\cdot} . b\rangle$ if and only if $\langle e. {}^{\cdot}a\rangle = \langle e. {}^{\cdot}b\rangle$. $\bar{S}$ is a group if and only if each $\langle e^{\cdot} . a\rangle$ has left multiplicatively maximal elements. The theory is applied to the study of certain subsemigroups of the semigroup of subsets of a semigroup.

W. C. Holland (Madison, Wis.)

Kopytov, V. M. 5934

On the completion of the center of an ordered group. (Russian)

Ural. Gos. Univ. Mat. Zap. **4**, *tetrad* **3**, 20–24 (1963).

It is shown that every fully ordered group G with center $Z(G)$ can be embedded in a fully ordered group G^* with divisible center $Z(G^*)$ such that $Z(G) \subseteq Z(G^*)$ and the factor groups $G/Z(G)$ and $G^*/Z(G^*)$ are isomorphic abstract groups. If G as an abstract group has the property that every partial order can be extended to a full order of G, then G^* has the same property. Corollaries: (1) A torsion-free group can be fully ordered if and only if the factor group modulo its center can be fully ordered; (2) If a group can be fully ordered, then it admits a full order in which the center is a convex subgroup.

L. Fuchs (Budapest)

Livčak, Ja. B. 5935

On orderable groups. (Russian)

Učen. Zap. Ural. Gos. Univ. **1959**, *vyp.* 23, 11–12.

An example is given for a group which is not locally nilpotent in which every partial order can be extended to a full order (in locally nilpotent groups this is always possible [see Mal'cev, Trudy Mat. Inst. Steklov. **38** (1951), 173–175; MR **14**, 13]). Another example exhibits an R-group ($a^n = b^n$ for some positive integer n implies $a = b$) that cannot be fully ordered. *L. Fuchs* (Budapest)

Kokorin, A. I. 5936

On a class of lattice-ordered groups. (Russian)

Ural. Gos. Univ. Mat. Zap. **3**, *tetrad* 3, 37–38 (1962).

An example is given for an R-group which does not admit any lattice-order. *L. Fuchs* (Budapest)

Kokorin, A. I. 5937

Methods for the lattice-ordering of a free Abelian group with a finite number of generators. (Russian)

Ural. Gos. Univ. Mat. Zap. **4**, *tetrad* 1, 45–48 (1963).

If a free Abelian group A with a finite number of generators is lattice-ordered, then it is either the cardinal sum of two of its proper l-ideals or the lexicographic extension of an l-ideal by a fully ordered group [cf. also P. Conrad, Michigan Math. J. 7 (1960), 171–180; MR **22** #6854]. By making use of this, the lattice-orders on A can be described.

L. Fuchs (Budapest)

Kokorin, A. I. 5938

Ordering a direct product of ordered groups. (Russian)

Ural. Gos. Univ. Mat. Zap. **4**, *tetrad* **3**, 95–96 (1963).

A group G is called an O^*-group if every partial order on G can be extended to a full order of G. Answering a question of the reviewer [*Partially ordered algebraic systems*, Pergamon, Oxford, 1963; MR **30** #2090], it is shown that direct products of O^*-groups are again O^*-groups.

L. Fuchs (Budapest)

Lévy-Bruhl, Jacques 5939

Le théorème de Jordan-Hölder dans certains groupoïdes ordonnés.

C. R. Acad. Sci. Paris **258** (1964), 1114–1116.

The author notes that with suitable hypotheses on semilattices and ordered semigroups one can deduce theorems of Jordan-Hölder type.

J. McLaughlin (Ann Arbor, Mich.)

Appel, K. I.; Djorup, F. M. 5940

On the group generated by a free semigroup.

Proc. Amer. Math. Soc. **15** (1964), 838–840.

The principal result of the paper is that a group generated by a two-generator free semigroup need not be free. This result is proved by showing that the free semigroup K generated by the elements a, b is imbeddable as a free subsemigroup of H/N, where H is the free group on the generators a, b, and N is a nontrivial normal subgroup of H (the one generated by $ab^{-1}ab^{-1}$).

V. S. Krishnan (Madras)

Fuchs, L.; Steinfeld, O. 5941

Principal components and prime factorization in partially ordered semigroups.

Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **6** (1963), 103–111.

Two new sets of conditions are obtained for unique prime factorisation in a partially ordered semigroup (not necessarily commutative), generalising the fundamental theorem or commutative ideal theory. The concept of principal component is introduced and its properties are studied. Let S be a partially ordered semigroup with more than one element satisfying the conditions: (i) S is negatively ordered and has a maximum element e which is a left identity; (ii) Every pair of elements a, b of S have a least upper bound $a \vee b$; (iii) If $a \neq 0$, the set of elements x in S with $a \leqq x \leqq e$ satisfies the minimum condition; (iv) For every $a, b \in S$, there exist right and left residuals $a \cdot b$ and $a::b$, such that $xb \leqq a$ if and only if $x \leqq a \cdot b$, and $bx \leqq a$ if and only if $x \leqq a::b$. Then, if p is a prime in S and $0 < a < e$, there exists a unique element $a(p) \in S$, called the right principal component of a belonging to p, such that $a : y \nleqq p$ if and only if $y \leqq a(p)$, and there exist a finite number of primes $p_1, \cdots, p_r \in S$ such that $a = a(p_1) \wedge \cdots \wedge a(p_r)$. If, further, S also satisfies the conditions: (v) If $a \neq 0$, then $ba = a$ implies $b = e$ and $ba = 0$ implies $b = 0$, (vi) If p is a prime, $a < b \leqq p$ implies $a : p < b : p$; then the above principal components of a are commuting prime powers, so that $a = p_1^{k_1} \cdots p_r^{k_r}$. Finally, if S satisfies conditions (i) and (v) and the further conditions: (III) $a < b$ implies $a = bc$ for some $c \in S$, (IV) S satisfies the maximum condition; (V) The right residual $a : b$ exists for all $a, b \in S$; then e is the identity of S and every element $a \in S$, $0 < a < e$, can be represented uniquely as the product of pairwise commuting primes.

J. A. H. Shepperd (Manchester)

Hedrlín, Z.; Pultr, A. 5942

Relations (graphs) with given finitely generated semigroups.

Monatsh. Math. **68** (1964), 213–217.

Let $S(\cdot)$ be a semigroup with unity element, and $A \subset S$ a set of generators of $S(\cdot)$. Then there exists a system of binary relations R_a over S, with one relation for each a in A, such that the set of all transformations compatible with the relations R_a forms, under composition, a semigroup isomorphic with $S(\cdot)$. Using this theorem, the authors then obtain the following graph-theoretic result. For each finitely generated semigroup $S(\cdot)$ with unity element there exist infinitely many non-isomorphic graphs G such that $S(\cdot)$ is isomorphic with the semigroup of all endomorphisms of G. The proofs use a construction similar to that used by R. Frucht [Compositio Math. **6** (1938), 239–250].

R. Artzy (New Brunswick, N.J.)

R.-Salinas, Baltasar [Rodríguez-Salinas, Baltasar] 5943

A class of groupoids: Tribes. Embedding in a tribe. (Spanish)

Collect. Math. **15** (1963), 153–167.

Let H be the union of a non-empty family $\{G_i \mid i \in I\}$ of disjoint groups, and for every $i \in I$ let σ_i be an isomorphism of G_i onto a group G. For every $a \in G_i$ and $b \in G_j$ there exists a single $c \in G_j$ such that $\sigma_i(a)\sigma_j(b) = \sigma_j(c)$. H is a semigroup under the operation $ab = c$. Such semigroups are called tribes. The author finds various properties of tribes. If S is a left-cancellable semigroup satisfying the properties: that $ac = bc$ implies $ad = bd$ for every a, b, c, $d \in S$ and that for every $a, b, c \in S$ there exist $x, y \in S$ such that $xac = ybc$, then S can be imbedded in a tribe.

To all appearances the author is not aware that his tribes are exactly the right groups whose properties have been repeatedly rediscovered by many authors since 1928 [cf. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Vol. I, Amer. Math. Soc., Providence, R.I., 1961; MR **24** #A2627].

B. M. Schein (**Šaĭn**) (Saratov)

Ševrin, L. N. 5944

Semigroups all of whose subsemigroups coincide with their own idealizers. (Russian)

Ural. Gos. Univ. Mat. Zap. **3**, *tetrad* 1, 85–87 (1961).

The [left] idealizer of a subsemigroup H of a semigroup Γ is the greatest subsemigroup of Γ in which H is a [left] ideal. It is shown that Γ has the property that each of its subsemigroups coincides with its [left] idealizer if and only if Γ is the set-theoretic union of pairwise disjoint groups and another simple condition is satisfied.

L. Fuchs (Budapest)

Šutov, È. G. 5945

Embedding of semigroups into simple semigroups with one-sided division. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, no. 5 (42), 143–148.

Let S be an infinite set and q an equivalence on S with infinitely many equivalence classes; for any transformation α of S (into itself) denote by S_α^* the set of all q-classes not meeting αS. Further, denote by Σ_S^* the set of transformations α such that, for any $a, b \in S$, (i) $a \equiv b\ (q)$ implies $\alpha a = \alpha b$, (ii) $\alpha a \equiv \alpha b\ (q)$ implies $a \equiv b\ (q)$, (iii) S_α^* is equipotent to the set S_q of all q-classes. Then Σ_S^* is a subsemigroup of the semigroup Σ_S of all transformations of S, with right division and no idempotents; moreover, the semigroups Σ_S^* generate the universal class of semigroups with this property. Given $\alpha, \beta \in \Sigma_S^*$, write $\omega(\alpha, \beta)$ for the cardinal of the set of q-classes M such that $\alpha M \neq \beta M$, and for any infinite cardinal λ, denote by $\sigma[\lambda]$ the relation on Σ_S^* for which $\alpha \equiv \beta\ (\sigma[\lambda])$ if and only if $\omega(\alpha, \beta) < \lambda$. The author proves that each $\sigma[\lambda]$ is a congruence on Σ_S^* and these are all the congruences. Taking λ_0 to be the cardinal of S_q one thus finds the quotient semigroup $\Sigma_S^*/\sigma[\lambda_0]$ to be simple (i.e., without proper quotients). As a consequence the author shows that a semigroup B without idempotents can be embedded in a simple semigroup with right divisibility if and only if for all $a, b, x, y \in B$, $ax = ay$ implies $bx = by$. This is used to give a simple proof of a result by the reviewer [J. London Math. Soc. **31** (1956), 169–181; MR **18**, 14].

P. M. Cohn (Chicago, Ill.)

Šutov, È. G. 5946

Translations of semigroups. (Russian)

Uspehi Mat. Nauk **19** (1964), no. 4 (118), 215–218.

Let S be a semigroup. Let $F(S)$, $F_1(S)$, $\Phi(S)$, $\Phi_1(S)$ mean, respectively, the semigroups of all left, inner left, right, inner right translations of S [T. Tamura, Kōdai Math. Sem. Rep. 7 (1955), 67–70; MR **18**, 318].

The author shows that any semigroup A is isomorphic to some $F_1(B)$ with $B \supset A$. He proves also that for any A the following conditions are equivalent: (1) $F(A) = F_1(A)$ and $\Phi(A) = \Phi_1(A)$; (2) there exists a two-sided ideal densely imbedded in A [L. M. Gluskin, Mat. Sb. (N.S.) **55** (**97**) (1961), 421–448; MR **25** #3106]; (3) there exists a one-sided ideal densely imbedded in A; (4) A is isomorphic to some $F(B)$; (5) A contains a unity element.

K. Drbohlav (Prague)

Švarc, V. Ja.; Jaroker, I. S. 5947

Extension elements of semigroups with one-sided unity elements. (Russian)

Uspehi Mat. Nauk **19** (1964), no. 4 (118), 209–214.

The authors construct a semigroup C with two generators u, v and a right unity element $e = vu$ with the following property. Let A be any semigroup with a right unity element $f \in A$. Then $x \in A$ is a right extension element of A (i.e., $A'x = A$ holds for some proper subset $A' \subset A$) if and only if there exists a homomorphism ψ of C into A with $\psi e = f$ and $\psi u = x$. In this case x generates an infinite subsemigroup in A. All homomorphic images of C are found and, among them, all which contain some right extension element. Those with some left extension element are also determined.

The authors' results generalize the investigations of E. S. Ljapin on extension elements in semigroups with a two-sided unity element [*Semigroups* (Russian), Fizmatgiz, Moscow, 1960; MR **22** #11054] and of L. M. Gluskin [Mat. Sb. (N.S.) **41** (**83**) (1957), 23–36; MR **19**, 836].

K. Drbohlav (Prague)

Weinert, H. J. 5948

Halbgruppen ohne Frattinische Unterhalbgruppe.

Acta Math. Acad. Sci. Hungar. **15** (1964), 309–323.

Let S be a semigroup. A totally ordered decomposition $D = \{X, Y, \cdots\}$ of S is called a chain of subsemigroups in S if $XY \subset Y$ and $YX \subset Y$ whenever $X \leq Y$. Call the chain D trivial if $X < Y$ implies $xy = yx = y$ for all $x \in X$, $y \in Y$. Let $\mathbf{C}$ be the class of all S whose Frattini subsemigroup is empty [see, e.g., S. Lajos, Mat. Lapok **10** (1959), 274–277; MR **24** #A1331]. The main results are the following. If D is a trivial chain in S, then $S \in \mathbf{C}$ if and only if $D \subset \mathbf{C}$. For any band $S \in \mathbf{C}$ there is a chain D in S such that all $X \in D$ are rectangular. A general structure theorem describes all such chains. For any semigroup S the assertion $S \in \mathbf{C}$ is equivalent to the existence of a chain in S of a special kind. Among such special chains, if $S \in \mathbf{C}$, a finest one can be found; it consists of all maximal simple subsemigroups. Further results are obtained under the assumption that S admits relative inverses. $S \in \mathbf{C}$ is true for any direct product $S = G \times L$ of a group G and a rectangular band L with $|L| > 1$. The author's results yield many examples of non-splitting semigroups $S \in \mathbf{C}$. All such $S \in \mathbf{C}$ with $|S| = 4$ are determined.

K. Drbohlav (Prague)

Hoehnke, Hans-Jürgen 5949

Zur Theorie der Gruppoide. VIII. (Russian summary)

Spisy Přírod. Fak. Univ. Brno **1963**, 195–222.

Für Teile I–VII, IX, siehe Math. Nachr. **24** (1962), 137–168 [MR **28** #4052]; ibid. **24** (1962), 169–179 [MR **28** #4053]; Acta Math. Acad. Sci. Hungar. **13** (1962), 91–100 [MR **26** #247]; Monatsb. Deutsch. Akad. Wiss. Berlin **4** (1962), 337–342 [MR **26** #6289a]; ibid. **4** (1962), 539–544 [MR **26** #6289b]; Math. Nachr. **25** (1963), 191–198 [MR **27** #3735]; ibid. **27** (1963/64), 289–298 [MR **29** #3568]; Monatsb. Deutsch. Akad. Wiss. Berlin **5** (1963), 405–411 [MR **30** #1204].

Aus Verfassers Einleitung: "In der vorliegenden Arbeit sollen daher die Grundlagen für eine Theorie der Kategorien mit Nullabbildungen geschaffen werden. Dabei spielt die Feststellung eine wesentliche Rolle, daß das System $^-\Omega$ der Nullabbildungen einer solchen Kategorie ^-K zusammen mit dem Nullelement 0 ein Ideal der $^+$Kategorie $K = {}^-K \cup \{0\}$ bildet, die ... eine Halbgruppe mit Null ist und als $^+$Kategorie mit Nullabbildungen bezeichnet werden soll. Das Ideal $\Omega = {}^-\Omega \cup \{0\}$ ist die Vereinigung aller minimalen (von 0 verschiedenen) ein- und zweiseitigen Ideale von K.... Mittels des halbgruppentheoretischen Nilradikals der Differenzhalbgruppe K/Ω von K mod Ω (die im allgemeinen keine $^+$Kategorie darstellt) läßt sich ein neues Radikal, das 'Ω-Nilradikal' von K bilden.

"Die in Teil IV bemerkte Tatsache, daß sich das Radikal einer beliebigen $^+$Kategorie K abspalten läßt, hat uns dazu veranlaßt, allgemeiner diejenigen Ideale $\mathfrak{a}$ von K zu betrachten, die eine Zerfällung $K = A \cup \mathfrak{a}$, $A \cap \mathfrak{a} = \Omega$, herstellen, wo A eine Teilstruktur von K ist. Solche Ideale nennen wir Ω-isoliert in K". Die Differenzhalbgruppe $K/\mathfrak{a}$ nach einem Ω-isolierten Ideal $\mathfrak{a}$ im allgemeinen keine $^+$Kategorie mehr ist. "Aus diesem Grund führen wir die '$\cap$-Differenz-$^+$Kategorie' $K/_{\kappa}\mathfrak{a}$ ein. Sie stellt für jedes Ideal $\mathfrak{a}$ eine $^+$Kategorie dar, die auch wieder Nullabbildungen besitzt, sobald das für K zutrifft, und die gerade die gewünschte Eigenschaft hat, daß $K_{\mathfrak{a}} = K/_{\kappa}\mathfrak{a} \simeq A$ wird. Damit läßt sich ein Satz über die Abspaltung des Ω-isolierten Bestandteils des Ω-Nilradikals einer $^+$Kategorie mit Nullabbildungen formulieren..., welcher dem Satz über die Abspaltung des Radikals völlig analog ist."

B. M. Schein (*Šain*) (Saratov)

Maniakowski, F. 5950

Sur les axiomes du pseudogroupe.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 197–201.

The author establishes three systems of independent axioms, one for the pseudogroup in the sense of Ch. Ehresmann [Ann. Inst. Fourier (Grenoble) **10** (1960), 307–332; MR **22** #8525], and the others for two extensions of this notion which are due to L. Dubikajtis [Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 469–471; MR **28** #154; Colloq. Math. **12** (1964), 163–185].

H.-J. Hoehnke (Berlin)

Ramos, Gerardo 5951

The notion of groupoids. (Spanish)

Univ. Nac. Ingen. Inst. Mat. Puras Apl. Notas Mat. **1** (1962/63), 117–131.

L'auteur expose un ensemble de propriétés des groupoïdes

de Brandt: Règles de calcul des produits, groupoïdes algébriquement connexes, éléments idempotents, relations d'équivalence compatibles avec une structure de groupoïde, groupoïdes d'opérateurs, de transformations d'un ensemble, sous-groupoïdes. Exemples.

A. Sade (Marseille)

Nishigôri, Noboru 5952

On loop extensions of groups and M-cohomology groups.

J. Sci. Hiroshima Univ. Ser. A-I Math. **27** (1963), 151–165.

The author develops a cohomology theory for Bol-Moufang loop extensions and shows that the extensions correspond with the elements of H^2 in the usual fashion.

S. Rubinstein (Seattle, Wash.)

TOPOLOGICAL GROUPS AND LIE THEORY

See also 5827, 6058, 6255, 6256, 6460, 6491, 6493, 6495a–b, 6502, 6862.

Bagley, R. W. 5953

Invariant uniformities for coset spaces.

Math. Scand. **14** (1964), 19–20.

A subgroup H of a topological group G (not necessarily Hausdorff) is said to satisfy condition A if for each neighborhood U of e there is a neighborhood V of e such that $HV \subset UH$. The author proves that if H in G satisfies condition A, then G/H has a right invariant uniformity. This uniformity is described and related results are discussed.

F. Hahn (New Haven, Conn.)

Heimberg, Gilbert 5954

Über eine Zerlegung des Haarschen Masses auf kompakten Gruppen.

Monatsh. Math. **68** (1964), 218–223.

Let X_1 be a compact group and X_2 and X_3 closed subgroups of X_1. Let μ_j denote normalized Haar measure on X_j. The following assertions are equivalent: (1) $\mu_1 = \mu_2 * \mu_3$; (2) $\mu_1 = \mu_3 * \mu_2$; (3) $\mu_2 * \mu_3 = \mu_3 * \mu_2$ and $X_2X_3 = X_1$. Nine other equivalent properties are listed as well, and the case in which X_2 and X_3 are monothetic is dealt with separately. See also the author's earlier communication [same Monatsh. **66** (1962), 417–423; MR **27** #1530].

Edwin Hewitt (Seattle, Wash.)

Rohlin, V. A. 5955

Metric properties of endomorphisms of compact commutative groups. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 867–874.

Let G be a compact commutative group with a countable basis. Let μ be the Haar measure in G. The endomorphisms of G onto G preserve μ and hence are measurable transformations of G with an invariant measure μ. The author proves that every ergodic endomorphism of G onto G has a completely positive entropy [M. S. Pinsker, Dokl. Akad. Nauk SSSR **133** (1960), 1025–1026; MR **27** #2603].

As corollaries one can obtain several previously known theorems concerning the spectrum and mixing properties of such endomorphisms [cf., e.g., P. R. Halmos, Bull. Amer. Math. Soc. **49** (1943), 619–624; MR **5**, 40; V. A. Rohlin, Izv. Akad. Nauk SSSR Ser. Mat. **13** (1949), 329–340; MR **11**, 40].

In particular, the above theorem includes the author's theorem on the positivity of an ergodic automorphism of G onto G [Teor. Verojatnost. i Primenen. **6** (1961), 351–352; MR **27** #2605]. This theorem also implies that every ergodic automorphism of G onto G is a K-automorphism. This fact has been known previously only for some special cases.

Y. N. Dowker (London)

Connell, E. H.; Montgomery, D.; Yang, C. T. 5956

Correction to "Compact groups in E^n".

Ann. of Math. (2) **81** (1965), 194.

The authors modify the proofs of the main theorems of an earlier paper [same Ann. (2) **80** (1964), 94–103; MR **29** #189]; the reviewer of the earlier paper indicated how Proposition 3 might be modified so as to be applicable to the proof of Theorem B, but the authors give an alternative method.

Antoine, J.-P.; Speiser, D. 5957a

Characters of irreducible representations of the simple groups. I. General theory.

J. Mathematical Phys. **5** (1964), 1226–1234.

Antoine, J.-P.; Speiser, D. 5957b

Characters of irreducible representations of the simple groups. II. Application to classical groups.

J. Mathematical Phys. **5** (1964), 1560–1572.

The authors present a geometrical version of the character formula for a compact simple Lie group G, which they obtain by geometrically carrying out the division in Weyl's character formula. In detail, let $\beta_1, \cdots, \beta_m$ be the simple negative roots of G, of which the first l are simple, let R_0 be half the sum of the positive roots, and let D_0 be the Weyl chamber containing R_0. Let $\mathscr{G}$ be the lattice $(2\pi Z)^l$ in E_l (corresponding to the center of G in the Stiefel diagram setting in which the authors work). For each $P, \varphi \in E_l$ let $[P] = [P](\varphi) = \exp i(P, \varphi)$. Then for each $K_0 \in \mathscr{G} \cap D_0$ there is an irreducible representation of G whose character χ is given by (Weyl's formula) $\chi = X/\Delta$, where $X = \sum \operatorname{sgn}[sK_0]$, $\Delta = \sum \operatorname{sgn}[sR_0]$, and the sum is over all s in the Weyl group of G. Δ can be written as $\Delta = [R_0] \prod_1^m (1 - [\beta_i])$, and by inverting formally (which the authors show can be justified after multiplication by X), one has $\Delta^{-1} = [-R_0] \sum_k [\sum_1^m k_i\beta_i]$, where the first sum is over all m-tuples $k = (k_1, \cdots, k_m)$ of positive integers. Assuming for the moment that the m roots β_i are linearly independent and thus span E_m, the authors regard the sum as defining all points with positive integral coordinates in the lattice spanned by the β_i, with each point being assigned a multiplicity $+1$. Δ^{-1} is then the same lattice translated by $-R_0$. When one takes into account the relations between the roots, the previous figure is projected onto points in a pyramid in E_l with l surfaces and vertex at $-R_0$. The points $\sum k_i\beta_i$ which occurred with multiplicity $+1$ in E_m correspond to points in the pyramid with multiplicity greater than one in general, and in fact the multiplicity of the point ω is

$P(-w)$, where P is the partition function introduced by Kostant [Trans. Amer. Math. Soc. 93 (1959), 53–73; MR 22 #80]. The final formula for χ is then

$$\chi = \sum_k \left(\sum_s \operatorname{sgs}\left[\sum_1^m k_i\beta_i + sK_0 - R_0 \right] \right),$$

which the authors show is, in fact, a finite sum.

In Part II the authors explicate this formula in detail for the four main classes of simple groups, and for G_2, and in even greater detail for groups of ranks 1 and 2 and for A_3. They also remark that the representations of each of these groups fall into classes which form a group isomorphic to the center of the group.

A. Kleppner (College Park, Md.)

Maurin, K.; Maurin, L. 5958

Enveloping algebra of a locally compact group and its selfadjoint representations.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 525–529.

From the introduction: "Let G be a connected separable locally compact group. Using the fact that such a G is a projective limit of Lie groups, the authors define an algebra $E(G)$ with involution which is a natural generalization of the universal enveloping algebra of a Lie group. Next they construct a generalization of the Gårding subspace of H for each unitary representation (U, H) of G, using the space $D(G)$ of 'generalized C^∞ functions with compact support' defined earlier [same Bull. 7 (1959), 471–479; MR 22 #4960]. Using an ingenious idea of Nelson and Stinespring [Amer. J. Math. 81 (1959), 547–560; MR 22 #907], they show that, among others, the representants $dU(M)$ of central symmetric elements $M^* = M \in E(G)$ are essentially self-adjoint and that their closures have commuting spectral resolutions and commute with U. The construction of a 'Gelfand triplet' $\Psi \subset H \subset \Psi'$ (Ψ being nuclear) adapted to the representation (U, H) then allows them to decompose (U, H) as a direct integral of representations $(U(\lambda), H(\lambda))$, and to interpret the spaces $H(\lambda)$ as common eigenspaces of the operators $dU(M)$. As an application, they are able to decompose any $f \in L^2(G/G_0, \sigma)$ into eigenfunctions of invariant 'differential' operators on a homogeneous space G/G_0." No proofs are given.

J. M. G. Fell (Seattle, Wash.)

Maurin, K. 5959

A theory of characters. Harmonic analysis on connected type I-groups.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 587–592.

This paper continues the preceding one [#5958]. Let G be a connected, separable, locally compact group, and $D(G)$ the nuclear space of "generalized C^∞ functions with compact support" on G. Let B be a kernel (i.e., a separately continuous bilinear form on $D(G) \times D(G)$) which is invariant (i.e., $B(g\varphi, g\psi) = B(\varphi, \psi)$ for $g \in G$, $\varphi, \psi \in D(G)$) and positive Hermitian. It is shown (Theorem 2.2) that B is a direct integral of positive invariant kernels B_λ which induce factor (or irreducible) representations and which are "eigenkernels" of the central symmetric elements of the generalized enveloping algebra $E(G)$ of G. In particular, if the kernel B is a trace (i.e., positive, invariant and central), we can make the B_λ be characters, that is, traces which induce factor representations (Theorem 3).

If G is unimodular and of Type I, the integral decomposition of B can be taken over the dual space $\hat{G}$ of G, that is, the Borel space of all equivalence classes of irreducible unitary representations (Theorem 4.1).

Proofs are given in compressed form.

J. M. G. Fell (Seattle, Wash.)

Paljutkin, V. G. 5960

On the equivalence of two definitions of a finite ring group. (Russian)

Ukrain. Mat. Ž. **16** (1964), 402–406.

G. I. Kac [Trudy Moskov. Mat. Obšč. **12** (1963), 259–301; MR **28** #164] asked whether two different definitions of finite ring groups are equivalent. The present paper answers this question affirmatively.

M. Greendlinger (Ivanovo)

Kobayashi, Shoshichi; Nagano, Tadashi 5961

On filtered Lie algebras and geometric structures. I.

J. Math. Mech. **13** (1964), 875–907.

Cet article donne les démonstrations d'une partie des résultats annoncés dans une note antérieure [Bull. Amer. Math. Soc. **70** (1964), 401–403; MR **29** #196]. Ces résultats concernent les algèbres de Lie L de dimension finie sur un corps k de caractéristique 0 filtrées par une suite de sous-algèbres $L = L_{-1} \supset L_0 \supset L_1 \supset L_2 \supset \cdots$, avec $[L_p, L_q] \subset L_{p+q}$; c'est la situation qui se présente dans l'étude en un point d'une algèbre de Lie de champs de vecteurs sur une variété différentiable. On suppose donné un groupe A d'automorphismes de L laissant chaque L_p stable. On suppose que l'algèbre L et le groupe A vérifient les conditions suivantes: (a) dans l'algèbre de Lie graduée associée à L, L/L_0 est une sous-algèbre commutative maximale, (b) dans L, considéré comme module sur L_0, L_0 est maximal dans l'ensemble des sous-modules stables par A et distincts de L, (c) $L_1 \neq (0)$. Les auteurs démontrent qu'avec ces hypothèses: (1) $L_p = (0)$ pour $p > 1$, (2) L est semi-simple et n'admet pas d'idéal non trivial stable par A, (3) il existe un élément $e \in L$ et un seul tel que $\operatorname{Ker}(\operatorname{ad}(e) - p)$ soit un supplémentaire de L_{p+1} dans L_p ($p = -1, 0, 1$). Si $k = \mathbf{R}$ et si A est contenu dans la composante connexe neutre du groupe des automorphismes de L, alors L est simple et contient une sous-algèbre compacte maximale K stable par $\operatorname{ad}(e)^2$. L'élément e est alors orthogonal à K pour la forme de Killing de L. Les auteurs donnent ensuite la liste des classes de couples (L, e), où L désigne une algèbre de Lie réelle simple et e un élément de L tel que $\operatorname{ad}(e)$ soit semi-simple et admette pour seules valeurs propres: -1, 0 et 1 (deux couples (L, e) et (L', e') sont dans la même classe s'il existe un isomorphisme θ de L sur L' tel que $\theta(e) = e'$). Pour une algèbre L classique, les éléments e sont donnés explicitement en termes de matrices. Dans le cas général, ils sont déterminés à partir du diagramme de Satake de L.

J. L. Koszul (Grenoble)

Kuranishi, M.; Rodrigues, A. M. 5962

Quotients of pseudo groups by invariant fiberings.

Nagoya Math. J. **24** (1964), 109–128.

A basic technical problem in the theory of pseudogroups is that of passing to the "quotient space" of a representation. Roughly speaking, let Γ be a pseudogroup acting on a fiber space (M, M', ρ), where M and M' are differentiable manifolds and ρ is the projection of M onto the base M'. Suppose every element of Γ preserves the fibration. Then each $f \in \Gamma$ induces a locally defined transformation f' on M'. Does the collection of such f' form a pseudogroup on M'? The difficulty is that $f' \circ g'$ may be defined while $f \circ g$ is not, because we can have range $g' \subset$ domain f' without having range $g \subset$ domain f. The authors show by example that one can have an analytic intransitive pseudogroup acting on an M such that the induced family on M' is not a pseudogroup. Their main theorem asserts that for a real analytic transitive continuous pseudogroup the induced family of transformations is indeed a pseudogroup. The proof uses the celebrated Cartan-Kuranishi prolongation theorem.

S. Sternberg (Cambridge, Mass.)

FUNCTIONS OF REAL VARIABLES

See also 5733a–b, 5846, 5976, 5994, 6283, 6301, 6803, 6961.

Alexandroff, P. S. [Aleksandrov, P. S.] 5963

★Einführung in die Mengenlehre und die Theorie der reellen Funktionen.

Zweite Auflage. Übersetzung aus dem Russischen: Manfred Peschel und Wolfgang Richter. Hochschulbücher für Mathematik, Band 23.

VEB Deutscher Verlag der Wissenschaften, Berlin, 1964. xii + 279 *pp.* *DM* 18.00.

Diese zweite Auflage der verdienstvollen Übersetzung aus dem russischen ist ein im Text unveranderter Nachdruck der rasch vergriffenen ersten Auflage, die ihrerseits wieder ein getreues Abbild des russischen Originales ist. Ergänzt und modernisiert wurde das Literaturverzeichnis, ohne den Anspruch auf Vollständigkeit zu erheben.

K. Bögel (Ilmenau)

Smirnov, V. I. 5964

★A course of higher mathematics. Vol. V [Integration and functional analysis].

Translated by D. E. Brown; translation edited by I. N. Sneddon. ADIWES International Series in Mathematics.

Pergamon Press, Oxford-New York; Addison-Wesley Publishing Co., Inc., Reading, Mass.-London, 1964. xiv + 635 *pp.* $17.50.

This is a translation of the 1959 Russian edition [Fizmatgiz, Moscow, 1959]. The translation into German [VEB Deutscher Verlag der Wissenschaften, Berlin, 1962] was reviewed in MR 25 #5141. An earlier Russian edition [OGIZ, Moscow, 1947] was reviewed in MR 9, 574.

W. H. Fleming (Providence, R.I.)

Burkill, H. 5965

A note on rearrangements of functions.

Amer. Math. Monthly 71 (1964), 887–888.

For $0<x<1$, let the real-valued functions f and g be non-increasing and left-continuous. If for all real numbers A, the intervals $\{x: f(x) \geq A\}$ and $\{x: g(x) \geq A\}$ have the same length, then the author shows $f(x) = g(x)$ for $0<x<1$. Thus left-continuity specifies uniquely that non-increasing function which is equimeasurable with a given measurable function.

G. Freilich (New York)

Bruckner, A. M. 5966

Stationary sets for certain classes of derivates of Darboux functions.

Michigan Math. J. 11 (1964), 305–309.

The author proves several interesting theorems that generalize the basic result of elementary calculus, which states that a differentiable function f whose derivative vanishes on an interval $[a, b]$ must be constant on $[a, b]$. A set is called totally imperfect if it contains no non-empty perfect subsets. Let $\mathscr{C}$ be a class of functions defined on $[a, b]$. A set $E \subset [a, b]$ is called a stationary set for $\mathscr{C}$ provided every function in $\mathscr{C}$ that is constant on E is constant on $[a, b]$. Let $\mathscr{A}$ denote the class of approximate derivatives, finite or infinite, of Darboux functions; $\mathscr{D}$ the class of upper right Dini derivatives of Darboux-Baire functions; and $\mathscr{U}$ the class of upper derivatives of Darboux functions. It is proved that if E is a subset of $[a, b]$ whose complement in $[a, b]$ is totally imperfect, then E is a stationary set for each of the classes $\mathscr{A}$, $\mathscr{D}$, and $\mathscr{U}$. It is also shown that the only stationary set for the class of approximate derivatives of arbitrary approximately differentiable functions on $[a, b]$ (infinite values allowed for f_{ap}'), or for the class of Dini derivatives of arbitrary finite functions on $[a, b]$, or for the class of upper derivatives of arbitrary functions on $[a, b]$, is the interval $[a, b]$ itself.

S. Marcus (Bucharest)

Šmidov, F. I. 5967

Properties of the derivative numbers and approximative derivative numbers of a finite function. (Russian)

Sibirsk. Mat. Ž. 5 (1964), 679–711.

Soit E un ensemble réel borné. Convenons de dire qu'une fonction réelle finie f, définie sur E, jouit de la propriété (L) si $E = \bigcup_{n=1}^{\infty} E_n$ et, pour chaque n, f remplit une condition de Lipschitz sur E_n. On donne des conditions suffisantes, exprimées à l'aide des nombres dérivés—ordinaires ou approximatifs—afin que f jouisse de la propriété (L) sur E. Théorème 1: Supposons que f remplit, en chaque point de E, à l'exception possible d'un ensemble dénombrable, l'une au moins des quatre conditions suivantes:

(1) $\underline{f}^-(x) > -\infty$ et $\bar{f}^+(x) < \infty$;

(2) $\bar{f}^-(x) < \infty$ et $\underline{f}^+(x) > -\infty$;

(3) $-\infty < \underline{f}^-(x) \leq \bar{f}^-(x) < \infty$;

(4) $-\infty < \underline{f}^+(x) \leq \bar{f}^+(x) < \infty$.

Alors, f jouit de la propriété (L) sur E. Théorème 2: Supposons que f est mesurable sur E et qu'elle remplit en chaque point de E, à l'exception possible d'un ensemble dénombrable, l'une au moins des six conditions suivantes:

(1) $-\infty < \bar{f}_{ap}^+(x) < \infty$ et $\underline{f}_{ap}^-(x) > -\infty$;

(2) $\bar{f}_{ap}^+(x) < \infty$ et $-\infty < \underline{f}_{ap}^-(x) < \infty$;

(3) $-\infty < \underline{f}_{ap}^+(x) < \infty$ et $\bar{f}_{ap}^-(x) < \infty$;

(4) $\quad f_{ap}{}^{+}(x) > -\infty \quad$ et $\quad -\infty < \bar{f}_{ap}{}^{-}(x) < \infty;$

(5) $\quad -\infty < \underline{f}_{ap}{}^{+}(x) \leq \bar{f}_{ap}{}^{+}(x) < \infty;$

(6) $\quad -\infty < \underline{f}_{ap}{}^{-}(x) \leq \bar{f}_{ap}{}^{-}(x) < \infty.$

Alors, f jouit de la propriété (L) sur E. Le théorème 3 établit l'existence, pour toute fonction f mesurable, de certaines parties du graphique de f qui sont réunions dénombrables de courbes satisfaisant une condition de Lipschitz.

Les démonstrations sont extrêmement délicates et laborieuses. *S. Marcus* (Bucharest)

Ivanov, L. D. 5968

An estimate for the growth of smooth functions. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 1131–1134.

Let G be an open r-dimensional cube of side $N\varepsilon$, where N is a positive integer and $\varepsilon > 0$. Further, let $\lambda = s + d$, where s is a positive integer and $0 \leq d < 1$. The functions f considered are defined in G and possess partial derivatives of orders $\leq s$; moreover, their partial derivatives of order s are subject to a Hölder-Lipschitz condition of order α with constant M. For suitable positive k, K depending only on r, λ, the author shows that if an inequality of the type $|f(x)| \leq A(N\varepsilon)^{\lambda}$ holds at not less than kN^{r-1} points whose mutual distances are not less than ε, then a similar inequality holds throughout G, with A replaced by $K\cdot(A+M)$. *L. C. Young* (Madison, Wis.)

Meyers, Norman G. 5969

Mean oscillation over cubes and Hölder continuity.

Proc. Amer. Math. Soc. **15** (1964), 717–721.

By modifying a method due to John and Nirenberg [Comm. Pure Appl. Math. **14** (1961), 415–426; MR **24** #A1348], the author establishes, in the same order of ideas, an extremely convenient criterion for Hölder continuity in terms of the mean oscillation over cubes.

L. C. Young (Madison, Wis.)

Goldman, A. J. 5970

A generalization of Rennie's inequality.

J. Res. Nat. Bur. Standards Sect. B **68B** (1964), 59–63.

Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ $(n > 1)$ be positive numbers with $\sum_{i=1}^{n} \alpha_i = 1$. For every sequence $(x_1, x_2, \cdots, x_n)$ with all $x_i > 0$ and for every real $t \neq 0$, consider the mean of order t, $M_t(x_1, x_2, \cdots, x_n) = (\sum_{i=1}^{n} \alpha_i x_i{}^t)^{1/t}$. Let $r < s$, $r, s \neq 0$, and consider the ratio (*) $M_s(x_1, \cdots, x_n)/M_r(x_1, \cdots, x_n)$. It is well known that this ratio is ≥ 1. Cargo and the reviewer [same J. **66B** (1962), 169–170; MR **26** #5110] have shown that if $0 < A < B$ and if each x_i is restricted to the closed interval $[A, B]$, then the above ratio is bounded above by

$$(**)\quad \left\{\frac{r(\gamma^s-\gamma^r)}{(s-r)(\gamma^r-1)}\right\}^{1/s}\left\{\frac{s(\gamma^r-\gamma^s)}{(r-s)(\gamma^s-1)}\right\}^{-1/r}, \qquad \gamma = B/A.$$

They have also given a necessary and sufficient condition for the equality of (*) and (**). The author gives an alternative proof of the results of Cargo and the reviewer, formulating them in a more general setting, where integration over a measure space of unit total measure replaces the finite sums defining the means. For his proof he gives first a generalization of a simple but elegant observation made by Rennie [Amer. Math. Monthly **70** (1963), 982], who gave a simple proof of (the integral form of) the Kantorovich inequality.

O. Shisha (Dayton, Ohio)

Beckenbach, E. F. 5971

On the inequality of Kantorovich.

Amer. Math. Monthly **71** (1964), 606–619.

Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ $(n > 1)$ be positive numbers with $\sum_{i=1}^{n} \alpha_i = 1$. For every sequence $(x_1, x_2, \cdots, x_n)$ with all $x_i > 0$ and for every t, $-\infty \leq t \leq \infty$, consider the mean of order t, $M_t(x_1, x_2, \cdots, x_n)$, defined as $(\sum_{i=1}^{n} \alpha_i x_i{}^t)^{1/t}$ if $0 < |t| < \infty$, as $\prod_{i=1}^{n} x_i{}^{\alpha_i}$ if $t = 0$, as $\min_{1 \leq i \leq n} x_i$ if $t = -\infty$, and as $\max_{1 \leq i \leq n} x_i$ if $t = \infty$. It is known that

$$(*)\qquad M_s(x_1, x_2, \cdots, x_n)/M_r(x_1, x_2, \cdots, x_n) > 1$$

if $-\infty \leq r < s \leq \infty$, unless all the x_i are equal. Suppose that $c_1, c_2, \cdots, c_m$ $(1 \leq m < n)$ are given positive numbers. The author gives a lower bound for the function

$$(**)\quad \varphi(x_{m+1}, \cdots, x_n) \equiv \frac{M_s(c_1, \cdots, c_m, x_{m+1}, \cdots, x_n)}{M_r(c_1, \cdots, c_m, x_{m+1}, \cdots, x_n)} \qquad (-\infty \leq r < s \leq \infty),$$

from which he also derives (*). Let $0 < A < B < \infty$. Cargo and the reviewer [J. Res. Nat. Bur. Standards Sect. B **66B** (1962), 169–170; MR **26** #5110] have given an upper bound for $M_s(x_1, \cdots, x_n)/M_r(x_1, \cdots, x_n)$ (r, s fixed, $-\infty < r < s < \infty$), where each x_i ranges over the closed interval $[A, B]$, and a necessary and sufficient condition for the attainment of this bound. The author considers the function (**) for given r, s $(-\infty \leq r < s \leq \infty)$ and given positive c_i, derives an upper bound for it when the x_i lie in $[A, B]$, and gives a necessary and sufficient condition for the attainment of this bound.

O. Shisha (Dayton, Ohio)

MEASURE AND INTEGRATION

See also 5766, 5955, 5964, 6143, 6295, 6297, 6353, 6402, 6459, 6539, 6962, 6963a–b.

Larman, D. G. 5972

The approximation of G_δ-sets, in measure, by F_σ-sets.

Proc. Cambridge Philos. Soc. **61** (1965), 105–107.

Let E and A $(E \supset A)$ be sets of type G_δ and F_σ, respectively, on the real line R. The author calls A a good approximation to E provided there exists a translation-invariant outer measure μ (on subsets of R) such that $\mu(E) > 0$ and $\mu(E - A) = 0$. Modifying a construction by Besicovitch [J. London Math. Soc. **29** (1954), 382–383; MR **15**, 943], he obtains a set E of type G_δ with the following property: If A is of type F_σ and $E \supset A$, then E can be covered by a countable family of translates of $E - A$. In particular, no set of type F_σ is a good approximation to E.

G. Piranian (Ann Arbor, Mich.)

Savel'ev, L. Ja. 5973

Extension of measures by continuity. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 639–650.

Let A be a Boolean subalgebra of a Boolean algebra B.

By a measure on A the author means a bounded, real-valued, finitely or countably additive function defined on A. He shows that such a measure determines a topology for the larger Boolean algebra B. If the measure has a certain property called regularity, then the operations of B are continuous in the topology determined by the measure. In this case the measure is uniformly continuous, as a real-valued function defined over A. Hence it may be extended in a unique manner to a continuous function on the closure of A in the topology of B. This extension is also bounded and additive.

The author shows that in some standard cases this extension coincides with the usual extensions of measures. In particular, the Jordan and Lebesgue measures and some of their generalizations are obtainable as continuous extensions. The author generalizes these results to measures which, instead of being real-valued, take their values either in a topological abelian group, or in the Boolean algebra of all subsets of a set, with the Stone topology. *O. Frink* (University Park, Pa.)

Helsel, R. G.; Pu, H. W. 5974

Transforming a set-valued integral.

Bull. Soc. Roy. Sci. Liège **33** (1964), 451–455.

Le présent article est un complément à un autre [même Bull. **33** (1964), 272–286; MR **30** #2116], où les auteurs ont défini pour des fonctions réelles une intégrale à valeurs dans l'ensemble $\underline{\mathfrak{X}}$ des parties non vides bornées (et semble-t-il, fermées, bien que ce ne soit pas dit explicitement) d'un espace de Banach $\mathfrak{X}$. Cette intégrale jouit des propriétés classiques de la convergence monotone et de la convergence dominée. Si T est une application linéaire et continue de $\mathfrak{X}$ dans un espace de Banach $\mathfrak{Y}$, on en déduit de manière naturelle une intégrale à valeurs dans $\underline{\mathfrak{Y}}$. Le résultat essentiel est que l'application $(f, T)\to I(f, T)$ est continue en ce sens que si $f_n\to f$ pour la convergence dominée et si $\|T_n-T\|\to 0$, $I(f_n, T_n)\to I(f, T)$. On souhaiterait connaître des applications de la théorie. *A. Revuz* (Poitiers)

Henstock, Ralph 5975

The integrability of functions of interval functions.

J. London Math. Soc. **39** (1964), 589–597.

This paper is concerned with the concepts (introduced by the author) of variational equivalence and variational integration of pairs of interval functions. In the first theorem ξ is a function on the plane or its positive quadrant; for $j=1, 2$, $\mathbf{h}_j=\{h_{lj}, h_{rj}\}$ is a pair of interval functions variationally equivalent to $\mathbf{k}_j=\{k_{lj}, k_{rj}\}$; and $a_s=\xi(h_{s1}, h_{s2})$, $b_s=\xi(k_{s1}, k_{s2})$ $(s=l, r)$. It is shown that $\mathbf{a}=(a_l, a_r)$ is variationally equivalent to $\mathbf{b}=(b_l, b_r)$ if either ξ is such that always $|\xi(y_1, y_2)-\xi(x_1, x_2)|\leq |y_1-x_1|+|y_2-x_2|$, or else $\mathbf{h}_1$, $\mathbf{h}_2$ are variationally integrable and ξ satisfies a less stringent, but more complicated, continuity condition. In the second theorem a condition is given for $\mathbf{a}$ to be variationally integrable. By using particular ξ's the author then improves a number of results given in his book *Theory of integration* [Butterworths, London, 1963; MR **28** #1274]. The final theorem, also based on the first two, is the following. If f_1, f_2 are non-negative point functions, h_l, h_r are non-negative interval functions and, for $j=1, 2$, $(V)\int_a{}^b f_j\,d\mathbf{h}$ exists, then $(V)\int_a{}^b f_1{}^t f_2{}^{1-t}\,d\mathbf{h}$ exists for every t in $(0, 1)$. When $\mathbf{h}$ is variationally integrable, then $(V)\int_a{}^b f_1{}^t\,d\mathbf{h}$ also exists for every t in $(0, 1)$. *H. Burkill* (Sheffield)

Marcus, Solomon 5976

On a paper by B. K. Lahiri.

Bull. Calcutta Math. Soc. **55** (1963), 127–129.

The author first gives an elementary proof that if f is differentiable and the set Z of zeros of f' is everywhere dense, then f' is Riemann integrable on no subinterval of the domain if and only if the complement of Z is everywhere dense. He then proves the analogue for the approximate derivative of an approximately continuous function that has a finite approximate derivative at each point. *R. P. Boas, Jr.* (Evanston, Ill.)

Chacon, R. V. 5977

Ordinary means imply recurrent means.

Bull. Amer. Math. Soc. **70** (1964), 796–797.

Let X be a σ-finite measure space, T a positive linear operator from $L_1(X)$ to $L_1(X)$ with $\|T\|\leq 1$. Let $\{w_k\}$ be a sequence of non-negative numbers with $\sum_1{}^\infty w_k=1$. Define $u_n=\sum_1{}^n w_i u_{n-i}$, $u_0=1$. For any f and p in $L_1(X)$, $p\geq 0$, define $Q_n(f, p)=Z_n(f)/Z_n(p)$, where $Z_n(g)=\sum_0^{n-1} u_k T^k g$. Theorem: The ratios $Q_n(f, p)$ have a finite limit a.e. on the set where $\sum_0{}^\infty u_k T^k p>0$. The same assertion, on the set where $p>0$, was announced by G. E. Baxter [Notices Amer. Math. Soc. **11** (1964), 464]. The stronger version stated above is obtained directly from the general ergodic theorem of R. V. Chacon and D. S. Ornstein [Illinois J. Math. **4** (1960), 153–160; MR **22** #1822], which is a special case of it. *J. C. Oxtoby* (Bryn Mawr, Pa.)

Forte, Bruno 5978

Sulle funzioni di fase isocrone. (English and French summaries)

Rend. Mat. e Appl. (5) **22** (1963), 371–381.

From the author's summary: "A single but expressive kinematical description is given for the time average of a phase function. This description leads to a sufficient condition for ergodicity. The main properties of the phase function which satisfies such a condition are studied." *G.-C. Rota* (Cambridge, Mass.)

Sarymsakov, T. A. 5979

On the general ergodic theory. (Russian)

Dokl. Akad. Nauk SSSR **159** (1964), 25–27.

Notation and terminology are as in Antonovskiĭ, Boltjanskiĭ and the author [*Topological semifields* (Russian), Izdat. Samarkand. Gos. Univ., Tashkent, 1960; MR **25** #2461]. Let X be a normed space over the semifield R_Δ, and let G be a semigroup of linear operators carrying X into itself. Let Ω be the family of all finite subsets of Δ. A set $\{A_\alpha\}_{\alpha\in\Omega}$ of linear operators carrying X into itself is called a basic Ω-system for G if for all $x\in X$ and $T\in G$,

$$\lim_\alpha \|TA_\alpha x-A_\alpha x\| = \lim_\alpha \|A_\alpha Tx-A_\alpha\| = 0.$$

The norm in X has values in K_Δ. The semigroup G is called ergodic if it admits at least one basic Ω-system.

Next, a norm $\|A\|$ for linear operators A on X is defined; this norm is a certain mapping of K_Δ into K_Δ.

Several theorems are announced without proof; the following is typical. If A is a commutative set of operators with a common bound on their norms, then the semigroup generated by A is ergodic.

Edwin Hewitt (Seattle, Wash.)

Roos, B. W. 5980

Note on generalized dynamical systems.

SIAM Rev. **6** (1964), 269–274.

This note presents a phase flow description of generalized dynamical systems. This description is advantageous for the theoretical investigation of these systems and for the proofs of general theorems. After introducing the concept of regional recurrence, a criterion is given for the ergodicity of a dynamical system which is reversible and measure-preserving.

W. Sangren (San Diego, Calif.)

Gähler, Werner 5981

Der innere Flächeninhalt.

Math. Nachr. **28** (1964), 67–110.

In der vorliegenden Arbeit werden die in einer früheren Publikation [dieselben Nachr. **26** (1963), 181–228; MR **29** #3610] begonnenen Untersuchungen fortgesetzt. Die dort getroffenen Vereinbarungen werden übernommen, die eingeführten Begriffe verwendet. Ein Polyeder $S: p|M_\pi$ ($\pi \in \Pi$) heisst "Unterpolyeder einer eigentlichen Fläche" $S': p'|M_{\sigma'}$ ($\sigma' \in \Sigma$) wenn folgendes gilt: Zu jeder offenen Überdeckung des Trägers $[S]$ von S gibt es ein Element $\pi' \in \Pi$ und eine Abbildung $f \in F_{\pi\pi'}$, so dass $\sigma_{\pi'} = \sigma$ ist und zu jedem Paar von Punkten $t \in M_\pi$, $t' \in M_{\pi'}$, mit tft' die Bildpunkte $p(t)$ und $p'(t')$ stets beide gemeinsam einer gewissen Menge aus der Überdeckung angehören. Sätze 1 und 2 und das Korollar zu Satz 2 liefern mehrere Bedingungen, die notwendig und hinreichend dafür sind, dass ein Polyeder Unterpolyeder einer eigentlichen Fläche ist. Ein Polyeder S^* heisst in der eigentlichen Fläche $S: p|M_\sigma$ ($\sigma \in \Sigma$) bezüglich $p|M_\pi$ einbeschrieben, wenn S^* als eine Darstellung die Einschränkung $p|M_\pi$ von p über einem Raum M_π ($\pi \in \Pi$) mit $\sigma_\pi = \sigma$ besitzt. Wie aus Satz 3 folgt, ist ein Polyeder genau dann Unterpolyeder einer eigentlichen Fläche S, wenn es in S bezüglich mindestens einer Darstellung von S einbeschrieben ist. Ähnlich wie mit den Flächen werden vier Kurvenarten erklärt, je nach dem ebenen Urbild der Kurvendarstellung: orientierte und nicht orientierte, berandete und geschlossene Kurven. Die orientierte berandete Kurve steht im engen Zusammenhang mit dem im Artikel von S. Gähler und dem Verfasser [ibid. **22** (1960), 175–203; MR **24** #A444] eingeführten Kurvenbegriff. Die meisten für Flächen aufgestellten Aussagen lassen sich auf die in der vorliegenden Arbeit definierten Kurven übertragen. Satz 11 drückt eine wichtige Konvergenzaussage für eine Folge von Polygonen gegen eine eigentliche Kurve aus, mit deren Hilfe in Satz 18 unter gewissen Voraussetzungen eine ähnliche Aussage für Flächen bewiesen wird. Ähnlich wie im Fall des äusseren Flächeninhalts definiert Verfasser eine allgemeine Inhaltsfunktion β und gibt für sie Aussagen an. Der innere Flächeninhalt wird wie der äussere Flächeninhalt mittels eines Tripelmasses definiert, mit dem einzigen Unterschied, dass jetzt lediglich Unterpolyeder der betreffenden Fläche zur Inhaltsdefinition zugelassen sind. Die Aussagen über β werden dann zu Aussagen über den inneren Flächeninhalt spezialisiert. Unter anderem ergibt sich dabei, dass der innere Flächeninhalt unter gewissen Voraussetzungen subadditiv ist.

Chr. Y. Pauc (Nantes)

Itô, Seizô; Sawashima, Ikubo 5982

On linear measure of projections of two-dimensional sets to arbitrary straight line.

Natur. Sci. Rep. Ochanomizu Univ. **15** (1964), 33–40.

For a set E in the plane let $f(\theta)$ denote the linear measure of the orthogonal projection of E onto a straight line having direction θ. It is shown that if E is a bounded open set, then $f(\theta)$ is lower semicontinuous but not necessarily upper semicontinuous; if E is a bounded closed set, then $f(\theta)$ is upper semicontinuous but not necessarily lower semicontinuous.

P. V. Reichelderfer (Columbus, Ohio)

Vinti, Calogero 5983

Perimetro—variazione.

Ann. Scuola Norm. Sup. Pisa (3) **18** (1964), 201–231.

For symmetric density functions (or "nuclei") the author gives a more general setting and stronger results on approximation theory along the lines of a recent joint paper with the reviewer [E. Baiada and C. Vinti, Ann. Mat. Pura Appl. (4) **62** (1963), 1–58; MR **28** #4076]. Moreover, if H is the class of "nuclei" $h(r, s, t)$, $0 \leq r < s < +\infty$, $t \in S_s$, such that (i) $h(r, s, t) \geq 0$, (ii) $\int_{S_s} h\, dt = 1$, (iii) for any s and positive δ, $\lim_{r \to s-0} \int_{|t| \geq \delta} h(r, s, t)\, dt = 0$, (iv) $f^* h$ is locally of class B_1 of J. W. Calkin and C. B. Morrey [J. W. Calkin, Duke Math. J. **6** (1940), 170–186; MR **1**, 208; C. B. Morrey, ibid. **6** (1940), 187–215; MR **1**, 209], then $\max \lim_{r \to s-0} \int_{S_s} |\text{grad } f^* h|\, dx$, called the "perimeter" of f, is independent of s in $(0, +\infty)$ and of h in H. This invariant coincides with Fréchet continuation of the variation functional to (L) and is then lower semicontinuous [cf. also E. De Giorgi, Ann. Mat. Pura Appl. (4) **36** (1954), 191–213; MR **15**, 945]. When nuclei are of compact support in S_2 an approximation theorem in area (in the generalized sense given by L. Cesari [Ann. Scuola Norm. Sup. Pisa (2) **5** (1936), 299–313]) and in S_1 an approximation theorem in length (in the generalized sense of C. Goffman [Rend. Circ. Mat. Palermo (2) **2** (1954), 203–235; MR **16**, 457]) are obtained. Also given are examples of some of the most important "nuclei" with their properties.

E. Baiada (Modena)

Federer, Herbert 5984

Some theorems on integral currents.

Trans. Amer. Math. Soc. **117** (1965), 43–67.

As the title suggests, this paper contains several different theorems concerning integral currents. We shall use the notation of the basic reference [the author and W. H. Fleming, Ann. of Math. (2) **72** (1960), 458–520; MR **23** #A588] throughout. Perhaps the most important results, which have already had significant applications, concern the slicing of a normal current T in a differentiable manifold X by a locally Lipschitzian map $f: X \to R^n$. With almost every point $y \in R^n$ there is associated, by means of relative differentiation of measures, a normal current $\langle T, f, y \rangle$ of dimension $\dim(T) - n$, which can be thought

of as the slice of T in $f^{-1}\{y\}$. In case T is integral, $\langle T, f, y\rangle$ will also be integral. Slicing has many other natural properties, e.g.,

$$\langle T, f, u\omega_n\rangle\ (= T \wedge f^{\#}(u\omega_n) \wedge \gamma) = \int_{R^n} \langle T, f, y\rangle(\gamma)\cdot u(y)\, dL_n$$

whenever $u: R^n \to R$ is a bounded Baire function and $\gamma \in \mathbf{E}^{k-n}(X)$. The author also establishes two sets of sufficient conditions to make the slicing function $\langle T, f, \cdot\rangle: R^n \to \mathbf{I}_k(X)$ M-bounded and F-continuous.

A second result of this paper is a new measure-theoretic characterization of integral currents. Suppose $T \in \mathbf{E}_k(R^n)$ and $U, V \in \mathbf{E}_{k-1}(R^n)$ with $\partial T = U + V$. If T is rectifiable, $\mathbf{M}(U) < \infty$, and $\mathbf{H}^{k-1}(\text{spt } V) = 0$, then $\mathbf{M}(\partial T) < \infty$; hence $T \in \mathbf{I}_k(R^n)$.

The author now gives a very short proof of Wirtinger's inequality, from which it follows that on a Kähler manifold integral currents with almost everywhere complex tangent spaces are minimal currents. Using the characterization of integral currents above, one sees that each complex algebraic variety of complex dimension k on a Kähler manifold X is naturally a minimal (locally) integral current of dimension $2k$ on X. The following striking fact is a consequence. In the higher-dimensional versions of the problem of Plateau (now formulated in terms of minimal currents), not only will singularities ordinarily arise in the solutions, but, in particular, every singularity of a complex algebraic variety can occur. The author solicits other examples of minimal currents with interesting singularities.

Finally it is proved that the plane sections of a variety depend continuously, as currents, on the cutting planes and that the algebraic geometric description of tangent cones at singular points is equivalent, for complex varieties, to the purely measure-theoretic device used in the theory of minimal currents.

F. J. Almgren, Jr. (Princeton, N.J.)

Young, L. C. 5985

Some extremal questions for simplicial complexes. II. On the "radius times periphery" problem for area.

Rend. Circ. Mat. Palermo (2) **11** (1962), 271–279.

Part I appeared in same Rend. (2) 11 (1962), 178–184 [MR 27 #2609]. Let Π be a 2-dimensional polyhedron, let h be the supremum of all ρ such that any relative cycle on Π of diameter less than ρ is null-homotopic, and let L be the length of the shortest system of cuts on Π whose complement is a polyhedral disk. Then

$$A \geq \frac{\varepsilon}{14} L^{1-\varepsilon} h^{1+\varepsilon},$$

where A is the area of Π and ε is arbitrary, subject to $0 < \varepsilon < \frac{1}{2}$.

W. H. Fleming (Providence, R.I.)

Young, L. C. 5986

Some extremal questions for simplicial complexes. III. Problems of the geometry and analysis of the higher Euclidean spaces.

Rend. Circ. Mat. Palermo (2) **12** (1963), 41–58.

Part II is reviewed above [#5985]. If Π is a closed, oriented k-polyhedron in n-space R^n and j_ν is the k-vector of the νth face of Π, then it is an elementary fact that $\sum_\nu j_\nu = 0$. The converse is shown to be nearly correct in the following sense. Let $j_1, \cdots, j_N$ be decomposable k-vectors such that $j_1 + \cdots + j_N = 0$. Then given $\varepsilon > 0$ there exist k-polyhedra Π_ε^1, Π_ε^2 such that $\Pi_\varepsilon = \Pi_\varepsilon^1 + \Pi_\varepsilon^2$ is closed, Π_ε^2 has k-area less than ε, the k-vector of every face of Π_ε^1 is a positive scalar multiple $a_{\mu\nu}$ of some j_ν, and $1 = \sum_\mu a_{\mu\nu}$. When $k = n-1$, one can arrange that Π_ε is the image of a k-sphere (for $n = 3$ see the author's paper [Bull. Soc. Math. France **79** (1951), 59–84, p. 71; MR **13**, 731]). It is an open question (when $2 \leq k \leq n-2$) under what general conditions one can take Π_ε to be the image of a k-sphere. However, this is shown to be possible in the special case $j_\nu = J_\nu - J_{\nu-1}$, where $J_0, J_1, \cdots, J_N$ are decomposable k-vectors and $J_0 = J_N$.

In a similar way, let $j_1, \cdots, j_N$ be decomposable k-vectors such that $j = j_1 + \cdots + j_N$ is decomposable. Let Δ be a k-simplex with k-vector j. Then given $\varepsilon > 0$ there exist Π_ε^1, Π_ε^2 as above and a positive integer m_ε such that $m_\varepsilon \partial\Delta = \partial\Pi_\varepsilon$, $m_\varepsilon = \sum_\mu a_{\mu\nu}$. When $2 \leq k \leq n-2$ it is an open question under what general conditions one may take $m_\varepsilon = 1$ and Π_ε an image of a k-disc. This problem is closely related to that of the appropriate Weierstrass condition for multiple integral variational problems in parametric form.

W. H. Fleming (Providence, R.I.)

Fleming, W. H.; Young, L. C. 5987

Some extremal questions for simplicial complexes. IV. The algebraic and the geometric resultant, an application of variational methods.

Rend. Circ. Mat. Palermo (2) **12** (1963), 200–210.

A k-integrand is a real-valued continuous function $f(x, \alpha)$ subject to the homogeneity condition $f(x, t\alpha) = tf(x, \alpha)$ for $t \geq 0$, where x is a point in Euclidean n-space and α is a k-vector in n-space. A generalized k-variety is a linear functional L on the space of k-integrands such that $L(f) \geq 0$ whenever $f \geq 0$. A micro-polytope concentrated at x_0 is a variety of the special form $L(f) = \int f(x_0, \alpha)\, d\mu$, where μ is a measure on the unit sphere $|\alpha| = 1$ which vanishes on the complement of a finite set of unit k-vectors.

If $F(x, \alpha) = \alpha$, write $L(F)$ for the k-vector with components $L(F_\lambda)$, where F_λ are the components of F. $L(F)$ is called the flux of L, and in the case $L(F)$ is a simple k-vector, it is termed the resultant flux.

The main result of the paper is the following theorem. Let L be a k-dimensional micro-polytope concentrated at x_0 with resultant flux α_0. For a suitable subsequence $n = n_1, n_2, \cdots$ of the positive integers, there exists a polytope Π_n, whose boundary coincides with that of n repetitions of an oriented k-simplex Δ_n with flux α_0/n, such that if $L_n(f) = \Pi_n(f_0)$, where $f_0(x, \alpha) = f(x_0, \alpha)$, then L is the strong limit of L_n in the sense that the total variation of the difference of the measures that define L and L_n tends to zero. This result then leads to the theorem announced above [#5986], namely, every system of multivectors which possesses the algebraic resultant α_0 possesses also the geometric resultant α_0, and vice versa.

W. P. Ziemer (Bloomington, Ind.)

FUNCTIONS OF A COMPLEX VARIABLE

See also 5744, 5821, 6056, 6208, 6217, 6235, 6236, 6261, 6336.

★**Studies of the modern problems of the constructive theory of functions [Исследования по современным проблемам конструктивной теории функций].** 5988

A collection of papers edited by V. I. Smirnov.

Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1961. 368 *pp*. 1.28 *r*.

This volume contains the reports presented at a conference on the constructive theory of functions held in 1959 in Leningrad. Some of the papers will be reviewed individually.

Günter, N. M. [Gjunter, N. M.]; Kusmin, R. O. 5989

★**Aufgabensammlung zur höheren Mathematik. Band II.**

Zweite, berichtigte Auflage. Hochschulbücher für Mathematik, Band 33.

VEB Deutscher Verlag der Wissenschaften, Berlin, 1963. vi+289 *pp*. *DM* 19.60.

Volume I (1957) was reviewed earlier [MR **19**, 236]. The present volume contains Chapters 11–17 dealing with partial differential equations, infinite series, approximation theory, functions of a complex variable, the equations of mathematical physics, the calculus of variations, and the theory of probability. Solutions of the problems are given at the end of the book.

Tomić, Boško S. 5990

Les relations asymptotiques pour les zéros croissant indéfiniment des polynomes à coefficients positifs.

Acad. Roy. Belg. Bull. Cl. Sci. (5) **49** (1963), 799–818.

By use of certain asymptotic relations, the author proves the following theorems for polynomials $P_p(z)=\sum_{\nu=0}^{p} a_\nu z^{p-\nu}$ with positive real a. (1) If P_p has some real zeros, the smallest real zero $r_p\to-\infty$ as $a_0\to+0$. (2) If P_p has no real zeros, certain pairs of conjugate imaginary zeros will approach real values of which the smallest will tend to $-\infty$ as $a_0\to+0$.

(3) $r_3<r_5<\cdots<r_{2n-1}<r_{2n}<r_{2n-2}<\cdots<r_2<0$.

M. Marden (Milwaukee, Wis.)

Rényi, Alfréd; Rényi, Catherine 5991

On "small" coefficients of the power series of entire functions.

Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **6** (1963), 27–38.

In a previous paper [Acta Math. Acad. Sci. Hungar. **7** (1956), 145–150; MR **18**, 201] the second author proved the following theorem (Theorem A), based on a conjecture of G. Pólya. Let $f(z)=\sum a_n z^n=\sum b_n(z-1)^n$ be the power series expansions of the entire transcendental function $f(z)$ around the points $z=0$ and $z=1$. Let $Z_0(n)$ and $Z_1(n)$ denote the number of zeros in the sequences $a_0, a_1, \cdots, a_{n-1}$ and $b_0, b_1, \cdots, b_{n-1}$, respectively. Then, as $n\to\infty$, $\liminf([Z_0(n)+Z_1(n)]/n)\le 1$.

In the present paper the authors generalize Theorem A to the following theorem (Theorem 1). Let $f(z)$ be as in Theorem A and let $g(z)=\sum c_n z^n$ be an arbitrary entire function. Let $\mu_f(r)$ and $\mu_g(r)$ denote $\max_n(|a_n|r^n)$ and $\max_n(|c_n|r^n)$, respectively. Let $S_0(n)$ and $S_1(n)$ denote the number of indices $k<n$ for which $|a_k|\le|c_k|$ and $|b_k|\le|c_k|$, respectively. If for some δ, $0<\delta<1$, $\mu_f(r^{1/\delta})/\mu_g(r)$ fails to approach ∞ as $r\to\infty$, then $\liminf([S_0(n)+S_1(n)]/n)\le 1+2\delta$. By specializing to $g(z)\equiv 0$, Theorem 1 becomes Theorem A. Furthermore, by the use of the well-known relations which connect $\mu_f(r)$ and $\mu_g(r)$ with the orders ρ_f and ρ_g of $f(z)$ and $g(z)$, respectively, Theorem 1 gives $1+2\rho_f/\rho_g$ as an upper estimate for the limes inferior of Theorem 1. However, the authors succeed by a special device in improving this upper estimate by replacing it by $1+\rho_f/\rho_g$ (Theorem 4). Two more theorems of the type of Theorem 1 are proved, in one of which two "comparison functions", $g_1(z)$ and $g_2(z)$, are used. Finally, an example of a pair of functions $f(z)$ and $g(z)$ is given which shows that, in general, the estimate $1+\rho_f/\rho_g$ of Theorem 4 cannot be replaced by a smaller number.

The proofs are elementary and in part based on Johansen's general interpolation formula for polynomials [Skand. Aktuarietidskr. **14** (1931), 231–237].

F. Herzog (E. Lansing, Mich.)

Staniszewska, J. 5992

Sur l'ensemble des points de divergence des séries entières continues sur la circonférence du cercle de convergence.

Fund. Math. **54** (1964), 305–324.

It was shown by J. Śladkowska [Fund. Math. **49** (1960/61), 271–294; MR **23** #A2700; C. R. Acad. Sci. Paris **250** (1960), 258–259; MR **22** #161] that if E is a set of type $G_{\delta\sigma}$ and a subset of a set Φ of logarithmic measure zero and type F_σ, then there exists a continuous function whose Fourier series has E as its set of divergence. (A set is of logarithmic measure zero if it can be covered by denumerably many open intervals of lengths $L_i<1$ such that $\sum_i 1/\log(1/L_i)$ is arbitrarily small.)

The present paper contains a result, similar to the above, for "continuous power series", that is, for power series $H(z)=\sum_n a_n z^n$, such that the power series converges for $|z|<1$ and $H(z)$ is continuous for $|z|\le 1$. The author first proves the following lemma: Given a set E on the circumference of the unit circle, E of type G_δ, $\bar{E}$ of logarithmic measure zero, there exists a continuous power series whose partial sums are uniformly bounded on $|z|=1$, which diverges on E and converges on the complement of E. From this lemma the author derives the following theorem. Let E be a set on the circumference of the unit circle, E of type $G_{\delta\sigma}$, $E\subset\Phi$, where Φ is of logarithmic measure zero and of type F_σ. There exists a continuous power series whose partial sums are uniformly bounded on $|z|=1$, which diverges on E and converges on the complement of E. *F. Herzog* (E. Lansing, Mich.)

Azpeitia, A. G. 5993

On the Ritt order of entire Dirichlet series.

Quart. J. Math. Oxford Ser. (2) **15** (1964), 275–277.

Soit $\sum a_n e^{-\lambda_n s}$ dont l'abscisse de convergence absolue est $-\infty$. Soit ρ son ordre de Ritt, et posons

$$\rho_1=\limsup \lambda_n \log\lambda_n/\log|a_n|^{-1},$$

$$\delta=\limsup \log n/(\lambda_n\log\lambda_n).$$

On a $0 \leq \rho_1 \leq \rho \leq \infty$, et si $\delta\rho_1 < 1$ $(0 \leq \delta < \infty;\ 0 \leq \rho_1 < \infty)$, on a $(1-\delta\rho_1)^{-1}\rho_1 \geq \rho \geq 0$. *S. Mandelbrojt* (Paris)

Kamthan, Pawan Kumar 5994

On a step function.

J. Gakugei Tokushima Univ. **14** (1963), 59–63.

The function of the title is an increasing step-function of a form suggested by the theory of Dirichlet series, where it appears as the rank of the maximum term. Given $\limsup x^{-1} \log f(x)$ and $\liminf x^{-1} \log f(x)$, the author obtains results about limits of $\int_0^x f(t)\,dt/f(x)$ and of $e^{-cx}\int_0^x e^{ct}\,df(t)/f(x)$; he applies these to obtain some known theorems about Ritt orders of entire functions defined by Dirichlet series.

R. P. Boas, Jr. (Evanston, Ill.)

Ahmad, M. 5995

On analytical continuation by the partial sums of a power series.

J. London Math. Soc. **39** (1964), 577–579.

It is shown that if $f(z)$ and $\phi(z)$ are regular in a neighbourhood of $z=0$ and $f(z)\phi(z)$ is regular in $|z|<R$, then if $\phi(z)\neq 0$ and $R^{2k}+z^k\bar{q}_k\neq 0$ $(k=1,2,\cdots)$ and $|z|<r_1$, we have

$$f(z) = \lim_{s\to\infty}\left\{\frac{p_s(0)}{p_s(z)}\sum_{n=0}^{s} a_n z^n\right\}.$$

Here the a_n are the Taylor coefficients of $f(z)$ and

$$p_s(z) = \phi(z)\prod_{k=1}^{s}\frac{R^{2k}+z^k\bar{q}_k}{R^k(z^k+q_k)},$$

where the q_k depend on $\phi(z)$, and $r_1 = R^2/\limsup |q_k|^{1/k}$.

M. E. Noble (Canterbury)

Nosenko, O. S. 5996

On the range of Stieltjes functionals with restrictions of equality type. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukrain. RSR **1963**, 1563–1567.

The author announces some theorems concerning the domain of variability of the integral $\int_a^b G(t)\,d\mu(t)$, where $G(t)$ is a fixed, continuous, complex-valued function and $\mu(t)$ is a (variable) distribution of the unit mass over $\langle a,b\rangle$ subject to the condition $\int_a^b \varphi(t)\,d\mu(t)=0$, where $\varphi(t)$ is a fixed continuous, real-valued function. Some applications concerning typically-real functions are stated.

Z. Lewandowski (Lublin)

Grunsky, Helmut 5997

Über Extremaleigenschaften gewisser konformer Normalgestalten mehrfach zusammenhängender Gebiete.

J. Math. Pures Appl. (9) **43** (1964), 27–47.

Consider a bounded domain D of connectivity p in the complex z-plane. We suppose that the outer boundary curve Γ_0 of D has capacity 1. Let $P(z)=z^{p-1}+\cdots$ be a normalized polynomial of degree $p-1$ which has in each bounded complementary component D_ρ of D $(\rho=1,\cdots,p-1)$ precisely one zero b_ρ. We denote the discriminant of $P(z)$ by d and the mapping radius of D_ρ relative to b_ρ by r_ρ; form the expression $E(P)=|d|^2\prod_{\rho=1}^{p-1} r_\rho$. This expression is bounded for all conformally equivalent domains D of this type and all such polynomials $P(z)$. The maximum is achieved for a domain $\hat{D}$ and a polynomial $\hat{P}(z)$ such that the outer boundary of $\hat{D}$ satisfies $|\hat{P}(z)|=1$ and all outer boundary curves of $\hat{D}$ satisfy $|\hat{D}(z)|=C_\rho$ $(\rho=1,\cdots,p-1)$. We also have $E(\hat{P})=\prod_{\rho=1}^{p-1} C_\rho$. The theorem characterizes by an extremal property the well-known canonical map of a multiply connected domain onto a domain bounded by a set of lemniscates. Such canonical domains were introduced by de La Vallée Poussin and Julia and have been more recently studied by Walsh, Jenkins and Landau. The proof of the above theorem is based on the Dirichlet integral technique for the Green's function and the harmonic measures which lead to an inequality between the Green's functions of a domain and an included domain. This inequality is used to estimate the capacity of multicomponent sets in terms of simple geometric quantities.

M. Schiffer (Stanford, Calif.)

Kleiner, W. 5998a

Sur la condensation des masses.

Ann. Polon. Math. **15** (1964), 85–90.

Kleiner, W. 5998b

Sur la détermination numérique des points extrémaux de Fekete-Leja.

Ann. Polon. Math. **15** (1964), 91–96.

Let C be a simple closed curve (or an arc) of bounded curvature in the z-plane, and let D be the exterior domain bounded by C. Let E_n be a set of n^2 points dividing C approximately into n^2 equal arcs: $\theta^{-1}\leq n^2L_i/L\leq\theta$, where L_i is the length of the ith subarc, L is the length of C and $\theta\geq 1$ is a constant. Let $z_{n1},\cdots,z_{nn}$ be the subset of n distinct points of E_n which minimizes $\prod_{i\neq k}|z_i-z_k|$ among all such subsets, and let $h_n(z)=[\prod_{k=1}^{n}(z-z_{nk})]^{1/n}$ ($h_n(z)/z\to 1$ as $z\to\infty$). In the second paper it is shown that, as $n\to\infty$, $h_n(z)$ converges to the normalized conformal mapping $f(z)$ of D onto the exterior of a circle. The result depends on potential-theoretic lemmas established in the first paper and on another paper by the author [Ann. Polon. Math. 14 (1963/64), 131–140; MR 28 #4132]. The author stresses the value of the main result for numerical methods of conformal mapping. The papers also contain estimates of error and approximate methods for the inverse mapping.

W. Kaplan (Ann Arbor, Mich.)

Suvorov, G. D. 5999

Metric properties of planar univalent mappings of closed regions. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 802–805.

The author considers plane homeomorphisms T of class BL_k, that is, with Dirichlet norm $\leq k$. Distortion theorems for regions D are obtained in terms of the relative distance $\rho(z_1,z_2;D)$ between points $z_1,z_2\in D$. For the definition of ρ the reader is referred to the author's paper [Sibirsk. Mat. Ž. 1 (1960), 492–522; MR 28 #210]. The basic theorem stated in the paper is of the following type. Suppose D and $\Delta=T(D)$ contain disks of radii δ_0, δ_1, respectively. Then whenever $\rho(t_1,t_2;\bar{D})$ is less than a certain number depending on k, δ_0 and δ_1, we have

$$\varphi_1[\rho(t_1,t_2;\bar{D})] < \rho[T(t_1),T(t_2);\bar{\Delta}] < \varphi_2[\rho(t_1,t_2;\bar{D})].$$

φ_1 are explicitly given. The tildes denote closure with respect to the prime-end topology. The author also states a result for the distortion of distance between sets, making use of the Hausdorff definition [*Mengenlehre*, pp. 145–146, Dover, New York, 1944; MR 7, 419].

E. Reich (Minneapolis, Minn.)

Wu, Xue-mou [Wu, Hsueh-mou] 6000

On Bieberbach polynomials.

Acta Math. Sinica **13** (1963), 145–151 (*Chinese*); *translated as Chinese Math.* 4 (1963), 161–168.

The author presents proofs of several results on the approximation by polynomials of analytic functions, in particular, on the approximation by the Bieberbach polynomials $B_n(z)$ of the normalized mapping function $\phi(z)$ of a simply connected domain D on a disc. A particular result is an estimate

$$|\phi(z) - B_n(z)| = Cn^{-(m+1/2+\alpha)} \log n;$$

here the boundary Γ of D is assumed to be a smooth Jordan curve $z = z(s)$, s being arc length, for which $(d^m/ds^m) \arg z'(s)$ satisfies a Lipschitz condition of order α, $0 < \alpha < 1$, and C is a constant independent of n; for $\alpha = 1$ there is a similar conclusion with $\log n$ replaced by $(\log n)^2$. The proofs depend, in particular, on a previous paper of the author [Acta Math. Sinica 7 (1957), 271–276 (Chinese with German summary)].

W. Kaplan (Ann Arbor, Mich.)

Charzyński, Zygmunt; Śmiałkówna, Henryka 6001

The general equation of extremal functions with respect to any differentiated functional.

Bull. Soc. Sci. Lettres Łódź **12** (1961), *no.* 13, 16 *pp.*

Let F denote the family of functions $f(z) = \sum_1 a_n z^n$, $a_1 > 0$, a_n real, which are analytic and univalent in the unit disc and which satisfy $|f(z)| < 1$ there. Let F_T denote the subfamily of F such that $a_1 \geq T$, where T is fixed, $0 < T < 1$. Then the authors' result is the following one. Let $K(f)$ be a real functional defined in F_T, and with a Fréchet differential at every point of F_T, such that the differential $L(h)$ of $K(f)$ does not vanish identically at any point of F_T. Then every extremal function for the problem $\max[K(f)|f \in F_T]$ satisfies a certain differential equation. The equation and additional properties of the extremals are too complicated to reproduce in a brief review. Suffice it to note that these results are similar to those already obtained by Charzyński and Janowski [Ann. Univ. Mariae Curie-Skłodowska Sect. A 4 (1950), 41–56; MR 13, 122].

M. Reade (Ann Arbor, Mich.)

Kir'jackiĭ, È. G. [Kirjackis, E.] 6002

Some extremal problems in the classes $K_n(E)$ and $P(E)$. (Russian. Lithuanian and German summaries)

Litovsk. Mat. Sb. **3** (1963), *no.* 2, 83–90.

Let $K_n(E)$ be the class of functions $F(z)$ that are regular and single-valued in the unit disk E and satisfy the condition that the nth divided difference of $F(z)$ is never zero for arbitrary pairs of distinct points $z_0, z_1, \cdots, z_n$ in E. The class $P(E)$ is the set of all functions $f(z)$, $f(0) = 0$, $f'(0) = 1$, that are regular in E and satisfy the condition that $z^{n-1}f(z)$ belongs to $K_n(E)$. The author has shown previously [same Sb. **3** (1963), no. 1, 157–168; MR **29** #1328] that $f(z)$ has many extremal properties possessed by convex normalized univalent functions. For example, $|z|(1+|z|)^{-1} \leq |f(z)| \leq |z|(1-|z|)^{-1}$ and the Maclaurin coefficients a_k satisfy $|a_k| \leq 1$. In this paper he shows, among other things, that these bounds are not only sharp, but that equality is attained only for $f(z) = z(1-e^{i\alpha}z)^{-1}$, α real.

W. C. Royster (Lexington, Ky.)

Kir'jackiĭ, È. G. [Kirjackis, E.] 6003

An extension of some theorems of Aksent'ev and Čakalov to the class $K_n(D)$. (Russian. Lithuanian and German summaries)

Litovsk. Mat. Sb. **3** (1963), *no.* 2, 91–96.

The main result of this paper concerns the maximal K_n-domain of a class of functions. D is a maximal K_n-domain for a class Q of functions if each member of Q is regular in D and the nth divided difference of this member is never zero for arbitrary pairs of distinct points $z_0, \cdots, z_n$ in D, and in no domain containing D do the members of Q possess this property.

Let $C(\zeta, R)$ be the class of rational functions $f(z) = \sum_{k=1}^m A_k(z-a_k)^{-1}$, where $A_k > 0$, $\zeta, a_1, \cdots, a_m$ are complex constants with $|a_k - \zeta| \leq R$, $k = 1, 2, \cdots, m$. The author proves that the maximal K_n-domain for $C(\zeta, R)$ is the domain $G: |z-\zeta| > \mathrm{Re}[\sin(\pi/2(n+1))]^{-1}$. This result is generalized to the case where $f(z)$ is given by a Stieltjes integral, $f(z) = \int_{-\pi}^{\pi} (z-e^{it})\, d\alpha(t)$, $t \in [-\pi, \pi]$, $\alpha(t)$ nondecreasing and increasing at least at one point. The maximal K_n-domain is $|z| > [\sin(\pi/2(n+1))]^{-1}$. For $n = 1$ we get the maximal domain of univalence, a result obtained by Čakalov [Bŭlgar. Akad. Nauk. Izv. Mat. Inst. 4 (1960), no. 2. 43–55; MR **23** #A1022].

W. C. Royster (Lexington, Ky.)

Kocur, M. F. 6004

Some special classes of analytic functions in a circular annulus. I. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 4 (41), 79–85.

Let $P(z)$ belong to the class $C_q(\alpha, \beta, \gamma, \delta)$ of functions which are regular in the circular ring $K_z(q^2, 1)$, $q^2 < |z| < 1$, and which are representable by the structural formula

$$(*) \quad P(z) = 1 - \alpha - \beta + \frac{\alpha}{2\pi} \int_{-\pi}^{\pi} \frac{1 + ze^{-i(t-\gamma)}}{1 - ze^{-it}}\, d\mu_1(t) + \frac{\beta}{2\pi} \int_{-\pi}^{\pi} \frac{1 + (q^2/z)e^{i(t+\delta)}}{1 - (q^2/z)e^{it}}\, d\mu_2(t)$$

where $\alpha \geq 0$, $\beta \geq 0$, $0 < \alpha + \beta \leq 1$, $-\pi < \gamma, \delta < \pi$, $0 < q < |z| < 1$. The integral is a Stieltjes integral, and $\mu_1(t)$ and $\mu_2(t)$ are real nondecreasing functions of t in $[-\pi, \pi]$ with $\mu_j(-\pi) = \mu_j(-\pi + 0)$, $\mu_j(\pi) = 2\pi$, $j = 1, 2$, and $P(z)$ is normalized so that $(1/2\pi i) \int_{|z|=\rho} (P(z)/z)\, dz = 1$, $q^2 < \rho < 1$. By letting $q \to 0$, $\gamma = 0$, $\alpha = 1$, the class $C_q(\alpha, \beta, \gamma, \delta)$ is the well-known class P of functions with positive real part in the unit disk.

The author obtains sharp bounds for $|P(z)|$ and $\arg P(z)$ for functions belonging to $C_q(\alpha, \beta, \gamma, \delta)$. He shows that the Laurent coefficients satisfy $|a_n| \leq 2\alpha \cos(\gamma/2)$ and $|a_{-n}| \leq 2q^{-n}\beta \cos(\delta/2)$.

W. C. Royster (Lexington, Ky.)

Kocur, M. F. 6005
Some special classes of analytic functions in a circular annulus. II. (Russian)
Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 5 (42), 30–40.

Let $S_q(\alpha, \beta, \gamma, \delta)$ denote the class of functions $f(z)$ that are regular in the circular ring $K(q^2, 1)$ and satisfy the condition $zf'(z)/f(z) = P(z)$, where $P(z)$ belongs to $C_q(\alpha, \beta, \gamma, \delta)$ and is given by (*) in the preceding review [#6004]. We see by taking $\alpha = \beta$, $0 \leq \alpha \leq \frac{1}{2}$, $\gamma = \delta$, and letting $q \to 0$ that we get the class of spiral-like functions of order α in $|z| < 1$, and if $\gamma = 0$, then we have the starlike functions of order α in $|z| < 1$. For the class $S_q(\alpha, \beta, \gamma, \delta)$ the author obtains sharp bounds for $|f(z)|$ and $\arg f(z)$.

Let $g(z)$ be regular in $K_z(q^2, 1)$ and satisfy $P(z) = 1 + zg''(z)/g'(z)$. Bounds for $|g'(z)|$ and $\arg g'(z)$ are obtained. All of the expressions for these bounds are too long and cumbersome to be displayed.

W. C. Royster (Lexington, Ky.)

Ozawa, Mitsuru 6006
On certain coefficient inequalities of univalent functions.
Kōdai Math. Sem. Rep. **16** (1964), 183–188.

In this note the author solves two extremal problems by means of Schiffer's variational method [Proc. London Math. Soc. (2) **44** (1938), 432–449]. The author's purpose in introducing these problems is to show that the "perfect square" technique used by Charzyński and Schiffer in a consideration of the fourth coefficient in the Bieberbach conjecture [Arch. Rational Mech. Anal. **5** (1960), 187–193; MR **22** #5746] is not always applicable.

Let S denote the family of all functions $f(z) = z + \sum_2 A_n z^n$, which are analytic and univalent in the unit disc $D: [z| \; |z| < 1]$. Then the author obtains the following results. (I) In S the inequality

$$|A_4 - 3A_2A_3 + 2A_2^3| \leq 2$$

holds, with equality only for the Koebe function. (II) In S the inequality

$$\operatorname{Re}\left\{A_5 - 2A_2A_4 - \frac{3}{2}A_3^2 + 4A_2^2A_3 - \frac{79}{54}A_2^4\right\} \leq \frac{1}{2}$$

holds, with equality only for functions in S that satisfy the equation

$$\frac{1}{w^2}\sqrt{\left(1 + \frac{4}{3}A_2 w\right)^3} = \frac{1}{z^2} - \left(2A_3 - \frac{4}{3}A_2^2\right) - z^2.$$

M. Reade (Ann Arbor, Mich.)

Pommerenke, Christian 6007
Linear-invariante Familien analytischer Funktionen. I, II.
Math. Ann. **155** (1964), 108–154; *ibid.* **156** (1964), 226–262.

In this most interesting work, the author generalizes certain results from the theory of univalent functions to a wider class of analytic functions. He not only obtains generalizations of important old results, but also obtains certain new ones which should lead to further work in the area he has opened for investigation. In this brief review we can only give some of the author's results; the papers must be read to be fully appreciated.

Let D denote the unit disc $[z| \, |z| < 1]$, and let L denote the collection of linear fractional transformations that map D onto itself. Then a non-empty collection M of functions $f(z)$, analytic in D, is said to be a linear-invariant family if and only if (i) each $f(z)$ in M satisfies $f'(z) \neq 0$ in D, (ii) each $f(z)$ in M has the form $z + a_2 z^2 + \cdots$ in D, and (iii) for each $f(z)$ in M and for each ϕ in L, the function

$$\Lambda_\phi[f(z)] = \frac{f(\phi(z)) - f(\phi(0))}{\phi'(0) f'(\phi'(0))}$$

is again in M. A family M of functions $f(z)$, analytic in D, is said to be "beschränktartig" if and only if there exists a constant $K = K(M)$ such that

$$\int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta \leq K, \qquad z = re^{i\theta},$$

holds for $0 \leq r < 1$, for each $f(z)$ in M. The order of a linear invariant family M is defined to be $\alpha = \sup[\frac{1}{2}|f''(0)| \, |f(z)$ in $M]$. In Chapter 1, the author obtains results reminiscent of results on normal families and distortion theorems. For example, he proves that the linear-invariant family M is normal if and only if the order of M is finite. He also proves that the family of locally univalent, analytic functions $f(z) = z + a_2 z^2 + \cdots$, each of which is areally mean p-valent, is linear-invariant, beschränktartig, and hence normal. The author shows that if the order of a linear-invariant family M is α, then

$$|\log[(1 - |z|^2) f'(z)]| \leq \alpha \log \frac{1 + |z|}{1 - |z|},$$

from which it follows that the image of the closed disc $[z| \, |z| \leq r < 1]$ under $w = f(z)$ contains a disc of radius

$$\frac{1}{2\alpha}\left[1 - \left(\frac{1 - r}{1 + r}\right)^\alpha\right].$$

In Chapter 2, the author obtains other distortion theorems, which lead to the following results (among many others). (i) If M is a compact linear-invariant family of order α, and if $f(z)$ in M has the form $z + \alpha z^2 + a_3 z^3 + \cdots$, then $a_3 = \frac{2}{3}\alpha^2 + 1$. (ii) If M is a compact linear-invariant family of order α, then each $f(z)$ in M is convex and univalent in the disc $[z| \, |z| < \alpha - \sqrt{(\alpha^2 - 1)}]$; this result is sharp. (iii) Let M be a compact linear-invariant family of order α, let r_0 denote the largest real number for which $f(z)/z$ does not vanish for $[z| \, |z| < r_0]$, for any member of M, and let r_1 denote the "radius of univalence" of M. Then $r_1 = r_0/(1 + \sqrt{(1 - r_0^2)})$, and this result is sharp. In this same chapter, the author considers images of non-euclidean lines, the distance from a point to the boundary in the image domain, growth properties of the maximum modulus, and the notions of accessible points and prime ends. In Chapter 3, the author introduces the notion of "boundary behavior" for a linear-invariant family. Indeed, let M be a linear-invariant family that is also a normal family, and let $f(D)$ denote the image of the disc D under $w = f(z)$. Now set

$$f(ze^{i\theta}, \rho e^{i\theta}) = \frac{f\left(\dfrac{e^{i\theta}z + \rho e^{i\theta}}{1 + \rho z}\right) - f(\rho e^{i\theta})}{(1 - \rho^2) f'(\rho e^{i\theta})},$$

where z and $\rho e^{i\theta}$ are in D. Then $C(e^{i\theta}, f)$ is the set of all functions $g(z)$ for which there exists a sequence of real numbers $\rho_n \to (1 - 0)$ such that $e^{-i\theta} f(ze^{i\theta}, \rho_n e^{i\theta}) \to g(z)$ locally

uniformly in D, and $C(f)$ is the set of all functions for which there exists a sequence $\rho_n e^{i\varphi_n}$ with $\rho_n \to (1-0)$ such that $e^{-i\varphi_n} f(ze^{i\varphi_n}, \rho_n e^{i\varphi_n}) \to g(z)$. The sets $C(e^{i\varphi}, f)$ and $C(f)$ are those that indicate the "boundary behavior" of the family M; for the case that M is the set of all normalized univalent functions in D, $C(e^{i\varphi}, f)$ and $C(f)$ reduce to known items in the theory of univalent functions. For example, if $C(1, f)$ reduces to just one function $g(z)$, then the author shows that $g(z) \equiv (1/2c)[((1+z)/(1-z))^c - 1]$, where c is a complex constant. If M is a normal and linear-invariant family in D, and if $f(z) \in M$, then the author says $f(z)$ has "gut-erreichbares Verhalten" at $e^{i\varphi}$ if and only if there exists x_1, $0 < x_1 < 1$, so that

$$(1 - x_1^2)|g'(x_1)| < 1$$

holds for all $g(z) \in C(e^{i\varphi}, f)$. If this last inequality holds for all $g(z)$ in $C(f)$, then $f(z)$ has uniformly "gut-erreichbares Randverhalten". These last two notions lead to "angle limits" and the classical idea of accessible boundary points. A further result is a criterion as to when a univalent function (in D) can be extended to a quasiconformal map of the plane onto itself; this is connected with a recent result due to Ahlfors [Acta Math. **109** (1963), 291–301; MR **27** #4921]. After many more results, the author concludes with a discussion of the boundary behavior of M, yielding results that generalize a "Faltensatz" due to Ostrowski [Prace Mat.-Fiz. **44** (1937), 371–471].

Again, the present author's results may be looked on as generalizations of well-known results on univalent functions associated with the names of Bieberbach, Gronwall, Pick, Koebe, Visser, Lavrentieff, Ferrand, Walsh, Gehring, Hayman, Gaier, et al.

M. Reade (Ann Arbor, Mich.)

Tamrazov, P. M. 6008

Some estimates in the theory of univalent conformal mappings of doubly connected domains. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 1160–1163.

Let $H(R)$ be the class of functions regular and univalent in the annulus $R < |z| < 1$ which map it onto a domain situated inside the unit disc, with the outer boundary corresponding to $|z| = 1$. Let $H^*(R) \subset H(R)$ be the subclass of functions leaving $|z| = 1$ invariant. Some distortion theorems for both classes are proved.

Z. Lewandowski (Lublin)

Fuller, D. J. H. 6009

Mappings of bounded characteristic into arbitrary Riemann surfaces.

Pacific J. Math. **14** (1964), 895–915.

This paper treats value distribution theory for analytic mappings of an arbitrary open Riemann surface R into an arbitrary Riemann surface S. The proximity function is defined in terms of a principal function which has two logarithmic poles with opposite coefficients ± 1 and is normalized there by additive constants. The author obtains a generalization of Nevanlinna's first main theorem. The generalization of H. Cartan's theorem is given for the characteristic function whose proximity function is defined in terms of the principal function which satisfies the above condition and has constant value on ∂S. Let MB be the class of analytic mappings of R into S whose characteristic functions are bounded. Some characterizations of MB functions are given. The class MB is identical with the class of Lindelöfian maps.

Y. Kusunoki (Kyoto)

Sario, Leo 6010

General value distribution theory.

Nagoya Math. J. **23** (1963), 213–229.

The author considers analytic mappings from arbitrary Riemann surfaces into arbitrary Riemann surfaces. He generalizes the first and second fundamental theorems of Nevanlinna in a manner that is of interest even in the classical setting. He then derives an analogue of the classical relation concerning defective and ramified values. The author's "principal functions" are an important tool in constructing the necessary proximity functions.

The paper is the text of two lectures delivered by the author in May 1963 at Stanford. The style is somewhat more discursive than is usual in mathematical articles, and some details are omitted in order to emphasize the main aspects of the theory. The author has happily succeeded in his exposition, and the paper is recommended to any one who wishes to understand the author's recent contributions to general value distribution theory.

For complete details one is referred to the author's articles noted in the bibliography.

R. Accola (Providence, R.I.)

Maeda, Fumi-Yuki [Maeda, Fumiyuki] 6011

Notes on Green lines and Kuramochi boundary of a Green space.

J. Sci. Hiroshima Univ. Ser. A-I Math. **28** (1964), 59–66.

Let Ω be a Green space in the sense of Brelot and Choquet [Ann. Inst. Fourier (Grenoble) **3** (1951), 199–263; MR **16**, 34], and let Ω^* be a compactification of Ω. A statement is said to hold for almost all curves if the family of exceptional curves has infinite extremal length. Theorem: Almost every Green line converges to one point of $\Omega^* - \Omega$ provided either (i) there exists a countable family of BLD functions which separates the points of $\Omega^* - \Omega$, or (ii) Ω^* is metrizable and the set of BLD functions which extend continuously to Ω^* separates the points of $\Omega^* - \Omega$ (f separates $a, b \in \Omega$ if $\liminf_{\Omega \ni x \to a} f(x) > \limsup_{\Omega \ni x \to b} f(x)$). Corollary: On a hyperbolic Riemann surface almost every Green line tends to one point of the Kuramochi boundary.

An analogue of the Kuramochi boundary of a Riemann surface is defined for a general Green space. It is shown that this boundary shares with Kuramochi's boundary the above property of Green lines as well as several other properties.

B. Rodin (La Jolla, Calif.)

Ohtsuka, Makoto 6012

On limits of BLD functions along curves.

J. Sci. Hiroshima Univ. Ser. A-I Math. **28** (1964), 67–70.

The main results of the preceding paper [#6011] are generalized to spaces $\mathscr{E}$ in the sense of Brelot and Choquet [Ann. Inst. Fourier (Grenoble) **3** (1951), 199–263; MR **16**, 34]. The author shows how one of his theorems—Every BLD function on $\mathscr{E}$ has a finite limit along almost every

open curve—can be used to derive a result of M. Godefroid [ibid. 9 (1959), 301–304; MR 22 #2806] on the limits of BLD functions along regular Green lines.

B. Rodin (La Jolla, Calif.)

Künzi, Hans P. 6013

★Quasikonforme Abbildungen.

Ergebnisse der Mathematik und ihrer Grenzgebiete. Neue Folge, Heft 26.

Springer-Verlag, Berlin-Göttingen-Heidelberg, 1960. viii+182*pp*. *DM* 39.00.

The author has assembled a remarkable amount of the research on quasiconformal mappings in a relatively short monograph; indeed, every major result up to 1960 is either described or referred to. It is natural that, in order to bring together so much material within a limited space, it was necessary for the author to sacrifice any hope of writing a self-contained monograph which might introduce the novice to the field: instead, the monograph, with its rich bibliography, is invaluable to the specialist in the field.

After a concise introductory chapter on conformal mapping, there follows a chapter on quasiconformal homeomorphisms à la Grötzsch, ending with the Teichmüller and Grötzsch extremal problems. Chapter 3 is brief, and is devoted to the classical applications of quasiconformal mappings in the theory of functions. Chapter 4 is concerned with general K-quasiconformal homeomorphisms, and Chapter 5 with K-quasiconformal mappings. Chapter 6 is devoted to quadratic differentials and extremal quasiconformal mappings, while Chapter 7 contains applications to differential equations and pseudo-analytic functions. *A. J. Lohwater* (Providence, R.I.)

Mitjuk, I. P. 6014

The reduced modulus in the spatial case. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 563–566.

The author uses the concept of the modulus of a family of surfaces, as defined by B. V. Šabat [Dokl. Akad. Nauk SSSR **130** (1960), 1210–1213; MR **22** #11101], to define a reduced modulus for a 3-dimensional domain G at a point $a \in G$. He obtains some properties of this modulus, and he compares the moduli of the domains G and G' at a point a, where G' is the image of G under a Q-quasiconformal mapping $y=y(x)$, normalized so that $y(a)=a$ and $|y(x)-a| \sim K|x-a|^{1/Q}$ in a neighborhood of a.

F. W. Gehring (Cambridge, Mass.)

Akaza, Tohru 6015

Poincaré theta series and singular sets of Schottky groups.

Nagoya Math. J. **24** (1964), 43–65.

This is a study of the convergence of the Poincaré theta series and the Hausdorff dimension of the singular set of a Schottky group. The $(-\nu)$-dimensional theta series converges together with $\sum |c|^{-\nu}$ (the transformations of the group are $(az+b)(cz+d)^{-1}$). Less trivially, in case of convergence the singular set has $(\nu/2)$-dimensional measure zero. There are many other results that are consequences of geometric conditions relative to the circles that define the Schottky group. The author proves the existence of a singular set of Hausdorff dimension >1. Such a set is not a Painlevé null set. *L. V. Ahlfors* (Cambridge, Mass.)

Gray, Alfred; Shah, S. M. 6016

A note on entire functions and a conjecture of Erdős. II.

J. Analyse Math. **12** (1964), 83–104.

As announced in their previous paper [Bull. Amer. Math. Soc. **69** (1963), 573–577; MR **26** #6409], the authors prove several inequalities relating U, u,

$$U^* = \limsup_{n\to\infty} (\mu(\rho_n)/M(\rho_n))$$

and

$$u^* = \liminf_{n\to\infty} (\mu(\rho_n)/M(\rho_n))$$

(for notations and terminology see their earlier paper). For example, for any entire function $U^* \leq 2/\pi$, $u^*=u$, and if $u<U$ every point of $[u, U]$ is a limit point of $\{\mu(r)/M(r)\}$. Several results on entire functions defined by gap power series are also established and finally, the conjecture of Erdős, which is proved not to hold for functions holomorphic in $|z|<R<\infty$, is modified for that type of function by the additional assumption that $\{\rho_n\}$ is an infinite sequence. If so, it is proved that if either

$$\limsup_{k\to\infty} (n_{k+1}-n_k) < \infty$$

or

$$\lim_{k\to\infty} [\rho(n_k)/\rho(n_{k+p})]^{n_{k+p}-n_{k+p-1}} = 1$$

for $p=1, 2, \cdots$, then $U=0$.

A. G. Azpeitia (Amherst, Mass.)

Hellerstein, S.; Korevaar, J. 6017

The real values of an entire function.

Bull. Amer. Math. Soc. **70** (1964), 608–610.

If $f(z)$ is a nonconstant entire function of order ρ and $\Phi(r)$ is the number of points of $|z|=r$ at which $f(z)$ is real, the authors show that

$$\kappa = \limsup \frac{\log \Phi(r)}{\log r} = \rho.$$

That $\kappa \geq \rho$ was shown by the reviewer [same Bull. **67** (1961), 488–489; MR **24** #A1398] by a more complicated method than that used here. In this paper, $\kappa \geq \rho$ depends on the elementary theory of Borel exceptional values, while $\kappa \leq \rho$ depends on Jensen's theorem applied to $\mathrm{Im}(f(re^{i\theta}))$ regarded as an entire function of θ.

[Reviewer's remark. The authors have pointed out that, before equation (10), it is not enough for $g_r(0) \neq 0$, but note that $|g_r(0)| \geq 1$ can be accomplished by a rotation, which is sufficient.] *H. S. Wilf* (Philadelphia, Pa.)

Rauzy, Gérard 6018

Fonctions entières prenant des valeurs entières sur des ensembles partiels d'entiers.

C. R. Acad. Sci. Paris **259** (1964), 19–21.

A sequence of natural numbers is said to have infinite frequency if it contains intervals (n, A^n) with arbitrarily large A. Pólya's theorem on integer-valued functions is generalized to the theorem that an entire function of exponential type $\sigma < \log 2$ which is integer-valued on a sequence of infinite frequency is a polynomial. The proof is reduced via the Borel transform to the following generalization of the Pólya-Carlson theorem: A function $f(z)=\sum a_n z^{-n}$, where the a_n are integers for a sequence of

indices of infinite frequency, is analytic for $|z-1| > \rho$ with $\rho < 1$ if and only if it is rational with no poles except at $z=1$.

E. G. Straus (Los Angeles, Calif.)

Hayman, W. K. 6019

On the characteristic of functions meromorphic in the unit disk and of their integrals.

Acta Math. **112** (1964), 181–214.

It is known that the derivative of a bounded analytic function in a disk need not be of bounded Nevanlinna characteristic [Frostman, Kungl. Fysiogr. Sällsk. i Lund Förh. **12** (1942), 169–182; MR **6**, 262]. Precise information concerning the growth of the Nevanlinna characteristic function of the derivative of a function of bounded type has been obtained by P. B. Kennedy [Quart. J. Math. Oxford Ser. (2) **15** (1964), 337–341].

The converse question whether a function meromorphic in a disk can have an unbounded Nevanlinna characteristic function while its derivative is of bounded type is settled by the present paper. The following theorems are established. (I) Let f be meromorphic and of bounded type in $|z| < R$, $0 < R < +\infty$, $f(0) \neq \infty$. Then

$$(2\pi)^{-1}\int_0^{2\pi} \log^+ f_1(re^{i\theta})\,d\theta \le T(R,f) + \pi^{-1}\log\frac{R+r}{R-r}\,m(R,f) + \psi\left(\frac{r}{R}\right)N(R,f)$$

$$\le \left[1+\psi\left(\frac{r}{R}\right)\right]T(R,f)$$

for $0<r<R$, where

$$\psi(t) = \left[(1-t)\log\left(1+\frac{2\pi\sqrt{t}}{1-t}\right)\right](\pi\sqrt{t}\log t^{-1})^{-1},$$

$f_1(re^{i\theta}) = \sup_{0\le t\le r}|f(te^{i\theta})|$ and T, m, and N have the standard meanings in the Nevanlinna theory. The first part of the inequality is sharp when f is analytic. This theorem has a subharmonic counterpart. (II) Given F meromorphic in $|z| < R$, $F(0)=0$, $f=F'$ of bounded type in $|z|<R$, then

$$m(r,F) \le T(R,f) + \pi^{-1}\log\frac{R+r}{R-r}\,m(R,f) + \psi\left(\frac{r}{R}\right)N(R,f) + \log^+ r$$

$$\le \left[1+\psi\left(\frac{r}{R}\right)\right]T(R,f) + \log^+ r.$$

One has $T(r,F) \le [2+\psi(r/R)]T(R,f) + \log^+ r$, $0<r<R$, and if F is analytic,

$$T(r,F) \le \left(1+\frac{1}{\pi}\log\frac{R+r}{R-r}\right)T(R,f).$$

(III) Given $C>0$, there exists f analytic, $|f|>1$, in $|z|<1$, and satisfying $T(1,f)-\log|f(0)|=C$, while $T(r,F) > .12C\log[1/(1-r)]$, $r_0<r<1$, where $F(z)=\int_0^z f(\zeta)\,d\zeta$, $r_0 = 1-[\min(\frac{1}{2},C)]^4$, A being an absolute positive constant. Applications of these theorems to entire and meromorphic functions will be given in a later paper.

M. H. Heins (Urbana, Ill.)

Džrbašjan, M. M. 6020

The parametric representation of some general classes of meromorphic functions in the unit circle. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 1024–1027.

For $\varphi \in L(0,1)$, $\alpha > 0$, define

$$D^{-\alpha}\varphi(r) = \frac{1}{\Gamma(\alpha)}\int_0^r (r-t)^{\alpha-1}\varphi(t)\,dt,$$

and extend the definition to α in the interval $(-1,0)$ by $dD^{-(1+\alpha)}\varphi(r)/dr = D^{-\alpha}\varphi(r)$. Write $g_+(r)$ for $\max(g(r),0)$. Let $F(z)$ be meromorphic in $|z|<1$, and write $n(t)$ for the number of its poles in $|z|\le t$. Put

$$N_\alpha(r,F) = r^{-\alpha}D^{-(1+\alpha)}\{[n(r)-n(0)]/r\} + n(0)\log r/\Gamma(1+\alpha).$$

Further, let

$$m_\alpha(r,F) = (r^{-\alpha}/2\pi)\int_0^{2\pi} D_+^{-\alpha}\log|F(re^{i\theta})|\,d\theta.$$

The function $T_\alpha(r,F) = m_\alpha(r,F) + N_\alpha(r,F)$ is called the α-characteristic of F. For $\alpha=0$ it is identical with the Nevanlinna characteristic. T_α is an increasing function of r. The class N_α of meromorphic functions contains those functions for which $T_\alpha(r,F)$ is $O(1)$ in $r<1$. The author has shown that $N_\alpha < N_\beta$ $(-1<\alpha<\beta)$. He proves $F\in N_\alpha$ if and only if

$$F(z) = K\frac{\pi_\alpha(z,a)}{\pi_\alpha(z,b)}\exp\left\{\int_0^{2\pi}(1-e^{-i\theta}z)^{-1-\alpha}\,d\psi(\theta)\right\},$$

where K is a constant, $\pi_\alpha(z,a)$ is an (explicitly defined) product analogous to a Blaschke product extended over the zeros of F, $\pi_\alpha(z,b)$ is the same type of product extended over the poles, and ψ is a function of bounded variation. For $\alpha=0$ this gives the well-known characterization of functions of bounded Nevanlinna characteristic.

W. H. J. Fuchs (Ithaca, N.Y.)

Džrbašjan, M. M. 6021

On the representation of certain classes of entire and quasi-entire functions. (Russian)

Dokl. Akad. Nauk SSSR **159** (1964), 9–12.

Let the z-plane be divided into sectors by a finite number of rays from the origin. Let $\arg z = \theta_1, \cdots, \theta_p$ be the middle lines of the largest sectors, each one of which has an angle π/ρ $(\rho \ge \frac{1}{2})$ at the origin. Let $W(\rho,\sigma)$ be the class of entire functions $f(z)$ of order ρ and finite type satisfying

$$(1) \qquad \int_0^\infty |f(z)|\,|z|^\omega|dz| < \infty$$

$(-1<\omega<1$, integration along any boundary of a sector) and

$$(2) \qquad \limsup_{r\to\infty}\log|f(re^{i\theta_k})|r^{-\rho} \le \sigma_k,$$

$$\sigma_k \ge 0 \quad (k=1,2,\cdots,p).$$

If $f(z)\in W(\rho,\sigma)$, then

$$f(z) = \sum_{k=1}^{p}\int_0^{\sigma_k} E_\rho(e^{-i\theta_k}zt^{1/\rho},\mu)\varphi_k(t)t^{\mu-1}\,dt$$

when $E_\rho(z,\mu) = \sum_{k=0}^\infty z^k/\Gamma(\mu+k\rho^{-1})$ and $\mu = (1+\omega+\rho)/2\rho$, $\varphi_k(t)\in L^2(0,\sigma_k)$. Vice versa, any such integral represents an entire function of class $W(\rho,\sigma)$. The functions φ_k can be written as integral transforms of $f(z)$, so that this

representation is a complete analogue of the Paley-Wiener theorem.

The paper also contains integral representations for "quasi-entire" functions, i.e., functions $f(z)$ analytic on the Riemann surface of $\log z$ and such that $f(z)$ tends to a limit as $z \to 0$ in any manner on this Riemann surface. The role of the function $E_\rho(z, \mu)$ is now taken over by $\int_0^\infty z^t/\Gamma(\mu + t\rho^{-1})\,dt$. *W. H. J. Fuchs* (Ithaca, N.Y.)

Collingwood, E. F.; Piranian, G. 6022

The mapping theorems of Carathéodory and Lindelöf.

J. Math. Pures Appl. (9) **43** (1964), 187–199.

Let B be a bounded simply connected plane region, and let D be the open unit disk. A sequence $\{c_n\}$ of cross-cuts of B is called a "weak" chain, provided that, for all positive integers n and k, (i) c_n separates c_{n+k} with respect to B from some fixed point of B; (ii′) rel dist $(c_n, c_{n+k}) > 0$ (rel dist is the infimum of diameters of arcs in B connecting c_n to c_{n+k}); and (iii) no closed subset of B meets every c_n. This modifies Carathéodory's notion of a chain of cross-cuts (here called a "C-chain") which required, in place of (ii′), the condition (ii) $\bar{c}_n \cap \bar{c}_{n+k} = \varnothing$ [Math. Ann. **73** (1913), 323–370]. Carathéodory's theorem is then proved by the authors in the form: Any conformal homeomorphism $B^* \to D$ can be extended to a homeomorphism $B^* \to \bar{D}$, where B^* is the compactification of B by means of "weak" prime ends. The latter are defined in terms of "weak" chains of cross-cuts, in precisely the same way Carathéodory used to obtain prime ends from his C-chains of cross-cuts. It is shown that every Carathéodory prime end is a "weak" prime end, and conversely. However, the proof of the mapping theorem is now more direct and perspicuous than in Carathéodory's version. The authors' argument rests on a length-area comparison by means of the Schwarz inequality.

In the last two sections, the theorem of Lindelöf [Acta Soc. Sci. Fenn. **46** (1915), no. 4] that the principal cluster set coincides with the cluster set along a Stolz path is proved by a similar method. It is here given in a setting due to Gross, under the hypothesis that the meromorphic function f with domain D maps some neighborhood (relative to D) of a boundary point onto a Riemann surface of finite spherical area.

E. C. Schlesinger (New London, Conn.)

Matsumoto, Kikuji 6023

Some notes on exceptional values of meromorphic functions.

Nagoya Math. J. **22** (1963), 189–201.

It was shown by the author [same J. **18** (1961), 171–191; MR **24** #A237] that, given any closed set K of capacity zero, there exists a closed set E of capacity zero and a function $f(z)$ meromorphic in the complement of E which possesses an essential singularity at each point of E and omits the set K. On the other hand, L. Carleson [Bull. Amer. Math. Soc. **67** (1961), 142–144; MR **24** #A238] established the existence of a linear closed set F of positive capacity such that every function $f(z)$ meromorphic in the complement of F and omitting four values there is rational. Carleson's set F is a Cantor set on $[0, 1]$, where the successive ratios ξ_ν decrease and satisfy the relation $\lim \log \xi_\nu/\log \nu = -\infty$. As a consequence of a more general theorem, too complicated to be stated here, the author shows that if F is a Cantor set on $[0, 1]$, where the successive ratios ξ_ν merely satisfy the condition $\lim \xi_\nu = 0$, then every function, meromorphic in the complement Ω of F, which has at least one essential singularity in F, omits at most three values in that subset of Ω which lies in some neighborhood of each essential singularity. Other interesting generalizations of Picard's theorem of this general character are obtained.

W. Seidel (Detroit, Mich.)

Vituškin, A. G. 6024

On a problem of Denjoy. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 745–756.

The following problem is considered. Determine a set e in the finite complex plane and a function $f(z)$ with the properties: (i) e is compact and totally disconnected, (ii) e has vanishing φ-Hausdorff measure, where $\varphi(\delta)$ is a given monotone function with $\varphi(\delta)/\delta \to 0$ for $\delta \to 0$, (iii) $f(z)$ is continuous in the finite plane, non-constant, and analytic outside e, (iv) $\int_\gamma f\,dz = 0$ for any contour γ disjoint to e.

The author gives an example for the solution (the reviewer was unable to follow the actual construction), notes the connection with Morera's theorem, and observes that the function $f(z)$ cannot be represented as a Golubev series $\sum_{k=0}^\infty \int_e (\zeta - z)^{-1}\,d\mu_k(\zeta)$. A related example shows finally that the integral of a small analytic function around a set of small analytic capacity need not be small.

F. Huckemann (E. Lansing, Mich.)

Aronszajn, N.; Donoghue, W. F. 6025

A supplement to the paper on exponential representations of analytic functions in the upper half-plane with positive imaginary part.

J. Analyse Math. **12** (1964), 113–127.

If $\varphi(\zeta)$ is analytic and has positive imaginary part in the upper half-plane, it may be represented in the form

$$\varphi(\zeta) = \alpha\zeta + \beta + \int_{-\infty}^{\infty} \left[\frac{1}{x-\zeta} - \frac{x}{x^2+1}\right] d\mu(x),$$

where $d\mu$ is a positive measure. Moreover, $\log \varphi(\zeta)$ (principal branch) has such a representation with $\alpha = 0$ and $d\mu(x) = f(x)\,dx$, $0 \le f(x) \le 1$. This leads to the "exponential representation" of $\varphi(\zeta)$. In an earlier paper [same J. **5** (1956/57), 321–388] the authors found various close connections between the growths of $d\mu(x)$ and $f(x)$ near $x = \infty$. These results are now extended to finer measurements of growth, the proofs are simplified, and a few oversights in the earlier paper are corrected.

P. L. Duren (London)

Curtiss, J. H. 6026

Harmonic interpolation in Fejér points with the Faber polynomials as a basis.

Math. Z. **86** (1964), 75–92.

Let $[a_{i,j}]_n$ represent a determinant of order n with $a_{i,j}$ the element in the ith row and jth column. There are $2n+1$ linearly independent harmonic polynomials of degree n. Let $\{h_k(z)\}$ denote a particular basis. A collection of $2n+1$ points is called a rank set of degree $2n+1$ if and only if $[h_i(z_j)]_{2n+1} \neq 0$. If $\{z_i\}$ is a rank set of degree $2n+1$, then

there exists a harmonic polynomial of degree n taking on arbitrarily preassigned values at the $\{z_i\}$.

Let C be an analytic curve. The normalized exterior mapping function maps C onto a circle Σ. For each integer n there is a unique set of $2n+1$ points on C determined as the images of $2n+1$ equally spaced points on Σ, where one point on Σ lies on the positive x-axis. Call this set the Fejér set F_n.

The author shows: (a) If C is a star-shaped curve, then F_n is a rank set for all n sufficiently large (say $n > n_0$); (b) If C is star-shaped and $g(z)$ is a continuous function on C, then if $H_n(z)$, a harmonic polynomial of degree n, interpolates $g(z)$ on F_n, the $H_n(z)$ converge uniformly on closed sets of the interior of C to the harmonic function having $g(z)$ as boundary values.

In the proof of (b) $\{h_n(z)\}$ are chosen as the real and imaginary parts of the Faber polynomials associated with C. The technique for showing $[h_i(z_j)]_{2n+1} \neq 0$ is to multiply by a non-singular determinant to obtain a determinant with a dominant principal diagonal for large n.

For a general analytic curve and $n \leq 4$, it is known that F_n need not be a rank set [Sobczyk, J. Soc. Indust. Appl. Math. **12** (1964), 499–514]. Information for the case $n > 4$ would thus seem to be of interest. The assumption of star-shapedness plays an essential role in the author's method, so new methods would apparently be needed.

J. L. Ullman (Ann Arbor, Mich.)

Ibragimov, G. I. 6027

On the completeness of subsystems of Faber polynomials on curves in the complex plane. (Russian)

Mat. Sb. (*N.S.*) **65 (107)** (1964), 3–17.

If $w = \Phi(z) = z + a_0 + a_1 z^{-1} + \cdots$ maps the outside of the simple closed curve L onto $|w| > \rho$, then the polynomial obtained by deleting the terms with negative powers of z in the expansion of $(\Phi(z))^n$ is called the nth Faber polynomial $\Phi_n(z)$ associated with z.

The author investigates the completeness of sets of Faber polynomials $\{\Phi_{\lambda_n}(z)\}$ in a Banach space of continuous functions defined on a system of curves C and with norm $\|f\| = \sup_{z\in C} e^{-p(|z|)} |f(z)|$. He also investigates the same problem for a Hilbert space with norm given by $\|f\|^2 = \int_C e^{-p(|z|)} |f(z)|^2 |dz|$. The weight function $p(t)$ is real-valued, and for $t > t_0$, $p(t) = \int_a^t y^{-1} q(y)\, dy$, where $q(y)$ tends monotonically to ∞ as $y \to \infty$. The system of curves C divides the plane into n domains $G_1, \cdots, G_n$. The domain G_j contains an angle $\gamma_j \leq \arg|z - z_j| \leq \gamma_j + (\pi/\alpha_j)$. In addition, C satisfies a smoothness condition. The curve L lies in one of the G_j (G_1, say), and special conditions have to be made about the width of G_1. A sufficient condition for completeness is then obtained under the condition that the sequence $\{\mu_n\} = \{n\} - \{\lambda_n\}$ satisfies $n/\mu_n \to \tau < 1$ as $n \to \infty$. The exact statement is too long to reproduce.

W. H. J. Fuchs (Ithaca, N.Y.)

Pommerenke, Ch. 6028

Über die Faberschen Polynome schlichter Funktionen.

Math. Z. **85** (1964), 197–208.

For $f(z) = z + a_0 + a_1 z^{-1} + \cdots$,

$$f'(z)/(f(z) - w) = \sum_0^\infty F_n(w) z^{-n-1}$$

is the generating function of the Faber polynomials $F_n(w) = w^n + b_n w^{n-1} + \cdots$. Let

$$F_n(f(z)) = z^n + n \sum_1^\infty a_{nk} z^{-k}.$$

The author proves the following for the case that $f(z)$ is schlicht for $|z| > 1$, and where E is the complement of the image of $|z| > 1$: (a) $\sum_1^\infty n \left|\sum_1^\infty a_{nk} c_k\right|^2 \leq \sum_1^\infty |c_k|^2/k$, c_k arbitrary; (b) $\log n < \max \sum_1^n |F_k(w)|/k < 4 \log n + 8$, the maximum being taken with respect to $w \in E$. The Grunsky inequality $\left|\sum_1^m \sum_1^m a_{jk} c_j c_k\right| \leq \sum_1^m |c_k|^2/k$ follows from (a). Refinements of (b) are made for the case when E is convex. Both (b) and its refinements are applied to the theory of transfinite diameter.

J. L. Ullman (Ann Arbor, Mich.)

Suetin, P. K. 6029

The basic properties of Faber polynomials. (Russian)

Uspehi Mat. Nauk **19** (1964), *no.* 4 (118), 125–154.

Nach einer ausführlichen Besprechung der vorhandenen Literatur über die speziellen und die verallgemeinerten Faberpolynome werden die letzteren einer eingehenden Betrachtung unterzogen. Ist $B_n(z)$ das n-te Faberpolynom für den Bereich G bei der Belegung $g(z)$, so werden in § 1 unter bestimmten Bedingungen Abschätzungen von $|B_n(z)|$ in G vorgenommen; in § 2 folgt eine asymptotische Formel für $B_n(z)$. Die §§ 3 und 4 enthalten hinreichende Bedingungen für die absolute und gleichmäßige Konvergenz von Faberreihen in G bzw. in $\bar{G}$, darauf § 5 solche für die eindeutige Darstellbarkeit einer analytischen Funktion durch die zugehörige Faberreihe. In § 7 werden die Nullstellen der Faberpolynome als Interpolationsknoten verwendet. Sodann behandelt § 8 den Fall, daß die Belegung $g(z)$ lokale Nullstellen besitzt, und § 9 den Spezialfall, in welchem die Faberpolynome mit denen von Tchebychev übereinstimmen. Schließlich enthält § 10 noch erläuternde Bemerkungen und Desiderata.

K. Bögel (Ilmenau)

Gromov, V. P. 6030

The completeness of systems of derivatives of an analytic function. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **25** (1961), 543–556.

The generalized derivative $D^n F$ of a function $F(z) = \sum b_n z^n$ which is regular in $|z| < R$ is defined by

$$D^n F(z) = \sum_{k=0}^\infty b_{k+n} \left(\frac{a_k}{a_{k+n}}\right) z^k,$$

where $\sum_0^\infty a_k z^k = f(z)$ and f is of order ρ, type $\sigma \neq 0$, and such that $a_n \neq 0$ and $\lim n^{1/\rho} |a_n|^{1/n} = (\sigma e \rho)^{1/\rho}$. The author investigates the completeness of systems $\{D^{n_\nu} F(z)\}$. A necessary and sufficient condition for completeness in $|z| < R$ is that $F(z)$ satisfies no linear differential equation with constant coefficients $\sum a_n D^n F(z) = \Phi(z)$ whose characteristic function $\varphi(z) = \sum a_n z^n$ is of growth at most order ρ, type less than $R^\rho \sigma$, where Φ is regular at the origin with $\Phi^{(n_\nu)}(0) = 0$. If completeness is considered in a simply connected region D, the condition is that $f(z)$ satisfies no equation of the form

$$\int_\gamma f(\xi + z) g(\xi)\, d\xi = \Phi(z),$$

where γ is a closed contour in D, $g(\xi) \neq 0$ is analytic on

and outside γ, and $g(\infty)=0$, and Φ is as before. As applications of the first theorem, the author shows that if F is entire, then $\{D^n F(z)\}$ is complete in the finite plane if $F(z)$ is of growth at most order ρ, type 0, not a polynomial, or if $F(z)$ is of order ρ, finite type, and not a finite linear combination of the functions $z^k f^{(k)}(\lambda z)$; that if $F(z)=z^{-1}\int_0^z f(t)\,dt$, the system $\{D^{n_\nu}F(z)\}$ is complete provided $\lim \nu/n_\nu>0$; and that if $F(z)=e^{h(z)}$, where $h(z)$ is a polynomial of degree at least 2, then the system of ordinary derivatives $F^{(n)}(z)$ is complete in the plane. An application of the second theorem is that if the periodic function $\varphi(z)=\sum_{-\infty}^{\infty} c_k e^{ikz}$ is analytic in the strip $\alpha<\Im(z)<\beta$ then the system $\{\varphi^{(n)}(z)\}$ is complete in the rectangle $|\Re(z)|<\pi\tau$, $\alpha<\Im(z)<\beta$, provided $c_k\neq 0$ for $k=\mu_n$ with $\lim n/\mu_n=\tau\neq 0$. For related but independent work see Kaz'min [Vestnik Moskov. Univ. Ser. I Mat. Meh. **1960**, no. 5, 3–13; ibid. **1960**, no. 6, 11–19; MR **23** #A3268].

R. P. Boas, Jr. (Evanston, Ill.)

Korevaar, Jacob 6031

Asymptotically neutral distributions of electrons and polynomial approximation.

Ann. of Math. (2) **80** (1964), 403–410.

Let D be a bounded simply connected region; the author solves the problem of determining all bounded sets Γ such that every zero-free holomorphic function in D can be approximated, uniformly on compact subsets, by polynomials whose zeros lie in Γ. The solution depends on the notion of asymptotically neutral families, which are families of finite sequences $\{z_{nk}\}_1^n$ of points of Γ such that $\sum_{k=1}^n (z_{nk}-z)^{-1}\to 0$ uniformly on compact subsets of D; in other words, the electrostatic field at z of point charges at the z_{nk} tends to zero uniformly on compact subsets of D. The solution is as follows. Let Γ be bounded and disjoint from D. The following four statements are equivalent. (i) Γ possesses the approximation property of the problem. (ii) Γ possesses the approximation property relative to $1/(\zeta-z)$ for some one $\zeta\in\Gamma$. (iii) Γ contains an asymptotically neutral family relative to D. (iv) The closure of Γ divides the plane and D belongs to a bounded component of its complement. In particular, the boundary of D can always serve as Γ.

R. P. Boas, Jr. (Evanston, Ill.)

Dračinskiĭ, A. È. 6032

On a Riemann-Privalov boundary-value problem for the circle. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **34** (1964), 31–35.

The author states several results concerning boundary-value problems for the unit disc, of which the following is typical: To find a function $f(z)=u+iv$ analytic for $|z|<1$, representable by a Cauchy integral, and such that $au-bv=c$ almost everywhere on $|z|=1$, where a, b, c are real functions defined on $|z|=1$. Sufficient conditions are given for the existence of a solution and for uniqueness, and an explicit formula for the solution is given.

W. Kaplan (Ann Arbor, Mich.)

Kuz'mina, A. L. 6033

The Hilbert boundary-value problem with arbitrary shift. (Russian)

Comment. Math. Univ. Carolinae **5** (1964), 117–120.

Let L be a simple, closed Liapunov curve enclosing a domain D. Let $\omega(t)$, $t\in L$, be a continuous function with non-vanishing derivative which maps L into itself biuniquely with preservation of direction. The Hilbert problem, wherein one seeks a function $\phi(z)=u+iv$, analytic in D and continuous in $\bar D$, satisfying on L the relation (*) $a(t)u(\omega(t))+b(t)v(t)=c(t)$, where a, b, c are given real functions, is considered. The author shows that when $a(t)\neq 0$, $b(t)$, $c(t)$ belong to the space C_α of functions $f(t)$, $t\in L$, where $f(t)$ is Hölder-continuous with exponent α, $0<\alpha\leq 1$, and the norm is given by

$$C_\alpha(f)=\max_{t\in L}|f(t)|+\sup_{t,t_0\in L}|f(t)-f(t_0)|/|t-t_0|^\alpha,$$

then (*) will have a solution belonging to C_α depending on a real arbitrary constant if $C_\alpha(b/a)$ is less than a specified constant whose value depends on the function ω. As an application of this result, the author cites a boundary-value problem for the equation of mixed type:

$$u_{xx}+\operatorname{sgn} y\,u_{yy}=0.$$

J. F. Heyda (King of Prussia, Pa.)

Mihaĭlov, L. G. 6034

The general conjugacy problem for analytic functions and its applications. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **27** (1963), 969–992.

Let Γ consist of the simple closed Liapunov contours $\Gamma_0,\Gamma_1,\cdots,\Gamma_m$ which enclose the domain D^+, and denote the planar complement of $D^++\Gamma$ by D^-. The author considers the problem of determining functions $\varphi^+(z)$, $\varphi^-(z)$, analytic in D^+, D^-, respectively, representable by Cauchy integrals, whose boundary values $\varphi^+(t)$, $\varphi^-(t)$ satisfy on Γ the relation

$$(*)\quad \varphi^+(t)=a(t)\varphi^-(t)+b(t)\overline{\varphi^-(t)}+c(t),\qquad \varphi^-(\infty)=0,$$

where a, b, c are given functions.

The author presents a systematic study of this problem when $a(t)\neq 0$ on Γ and $|a(t)|>|b(t)|$ (elliptic case), and also when $|a(t)|=|b(t)|$ (parabolic case). No such study for the hyperbolic case ($|a(t)|<|b(t)|$) exists, although partial results are available [see, e.g., I. H. Sabitov, Sibirsk. Mat. Ž. **5** (1964), 124–129; MR **28** #3164].

J. F. Heyda (King of Prussia, Pa.)

Serbin, A. I. 6035

A boundary-value problem on a finite Riemann surface. (Russian)

Boundary-value problems in the theory of functions of a complex variable, pp. 25–39. Izdat. Kazan. Univ., Kazan, 1962.

Čibrikova, L. I. 6036

The Hilbert boundary-value problem on a finite Riemann surface. (Russian)

Boundary-value problems in the theory of functions of a complex variable, pp. 59–72. Izdat. Kazan. Univ., Kazan, 1962.

Żakowski, W. 6037

Sur un problème non linéaire et discontinu de Hilbert-Haseman.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 287–292.

Let L be a smooth Jordan curve in the plane and $c_1, \cdots, c_p$ distinguished points on L; let α be a smooth transformation of L onto itself which leaves the c_s fixed. It is desired to find functions $\Phi_i(z)$ $(i=1, \cdots, n)$ analytic in the complement of L and whose limiting values $\Phi_i^{\pm}(t)$ on L satisfy the system of equations

$$\Phi_k^+[\alpha(t)] = \sum_{i=1}^{n} G_{ki}(t)\Phi_i^-(t) + F_k\{t, \Phi_1^+[\alpha(t)], \Phi_1^+(t), \cdots, \Phi_n^-[\alpha(t)], \Phi_n^-(t)\}.$$

The G_{ki} satisfy Hölder conditions on L; the F_k as functions of t have singularities at worst of a prescribed type at the c_s and as functions of their other arguments satisfy Hölder conditions. (The exact conditions are too complicated to state here.) The $\Phi_k(z)$ are to have singularities at the c_s at worst of a prescribed type. It is shown that this problem is equivalent to the solution of a certain nonlinear integral equation, which in turn is shown, by using the Schauder fixed-point theorem, to have at least one solution if certain constants involved in the conditions on the F_k are sufficiently small.

H. Widom (Ithaca, N.Y.)

POTENTIAL THEORY

See also 5998a–b, 6927, 6928, 6929.

Gegelia, T. G. 6038

Behavior of a generalized potential near the boundary of the region of integration. (Russian)

Akad. Nauk Gruzin. SSR Trudy Tbiliss. Mat. Inst. Razmadze **26** (1959), 189–193.

In an earlier paper [Soobšč. Akad. Nauk Gruzin. SSR **18** (1957), 257–264; MR **20** #4710], the terminology of which is assumed here, it was shown that the integral

$$(1) \qquad \Phi(R) = \int_S \frac{M(Q, R)}{r^n(Q, R)} \varphi(Q)\, dS_Q$$

possesses angular boundary values at almost all points of the manifold S under certain restrictions on $M(Q, R)$ and on S. In the present paper the author establishes the rate at which the function $\Phi(R)$ tends to infinity as $R \to P \in S$ from a neighborhood $G(P, \varepsilon)$ of an arbitrary point P of S, including the boundary L of the manifold S. Theorem: If $\varphi(Q) \in L_p(S, r^\alpha(Q, P))$, $p \geq 1$, $-n < \alpha < n(p-1)$, and $P \in S$, then

$$|\Phi(R)| \leq C_{\varepsilon,\alpha,p,n} \|Q\|_p [r(R, P)]^{-(\alpha+n)/p}$$

for $R \in G(P, \varepsilon)$.

A. J. Lohwater (Providence, R.I.)

Dolženko, E. P. 6039

On the representation of continuous harmonic functions in the form of potentials. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 1113–1130.

Let u be a differentiable function in an open set dom $u \subset E_n$ and, for $M \subset$ dom u, denote by $\omega(\delta, M)$ and $\omega_i(\delta, M)$ the moduli of continuity on M of u and $\partial u/\partial x_i$, respectively. Let $f(\delta)$ be non-decreasing in $\delta \geq 0$, positive for $\delta > 0$, and denote by H_φ the Hausdorff measure determined by $\varphi(\delta) = \delta^{n-1}f(\delta)$. Suppose that G is a domain in E_n and E is a subset in G such that $G-E$ is dense in G and the inner measure (H_φ) of E is finite. Let dom $u \supset G-E$ and u be harmonic in an open set containing $G-E$. If $\omega(\delta, G-E) \leq f(\delta)$ and $\max_i \omega_i(\delta, G-E) \leq f(\delta)$, then there is a v, harmonic in G, a closed $F \subset E$, and bounded Borel measurable functions λ, μ_i on F such that, for $x \in G-E$,

$$(1) \quad u(x) = v(x) + \int_F \lambda(y) p_n(|x-y|) H_\varphi(dy) + \sum_i \int_F \mu_i(y) \frac{\partial p_n(|x-y|)}{\partial x_i} H_\varphi(dy),$$

where $p_n(r) = r^{2-n}$ or $p_n(r) = \log r$ according as $n > 2$ or $n = 2$. If dom $u \supset G$, $H_\varphi(E) < \infty$ and $\max_i \omega_i(\delta, G) \leq f(\delta)$, then (1) holds for $x \in G-F$ with all $\mu_i \equiv 0$. Hence some corollaries are derived; in particular, under the above assumptions, the singular set E is removable for the harmonic function u provided $H_\varphi(E) = 0$.

J. Král (Prague)

Kreyszig, Erwin 6040

Kanonische Integraloperatoren zur Erzeugung harmonischer Funktionen von vier Veränderlichen.

Arch. Math. **14** (1963), 193–203.

If $f(u, v, w)$ is complex-analytic in u and continuous for complex v and w, the operator

$$P[f](\mathbf{x}) = -\frac{1}{4\pi^2} \iint f(u, v, w) v^{-1} w^{-1}\, dv dw,$$

integrated over $|v| = 1$, $|w| = 1$, sends f into a harmonic function $P[f]$ of the vector $\mathbf{x} = (x_1, x_2, x_3, x_4)$ provided $u = \mathbf{a} \cdot \mathbf{x}$, where $\mathbf{a}(v, w)$ is a vector function satisfying $\mathbf{a} \cdot \mathbf{a} = 0$. Operators of the above type in R^3 have been studied by S. Bergman [*Integral operators in the theory of linear partial differential equations*, Springer, Berlin, 1961; MR **25** #5277]. The author proves that, with suitable choices for f and fixed $\mathbf{a}$, P generates a family of harmonic polynomials $\{H\}$ expressible in terms of Jacobi polynomials, that the H's are orthogonal on the unit sphere in R^4, and that any function harmonic in a neighborhood of the origin can be represented by a series in the H's. He also gives an integral formula for recovering f from the corresponding H.

R. E. Williamson (Norwich, Vt.)

Murdoch, B. H. 6041

A theorem on harmonic functions.

J. London Math. Soc. **39** (1964), 581–588.

The main theorem is to the effect that if $P(x)$, $x = (x_1, \cdots, x_k)$, is a homogeneous harmonic multinomial and $f(x)$ is a function harmonic everywhere in E_k such that $P(x)f(x) \geq 0$, all $x \in E_k$, then $f(x) = CP(x)$, where C is a constant ≥ 0. The proof is based on Fourier expansions in spherical harmonics, and on the following results: If $P(x)$ (as above) is of degree n and $R(x)$ is a homogeneous multinomial of degree m such that $P(x)R(x) \geq 0$, all x, then $m \geq n$; a harmonic multinomial is always "well-signed" in the sense that the factors in its factorization into a product of irreducible multinomials are distinct and each takes on both positive and negative values; if $P(x)$ and $Q(x)$ are well-signed multinomials such that $P(x)Q(x) \geq 0$ for all $x \in E_k$, then $Q(x) = CP(x)$, C a constant ≥ 0.

J. H. Curtiss (Coral Gables, Fla.)

Januškauskas, A. 6042

The Cauchy problem for the Laplace equation and the multiplication operation for harmonic functions. (Russian)

Dokl. Akad. Nauk SSSR **159** (1964), 286–289.

Laplace's equation in three-dimensional space is studied by applying the theory of analytic functions in two complex variables. The method leads to a multiplicative structure on the space of solutions of Laplace's equation.

L. de Branges (Lafayette, Ind.)

Tilljašaihova, R. 6043

An example of computing the Green's function. (Russian. Uzbek summary)

Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk **1962**, *no.* 6, 44–52.

En se basant sur les recherches de N. I. Buleev et G. I. Marčuk [Trudy Inst. Fiz. Atmosfery **1958**, no. 2, 66–104] et en utilisant les formules de Gubin [*La théorie hydrodynamique du frontogenèse* (en russe), Izdat. Akad. Nauk UzSSR, Tashkent, 1960] l'auteur donne des exemples pour le calcul de la fonction de Green qui caractérise le domaine de dépendance de la solution des champs d'action des éléments météorologiques dans l'espace environnant.

M. Kiveliovitch (Paris)

Dynkin, E. B. 6044

Martin boundaries and non-negative solutions of a boundary-value problem with inclined derivative. (Russian)

Uspehi Mat. Nauk **19** (1964), *no.* 5 (119), 3–50.

Proofs of results described in another paper [Dokl. Akad. Nauk SSSR **157** (1964), 1028–1030; MR **29** #3663].

J. L. Doob (Princeton, N.J.)

Fukushima, Masatoshi 6045

On Feller's kernel and the Dirichlet norm.

Nagoya Math. J. **24** (1964), 167–175.

The author first considers Brownian motion on a Green space (he assumes, without so noting, that the rigid motions are the local parameter transformations for all dimensions). He defines the analogue of a kernel used by Feller [Ann. of Math. (2) **65** (1957), 527–570; MR **19**, 892] and shows that it is a constant multiple of Naïm's [Ann. Inst. Fourier (Grenoble) **7** (1957), 183–281; MR **20** #6608] θ kernel, under slight restrictions on the Martin boundary of R. He then considers a diffusion on a bounded smooth domain of N-space, with infinitesimal operator a smooth self-adjoint differential operator A. Defining a version U of the Feller kernel for this operator he follows the method of the reviewer [ibid. **12** (1962), 573–621] to obtain a generalized Dirichlet integral of a sufficiently smooth solution of the first boundary-value problem for A in terms of U and the boundary function.

J. L. Doob (Princeton, N.J.)

Gilbert, R. P. 6046

Harmonic functions in four variables with rational and algebraic p_4-associates.

Ann. Polon. Math. **15** (1964), 273–287.

Author's summary: "Integral representations for harmonic functions in four variables are investigated by means of an operator bearing close resemblance to the Whittaker-Bergman operator. The case where the p_4-associate is algebraic is considered by means of the theory of double integrals on algebraic three-folds. When the p_4-associate is rational, one obtains particularly interesting representations by considering the connections with Weierstrass integrals of the first, second, and third kinds defined over a Riemann surface. In addition, a residue theorem is given for a class of harmonic vectors $U=(u_1, u_2, u_3, u_4)$ satisfying the relations

$$\varepsilon_{mnrs}\frac{\partial u_r}{\partial x_s}=0, \qquad \frac{\partial u_\nu}{\partial x_\nu}=0,$$

which are analogous to the vanishing of the curl and divergence in three dimensions."

A. M. White (Claremont, Calif.)

Šafeev, M. N. 6047

A boundary-value problem for biharmonic functions of two complex variables. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 5 (42), 109–114.

Die reelle Funktion $f(\alpha_1, \alpha_2)$ sei in $\Delta_0=[0, 2\pi]\times[0, 2\pi]$ integrierbar und besitze in jeder Veränderlichen die Periode 2π. Verfasser zeigt, daß im Bizylinder $\{(z_1, z_2): |z_k|<R_k\}$ die Funktion

$$\frac{1}{4\pi^2}\iint_{\Delta_0} f(\alpha_1, \alpha_2)\,\mathrm{Re}\prod_{k=1}^{2}\frac{R_k e^{i\alpha_k}+z_k}{R_k e^{i\alpha_k}-z_k}\,d\alpha_1 d\alpha_2$$

biharmonisch ist und unter Annahme einer Lipschitzbedingung für f bei radialer Annäherung an einen Punkt $(R_1 e^{i\alpha_1^{(0)}}, R_2 e^{i\alpha_2^{(0)}})$, $\alpha_k^{(0)}\neq 0$, den Randwert

$$f(\alpha_1^{(0)}, \alpha_2^{(0)})-\frac{1}{4\pi^2}\iint_{\Delta_0} f(\alpha_1, \alpha_2)\prod_{k=1}^{2}\mathrm{ctg}\,\frac{\alpha_k^{(0)}-\alpha_k}{2}\,d\alpha_1 d\alpha_2$$

annimmt. Der Beweis beruht auf elementaren Eigenschaften trigonometrischer Reihen.

J. Spilker (Freiburg)

Kaniev, S. 6048

An exact bound for functions biharmonic in a circle and their boundary values. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 451–454.

Denote by $W^{(p)}LM$ the class of functions $f(\varphi)$ which are of period 2π, have an absolutely continuous $(p-1)$st derivative $(p\geq 1)$, and whose pth derivative satisfies $\int_0^{2\pi}|f^{(p)}(\varphi)|\,d\varphi\leq M=\text{const}$. If, instead of the latter inequality, we have $|f^{(p)}(\varphi)|\leq M$ almost everywhere, we denote the class of functions by $W^{(p)}M$. Consider functions $f(r, \varphi)$ which are biharmonic in the unit circle, assume values $f(\varphi)$ on $r=1$ and for which also $\partial f(r, \varphi)/\partial r|_{r=1}=0$. In a previous paper [Dokl. Akad. Nauk SSSR **153** (1963), 995–998; MR **28** #240] the author evaluated

$$\sup\max_{0\leq\varphi\leq 2\pi}|f(r, \varphi)-f(1, \varphi)|$$

for $f\in W^{(p)}M$. In the present paper he shows that one obtains the same upper bound if one considers $f\in W^{(p)}LM$.

J. F. Heyda (King of Prussia, Pa.)

SEVERAL COMPLEX VARIABLES

See also 5847, 6042, 6047, 6402, 6502, 6503.

Fuks, B. A. 6049

★Theory of analytic functions of several complex variables.

Translated by A. A. Brown, J. M. Danskin and E. Hewitt.

American Mathematical Society, Providence, R.I., 1963. vi + 374 *pp.* $13.00.

This is a translation of the Russian book *Introduction to the theory of analytic functions of several complex variables* [Fizmatgiz, Moscow, 1962; MR **27** #4945]. A more modern version [*Special chapters in the theory of analytic functions of several complex variables* (Russian), Fizmatgiz, Moscow, 1963] has recently appeared.

Michiwaki, Yoshimasa 6050

Remarks on the minimum problems of several complex variables.

Res. Rep. Nagaoka Tech. College **1**, 145–150 (1963).

The paper asserts a formula for mappings between domains in several complex variables. However, it is very difficult to make out the proof, or even the meaning of the formula. *S. Bochner* (Princeton, N.J.)

Ibragimov, I. I. 6051

Inequalities for entire functions of finite degree in the metric of a generalized Lebesgue space. (Russian. Azerbaijani summary)

Akad. Nauk Azerbaidžan. SSR Dokl. **20** (1964), *no.* 4, 13–18.

The author compares different norms of an entire function of exponential type in n variables, where the function has finite $(p_1, \cdots, p_n)$ norm on the real hyperaxis (norm defined as in Dokl. Akad. Nauk SSSR **152** (1963), 1054–1057 [MR **27** #4949]). He gives bounds for the supremum in terms of the $(p_1, \cdots, p_n)$ norm; for the $(p_1, \cdots, p_k)$ norm in terms of the $(p_1, \cdots, p_m)$ norm, $1 \leq k < m \leq n$; and for the $(q_1, \cdots, q_n)$ norm in terms of the $(p_1, \cdots, p_n)$ norm, with $p_k < q_k$. *R. P. Boas, Jr.* (Evanston, Ill.)

Mamedhanov, D. I. 6052

Inequalities for positive entire functions in a generalized Lebesgue space. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 526–528.

The author continues the work of Ibragimov [#6051]. He obtains bounds for the $(q_1, \cdots, q_n)$ norm of $f(x+iy)$ in terms of the $(p_1, \cdots, p_n)$ norm of $f(x)$. He also obtains the analogous bounds for the classes of functions that satisfy $|f(x+iy)| \leq |f(x-iy)|$, $y \geq 0$, or are non-negative on the real hyperaxis. *R. P. Boas, Jr.* (Evanston, Ill.)

Andreotti, Aldo; Vesentini, Edoardo 6053

Disuguaglianze di Carleman sopra una varietà complessa.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **35** (1963), 431–434.

In this paper the authors announce certain results presented at the Centro Internazionale Matematico Estivo in July, 1963. By establishing certain inequalities of the type of Carleman [Ark. Mat. Astronom. Fys. **26B** (1939), no. 17; MR **1**, 55] the authors refine the finiteness theorems [Andreotti and Grauert, Bull. Soc. Math. France **90** (1962), 193–259; MR **27** #343] for the cohomology of locally free sheaves on q-concave or q-convex complex manifolds.

In particular the following theorem is stated. Let X be a q-complete complex manifold, and F a locally free sheaf on X. Then the cohomology groups $H_k^s(X, F)$ with compact support vanish for $s \leqq n-q$. $H_k^{n-q+1}(X, F)$ has a Hausdorff topology (i.e., the mapping $\bar\partial$ on $n-q$ forms of compact support has closed range). The finiteness theorems above follow from this by dualization and the use of the theorem of L. Schwartz [C. R. Acad. Sci. Paris **236** (1953), 2472–2473; MR **15** 233] on the Čech cohomology groups of a suitable covering. *H. Rossi* (Waltham, Mass.)

Grauert, Hans 6054

Bemerkenswerte pseudokonvexe Mannigfaltigkeiten.

Math. Z. **81** (1963), 377–391.

In einer früheren Arbeit [Ann. of Math. (2) **68** (1958), 460–472; MR **20** #5299] zeigte der Autor, dass in einer komplexen Mannigfaltigkeit X ein Gebiet $G \Subset X$ mit glattem Rand ∂G holomorph-konvex ist, wenn ∂G streng pseudokonvex ist. In der vorliegenden Arbeit wird durch ein Beispiel $T \Subset F$ zunächst gezeigt, dass dabei streng pseudokonvex nicht durch pseudokonvex ersetzt werden darf, selbst wenn G ein Meromorphiegebiet ist: Es sei R eine kompakte Kählersche Mannigfaltigkeit und F ein komplex analytisches Geradenbündel über R, das topologisch trivial sei. Es wird gezeigt, dass bereits eine geeignete Tensorpotenz F^k von F analytisch trivial ist, wenn in einer Umgebung der Nullschnittfläche von F eine nicht lokalkonstante holomorphe Funktion existiert. Andererseits kann das Bündel F durch Übergangsfunktionen g_{ij} mit $|g_{ij}| \equiv 1$ beschrieben werden, man hat damit eine Norm $\| \ \|$ auf F, d.h. auf jeder Faser F_x eine Norm $\| \ \|_x$, die reell-analytisch von $x \in R$ abhängt. Durch $\| \ \| < 1$ wird eine relativ-kompakte Umgebung T der Nullschnittfläche in F definiert, deren Rand glatt, reell-analytisch und pseudokonvex, jedoch nirgendwo streng pseudokonvex ist. Ist F speziell so gewählt $(b_1(R) > 0)$, dass keine Tensorpotenz F^k analytisch trivial ist, lässt T nur konstante holomorphe Funktionen zu und ist nicht holomorph-konvex. Ist R insbesondere noch projektiv algebraisch, so ist T ein Meromorphiegebiet und lässt sich als verzweigtes Gebiet einem komplexprojektiven Raum P_n überlagern. Ferner wird ein Gebiet $G \Subset X$ konstruiert, dessen Rand pseudokonvex und bis auf eine 2-codimensionale Teilmenge streng pseudokonvex ist. G ist nicht holomorph-konvex; die Algebra $I(G)$ aller holomorphen Funktionen auf G ist nicht isomorph zur Funktionenalgebra eines Steinschen Raumes. Denn $I(G)$ besitzt Charakterideale, die nicht endlich erzeugbar sind. Es existiert sogar eine 1-codimensionale analytische Menge $A \subset G$, so dass $G - A$ über dem 3-dimensionalen komplexen Zahlenraum C_3 ausgebreitet werden kann und $I(G-A)$ isomorph ist zu $I(G)$. Innerhalb der Kategorie der komplexen Räume lässt sich für $G - A$ keine Holomorphiehülle konstruieren. Der Autor verallgemeinert deshalb den Begriff des komplexen Raumes zum Begriff des komplexen Schemas und spricht die Hoffnung aus, dass innerhalb der Kategorie der komplexen Schemata Holomorphiehüllenbildung stets möglich sei. *W. Kaup* (Erlangen)

Lu, Qi-keng [Look, K. H.] 6055

The elliptic geometry of extended spaces.

Acta Math. Sinica **13** (1963), 49–62 (*Chinese*); *translated as Chinese Math.* **4** (1963), 54–69.

In this paper the author begins by considering the classical space $\Re_I$ as an absolutely extended space, then introduces an elliptic distance between two elements, and shows that the corresponding differential metric is Kählerian. For instance, let $\mathfrak{z}_1, \mathfrak{z}_2$ be two elements in the space of $m\times(m+n)$ matrices, $E(m, m+n)$, and let $\mathfrak{P}(m, n)$ be the space of equivalence classes of elements $\mathfrak{z}_1$ or $\mathfrak{z}_2$ which are related by $\mathfrak{z}_1 = Q\mathfrak{z}_2$, where Q is a nonsingular $m\times m$ matrix. $\Re_I$ is the space $\mathfrak{P}(m, n)$; for a detailed discussion of the other classical domains see L. K. Hua [*Harmonic analysis of functions of several complex variables in the classical domains*, Amer. Math. Soc., Providence, R.I., 1963; MR **30** #2162].

He introduces as a distance in $\mathfrak{P}(m, n)$ between two points $\mathfrak{z}_1, \mathfrak{z}_2$,

$$\sigma(\mathfrak{z}_1, \mathfrak{z}_2) = \arccos \frac{|\det \mathfrak{z}_1\bar{\mathfrak{z}}_2'|}{\sqrt{(\det \mathfrak{z}_1\bar{\mathfrak{z}}_1' \cdot \det \mathfrak{z}_2\bar{\mathfrak{z}}_2')}}$$

(where $\mathfrak{z}'$ denotes the transpose of $\mathfrak{z}$, and $\bar{\mathfrak{z}}$ is the complex conjugate), and shows it satisfies the required conditions for a metric.

In Section II the author introduces an analytic structure on the compact complex analytic manifold $\mathfrak{P}(m, m)$, and derives the differential metric of $\mathfrak{P}(m, n)$ from the definition of the distance $\sigma(\mathfrak{z}_1, \mathfrak{z}_2)$ given in Section I. He obtains as a differential metric the form

$$ds^2 = \operatorname{tr}[d\bar{Z}'(I+Z\bar{Z}')^{-1}\,dZ(I+\bar{Z}'Z)^{-1}] = d\bar{d}\log\det(I+Z\bar{Z}'),$$

where

$$d \equiv \sum_{j=1}^{m}\sum_{\alpha=1}^{n} dz_{j\alpha}\frac{\partial}{\partial z_{j\alpha}}, \qquad \bar{d} \equiv \sum_{j=1}^{m}\sum_{\alpha=1}^{n} d\bar{z}_{j\alpha}\frac{\partial}{\partial \bar{z}_{j\alpha}},$$

where Z is a local coordinate given by

$$Z = (\mathfrak{z}_{i_1}, \cdots, \mathfrak{z}_{i_m})^{-1}(\mathfrak{z}_{i_{m+1}}, \cdots, \mathfrak{z}_{i_{m+n}}),$$

$$\mathfrak{z}_\alpha \equiv \begin{pmatrix}\mathfrak{z}_{1\alpha}\\ \vdots \\ \mathfrak{z}_{m\alpha}\end{pmatrix}, \qquad \mathfrak{z} \equiv (\mathfrak{z}_1, \cdots, \mathfrak{z}_{m+n});$$

$$Z = \begin{pmatrix} z_{11} & \cdots & z_{1n}\\ \cdots & \cdots & \cdots \\ z_{m1} & \cdots & z_{mn}\end{pmatrix}.$$

Using the form of the unitary curvature of $\mathfrak{P}(m, n)$ as given by the author in an earlier work [Advancement in Math. **2** (1956), 567–662; MR **20** #6535] and the fact that the unitary curvature is invariant under analytic transformations, he shows that it is always positive. This result is a special case of a result due to É. Cartan [Bull. Soc. Math. France **54** (1926), 214–264]; however, the curvature is completely written out here.

In Section III the author writes out the expressions for the differential metric and unitary curvature for the submanifolds $\Re_{II}$, $\Re_{III}$, and $\Re_{IV}$.

R. P. Gilbert (College Park, Md.)

Bers, Lipman; Ehrenpreis, Leon 6056

Holomorphic convexity of Teichmüller spaces.

Bull. Amer. Math. Soc. **70** (1964), 761–764.

B is the Banach space of analytic functions φ in the lower half-plane $y<0$ with norm $\|\varphi\| = \sup|y^2\varphi(z)|$. The universal Teichmüller space T can be identified with an open subset of B; T is complete in the Teichmüller metric.

If $Q \subset T$, let $h(Q)$ denote the hull of Q with respect to holomorphic functions in T. The authors prove that if Q is bounded in the Teichmüller metric, so is $h(Q)$. As a consequence, finite-dimensional Teichmüller spaces $T(G)$ (G is a finitely generated Fuchsian group of the first kind) are domains of holomorphy.

The result is applied to obtain some refinements of Baily's theorem on the boundary of Teichmüller space [W. L. Baily, Jr., Ann. of Math. (2) **71** (1960), 303–314; MR **22** #1583].

L. V. Ahlfors (Cambridge, Mass.)

Bishop, Errett 6057

Conditions for the analyticity of certain sets.

Michigan Math. J. **11** (1964), 289–304.

Results about the limit and the continuation of analytic sets with bounded volume are obtained by using methods of functional analysis. They go much further and complete previous partial results.

Let v be the $2k$-dimensional Hausdorff measure in $\mathbf{C}^n$, where $n>k>0$. Suppose that $\{A_m\}$ is a sequence of analytic sets of pure complex dimension k in the open subset U of $\mathbf{C}^n$. Suppose that $A_m\to A$ for $m\to\infty$ and that $v(A_m)$ is bounded. Then $\bar{A}\cap U$ is analytic in U. This theorem was conjectured by Oka [J. Sci. Hiroshima Univ. A **4** (1934), 93–98] and proved by Rutishauser [Acta Math. **83** (1950), 249–325; MR **12**, 90] and Nishino [J. Math. Kyoto Univ. **1** (1961/62), 357–377; MR **26** #6441] in the case $k = n-1 = 1$, and by the reviewer [Math. Z. **84** (1964), 154–218; MR **29** #2431] in the case $k = n-1$.

Suppose that B is an analytic subset of dimension p of the open subset U of $\mathbf{C}^n$. Let A be an analytic subset of pure complex dimension k in $U-B$. Suppose that $v(A)<\infty$. Then $U\cap\bar{A}$ is analytic in U. This theorem generalizes a theorem of Remmert and Stein [Math. Ann. **126** (1953), 263–306; MR **15**, 615]. In the case $p=k$, the theorem was proved by the reviewer [Arch. Math. **9** (1958), 167–175; MR **21** #729].

The Remmert-Stein theorem can be generalized further with the introduction of sets of capacity zero. Suppose that B is a relative closed subset of the open bounded set U of $\mathbf{C}^n$. Let A be an analytic subset of pure dimension k on U with $B \subseteq \bar{A}$. Suppose that B has capacity zero relative to the algebra of all continuous functions on $\bar{A}$ which are holomorphic on A. Let S be an open and connected subset of $\mathbf{C}^k$. Let $\pi: U\to S$ be surjective and holomorphic. Suppose that $\pi\,|\,B: B\to S$ is proper, with $\pi(B)\neq S$. Then $\bar{A}\cap U$ is analytic in U.

It should be noted that the constant c of Lemma 3 was introduced by P. Lelong [Bull. Soc. Math. France **85** (1957), 239–262; MR **20** #2465], that c depends on k but not on n, and that $c=\pi^k/k!$ is the best possible choice of c.

W. Stoll (Notre Dame, Ind.)

Baily, W. L., Jr.; Borel, A. 6058

On the compactification of arithmetically defined quotients of bounded symmetric domains.

Bull. Amer. Math. Soc. **70** (1964), 588–593.

This is a brief announcement of the results extending earlier results [*Séminaire H. Cartan*, 1957/58, Secrétariat

mathématique, Paris, 1958; MR **21** #2750; W. L. Baily, Jr., Amer. J. Math. **81** (1959), 846–874; MR **22** #12244; I. I. Pjateckiĭ-Šapiro, *Geometry of classical domains and theory of automorphic functions*, Fizmatgiz, Moscow, 1961; MR **25** #231] to the most general case, of which a full account is supposed to be published elsewhere. Let X be a hermitian symmetric space of non-compact type. One considers a (connected) semi-simple linear algebraic group G $(\subset GL(m, \mathbf{C}))$ defined over $\mathbf{Q}$ such that the symmetric space associated with $G_{\mathbf{R}}$ is equal to X, i.e., $X = K\backslash G_{\mathbf{R}}$, where K is a maximal compact subgroup of $G_{\mathbf{R}}$. (As usual, for every subring B of $\mathbf{C}$, one puts $G_B = G \cap GL(m, B)$.) Let Γ be an "arithmetic subgroup" of G, i.e., a subgroup of $G_{\mathbf{Q}}$ commensurable with the group $G_{\mathbf{Z}}$ of units of G; then Γ acts on X (from the right) in a properly discontinuous manner, and the quotient space X/Γ carries a natural ringed structure with which it becomes an irreducible normal analytic space. Since the main purpose of the paper is to construct a suitable compactification of X/Γ, one may assume that X/Γ is not compact, which implies, in particular, that $G_{\mathbf{R}}$ has no compact simple factor $\neq (e)$; for simplicity, one further assumes that G is strictly simple over $\mathbf{Q}$ (i.e., that it has no proper invariant subgroup $\neq (e)$ over $\mathbf{Q}$). Now the first step is to define the notion of "rational boundary component" for the natural compactification $\bar{X}$ of X (i.e., the closure of X in the Harish-Chandra realization as a bounded symmetric domain). This being done in a very ingenious way, it turns out that, for a rational boundary component F of X (with respect to Γ), the complexification $\mathfrak{N}(F)_{\mathbf{C}}$ of the normalizer $\mathfrak{N}(F) = \{g \in G_{\mathbf{R}} \mid Fg = F\}$ is a proper maximal parabolic subgroup defined over $\mathbf{Q}$ of G, and actually the map $F \to \mathfrak{N}(F)_{\mathbf{C}}$ gives a one-to-one correspondence between the rational boundary components of X (with respect to Γ) and the proper maximal parabolic subgroups over $\mathbf{Q}$ of G (Theorem 1). Then, denoting by X^* the union of X and all rational boundary components, one introduces a topology on X^* in a natural way (but still making use of a fundamental set for Γ) so that $V^* = X^*/\Gamma$ becomes a compact Hausdorff space. From the construction, one has $V^* = V \cup V_1 \cup \cdots \cup V_t$ with $V_i = F_i/\Gamma(F_i)$, where $V = X/\Gamma$ is open, everywhere dense, the F_i's are such that the corresponding groups $\mathfrak{N}(F_i)_{\mathbf{C}}$ form a system of representatives for the equivalence-classes, modulo inner automorphisms by elements of Γ, of proper maximal parabolic subgroups over $\mathbf{Q}$ of G, and $\Gamma(F_i)$ is an arithmetic group acting on F_i, obtained as a homomorphic image of $\mathfrak{N}(F_i) \cap \Gamma$. Next, to define an analytic structure on V^*, one calls a complex-valued function f defined on an open subset U of V^* an $\mathscr{K}$-function on U if it is continuous and if its restriction to $V \cap U$ and to $V_i \cap U$ is analytic in the given analytic structure $(1 \leq i \leq t)$. Then, by arguments similar to those used in the previous papers [loc. cit.] and by considerations of Poincaré-Eisenstein series, the constructions of which depend essentially on the realization of X as a Siegel domain of the third kind for a given boundary component [cf. Pjateckiĭ-Šapiro, loc. cit.; and A. Korányi and J. Wolf, "Generalized Cayley transforms of bounded symmetric domains", Ann. of Math. (2) (to appear)], the authors show finally that V^*, provided with the ringed structure defined by the sheaf of $\mathscr{K}$-functions, becomes a normal analytic space projectively embeddable as a projectively normal algebraic variety by means of a set of automorphic forms for Γ of some suitably high weight (Theorem 3). As a corollary, a generalization of Köcher's theorem and the finiteness of the dimension of the space of automorphic forms (for G with dim $G > 3$) are mentioned, together with some comments on the other approaches to the same results.

I. Satake (Chicago, Ill.)

Gunning, R. C. 6059

Differential operators preserving relations of automorphy.

Trans. Amer. Math. Soc. **108** (1963), 326–352.

Let $\mathfrak{H}$ be a domain in C^n (or more generally a complex manifold) and let G be a group of complex analytic automorphisms of $\mathfrak{H}$. Let V be a (finite-dimensional) complex vector space and denote by $\Phi(V)$ $[\Theta(V)]$ the space of C^∞-mappings [holomorphic mappings] of $\mathfrak{H}$ into V. If $\rho(M, z)$ $(M \in G, z \in \mathfrak{H})$ is a $GL(V)$-valued factor of automorphy for the action of G on $\mathfrak{H}$, the group G acts as a group of linear transformations on the space $\Phi(V)$ by putting

$$(T_\rho(M^{-1})f)(z) = \rho(M, z)^{-1}f(Mz).$$

Under this action of G, the subspace $\Theta(V)$ of $\Phi(V)$ is stable under G. Let W be another complex vector space. A linear mapping $D: \Phi(V) \to \Phi(W)$ is called a holomorphic differential mapping of order ν if we have

$$(DF)(z) = \sum_{s=0}^{\nu} \sum_{i_1, \cdots, i_s} A_{i_1 \cdots i_s}(z) \cdot \partial^s F(z)/\partial z_{i_1} \cdots \partial z_{i_s}$$

for all $F \in \Phi(V)$, where $A_{i_1, \cdots, i_s}(z)$ are the holomorphic mappings from $\mathfrak{H}$ into the complex vector space Hom(V, W). The restriction of D defines a linear mapping $D: \Theta(V) \to \Theta(W)$. When $\hat{\rho}(M, z)$ is a $GL(W)$-valued factor of automorphy, we can also define the action $f \to T_{\hat{\rho}}(M)f$ of G on $\Phi(W)$, and we call a holomorphic differential mapping $D: \Phi(V) \to \Phi(W)$ a G-homomorphism if we have

$$T_{\hat{\rho}}(M) \cdot D = D \cdot T_\rho(M)$$

for all $M \in G$.

In this paper the author studies the problem of determining the holomorphic differential G-homomorphisms in the following situation: $\mathfrak{H}$ is the Siegel upper half-space, G is the symplectic group or a general subgroup thereof and the factors of automorphy are of the form $\rho(Cz + D)$. After obtaining various general results on the holomorphic differential G-homomorphisms, the author determines the holomorphic differential G-homomorphisms of orders 1 and 2 for the Siegel upper half-space of rank 2 and indicates some of the applications of these results.

Y. Matsushima (Osaka)

Maass, Hans 6060

Über die gleichmässige Konvergenz der Poincaréschen Reihen n-ten Grades.

Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II **1964**, 137–144.

Let $S(Z) = (AZ + B)(CZ + D)^{-1}$ be a transformation of the Siegel modular group of degree n. The Poincaré series of dimension $-k$ is defined as

$$g_{-k}(Z, T) = e^{2\pi i\sigma(TS(Z))}|CZ + D|^{-k},$$

where k is an even positive integer, $Z = X + iY$, $Y > 0$, $T \geq 0$ is a semi-integral symmetric matrix of rank r, S runs over a certain reduced set of modular matrices, and σ is the trace. The object of this paper is to prove the Theorem:

The series $g_{-k}(Z, T)$ for $k > n+1+r$ converges absolutely and uniformly in each region $Y \geq (1/m)E$, $\sigma(X^2) \leq m$. Here $m > 0$ and E is the identity matrix.

Up to now only the absolute convergence of the Eisenstein series ($T = 0$) had been known. Both the plan of the proof and Lemma 1 are attributed to C. L. Siegel.

The author proceeds by introducing a majorant for g_{-k}, which is essentially

$$\sum_{(C,D)} \varphi(Z; C, D)\|CZ+D\|^{-k},$$

with

$$\varphi(Z; C, D) = \sum_{P} e^{-2\pi\delta\sigma(Y_S(P))},$$

where δ, $0 < \delta < 1$, depends on T, Y_S is the imaginary part of $S(Z)$, and C, D, P are matrices with certain ranges. Lemma 1: $|Y|\Delta_k(Z) \geq c\Delta_k(iE)$, where Δ_k is a principal subdeterminant of the matrix $\begin{pmatrix} Y & 0 \\ 0 & Y^{-1}\end{pmatrix}\begin{pmatrix} C' \\ XC'+D'\end{pmatrix}$; C and D form the lower row of a transformation $S(Z)$, and c depends only on m and n. Lemma 2: For $k > n+1$ the Eisenstein series $g_{-k}(Z, 0)$ converges uniformly in each domain of the type stated in the Theorem.

The proof of Lemma 2 is made from the estimate

$$\varphi_p(Z; C, D)\|CZ+D\|^{-k} \leq c_1 p^{-2(k-r)},$$

where φ_p is φ restricted to $\sigma(P'P) \geq p^2$ and c_1 depends on m, n, r, δ, C, D.

The Theorem follows from the above three results.

J. Lehner (College Park, Md.)

Igusa, Jun-ichi 6061

On Siegel modular forms of genus two. II.

Amer. J. Math. **86** (1964), 392–412.

The author continues his intrepid investigation [Part I appeared in same J. **84** (1962), 175–200; MR **25** #5040] of the modular varieties in genus 2, using in an essential way the ideas and results of his work on theta constants [ibid. **86** (1964), 219–246; MR **29** #2258].

Let $\Gamma(n)$ be the modular group of level n acting on the Siegel upper half-plane $\mathfrak{S}_2$ of genus 2. Thus $\Gamma(1) = \mathrm{Sp}(2, \mathbf{Z})$ and there is a chain of natural subgroups $\Gamma(1) \supset \Gamma(2) \supset \Gamma(2, 4) \supset \Gamma(4, 8)$. Let $A(\Gamma)$ be the graded ring of modular forms with respect to Γ of different weights, and let $V(\Gamma) = \mathrm{proj}\, A(\Gamma)$ denote the corresponding projective variety. The main geometric results then are that $V(\Gamma(2, 4))$ is nonsingular; thus $V(\Gamma(1))$ and $V(\Gamma(2))$ have a nonsingular covering. On the other hand, it is proved that there is a singular point on $V(\Gamma(4))$ which has no local nonsingular covering, and this result easily extends to all $V(\Gamma(n))$, $n \geq 3$.

The general line of argument uses the ten theta constants θ_m for genus 2. He proves $\mathbf{C}(\theta_m\theta_n)$ is integrally closed, which shows by the second paper cited above that it coincides with $A(\Gamma(4, 8))$; this gives a hold on the structure of the latter. One then obtains the other $A(\Gamma)$ as the subrings of $A(\Gamma(4, 8))$ invariant under the corresponding finite groups; the passage from $A(\Gamma(2))$ to $A(\Gamma(1))$ requires an analysis of the representation of $\Gamma(2)/\Gamma(1)$ on the former ring. The result about $V(\Gamma(4))$ is obtained by studying it locally as a covering of the nonsingular $V(\Gamma(2, 4))$.

Along the way, a new proof is given for the structure of $A(\Gamma(1))$, and explicit formulas for the basic Eisenstein series of level one in terms of the θ_m are given. The paper is almost entirely algebraic.

A. Mattuck (Cambridge, Mass.)

Christian, Ulrich 6062

Bestimmung des Körpergrades der Siegelschen Modulfunktionen über den Eisensteinreihen.

Abh. Math. Sem. Univ. Hamburg **27** (1964), 171–172.

Es sei $\Gamma(q)$ die Hauptkongruenzuntergruppe q-ter Stufe der Siegelschen Modulgruppe n-ten Grades Γ. Früher [Math. Ann. **152** (1963), 275–341; MR **28** #2256] hatte der Autor bemerkt, dass der Körper $m(q)$ der zugehörigen Modulfunktionen nur in den Fällen $n \equiv 1$ (2) und $n \equiv 0$ (2), $q = 1, 2, 4, p^\nu, 2p^\nu$ (p ungerade Primzahl, $\nu = 1, 2, \cdots$) durch die Eisensteinreihen erzeugt wird. In der vorliegenden Note wird der Körpergrad von $m(q)$ über dem Körper $e(q)$ derjenigen Modulfunktionen, welche durch die Eisensteinreihen rational dargestellt werden können, ergänzend angegeben. Es sei $n \equiv 0$ (2), $q \geq 3$. Die zu $\Gamma(q)$ gebildeten Eisensteinreihen sind nämlich in Wahrheit sogar Modulformen bezüglich der Kongruenzgruppe $\Gamma^*(q) = \{M \in \Gamma \mid M \equiv uE\ (q),\ u^2 \equiv 1\ (q)\}$, und nach früheren Resultaten ist einerseits der zugehörige Funktionenkörper $m^*(q) = e(q)$ und andererseits $[m(q) : m^*(q)] = \frac{1}{2}[\Gamma^*(q) : \Gamma(q)]$. Es ergibt sich also durch elementare Bestimmung des Gruppenindex $[\Gamma^*(q) : \Gamma(q)]$ das Resultat $[m(q) : e(q)] = 2^{r+s-1}$, wobei r die Anzahl der ungeraden Primteiler von q und $s = 0, 1, 2$ je nachdem $4 \nmid q$, $4 \mid q$ und $8 \nmid q$, $8 \mid q$ ist.

H. Klingen (Freiburg)

Christian, Ulrich 6063

Über die Modulgruppe zweiten Grades. I, II.

Math. Z. **85** (1964), 1–28; *ibid.* **85** (1964), 29–39.

Seien $Z(n)$ der Siegelsche Halbraum und $\Gamma(n)$ die Siegelsche Modulgruppe n-ten Grades. Der Quotientenraum $Q(n) = Z(n)/\Gamma(n)$ lässt sich nach Satake, Siegel, und Verfasser in der Gestalt

$$\tilde{Q}(n) = Q(n) \cup Q(n-1) \cup \cdots \cup Q(0)$$

kompaktifizieren. Die Frage nach der Uniformisierbarkeit der Punkte von $\tilde{Q}(n)$ wurde früher für $n \geq 3$ vom Verfasser [Math. Ann. **152** (1963), 275–341; MR **28** #2256] und für $n = 2$ von Igusa [Amer. J. Math. **84** (1962), 175–200; MR **25** #5040] mit recht tiefliegenden Hilfsmitteln behandelt. Die vorliegende Arbeit betrifft den Fall $n = 2$ und gibt einfache neue Beweise für die Ergebnisse von Igusa in expliziter Darstellung. Das Hauptresultat von Teil I ist der Nachweis der Uniformisierbarkeit des Punktes $Q(0)$ in $\tilde{Q}(2)$. Verfasser benutzt seine verwendete Beschreibung [loc. cit.] der Umgebungen von $Q(0)$. In die Rechnungen geht wesentlich ein die einfache Gestalt der Minkowskischen Reduktionsbedingungen im binären Fall. Als Ortsuniformisierende von $Q(0)$ werden Funktionen folgender Art verwendet:

$$F(Z) = \sum e^{2\pi i(\mathrm{Sp}\, SZ)},$$

wobei über alle symmetrischen S summiert wird, die zu einer festen Kantenmatrix der Minkowskischen Pyramide unimodular äquivalent sind. Anschliessend werden einige Folgerungen aus diesem Uniformisierungssatz gezogen. Im zweiten Teil werden Thetafunktionen benutzt, um die Uniformisierbarkeit der Punkte von $Q(1)$ in $\tilde{Q}(2)$ zu untersuchen. Sämtliche Punkte von $Q(1)$ sind uniformisierbar

mit der genauen Ausnahme von denjenigen, die i und $1/2+i\sqrt{3}/2 \in Z(1)$ entsprechen. Dort liegen Singularitäten vor. *H. Klingen* (Freiburg)

Helwig, Karl-Heinz 6064

Automorphismengruppen des allgemeinen Kreiskegels und des zugehörigen Halbraumes.

Math. Ann. **157** (1964), 1–33.

Ein n-dimensionaler Kreiskegel Y_n ist eine Teilmenge eines normierten Vektorraumes X über dem Körper R der reellen Zahlen, die in einer geeigneten Basis von X durch die Bedingungen

$$(y_1, \cdots, y_n) \in X, \quad y_1^2 - y_2^2 - \cdots - y_n^2 > 0, \qquad y_1 > 0,$$

beschrieben wird. Y_n ist einer der vier Haupttypen der von M. Koecher [Amer. J. Math. **79** (1957), 575–596; MR **19**, 867] eingeführten homogenen Positivitätsbereiche mit positiver Charakteristik. In der vorliegenden Arbeit werden alle linearen Automorphismen von Y_n sowie alle holomorphen Automorphismen des zugehörigen Halbraumes $Z_n = R^n + iY_n$ bestimmt. Ferner werden durch einen arithmetischen Prozeß diskontinuierliche Automorphismengruppen von Z_n ("Modulgruppen") konstruiert, die einen ähnlichen Fundamentalbereich besitzen wie die bekannten Modulgruppen im Siegelschen Halbraum. Dadurch ist es möglich geworden, die Theorie Siegelscher Modulformen und Modulfunktionen auf Z_n zu übertragen.

J. Spilker (Freiburg)

SPECIAL FUNCTIONS

See also 5796, 5798a–b, 6244, 6608, 6609, 6610a–b, 6800.

Mitrinović, Dragoslav S.; Doković, Dragomir Ž. 6065

★Specijalne funkcije [Special functions].

Izdavačko Preduzeće Građevinska Knjiga, Belgrade, 1964. 267 *pp.*

This exposition of special functions is based upon Mitrinović's lectures to engineering and physics students in Belgrade. Each chapter contains the definition and fundamental properties of the class of functions in question, a set of problems, and further references. The book concludes with a collection of tables of numerical values and of graphs. The treatment is more expansive than a simple collection of formulas, but does have a little of this character and could be used in the same way as Magnus and Oberhettinger [*Formeln und Sätze für die speziellen Funktionen der mathematischen Physik*, second edition, Springer, Berlin, 1948; MR **10**, 38] and Jahnke, Emde and Lösch [*Tables of higher functions*, sixth edition, McGraw-Hill, New York, 1960; MR **22** #5140]. The chapter headings are as follows: The gamma and beta functions; Legendre polynomials; Laguerre polynomials; Hermite polynomials; Chebyshev polynomials; Orthogonal polynomials; Bessel functions; The Laplace equation and special functions; Elliptic functions; Numerical tables; Graphs of some special functions. *J. V. Wehausen* (Berkeley, Calif.)

Bhowmic, K. N. [Bhowmick, K. N.] 6066

A generalized Struve's function and its recurrence formula. (Hindi. English summary)

Vijnana Parishad Anusandhan Patrika **6** (1963), 1–11.

Define the generalized Struve function $H_\nu^\lambda(z)$ by the formula

$$(*) \qquad H_\nu^\lambda(z) = \sum_{r=0}^{\infty} \frac{(-1)^r (z/2)^{\nu+2r+1}}{\Gamma(r+3/2)\Gamma(\nu+\lambda r+3/2)}, \qquad \lambda > 0.$$

The author discusses the absolute convergence of the series on the right-hand side of (*) and then proves nine recurrence relations. The recurrence relations can also be found stated in the author's paper [Gaṇita **14** (1963), 89–97; MR **29** #4922]. *W. A. Al-Salam* (Lubbock, Tex.)

Smith-White, W. B.; Buchwald, V. T. 6067

A generalization of $z!$.

J. Austral. Math. Soc. **4** (1964), 327–341.

Authors' summary: "A generalized factorial function $(z:k)!$ is defined as an infinite product similar to the Euler product for $z!$, but with the sequences of integers replaced by the roots of $F(z) = \sin \pi z + k\pi z$. It is proved that, apart from poles in $\mathscr{R}(z) < 0$, $(z:k)!$ is analytic in both variables, and that $F(z)$ may be expressed in the form $F(z) = \pi z/(z:k)!(-z:k)!$. As $|z| \to \infty$, it is shown that the function satisfies a Stirling formula $(z:k)! \sim \sqrt{(2\pi z)} z^z e^{-z}$."

{Reviewer's remarks: This paper gives a concise and rigorous demonstration of the results announced in the summary. As expected, most of the properties of $(z:k)!$ are analogues of the corresponding features of the gamma function. For a special case see #6068 below.}

T. Erber (Chicago, Ill.)

Doran, H. E. 6068

Computation of the generalised factorial function.

J. Austral. Math. Soc. **4** (1964), 342–346.

This paper gives some numerical orientation for the special case $(z:1)!$ of the function $(z:k)!$ in the preceding review [#6067]. *T. Erber* (Chicago, Ill.)

Fempl, Stanimir 6069

Sur certaines séries qui jouent un rôle dans la théorie des fonctions doublement périodiques.

Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 70–76 (1962), 15–18.

Es wird gezeigt:

$$\sum q^m/(1-q^m)^2 = K(K-E)/2\pi^2,$$

$$\sum q^m/(1+q^m)^2 = K(E-k'^2K)/2\pi^2,$$

$$\sum q^n/(1+q^n)^2 = K(2E-K)/2\pi^2 - 1/8.$$

Dabei soll m alle ungeraden und n alle geraden natürlichen Zahlen durchlaufen. Beweismethode: Auf die von Schlömilch angegebenen Fourierreihen für sn x, cn x und dn x wird die Parsevalsche Formel angewandt, und zur Berechnung der Integrale über die Quadrate dieser Funktionen wird die Fourierreihe für $\mathrm{sn}^2 x$ gliedweise integriert. *G. Lochs* (Zbl **105**, 285)

Krause, Karl-Heinz 6070

Über die Koeffizienten einer speziellen asymptotischen Entwicklung.

Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe **13** (1964), 439–441.

Recurrence relations and other properties are obtained for

$$\int_0^\infty (t-s^{-2})^n \exp(-st)\sinh(2t^{1/2})\,dt.$$

T. E. Hull (Toronto, Ont.)

Mitrinović, Dragoslav S.; 6071
Pop-Stojanović, Zoran R.

About integrals expressible in terms of hyperelliptic integrals. (Serbo-Croatian summary)

Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hrvatske Ser. II **18** (1963), 235–239.

Integrals of the form $J=\int_{-\pi}^{+\pi}(\cos nx)(P(\cos x))^{1/2}\,dx$ (n a natural number), where $P(t)=a_0t^{2r}+a_1t^{2r-1}+\cdots+a_{2r}$, $a_0, a_1, a_2, \cdots, a_{2r}$ real numbers with $a_{2r}\neq 0$, are shown to be expressible as hyperelliptic or elliptic integrals depending on the natural number r.

M. D. Friedman (San Jose, Calif.)

Varma, V. K. 6072

On some infinite integrals involving the E-function of MacRobert and operational images.

Bull. Calcutta Math. Soc. **53** (1961), 185–192.

Chatterjea, S. K. 6073

New class of polynomials.

Ann. Mat. Pura Appl. (4) **65** (1964), 35–48.

The new class of polynomials is defined by a Rodrigues type formula:

$$M_n(x,k) = x^{(2-k)n}k^{-n}e^{kx}D^n(x^{kn}e^{-kx}), \qquad k=2,3,\cdots.$$

where k is a parameter. These become the Bessel polynomials when $k=2$. An explicit formula is derived for $M_n(x,k)$. For $m_n(x,k)=x^nM_n(1/x,k)$, it is shown that $(-k)^nm_n(x/k,k)=n!L_n^{(-kn-1)}(x)$, where $L_n^{(\alpha)}$ is the Laguerre polynomial. The differential equation and operational formulas for $m_n(x,k)$ are derived.

It is shown that $M_n(x,3)=y_n(x,n+2,3)$, where $y_n(x,a,b)$ is the generalized Bessel polynomial.

A. E. Danese (Buffalo, N.Y.)

Janić, Radovan R. 6074

Sur les fonctions de Bessel modifiées de première espèce d'ordre entier de plusieurs variables.

Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 122–129 (1964), 27–29.

As a generalization of Jekhowsky's definition [J. Math. Pures Appl. (9) **41** (1962), 319–337; MR **26** #3940], the author introduces the modified Bessel function of the first kind in n variables by the coefficients of the series

$$\exp\left\{\sum_{m=1}^{n}\frac{x_m}{2}(u^m+u^{-m})\right\} = \sum_{k=-\infty}^{+\infty} I_k(x_1,x_2,\cdots,x_n)u^k.$$

Then the author discusses several properties of the function $I_k(x_1,\cdots,x_n)$, e.g., integral representations, recursion formulas, etc., and announces the addition theorem and the differential equations which are satisfied by the function I_k for $n=3,4$. The demonstrations and further development will be published in Mat. Vesnik **1** (1964).

S. Hitotumatu (Tokyo)

Olver, Frank W. J. 6075

Error bounds for asymptotic expansions, with an application to cylinder functions of large argument.

Asymptotic Solutions of Differential Equations and Their Applications (Proc. Sympos., Math. Res. Center, U.S. Army, Univ. Wisconsin, Madison, Wis., 1964), pp. 163–183. *Wiley, New York*, 1964.

In several recent papers [Proc. Cambridge Philos. Soc. **57** (1961), 790–810; MR **24** #A313; J. Soc. Indust. Appl. Math. **11** (1963), 748–772; MR **28** #4190; ibid. **12** (1964), 200–214; MR **29** #306] the author gave sharp estimates for the error bounds in asymptotic expansions of solutions of differential equations. He now reviews his methods and applies them to Hankel's asymptotic expansion of Bessel functions. He states that the results are well-adapted to the control of accuracy in the construction of general-purpose automatic computing routines for the cylinder functions.

A. Erdélyi (Edinburgh)

Singh, S. P. 6076

Certain recurrence relations.

Ganita **13** (1962), 1–8.

In this paper the author investigates certain recurrence relations satisfied by the Fourier kernels $\bar\omega_{\nu,\lambda,\xi}(x)$ [K. P. Bhatnagar, Acad. Roy. Belg. Bull. Cl. Sci. (5) **39** (1953), 42–69; MR **14**, 977] and $W_{\nu,\lambda}^{*}(x)$ and $W_{\nu,\lambda,\xi}^{*}(x)$ [V. K. Verma, Ph.D. Dissertation (unpublished)]. He also calculates the $\bar\omega_{\nu,\lambda,\xi}(x)$ transform of the derivative of a function.

The line of treatment is the same as followed by the reviewer in one of his earlier papers [Bull. Calcutta Math. Soc. **1958**, suppl., 76–93; MR **25** #400].

V. P. Mainra (Pilani)

Keane, A.; Clancy, B. E. 6077

Doppler broadened contour functions in the complex domain.

J. Austral. Math. Soc. **4** (1964), 354–362.

Starting from the known definition of the Doppler broadened contour functions (Voigt profiles), $\Psi(x,t)$ and $\Phi(x,t)$, and the differential equations for them, as well as for the function $\chi(x,t)=\Psi+i\Phi$, the authors deduce integrals representing them, the asymptotic expansions for the upper and lower half-planes being added. Then, taking a contour consisting of the real axis and the semicircle ($R\to\infty$) in the upper half-plane, they calculate many integrals taken along the real axis, especially for the functions χ^n, $x^k\chi^n$, $\chi^n/(x-a)$, $\chi^n/(x^2+a^2)$.

Z. Janković (Zagreb)

Ilieff, L. [Iliev, Ljubomir] 6078

Turánsche Ungleichungen.

C. R. Acad. Bulgare Sci. **17** (1964), 693–696.

An infinite sequence of real numbers $\{\alpha_n\}$ is called a sequence of the first kind if (A) for any polynomial $a_0+a_1z+\cdots+a_nz^n$ having only real zeros it follows that (B) $\alpha_0a_0+\alpha_1a_1z+\cdots+\alpha_na_nz^n$ has only real zeros.

If in (A) the zeros are all of one sign or zero, then $\{\alpha_n\}$ is called a sequence of the second kind.

Pólya and Schur [J. Reine Angew. Math. **144** (1914), 89–113] have shown that sequences of the first and second kinds satisfy Turán's inequality: $\alpha_n^2-\alpha_{n-1}\alpha_{n+1}\geq 0$, $n=1,2,\cdots$. In the present paper, it is shown that sequences of the first kind satisfy

$$\alpha_n^2-\alpha_{n-2}\alpha_{n+2}\geq 0, \qquad n=2,3,\cdots.$$

$$\left(\frac{\alpha_{n-2}}{\alpha_n}\right)^2-\left(\frac{\alpha_{n-2}}{\alpha_n}\right)^2\leq 0, \qquad n=3,4,\cdots.$$

An entire function

$$\varphi(z)=\alpha_0+\alpha_1 z+\frac{\alpha_2 z^2}{2!}+\cdots$$

with only real zeros of the same sign is called a function of type I if either $\varphi(z)$ or $\varphi(-z)$ can be represented in the form:

$$\varphi(z)=\frac{\alpha_r}{r!}z^r e^{\gamma z}\prod_{k=1}^{\infty}(1+\gamma_k z), \qquad \alpha_r\neq 0,\ \gamma\geq 0,\ \gamma_k>0.$$

An entire function

$$\psi(z)=\beta_0+\beta_1 z+\frac{\beta_2 z^2}{2!}+\cdots$$

with only real zeros is called a function of type II if

$$\psi(z)=\frac{\beta_r}{r!}z^r e^{-\gamma z^2+\delta z}\prod_{k=1}^{\infty}(1+\delta_k z)e^{-\delta_k z},$$

$\beta_r,\delta,\delta_1,\delta_2,\cdots$ real, $\gamma\geq 0$.

Pólya and Schur [loc. cit.] have shown that $\{\alpha_n\}$ is a sequence of the first kind if and only if φ is a function of type I, and $\{\beta_n\}$ is a sequence of the second kind if and only if ψ is a function of type II. It follows readily that if $f_k(z)$, $k=1,2,\cdots,s$, are of type II and

$$\text{(C)}\qquad F(z)=\prod_{k=1}^{s}f_k(x_k z)=\sum_{n=0}^{\infty}R_n(x_1,x_2,\cdots,x_s)\frac{z^n}{n!},$$

then $\{R_n\}$ is a sequence of the second kind, where R_n is a polynomial in the real variables $x_1,x_2,\cdots,x_s$. If $f_k(z)$ are of type I, then $\{R_n\}$, with $x_1,x_2,\cdots,x_s$ assuming values of one sign only, is of the first kind.

Using known representations of the form (C) for some of the classical orthogonal polynomials, sequences of these polynomials are classified as being of first or second kind.

A. E. Danese (Buffalo, N.Y.)

Kishore, Nand 6079

The Rayleigh polynomial.

Proc. Amer. Math. Soc. **15** (1964), 911–917.

The Rayleigh function $\sigma_{2n}(\nu)$ and the Rayleigh polynomial $\phi_{2n}(\nu)$ are defined by

$$\sigma_{2n}(\nu)=\sum_{m=1}^{\infty}(j_{\nu,m})^{-2n}$$

$$(n=1,2,3,\cdots),$$

$$\phi_{2n}(\nu)=4^n\prod_{k=1}^{n}(\nu+k)^{[n/k]}\cdot\sigma_{2n}(\nu),$$

where the $j_{\nu,m}$ are the zeros of $z^{-\nu}J_\nu(z)$. Making use of some properties of $\sigma_{2n}(\nu)$ proved in an earlier paper [same Proc. **14** (1963), 527–533; MR **27** #1633], the author now obtains the following results. The degree d_n of $\phi_{2n}(\nu)$ is determined by

$$d_n=1-2n+\sum_{k=1}^{n}\left[\frac{n}{k}\right].$$

Put $\phi_{2n}(\nu)=\sum_1^{d_n}a_{nk}\nu^k$; then the a_{nk} are positive integers. It is proved that the real roots of $\phi_{2n}(\nu)=0$ lie in the interval $(-n,-2)$. Finally, the author obtains some congruences satisfied by $\phi_{2n}(\nu)$. The general formulas are rather complicated; the following special cases may be noted:

$$\phi_{2n}(\nu)\equiv 2\phi_{2n-2}(\nu)\pmod{\nu+n-1},$$

where n is an odd prime,

$$(-1)^{n-1}(n-1)!\,\phi_{2n}(\nu)\equiv\prod_{r=1}^{[n/2]}(\nu+r)^{[n/r]-1}\pmod{\nu+n}.$$

L. Carlitz (Durham, N.C.)

Medvedev, V. A. 6080

On the convergence of orthogonal series of Laguerre, Hermite and Jacobi. (Russian)

Dokl. Akad. Nauk SSSR **151** (1963), 1031–1034.

The Laguerre, Hermite and Jacobi polynomials are shown to be complete with respect to the set of functions $\varphi(x)$ which, together with their first $k-1$ derivatives, are absolutely continuous and whose kth derivative is in L_2, all for the interval (a,b) appropriate for the respective polynomials, and for which the norm is taken as $\|\varphi\|^2=\int_a^b\sum_{m=0}^{k}|d^m\varphi/dx^m|^2\,dx$. *R. G. Langebartel* (Urbana, Ill.)

Rangarajan, S. K. 6081

On some infinite series involving Appell-polynomials and the functions $F(z,\alpha)$. I.

Proc. Indian Acad. Sci. Sect. A **60** (1964), 153–158.

The identity

$$\sum\psi_r(x)F(y,\alpha+r)\frac{z^r}{r!}=\sum\psi_r(0)F(y+zx,\alpha+r)\frac{z^r}{r!}$$

is established, where $\psi_r(x)/r!$ is the Appell polynomial defined by $\psi_n(x)=\sum_{r=0}^{n}\binom{n}{r}\psi_r(0)x^{n-r}$ and $F(z,\alpha)$ satisfies the functional relation

$$\frac{\partial}{\partial z}F(z,\alpha)=F(z,\alpha+1)$$

(see C. Truesdell, *An essay toward a unified theory of special functions* [Princeton Univ. Press, Princeton, N.J., 1948; MR **9**, 431] for a detailed treatment of this functional equation).

The foregoing identity is used to sum a large number of series. For example:

$$\sum\frac{n!}{(\alpha+1)_n}L_n^{\alpha}(x)L_n^{\beta-n}(y)z^n=$$

$$e^{-yz}(1+z)^{\beta}\Phi_2\left(-\beta,\alpha+1,\frac{xz}{1+z};zyz\right),$$

where L_n^{α} is the Laguerre polynomial and Φ is a confluent hypergeometric function.

A. E. Danese (Buffalo, N.Y.)

ORDINARY DIFFERENTIAL EQUATIONS

See also 6196, 6205a–b, 6266, 6308, 6341, 6346, 6347, 6349, 6629, 6630, 6631, 6971, 6978, 6982, 6988.

Campbell, Robert; Reeb, Georges 6082

★Équations différentielles.

Formulaire de Mathématiques à l'Usage des Physiciens et des Ingénieurs, Fasc. VI. Centre d'Études Mathématiques en Vue des Applications. Institut Henri Poincaré.

Centre National de la Recherche Scientifique, Paris, 1964. 78 *pp*.

This booklet consists of a summary of results from the theory of ordinary differential equations intended for the use of physicists and engineers. After an introduction dealing with elementary methods of integration, Part I covers existence theorems, linear differential equations, and eigenvalue problems. Part II deals with differential equations in the complex domain. The material up to this point may be found in the well-known treatise by Ince. However, Part III by Reeb contains some results not readily available elsewhere. It covers dynamical systems, topological theory of 2-dimensional autonomous systems, application of fixed-point theorems, and perturbation procedures. {The book contains several printing errors.}

The early sections seem based on a traditional course in differential equations without regard for applications. Thus Clairaut's equation is included, but there is no mention of numerical methods or dependence of solutions on parameters. One could have wished for more sections like Part III, dealing with stability, singular perturbation problems, and so on. *W. A. Coppel* (Canberra)

Bandić, I. [Bandić, Ivan] 6083

Sur quelques nouvelles classes d'équations différentielles intégrables non-linéaires du premier et du deuxième ordre.

Bol. Soc. Mat. São Paulo **15** (1960), 81–93 (1964).

The second-order differential equation

$$(1)\quad f(y, y')y'' + \phi(y, y') = e^{\beta x}\psi(y, y') \qquad (\beta = \text{const}),$$

where as functions of y, y' the functions f, ϕ and ψ are homogeneous of respective degrees m, $m+1$ and n, is shown to be reducible to a first-order Abel equation of the form

$$(2)\qquad u' = a_0(x)u^3 + a_1(x)u^2.$$

In particular, equation (2) is utilized in determining new conditions of integrability for certain second-order nonlinear differential equations, as well as for the specific solution of some first-order equations that are obtainable from (1) by a suitable change of variable in case $\beta = 0$. *W. T. Reid* (Norman, Okla.)

Bandić, Ivan 6084

Sur le critère d'intégrabilité d'une classe d'équations différentielles non-linéaires du deuxième ordre qui apparaît dans la physique théorique.

Acad. Roy. Belg. Bull. Cl. Sci. (5) **50** (1964), 275–286.

L'auteur trouve quelques formes de l'équation $y'' + f(x)y' = \varphi(x)\Theta(y)$ apparaissant dans divers problèmes de la physique théorique et résolubles au moyen des quadratures. *M. Ráb* (Brno)

Dudašková, K. 6085

A remark on the transformation of solutions of linear, first-order differential equations. (Slovak)

Acta Fac. Natur. Univ. Comenian. **9**, 99–102 (1964).

L'auteur résout le problème suivant à peu près évident: Dans quelles conditions les intégrales de l'équation $y' + p(t)y = q(t)$ sont de la forme $y = Z(t)Y[X(t)]$, Z étant une intégrale de l'équation $Y' + P(T)Y = Q(T)$ et Z, X étant des fonctions convenablement choisies.

{Dans le texte il y a beaucoup de fautes d'impression.} *M. Ráb* (Brno)

Baranov, A. V. 6086

A new method of solving differential equations of the type

$$f_n(x)\frac{d}{dx}f_{n-1}(x)\frac{d}{dx}\cdots f_1(x)\frac{dy}{dx} - y(x) = f(x).$$

(Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 920–926.

The general solution of this equation is given in the form of a series. The terms of the series can be found by a recurrence formula. The expansion is valid under sufficiently weak conditions: boundedness of certain functions involving $f, f_1, f_2, \cdots, f_n$. It is shown that, in certain cases, the partial sums are successively the upper and lower bounds of the exact particular solution. The same method of integration was employed by P. A. Murav'ev [Bul. Inst. Politehn. Bucureşti **24** (1962), no. 3, 13–34; MR **27** #372] for a more general differential equation. *D. Ž. Djoković* (Belgrade)

Iwano, Masahiro 6087

Asymptotic solutions of Whittaker's differential equation as the moduli of the independent variable and two parameters tend to infinity.

Japan. J. Math. **33** (1963), 1–92.

Consider Whittaker's differential equation

$$\frac{d^2W}{dt^2} + \left[-\frac{1}{4} + \frac{k}{t} + \frac{\frac{1}{4} - n^2}{t^2}\right]W = 0$$

for large complex values of t and of the two parameters k and n. The author's aim is to solve this equation by means of asymptotic series that are valid as all three variables t, k and n tend to infinity. The nature of the series depends decisively on the relative sizes of these three variables in the passage to the limit. The author succeeds in dividing the neighborhood of the point $t = \infty$, $k = \infty$, $n = \infty$ of the (t, k, n)-space into a finite number of subdomains such that in each of them a pair of formally linearly independent formal series solutions can be constructed. It is then shown that these formal series are asymptotic expansions of true solutions, at least in sufficiently narrow subsectors of the respective subdomains. Since each subdomain can be covered by finitely many such sectors, the neighborhood of $t = k = n = \infty$ is completely covered by a finite set of domains in each of which the differential equation can be

olved by asymptotic series. There remains the problem of inding the linear relations between these several expansions. This "connecting problem" is not discussed in the paper. In fact, in view of the complexity of the situation, as revealed by the author's analysis, an exhaustive description of the connecting relations may well be an unrewarding, if not an impossible, task. There are altogether no less than 46 subdomains, each of which requires a separate analysis, and every one of these subdomains must be subdivided into an unspecified number of sectors.

The method of the paper resembles that of a paper by the author and Sibuya [Kôdai Math. Sem. Rep. **15** (1963), -28; MR **26** #6530]. In that article differential equations with one parameter only were considered. The presence of two independent parameters leads to enormous complications.

The construction of the subregions is based on a geometric argument involving convex polyhedra that may be regarded as three-dimensional generalizations of the Newton-Puiseux polygons. *W. Wasow* (New York)

Ahundov, A. M.; Toraev, A. 6088

On the solutions of a differential equation. (Russian)

Izv. Akad. Nauk Turkmen. SSR Ser. Fiz.-Tehn. Him. Geol. Nauk **1964**, no. 1, 21–23.

Consider the differential equation

$$(1)\qquad y''+P_1(x)y'+P_2(x)y=0,$$

where $P_1(x)$ and $P_2(x)$ are continuous functions on $x_0\le x<\infty$. The authors prove five theorems concerning oscillatory solutions of (1), the basic theorem being the following: If the coefficients of (1) are of fixed signs and if $P_1(x)<0$, $P_2(x)>0$ or $P_1(x)<0$, $P_2(x)<0$, then between two consecutive roots of a solution of (1) there are at most two roots of any other solution. The authors claim that their theorems remain in force also for the general equation of order three and for the case where $P_1(x)$ vanishes at certain points [cf. also L. D. Nikolenko, Dokl. Akad. Nauk SSSR **114** (1957), 483–485; MR **19**, 960; M. I. El'šin, ibid. **68** (1949), 813–816; MR **11**, 247; N. V. Adamov, Mat. Sb. (N.S.) **23** (**65**) (1948), 187–228; MR **10**, 250; V. A. Kondrat'ev, Trudy Moskov. Mat. Obšč. **8** (1959), 259–281; MR **21** #3628]. *E. Leimanis* (Vancouver, B.C.)

Aliev, R. G. 6089

On concrete criteria for estimates of lengths of subcritical intervals. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, no. 5 (42), 3–7.

On démontre deux théorèmes concernant le problème différentiel

$$L(y)=y^{(iv)}-g_3(x)y'''-g_2(x)y''-g_1(x)y'-g_0(x)y=f(x),$$

$$\begin{aligned}
&y(\alpha)=A,\quad y'(\alpha)=B,\quad y''(\alpha)=C,\quad y(\beta)=D,\\
&y(\alpha)=A,\quad y(\beta)=B,\quad y'(\beta)=C,\quad y''(\beta)=D,\\
&y(\alpha)=A,\quad y'(\alpha)=B,\quad y(\beta)=C,\quad y'(\beta)=D,\\
&y(\alpha)=A,\quad y(\gamma)=B,\quad y'(\gamma)=C,\quad y(\beta)=D,\\
&y(\alpha)=A,\quad y'(\alpha)=B,\quad y(\gamma)=C,\quad y(\beta)=D,\\
&y(\alpha)=A,\quad y(\gamma)=B,\quad y(\beta)=C,\quad y'(\beta)=D,\\
&y(\alpha)=A,\quad y(\gamma)=B,\quad y(\delta)=C,\quad y'(\beta)=D
\end{aligned}$$

$$(\alpha<\gamma<\delta<\beta),$$

c'est-à-dire la question d'unicité de ses solutions. La fonction $r(s)$ définit l'intervalle maximal $[s, r(s))$ tel que pour tout couple $\alpha,\beta\in[s,r(s))$ chacun des problèmes cités ait la solution unique. Les théorèmes donnent les conditions nécessaires et suffisantes d'existence des solutions uniques, utilisant la fonction de Cauchy $K(x,s)$ de l'opération $L[y]$, l'équation intégrale de Volterra ainsi que quelques résultats de N. V. Azbeleff, Ž. B. Caljuk, J. Mikusiński et L. D. Nikolenko. *M. Bertolino* (Belgrade)

Greguš, M. 6090

Über das Randwertproblem der n-ten Ordnung in m-Punkten. (Slovak and Russian summaries)

Acta Fac. Natur. Univ. Comenian. **9**, 49–55 (1964).

The author considers the linear operator of the form

$$Ly\equiv p_0(x)y^{(n)}+p_1(x)y^{(n-1)}+\cdots+p_n(x)y$$
$$(n\ge 2,\ p_0(x)\ne 0 \text{ in } \langle a_1,a_m\rangle)$$

with the boundary conditions $\bigcup_i y=0$ $(i=1,\cdots,n)$ prescribed at m distinct points $a_1<a_2<\cdots<a_m$. The $\bigcup_i y$ are linearly independent combinations of $y(a_1),\cdots,y^{(n-1)}(a_1),\ y(a_2),\cdots,y^{(n-1)}(a_2),\cdots,y(a_m),\cdots,y^{(n-1)}(a_m)$. Under the assumption that $u\equiv 0$ is the only solution of the corresponding homogeneous boundary-value problem, the author constructs the so-called particular Green's functions $G_k(x,\xi)$ which represent the generalization of the classical Green's functions. The solution of the nonhomogeneous problem $Ly=r(x)$, $\bigcup_i y=0$ $(i=1,\cdots,n)$ is given by the formula

$$y(x)=\sum_{k=1}^{m-1}\int_{a_k}^{a_{k+1}}G_k(x,\xi)r(\xi)\,d\xi.$$

M. Zlámal (College Park, Md.)

Hukuhara, Masuo 6091

Une propriété de l'application $f(x,y,y',\cdots,y^{(n)})$.

Funkcial. Ekvac. **5** (1963), 135–144.

L'auteur prend pour point de départ le fait suivant: Une fonction $\varphi\in C^n[a,a']$ est déterminée d'une manière unique si l'on donne la dérivée d'ordre n, ainsi que ses valeurs $b_1,\cdots,b_n$ en n points $a_1,\cdots,a_n$ de l'intervalle $[a,a']$; si la suite de fonctions $\varphi_\nu\in C^n[a,a']$ est telle que la suite $\{\varphi_\nu^{(n)}\}$ converge uniformément vers $q(x)$ sur $[a,a']$ et $\{\varphi_\nu(x)\}$ converge vers $b_1,\cdots,b_n$ respectivement pour les valeurs $a_1,\cdots,a_n$ de x, alors la suite $\{\varphi_\nu\}$ converge dans l'espace $C^n[a,a']$. Soit maintenant $f(x,y_0,y_1,\cdots,y_n)$ une fonction sommable par rapport à x, continue en $(y_0,y_1,\cdots,y_n)$ et telle que

$$|f(x,y_0,\cdots,y_n)-f(x,z_0,\cdots,z_n)|\le\sum_{k=0}^{n-1}P_{n-k}(x)|y_k-z_k|,$$

$$\frac{f(x,y_0,\cdots,y_n)-f(x,z_0,\cdots,z_n)}{y_n-z_n}\ge\lambda>0,$$

où $P_k(x)$ sont des fonctions sommables sur $[a,a']$. Supposons de plus $\|P_1\|l+\|P_2\|l^2+\cdots+\|P_n\|l^n<\lambda$ où $\|P_k\|=l^{-1}\int_a^{a'}|P_k(x)|\,dx$, $l=a'-a$. Alors, pour assurer la convergence de la suite $\varphi_\nu\in C^{n-1}[a,a']$ dans cet espace, il suffit que l'on ait la convergence de cette suite en n points de $[a,a']$ et la convergence en moyenne (c'est-à-dire en $L[a,a']$) de la suite $\{f(x,\varphi_\nu(x),\cdots,\varphi_\nu^{(n)}(x))\}$, avec $\varphi_\nu^{(n-1)}$ absolument continues. La fonction limite φ satisfait à

l'équation $f(x, y, y', \cdots, y^{(n)}) = \varphi(x)$, où $\varphi(x)$ est la limite de la suite $\{f(x, \varphi_\nu(x), \cdots, \varphi_\nu^{(n)}(x))\}$ dans $L[a, a']$.

C. Corduneanu (Iaşi)

Murphy, E. L.; Good, R. H., Jr. 6092

WKB connection formulas.

J. Math. and Phys. **43** (1964), 251–254.

The authors discuss the WKB connection formulas which relate the WKB approximate solutions of $\psi''(x) + p^2(x)\psi(x) = 0$ on either side of a simple zero of $p^2(x)$. They mention the well-known fact that the connection formulas may be derived by the asymptotic expansion of the uniform approximate solutions in terms of Bessel functions of order $\frac{1}{3}$ (or Airy functions) due to R. E. Langer. They conclude from this derivation that the connection formulas preserve linear combinations of solutions and hence can be used to connect approximate solutions across several successive turning points. These are well-known results.

C. H. Wilcox (Madison, Wis.)

Bellman, Richard 6093

Oscillatory and unimodal properties of solutions of second order linear differential equations.

Boll. Un. Mat. Ital. (3) **19** (1964), 306–310.

Author's summary: "In previous papers [same Boll. (3) **12** (1957), 520–523; MR **19**, 885; ibid. (3) **16** (1961), 164–166; MR **24** #A290] we have shown how to derive fundamental properties of the solutions of

$$u'' + p(x)u = 0$$

directly from the fact that it is the Euler equation associated with the functional

$$J(u) = \int_0^a [u'^2 - p(x)u^2]\,dx.$$

Here we wish to demonstrate a Sturm oscillation theorem and the unimodal property of the solution in analogous fashion, under the assumption that $p(x) > 0$."

Alekperov, N. I. 6094

A boundary-value problem with a complex weight function. (Russian)

Dokl. Akad. Nauk SSSR **159** (1964), 479–481.

Dans l'espace $L^2(-\infty, +\infty)$, on considère l'équation différentielle $-y'' + q(x)y = \lambda p(x)y$, où $q(x)$ et $p(x) = q_1(x) + iq_2(x)$ sont des fonctions sommables sur tout intervalle fini de l'axe réel. L'équation considérée s'écrit sous la forme $Ay = \lambda p(x)y$. Si l'on a $q(x) \geq 1$, l'opérateur A^{-1} est auto-adjoint et positif. En posant $y = A^{-1/2}z$, l'équation devient $z = \lambda Lz$. Supposons de plus que les fonctions $q_1(x)$ et $q_2(x)$ sont non-négatives et bornées et que

$$\lim_{|x|\to\infty} \left\{\frac{q(x)}{|x|^\alpha}\right\} \geq C > 0$$

pour $3\alpha \geq 2$. Alors les "autofonctions" de L associées aux points $\neq 0$ du spectre constituent un système complet dans l'espace des valeurs de L. Dans le cas $q_1(x) > 0$, $q_2(x) > 0$ le système est complet dans $L^2(-\infty, +\infty)$. Par "autofonctions" il faut comprendre les autofonctions au sens habituel ainsi que les fonctions attachés aux autovaleurs multiples dans un sens précisé. Voir aussi le travail de V. B. Lidskiĭ [Trudy Moskov. Mat. Obšč. **9** (1960), 45–79; MR **25** #3984].

C. Corduneanu (Iaşi)

Brauer, Fred 6095

On the completeness of biorthogonal systems.

Michigan Math. J. **11** (1964), 379–383.

In this paper the author obtains a Hilbert space theorem dealing with biorthogonal sequences to combine with the asymptotic estimates to yield a proof of completeness of the eigenfunctions of a non-selfadjoint boundary-value problem. In particular, he states Theorem 1: If $\{\phi_n\}$ is a complete orthonormal sequence, and if $\{x_n\}$, $\{y_n\}$ is a normalized biorthogonal system such that

$$(1) \qquad \sum_{n=1}^\infty \|\phi_n - x_n\|^2 < \infty, \qquad \sum_{n=1}^\infty \|\phi_n - y_n\|^2 < \infty,$$

then the system $\{x_n\}$, $\{y_n\}$ is complete. He then uses this theorem to prove Theorem 2: The eigenfunctions of the boundary-value problem $Lx = \lambda x$, $Ux = 0$, with regular boundary conditions, together with the eigenfunctions of the adjoint problem $L^+y = \lambda y$, $U^+y = 0$, can be chosen so as to form a normalized biorthogonal system that is complete in $L^2(a, b)$.

{The author has communicated to the reviewer that the proof of Theorem 1 must be modified.}

F. M. Stein (Fort Collins, Colo.)

McKelvey, Robert W. 6096

Asymptotic solutions and indefinite boundary value problems.

Asymptotic Solutions of Differential Equations and Their Applications (Proc. Sympos., Math. Res. Center, U.S. Army, Univ. Wisconsin, Madison, Wis., 1964), pp. 109–127. *Wiley, New York*, 1964.

The author considers several differential boundary-value problems, each of which in a certain sense is self-adjoint relative to an indefinite inner product. Pertinent features revealed in each case by asymptotic studies are described, and then there is presented an abstract vector-space setting for the problems which provides an explanation for their common properties. There is pointed out the significance in the abstract setting of some individual features observed in the separate problems, as well as phenomena present in the special cases which as yet are unexplained by any general theory.

W. T. Reid (Norman, Okla.)

Dautov, M. A.; Muratov, L. M. 6097

Asymptotic representation of solutions of a first-order polynomial differential equation. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 4 (41), 61–68.

The authors study a polynomial equation of the form $f(x, y, y') = \sum_{m,n,p} a_{mnp}(x)x^m y^n (y')^p = 0$. They determine some rather complicated conditions under which it is possible to find constants α_1, c_1 such that this equation has a solution of the form $y = c_1x^{\alpha_1} + o(x^{\alpha_1})$.

F. Brauer (Madison, Wis.)

Bureau, F. J. 6098

Differential equations with fixed critical points.

Ann. Mat. Pura Appl. (4) **64** (1964), 229–364.

Consider the differential equation (1) $y' = R(x, y, y')$, where $'$ denotes derivation with respect to the complex variable x and R denotes a rational function in y and y' whose coefficients $a_i(x)$ are analytic in x. The solution $y(x)$ of (1) with initial data x_0, y_0, y_0' is an analytic function whose singularities depend, in general, not only on the singularities of the $a_i(x)$ but also on the initial data y_0, y_0'. When (1) is such that the singularities of $y(x)$ depend only on those of the $a_i(x)$, then it is said to have fixed singularities. In the present paper the author determines all forms of R for which (1) has fixed singularities, using a method that is somehow inspired by the classical work of Painlevé. The results are too lengthy to be given here.

M. M. Peixoto (Providence, R.I.)

Čerdak, B. M. 6099

Some properties of systems of differential equations. (Russian)

Leningrad. Gos. Ped. Inst. Učen. Zap. **238** (1962), 149–156.

The author considers a system of differential equations:

$$(1) \qquad \frac{dy}{dt} = A(t)y + f(t),$$

where $f = (f_1, \cdots, f_n)$ and $A(t) = (a_{ij}(t))$, $i, j = 1, \cdots, n$, is a continuous $n \times n$ matrix. The author examines the conditions which $A(t)$ must satisfy in order that the boundedness of all solutions of (1) for any (continuous bounded) vector $f(t)$ with (2) $f_{k+1}(t) = \cdots = f_n(t) = 0$ implies the boundedness of all solutions of (1) for any continuous bounded vector $f(t)$. Let $B(t)$ be the $(n-k) \times (n-k)$ matrix $B(t) = (a_{ij}(t))$, $i, j = k+1, \cdots, n$, and $C(t)$ the $k \times (n-k)$ matrix $C(t) = (a_{ij}(t))$, $i = 1, \cdots, k$, $j = k+1, \cdots, n$. (I) In order that the boundedness on the positive t-axis of all solutions of (1) for any continuous and bounded vector function f satisfying (2) implies the boundedness on the positive t-axis of all solutions of (1) for any continuous and bounded vector f it is sufficient that the system $dz/dt = B(t)z + g(t)$ have at least one bounded solution for every continuous and bounded vector function $g(t) = (g_{k+1}, \cdots, g_n)$. (II) This condition is also necessary if $C(t) \equiv 0$. (III) Under the hypotheses of (I) and for every solution $x(t)$ of $dx/dt = A(t)x$, an inequality of the form $\|x(t)\| \leq Ne^{-\nu(t-t_0)}\|x(t_0)\|$, $0 \leq t_0 \leq t < +\infty$, always occurs for some constants N and ν. Results of M. G. Kreĭn [Uspehi Mat. Nauk **3** (1948), no. 3 (25), 166–169; MR **10**, 128] are used.

L. Cesari (Ann Arbor, Mich.)

Conti, Roberto 6100

Una relazione di equivalenza tra matrici e sue applicazioni alle equazioni differenziali lineari.

Math. Notae **19** (1964), 93–98.

Let $\mathscr{M}$ be the linear complex space of $n \times n$ matrices, whose elements are complex functions of t, which are continuous in a fixed open interval (α, β), $-\infty \leq \alpha < \beta \leq +\infty$. Let $\mathscr{L}^+$ be the subset of $\mathscr{M}$ of matrices of class $C^1(\alpha, \beta)$, nonsingular, bounded together with their inverse matrices in (γ, β), for any γ such that $\alpha < \gamma < \beta$. Finally, let $\mathscr{F}^+$ be a linear variety of $\mathscr{M}$ such that if $L \in \mathscr{L}^+$ and $F \in \mathscr{F}^+$, then LF and FL also belong to $\mathscr{F}^+$. The author defines $A \in \mathscr{M}$ and $B \in \mathscr{M}$ to be $\mathscr{F}^+$-similar if there exist $L \in L^+$ and $F \in \mathscr{F}^+$ such that

$$\dot{L} + LA - BL = F, \qquad \dot{L} = dL/dt, \; t \in (\alpha, \beta).$$

The author proves that $\mathscr{F}^+$-similarity is an equivalence relation. For particular choices of $\mathscr{F}^+$ the resulting relation has already been considered; for instance, taking $\mathscr{F}^+$ to be the subspace of $\mathscr{M}$ of absolutely integrable matrices in $(\gamma, +\infty)$, $\mathscr{F}^+$-similarity turns out to be t_∞-similarity, and the author gives a new, simple proof of his own previous result [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Natur. (8) **19** (1955), 247–250; MR **18**, 483; Riv. Mat. Univ. Parma **8** (1957), 43–47; MR **20** #5322]. The author defines two systems $\dot{x} = A(t)x$ and $\dot{y} = B(t)y$ to be $\mathscr{F}^+$-similar if A and B are $\mathscr{F}^+$-similar, and proves that the linear homogeneous differential systems which are uniformly and asymptotically stable form an equivalence class of F^+-similarity, provided that $\mathscr{F}^+$ is taken to be the subspace of $\mathscr{M}$ such that $F \in \mathscr{F}^+$ if and only if the norm of $F(t)$ tends to zero when $t \to \infty$. {Misprint: If (1) and (2) are correctly written, then they should be $\dot{x} = B(t)x$ and $\dot{y} = A(t)y$.}

A. Dou (Madrid)

Flatto, Leopold 6101

Limit cycle studies for circuits containing one Esaki diode.

J. Math. Anal. Appl. **9** (1964), 360–383.

The author studies the existence of limit cycles of a standard single tunnel diode electrical circuit described by

$$L\dot{i} = E - Ri - v,$$

$$C\dot{v} = i - f(v).$$

It is known [J. Moser, IBM J. Res. Develop. **5** (1961), 226–240; MR **23** #B1614] that $f'/C + R/L > 0$ for all v precludes the existence of limit cycles. This paper deals with the case $f'/C + R/L < 0$ for an interval (v_1, v_2). Let P_1, P_2 denote the two points $(v_1, f(v_1))$, $(v_2, f(v_2))$ and P the singular point whose position depends on E. It is shown that if $f'^2 - (1/R - CR/L)f'' > 0$ $[<0]$ at P_i $(i = 1, 2)$ then an unstable [stable] limit cycle grows out of P as it passes through P_i by varying E. A power series for the generated limit cycle is given.

R. Liu (Notre Dame, Ind.)

Halvorsen, Sigbjørn 6102

On the quadratic integrability of solutions of $d^2x/dt^2 + f(t)x = 0$.

Math. Scand. **14** (1964), 111–119.

On a half-open (finite or half-infinite) interval let $f(t)$ and $g(t)$ be real and locally integrable, and possess (possibly) non-integrable singularities at the open end. There are obtained certain comparison theorems for the pair of differential equations (1) $x'' + f(t)x = 0$ and (2) $y'' + g(t)y = 0$. In particular, the author generalizes a result, Theorem 2, p. 513, of R. Bellman [Duke Math. J. **11** (1944), 513–516; MR **6**, 66] and shows that if $|g-f|^{1/2}x \in L^2$ for every solution x of (1), then also $|g-f|^{1/2}y \in L^2$ for every solution y of (2), and (1) and (2) are of the same (Weyl) type, limit-point or limit-circle. In addition, there is derived a condition under which (2) is of the limit-circle type whenever (1) is. Additional limit-circle criteria, together with some results on non-oscillatory equations, are also obtained.

C. R. Putnam (Lafayette, Ind.)

Jean, Michel; Sideriades, Lefteri 6103

Circuits non autonomes du premier ordre.

C. R. Acad. Sci. Paris **256** (1963), 5505–5507.

The content of this note can be described in terms of an example. The equation of a circuit with a nonlinear tube, resistance, capacitance, and a periodic applied voltage is $f(x, y, t) = x + g(y) - F(t) = 0$, where x is charge, $\dot{x} = y$ is current, and $F(t)$ is periodic. This equation defines a surface S in the space (x, y, t). This surface is an integral surface of the system (*) $\dot{x} = y$, $g'(y)\dot{y} = -y + F'(t)$. The curve C_λ obtained by the intersection of S with the plane $t = \lambda$, a constant, is introduced. Projected on the (x, y)-plane, C_λ is an integral curve of (**) $dx/d\tau = y$, $dy/d\tau = -y/g'(y)$. This is an autonomous system (and in this case does not depend on λ) and is related to (*). This relationship is then used, it would appear, to construct geometrically the forced oscillations of the circuit. The method is not clearly explained and its value is dubious.

J. P. LaSalle (Providence, R.I.)

Leighton, Walter 6104

Erratum: "Behavior of solutions of a linear differential equation of second order."

Proc. Nat. Acad. Sci. U.S.A. **52** (1964), 1129.

"In the original article [same Proc. **52** (1964), 830–832; MR **29** #3718] the proof of the theorem is incomplete. The hiatus was discovered just as the article appeared in print. Whether or not the stated theorem is valid remains, at least for the time being, an open question."

Morris, G. R. 6105

Unbounded solutions of a second-order differential equation with non-negative damping.

Contributions to Differential Equations **3** (1964), 421–433.

In the differential equation $\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = p(t)$, with everywhere continuous coefficients, assume that $f(x, y) \geq 0$ and that $g(x) = x$ for all large x, while $p(t)$ is periodic. On the basis of the physical interpretations of such differential equations it is tempting to conjecture that every solution, as well as its derivative, remains bounded as $t \to \infty$, except possibly for certain equations close to linear ones with resonance. The author shows that this is not true. He constructs explicit examples of such equations that possess a one-parameter family of solutions for each of which $|x(t)| + |\dot{x}(t)| \to \infty$ as $t \to \infty$. The equation can even be subjected to the additional restrictions that $f(x, y) \leq \varepsilon$, $|x - g(x)| \leq \varepsilon$, where ε is arbitrarily small. The corresponding linear equation $\ddot{x} + x = p(t)$ does not exhibit resonance. The construction requires a detailed, fairly explicit analysis of several auxiliary simpler differential equations from which the counter-example is pieced together.

W. Wasow (New York)

Munteanu, I. 6106

Sur un cycle limite.

Mathematica (Cluj) **4** (**27**) (1962), 293–307.

Consider the differential equation in the plane

$$\varepsilon \frac{dx}{dt} = h(y) - F(x),$$

$$\frac{dy}{dt} = -\varepsilon g(x),$$

where F, g, and h are odd functions, F has a unique positive root and is negative for small positive x and positive for large positive x, $xg(x) > 0$ for $x \neq 0$, and h is monotone and goes to ∞ with y.

Theorem 1: If $\varepsilon > 0$, this sytem has a unique limit cycle Γ_ε which is orbitally stable. Theorem 2: If F, g, and h satisfy monotonicity conditions on certain intervals, then the limit cycles Γ_ε tend to a simple closed curve Γ_0 as ε tends to 0. The proofs are similar to those for related theorems by earlier authors.

B. L. Reinhart (College Park, Md.)

Reizin', L. È. [Reiziņš, L. È.] 6107

Systems of differential equations in local coordinates in the neighbourhood of a closed trajectory. (Russian. Latvian and English summaries)

Latvijas PSR Zinātņu Akad. Vēstis Fiz. Tehn. Zinātņu Sēr. **1964**, no. 1, 59–66.

This paper provides a new proof of the theorem that, in the neighborhood of a closed trajectory to $dx/dt = P(x)$, coordinates (y, s) may be introduced so that the differential equation reads $dy/ds = Q(y, s)$, where Q is periodic in s and $Q(0, s) \equiv 0$ [see Diliberto and the reviewer, *Contributions to the theory of nonlinear oscillations*, Vol. 3, pp. 207–236, Princeton Univ. Press, Princeton, N.J., 1956; MR **18**, 653]. This new proof gives the coordinate transformations more explicitly than do older proofs.

G. Hufford (Boulder, Colo.)

Reizin', L. È. [Reiziņš, L. È.] 6108

Invariant sets of point-wise mappings. (Russian. Latvian and English summaries)

Latvijas PSR Zinātņu Akad. Vēstis Fiz. Tehn. Zinātņu Sēr. **1964**, no. 3, 31–35.

Let the system $dx/dt = P(x)$ have a closed trajectory γ. Following Liapounov, the equation of first variation associated with γ is closely related to an autonomous equation $dv/ds = Bv$, where v has one less dimension than x. The present paper shows that if none of the "multipliers" associated with γ has absolute value one, then there exists a homeomorphism of $v = 0$ with a cross-section of γ which carries trajectories (or pairs of trajectories) of $dv/ds = Bv$ into sets which are invariant under the transformation induced by the motion of the original system as it goes once around γ. This generalizes a theorem of Hadamard [Bull. Soc. Math. France **29** (1901), 224–228] and others.

G. Hufford (Boulder, Colo.)

Kill', I. D. [Kill', I. D.] 6109

On periodic solutions of a certain nonlinear equation.

Prikl. Mat. Meh. **27** (1963), 1107–1110 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1699–1704.

The author studies the question of the existence of periodic solutions of the equation

$$\frac{d^2z}{dt^2} - \frac{2e \sin t}{1 + e\cos t}\frac{dz}{dt} + \frac{\mu}{1 + e\cos t}\sin z = \frac{4e \sin t}{1 + e\cos t}$$

$$(0 \leq e < 1)$$

by the method due to L. Cesari [Active Networks and Feedback Systems (Proc. Sympos., New York, 1960), pp. 545–560, Polytechnic Press of Polytechnic Inst. Brooklyn, Brooklyn, New York, 1961].

The author first derives a sufficient condition for the existence of periodic solutions. Then the existence of a periodic solution is proved for the case when $0 \leq \mu \leq 1$ and $e = 0.6$ [cf. V. V. Beleckiĭ, Iskusstvennye Sputniki Zemli No. 3 (1959), 13–31].

Y. Sibuya (Minneapolis, Minn.)

Reid, William T. 6110

Principal solutions of nonoscillatory linear differential systems.

J. Math. Anal. Appl. **9** (1964), 397–423.

The author presents generalizations of previous results obtained by him [Pacific J. Math. **13** (1963), 665–685; MR **27** #4991] on the principal solution at ∞ of a nonoscillatory linear system and the corresponding distinguished solution of the associated Riccati matrix differential equation. Besides replacing the previously required condition of identical normality by the weaker condition that both the given system and its adjoint be nonoscillatory, he includes necessary and sufficient conditions for the existence of a distinguished solution (formerly only a necessary condition was given). Following an extensive treatment of similar considerations for self-adjoint systems is a section containing two theorems on the general reciprocal of a finite matrix.

J. Stuelpnagel (Baltimore, Md.)

Reissig, Rolf 6111

Stabilität und asymptotisches Verhalten dynamischer Systeme. (Russian, English and French summaries)

Wiss. Z. Humboldt-Univ. Berlin Math.-Natur. Reihe **12** (1962/63), 527–532.

The paper is an exposition of some of the methods available for studying the asymptotic behavior of systems described by an n-dimensional differential equation $\dot{x} = F(x, t)$, where x and $F(x, t)$ are n-vectors and $\dot{x} = dx/dt$. The two principal problems discussed are (1) system stability and (2) stability of solutions. By system stability is meant that solutions are defined in the future and perhaps also that they be bounded in the future. For the most part the methods are those of Liapunov and are illustrated by the study of n coupled oscillators, $\ddot{x} + f(x, \dot{x}, t) + g(x) = e(t)$, where again x is an n-vector.

J. P. LaSalle (Providence, R.I.)

Reissig, R. [Reissig, Rolf] 6112

Ein funktionalanalytischer Existenzbeweis für periodische Lösungen.

Monatsb. Deutsch. Akad. Wiss. Berlin **6** (1964), 407–413.

The system (1) $x' = f(x) + p(t)$, where f is Lipschitzian and p is periodic with period ω, is compared with the linear system (2) $x' = Ax + p(t)$, where the eigenvalues of A all have negative real parts. This is done by setting $f(x) = Bx + f_1(x)$, $A = B + A_1$ and considering the system (3) $x' - Ax = \mu[f_1(x) - A_1x] + p(t)$ for $0 \leq \mu \leq 1$, which reduces to (1) when $\mu = 1$ and to (2) when $\mu = 0$. It is assumed that all solutions of (3) are bounded independently of μ. The author uses a Lyapunov function to deduce the existence of a periodic solution of (2). He then uses a fixed-point theorem for the Banach space of periodic functions to deduce the existence of a periodic solution of (3) for $0 < \mu \leq 1$ and hence the existence of a periodic solution of (1).

F. Brauer (Madison, Wis.)

Sansone, G. 6113

L'équation

$$\ddot{\theta} + f(\theta, \alpha)h(\dot{\theta}) = g(\theta) + p(t).$$

J. Math. Pures Appl. (9) **43** (1964), 149–175.

Continuing his research on "pendulum-type" equations [e.g., same J. (9) **40** (1961), 363–384; MR **25** #3226], the author now considers the solutions of the equation of the title, with forcing term $p(t)$. Under rather weak conditions on f, h, g, p, too variable to be reported in detail here, but much weaker than those considered by Seifert [Ann. of Math. (2) **67** (1958), 83–89; MR **19**, 960; ibid. (2) **69** (1959), 75–87; MR **20** #7132], he establishes the existence of a solution defined for all future time, the existence of a region in the phase plane, bounded by a smooth Jordan curve, such that any solution that touches it cannot leave (and every such solution is defined for all future time), and, for period p, the existence of a periodic solution of the equation and, under somewhat stronger conditions, the asymptotic stability of the periodic solution.

J. J. Schäffer (Pittsburgh, Pa.)

Sansone, G. 6114

Nonlinear differential systems of the third and fourth order.

Differential Equations and Their Applications (*Proc. Conf., Prague*, 1962), pp. 143–165. *Publ. House Czechoslovak Acad. Sci., Prague; Academic Press, New York*, 1963.

An expository paper outlining, with 88 supporting references, the important contributions to date in the area of third- and fourth-order non-linear systems of ordinary differential equations. *J. O. C. Ezeilo* (Ibadan)

Sirčenko, Z. F. 6115

On the existence and properties of an almost periodic solution in standard form in a Hilbert space near an equilibrium point. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 1132–1135.

The equation of the form (1) $dx/dt = \varepsilon X(t, x)$, where ε is a small positive parameter, is studied in a Hilbert space. The author proves theorems on the existence and on the properties of an almost periodic solution of (1) in the neighborhood of the position of equilibrium of the equation $d\xi/dt = \varepsilon X_0(\xi)$, where $X_0(\xi) = \lim_{T\to\infty} (1/T) \int_0^T X(t, \xi)\, dt$.

M. Zlámal (College Park, Md.)

Struble, Raimond A. 6116

A note on periodic solutions of the Duffing equation.

J. Math. Anal. Appl. **9** (1964), 498–501.

This note is concerned with periodic solutions of Duffing's equation

$$\ddot{x} + n^2x - \beta x^3 = \beta F_0 \cos \lambda t,$$

whose period is an integral multiple of λ. They can be obtained by intersecting the solutions with an appropriate line using the reversibility of the above equation.

J. Moser (New Rochelle, N.Y.)

Minorsky, Nicolas 6117

Sur la synchronisation.

C. R. Acad. Sci. Paris **259** (1964), 3421–3422.

It is shown that the stroboscopic method, applied to the equation $\ddot{x}+(a+bx^2)\dot{x}+\omega^2x=\varepsilon\cos t$, where $-a$, b, and ε are small positive constants, yields the principal known results concerning the phenomena of synchronization. No essential novelties appear in the methods used in the discussion. *L. A. MacColl* (New York)

Minorsky, N. [Minorsky, Nicolas] 6118

On some aspects of non-linear oscillations.

Studies in mathematical analysis and related topics, pp. 245–255. *Stanford Univ. Press, Stanford, Calif.*, 1962.

The author considers the phase portrait of non-linear differential equations. The bifurcation points of Liénard's equations are obtained by studying a stroboscopic system, and its phase portrait is studied. He indicates methods of studying the phase portrait of several other autonomous equations. Finally, it is shown how his method can be applied to certain non-autonomous systems. *B. Bernstein* (Washington, D.C.)

Bogdanov, Ju. S. 6119

An application of generalized characteristic numbers for studying the stability of a stationary point. (Russian)

Dokl. Akad. Nauk SSSR **158** (1964), 9–12.

The author considers a system of the form (1) $dx/dt=f(x)$, x an n-vector, f Lipschitzian around the origin, and the origin an isolated equilibrium point of (1). The generalized characteristic numbers in the title are defined in terms of two scalar functions v and d, where among other assumptions, the level surfaces of v fill out a neighborhood of 0, and d is very nearly a metric, except for the condition $d(\gamma_1,\gamma_2)=-d(\gamma_2,\gamma_1)$ for γ_1,γ_2 positive. Given the functions v and d, the author defines two types of generalized characteristic numbers, and shows that these give necessary and sufficient conditions for either the instability or the asymptotic stability of the origin. The characteristic numbers of Lyapunov are given by choosing $v(x)=\|x\|$, $d(\gamma_1,\gamma_2)=\ln(\gamma_1/\gamma_2)$. For linear systems, both generalized characteristic numbers of Bogdanov (with the above choices of v and d) coincide with the characteristic numbers of Lyapunov. For non-linear systems, this is no longer true, and the author determines their relationship in this case as well. *A. Stokes* (Washington, D.C.)

Brayton, R. K.; Miranker, W. L. 6120

A stability theory for nonlinear mixed initial boundary value problems.

Arch. Rational Mech. Anal. **17** (1964), 358–376.

The authors envisage a generalization to certain mixed initial-boundary-value problems in partial differential equations of the criterion for global asymptotic stability of Ljapunov and its converse, proved by Massera [Ann. of Math. (2) **64** (1956), 182–206; MR **18**, 42]. For this purpose, they consider, with every vector-valued function $u(t,x)$ defined in the x-interval $[0,1]$, the vector $[u]$, consisting of three blocks, the first being u itself, the second $u(t,0)$, and the third $u(t,1)$, and analyse equations which may be written in the form $J(u)[u_t]=P_{[u]}$, where J is a diagonal block matrix and P is a functional of the general form $\int_0^1 F(u,u_x)\,dx+\varphi(u)|_{x=0}+\psi(u)|_{x=1}$. They establish conditions under which this equation represents a dynamical system and examine the existence of appropriate Ljapunov functionals in several generic examples. The details cannot be reproduced here. An important application concerns networks with non-linear elements and transmission lines. *J. J. Schäffer* (Pittsburgh, Pa.)

Jean, Michel 6121

Sur un critère d'existence et de stabilité de solutions périodiques: Application à des problèmes du premier et du second ordre.

C. R. Acad. Sci. Paris **259** (1964), 2058–2060.

A sketch, not very clear to the reviewer, of a general criterion for the existence and asymptotic stability of a periodic solution of a periodic differential equation. {In Section 3.2, the equation should be $\ddot{x}+F(\dot{x})+x=p(t)$.} *J. J. Schäffer* (Pittsburgh, Pa.)

Segal, Martin 6122

Stability problems for systems of linear differential equations.

Comm. Pure Appl. Math. **17** (1964), 401–414.

The author considers pairs (T,Y) of matrix-valued functions of a real variable t, T periodic of fixed period w, related by $Y'=TY$, $Y(0)=I$. The pair is stable if the powers of $Y(w)$ are bounded, strongly stable if every neighbouring pair—in the obvious sense—is stable. If A is a fixed symmetric matrix, $G(A)$ is the group of invertible matrices Z such that $Z'AZ=A$. $D(A,w)$ is the set of pairs, in the above-defined sense, such that the values of Y are in $G(A)$ (this implies $T=A^{-1}H$, with H skew-valued and periodic); strong stability is now to be understood with respect to neighbouring pairs in $D(A,w)$. Theorem: (T,Y) in $D(A,w)$ is strongly stable if and only if $Y(w)$ is of definite type with respect to A, i.e., A is definite on the eigenspaces of $Y(w)$. The set of such strongly stable pairs turns out to be open. The author then determines its connected components. Similar problems for skew A were studied by Moser [same Comm. **11** (1958), 81–114; MR **20** #3354] and Gel'fand and Lidskiĭ [Uspehi Mat. Nauk **10** (1955), no. 1 (63), 3–40; MR **17**, 482; Amer. Math. Soc. Transl. (2) **8** (1958), 143–181; MR **19**, 960]. *J. J. Schäffer* (Pittsburgh, Pa.)

Kodnár, R. 6123

A remark on the stability of the solutions of linear differential equations. (Slovak. Russian and German summaries)

Acta Fac. Natur. Univ. Comenian. **9**, 75–81 (1964).

En partant du fait connu que les premières n composantes de chaque solution du système

$$y_i'=\sum_{j=1}^{n+1}a_{ij}(x)y_j,\qquad i=1,2,\cdots,n,$$

$$y_{n+1}'=0$$

fournissent une solution du système non homogène

$$y_i'=\sum_{j=1}^{n}a_{ij}(x)+ca_{i,n+1}(x),$$

où c signifie une constante convenable, l'auteur modifie

les conditions pour la stabilité des solutions d'un système des équations différentielles homogènes pour les solutions d'un système non homogène. Il s'agit des théorèmes établis dans la monographie de R. Bellman [*Stability theory of differential equations*, McGraw-Hill, New York, 1953; MR 15, 794]. *M. Ráb* (Brno)

Ezeilo, J. O. C. 6124

A boundedness theorem for some non-linear differential equations of the third order.

J. London Math. Soc. **37** (1962), 469–474.

Verfasser behandelt die Differentialgleichung $x''' + f(x, x')x'' + g(x') + h(x) = p(t)$, wobei die Funktionen f, f_x, h', p für alle Werte ihrer Argumente stetig sein sollen. Folgender Satz wird aufgestellt: (a) Es sei $g(0) = 0$, und es gebe positive Konstanten δ_0, δ_1, so daß $g(y)/y \geq \delta_1$ für $y \neq 0$ und $f(x, y) \geq \delta_0$ für alle x, y; (b) $h(x) \operatorname{sgn} x \geq 0$ für $|x| \geq 1$; (c) $h'(x) \leq c$ für alle x mit einer Konstante $0 < c < \delta_0\delta_1$; (d) $yf_x(x, y) \leq \Delta$ für alle x, y und eine Konstante $0 < \Delta < (\delta_0\delta_1 - c)/\delta_0$. Unter den Bedingungen (a) bis (d) existiert die Lösung $x(t)$ mit den Anfangswerten $x(0) = x_0$, $x'(0) = y_0$, $x''(0) = z_0$ für $t \geq 0$ und genügt den Abschätzungen $H(x(t)) = \int_0^{x(t)} h(\xi)\, d\xi \leq K_0\Phi(t)$, $[x'(t)]^2 \leq K_0\Phi(t)$, $[x''(t)]^2 \leq K_0\Phi(t)$; darin ist K_0 eine von den Funktionen f, g, h und den Anfangswerten x_0, y_0, z_0 abhängige Konstante und $\Phi(t) = [1 + \int_0^t \{|p(\tau)|\}^{2(1-\lambda)}\, d\tau]^{1/(1-\lambda)}$, $0 \leq \lambda \leq \frac{1}{2}$. Hat man $H(x) \to \infty$ für $|x| \to \infty$ sowie $\int_0^\infty \{|p(\tau)|\}^{2(1-\lambda)}\, d\tau < \infty$, so gibt es eine Konstante $K_1 = K_1(f, g, h, p; x_0, y_0, z_0)$, mit der man für $t \geq 0$ abschätzen kann: $|x(t)| \leq K_1$, $|x'(t)| \leq K_1$, $|x''(t)| \leq K_1$. Zum Beweis des Satzes bildet der Verfasser eine Ljapunovsche Funktion $V(x, y, z)$: $2V = 2H(x) + \alpha\{2\int_0^y g(\xi)\, d\xi + z^2\} + 2\alpha y h(x) + 2\int_0^y \eta f(x, \eta)\, d\eta + 2yz$, wo $\alpha = \varepsilon + 1/\delta_0$ und $\varepsilon > 0$ eine passend gewählte Zahl ist, und schätzt ihren Verlauf längs der betrachteten Lösung $x(t)$ ab, wobei $y = x'(t)$ und $z = x''(t)$ zu setzen ist (Überführung der Differentialgleichung in ein äquivalentes System von Gleichungen). *R. Reissig* (Zbl 108, 89)

Čičkin, E. S. 6125

On the non-oscillation of solutions of non-linear differential equations of third and fourth order. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1959**, no. 5 (12), 219–221.

Soit l'équation (E) $y^{(n)} = f(x, y, y', \cdots, y^{(n-1)})$ dans laquelle f est continue pour $a \leq x \leq b$, $a_k \leq y^{(k)} \leq b_k$, $k = 0, 1, \cdots, n-1$, et telle que par tout point du domaine de définition il passe une seule solution du problème de Cauchy. L'intervalle $[a, c) \subset [a, b]$ est (par définition) un intervalle de non-oscillation de l'équation (E) si la différence de deux solutions arbitraires de (E) admet tout au plus $n-1$ zéros sur $[a, c)$. Si l'on a $q(x) \leq \partial f/\partial y \leq 0$, où $q(x)$ est continue sur $[a, b]$, alors un intervalle de non-oscillation pour l'équation $v^{(n)} = q(x)v$ est aussi un intervalle de non-oscillation pour (E). Pour $n = 3, 4$, on obtient pour l'équation linéaire des conditions de non-oscillation à l'aide d'une fonction associée d'une manière convenable à cette équation. *C. Corduneanu* (Iași)

Geraščenko, E. I. 6126

On the degree of stability of non-linear systems in a moving regime. (Russian)

Izv. Akad. Nauk SSSR Tehn. Kibernet. **1964**, no. 2, 114–120.

The characteristic equation of the system is $p_r(\lambda) = \lambda^n + c_1(r)\lambda^{n-1} + \cdots + c_n(r) = 0$, where $c_{k+1} = rc_k(r) + a_{k+1}$, $c_0 = 1$ and $a_1, \cdots, a_n$ are given. Thus the coefficients can be varied with constraints on the allowable values of r. The measure $\delta(r)$ of the degree of stability is $\delta(r) = \max\{\alpha_1(r), \cdots, \alpha_n(r)\}$, where $\alpha_j(r)$ denotes the real part of the jth characteristic root. An optimal choice of r is one that minimizes $\delta(r)$. Necessary conditions for minima of $\delta(r)$ are given, and their application is illustrated by an example with $n = 3$. The reason things work out well is that $(\lambda - r)p_r(\lambda) = f_n(\lambda) - f_n(r)$, where $f_n(r) = rc_n(r) = r^{n+1} + a_1r^n + \cdots + a_nr$, and the conditions are in terms of the polynomial $f_n(\lambda)$ and its derivatives. *J. P. LaSalle* (Providence, R.I.)

Olech, Czesław 6127

Global phase-portrait of a plane autonomous system.

Ann. Inst. Fourier (Grenoble) **14** (1964), *fasc.* 1, 87–97.

Consider the autonomous system defined on the plane R^2, (1) $dx/dt = f(x)$, where $x = (x_1, x_2)$, $f = (f_1, f_2)$, and it is assumed that $\operatorname{div} f(x) = \partial f_1/\partial x_1 + \partial f_2/\partial x_2 \leq 0$ on R^2. Among others, the following results about the behaviour of trajectories of (1) are given: (i) If a non-singular trajectory I of (1) is such that its α-limit set $\alpha(I) \neq \varnothing$, then either I is a closed orbit, or else $\alpha(I)$ is composed of singular points; (ii) If the singular points of (1) are isolated, then they are either points of attraction, centers or else generalized (in a certain sense) saddle points. *M. M. Peixoto* (Providence, R.I.)

Ronsmans, P. 6128

Sur la stabilité et l'index des singularités multiples d'un système différentiel autonome.

Acad. Roy. Belg. Bull. Cl. Sci. (5) **50** (1964), 142–173.

From the author's summary: "The system considered is of the type

$$\frac{dx}{dt} = A(x, y), \qquad \frac{dy}{dt} = B(x, y).$$

Simple singularities of this system are always totally stable or totally unstable. Semi-stability appears with multiple singularities. The notion of conditional stability is applied to the study of singular points, and a measure is associated with the stability of the corresponding equilibrium states. The notion of the index, in the sense of Poincaré, is extended to singularities of arbitrary order. Several mathematical examples are studied, and the results are verified by analog computer. Finally, the theory is applied to the study of an electronic circuit." *F. Brauer* (Madison, Wis.)

Lakshmikantham, V. 6129

Differential inequalities and extension of Lyapunov's method.

Proc. Cambridge Philos. Soc. **60** (1964), 891–895.

The author continues his study of differential inequalities which he has previously used in a series of papers to study stability properties of solutions of differential equations. Here he studies the stability properties of solutions of a differential inequality of the form $|x' - f(t, x)| \leq w_1(t, d(x, M))$, where M is a non-empty subset of R^n,

$d(x, M) = \inf_{a \in M} |x-a|$, and $w_1(t, r)$ is continuous for $0 \leq t < \infty$, $r > 0$. The methods and results are analogous to those for differential equations.

F. Brauer (Madison, Wis.)

Salvadori, Luigi 6130

Sulla stabilità asintotica delle posizioni d'equilibrio di un sistema olonomo in presenza di forze dissipative anche non lineari. (English summary)

Rend. Accad. Sci. Fis. Mat. Napoli (4) **30** (1963), 145–152.

The author considers a holonomic autonomous mechanical system, subject to a system of conservative forces of total potential energy U and a system of dissipative forces with Lagrangian components $Q_i(q, \dot{q})$. A configuration is considered for which U has a minimum (hence the system, even without the dissipative forces, has a stable equilibrium there) and actually the matrix of second derivatives of U is positive. The author generalizes a result of Četaev [*Stability of motion. Papers on analytical mechanics* (Russian), Izdat. Akad. Nauk SSSR, Moscow, 1962; MR **28** #1358; English transl., Pergamon, Oxford, 1961] by showing that the addition of the dissipative forces ensures that the equilibrium is asymptotically stable (and hence totally stable) even if these dissipative forces are of the form $Q_i = Q_i^* + \eta_i$, where $Q_i^* = Q_i^*(\dot{q})$ is homogeneous of some degree $s \geq 1$ (with constant coefficients) in the $\dot{q}$ and such that the form $\sum_i Q_i^* \dot{q}_i$ is negative definite, and the $\eta_i(q, \dot{q})$ are small of order $> s$ for states near the state of rest at the considered configuration. The proof makes ingenious use of Ljapunov's criterion for asymptotic stability.

J. J. Schäffer (Pittsburgh, Pa.)

Salvadori, Luigi 6131

Sulla stabilità dei moti merostatici di un sistema olonomo in presenza di forze dissipative anche non lineari. (English summary)

Rend. Accad. Sci. Fis. Mat. Napoli (4) **30** (1963), 153–160.

Extension of the results of the preceding paper [#6130] of the author to the case where the state of rest at an equilibrium point is replaced by a "merostatic" motion, i.e., a motion where all non-ignorable coordinates are held constant (and there are ignorable ones) when the dissipative forces are such that the work along a merostatic motion vanishes, and the other assumptions in the preceding paper are suitably modified.

J. J. Schäffer (Pittsburgh, Pa.)

Stepanov, S. Ja. 6132

On the stability of dissipative systems. (Russian. English summary)

Vestnik Moskov. Univ. Ser. I Mat. Meh. **1964**, *no.* 4, 59–66.

Consider a Lagrangian system with generalized coordinates $q_1, \cdots, q_n$, with $q_{k+1}, \cdots, q_n$ cyclic, and with conservative forces F_i and other forces of a dissipative type derived from a definite quadratic form in the $\dot{q}_i$ with coefficients depending upon the q_i. For the F_i constant, G. K. Požarickiĭ [Prikl. Mat. Meh. **21** (1957), 503–512; MR **19**, 1100] derived necessary and sufficient conditions for the stability of those motions characterized by $q_i = \text{const}$, $i = 1, 2, \cdots, k$, $\dot{q}_i = \text{const}$, $i = k+1, \cdots, n$. The present author discusses the stability of the same motions when the F_i are allowed to vary with the disturbed motion but reduce to constants at the undisturbed motion. An application is given for the motion of a gyroscope. The results are obtained by the construction of an appropriate Lyapunov function.

J. K. Hale (Providence, R.I.)

Brullinskaja, N. N. 6133

Limit cycles of the equations of motion of a rigid body and for the Galerkin equations of hydrodynamics. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 1017–1020.

The object of study is a certain class of systems of nonlinear differential equations involving a parameter ν. This class is of great interest because it includes the Galerkin approximations to the time-dependent Navier-Stokes system of partial differential equations governing viscous fluid motion. Also, when the order is 3, the system considered represents the equations of damped motion of a rigid body. In the first case ν represents the viscosity; in the second, a damping factor. The author states several results about the behavior of the solution as the independent variable $t \to \infty$. For example, although for large ν there exists a stable equilibrium point which all solutions approach, there is a critical value ν_{cr} such that stable or unstable limit cycles arise as ν passes through ν_{cr}, provided the coefficients of the system satisfy a certain inequality. These results are what one might expect of solutions of the Navier-Stokes system. It is indicated that the proofs (not given) are based largely on the author's previous note [same Dokl. **139** (1961), 9–12; MR **26**, 5212].

P. C. Fife (Minneapolis, Minn.)

Kupka, Ivan 6134

Addendum et corrections au mémoire: "Contributions à la théorie des champs génériques".

Contributions to Differential Equations **3** (1964), 411–420.

In this addendum the author makes some corrections to his previous paper [Contributions to Differential Equations **2** (1963), 457–484; MR **29** #2818a], mostly of the misprint type, and suggests also some surgery that has to be done on it "dû à la mauvaise manipulation des épreuves de ce mémoire".

M. M. Peixoto (Providence, R.I.)

Likova, O. B. [Lykova, O. B.] 6135

On the behaviour of the solutions of a system of $n+m$ differential equations in the neighbourhood of an equilibrium point. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 569–573.

The author considers the following system (1) $dx/dt = X(y) + \varepsilon X^*(t, x, y)$, $dy/dt = \varepsilon Y(t, x, y)$, where ε is a small parameter, x, y are n- and m-vectors, respectively, X^* and Y are periodic in t of period 2π, and in addition, the following assumptions hold: Let $Y_0(x, y)$ be the time-average of $Y(t, x, y)$, then assume $Y_0(0, 0) = 0$, $(\partial Y_0/\partial x)(0, 0) \neq 0$, and each of the matrices $X(0)$ and $(\partial Y_0/\partial y)(0, 0)$ has a pair of purely imaginary (non-zero) eigenvalues, with the remaining $n-2$ and $m-2$ eigenvalues with

negative real parts, respectively. Then the author asserts that there exists, for ε sufficiently small, an asymptotically stable manifold S of (1) in a neighborhood of $x=y=0$, which is described by two n- and m-vector-valued functions of t, ε, and four real variables, periodic in t of period 2π, Lipschitzian in the last four variables. No proofs are given, merely an indication of the existence of transformations, using the method of averaging, which allow the above results to be deduced. *A. Stokes* (Washington, D.C.)

Klein, Joseph 6136

Les systèmes dynamiques abstraits.

Ann. Inst. Fourier (Grenoble) **12** (1963), *fasc.* 2, 191–202.

A nonconservative dynamical system may be defined by means of a two-form Ω on the tangent sphere bundle of the underlying space-time, such that $\Omega=d(\dot{d}L)+S$ with S a semi-basic two-form called the force tensor [the author, same Ann. **12** (1962), 1–124]. "Semi-basic" means that at each point the form is equal to a form induced from the base space, and $\dot{d}$ is the obvious exterior derivative that takes semi-basic forms into semi-basic forms. In this paper it is proved that, given a two-form Ω on the sphere bundle, its pull-back to the bundle of nonzero tangent vectors can be written in the above form if and only if $d\Omega=0$. In making this statement, we use the fact that $\dot{d}$ may be extended so that it operates on any form.

B. L. Reinhart (College Park, Md.)

Korobeinik, Ju. F. 6137

Entire solutions of a differential equation of infinite order. (Russian. Lithuanian and English summaries)

Litovsk. Mat. Sb. **4** (1964), 203–227.

From the author's summary: "The author considers entire solutions of the differential equations of infinite order of the form

$$(1)\quad y(x)+\sum_{k=1}^{\infty}P_k(x)y^{(k)}(x)=f(x),\qquad P_k(x)=\sum_{s=0}^{N_k}a_s^kx^s,$$

where

$$\sup_{k\geq 1}\frac{N_k}{k}=\alpha<1,\qquad \limsup_{k\to\infty}\sqrt[k]{\left(\sup_{0\leq s\leq N_k}|a_s^k|\right)}<\infty.$$

The author obtains a theorem on the existence and uniqueness of solutions of (1) in the class of entire functions $[1-\alpha,\sigma]$, $\sigma<\infty$, and indicates a method of approximating such solutions. The asymptotic behavior of such entire solutions is also investigated, and a lower bound for the entire transcendental solutions of the homogeneous equation is obtained."

A. Stokes (Washington, D.C.)

Korobeinik, Ju. F. 6138

Entire analytic solutions of equations of infinite order with polynomial coefficients. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 1031–1034.

The author continues his investigations of equations of the form (1) $\sum_{k=0}^{\infty}P_k(x)y^{(k)}(x)=f(x)$. In the paper the author announces some results which extend some of his earlier results. However, the hypotheses and conclusions of the five theorems given are too complicated to state here. In addition to announcing these new results, the author makes a valuable contribution by providing an extensive bibliography (14 references) of his own work and related work by other authors, with a brief expository account of the development of the theory of such equations. At the end of the paper, the author gives an application of the theory developed for equations of the form (1) to the functional equation (2) $y(x+a\sqrt{x})+y(x-a\sqrt{x})=f(x)$.

A. Stokes (Washington, D.C.)

Valeev, K. G. 6139

Linear differential equations with a time lag depending linearly on the argument. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 290–300.

The author applies Laplace transform methods to the linear functional-differential system (generally of neutral type)

$$(1)\qquad \sum_{q=0}^{l}\sum_{k=0}^{n}A_{qk}\frac{d^kY}{dt^k}(\alpha_{qk}t-\beta_{qk})=\sum_{j=1}^{\lambda}C_jt^{\nu_j}e^{\omega_jt},$$

where $Y(t)$ is the unknown vector, the A_{qk} are constant matrices, $\alpha_{0k}=1,\beta_{0k}=0,0<\alpha_{qk}<1,\beta_{qk}\geq 0$ (for $q=1,\cdots,l$), $A_{0n}=E$, $\sum_{q=1}^{l}|A_{qn}(\alpha_{qn})^{-n-1}|<1$, the C_j are constant vectors, the ν_j are non-negative integers, and the ω_j are complex numbers. Specifying $d^kY(0)/dt^k=Y_0^{(k)}$ for $k=0,\cdots,n-1$ and $d^kY(t)/dt^k=\Theta^{(k)}(t)$ for $-\beta\leq t<0$, $k=0,\cdots,n$, where the $\Theta^{(k)}(t)$ satisfy Dirichlet conditions, the author obtains infinite-series expansions for the solution of (1) for $t>0$. Special results are obtained in case each $\beta_{qk}=0$, and the case of complex α_{qk} is considered. A number of examples of delay-differential systems ($A_{qn}=0$ for $q=1,\cdots,l$) are given, including two in which the results are applied to other special forms of such equations:

$$v'(x)+\mu e^xv(x)=\gamma e^xv(x-\beta)\ \text{ and }\ zv'(z)+\mu v(z)=\gamma v(\sqrt{z}).$$

R. D. Driver (Albuquerque, N.M.)

Baumgärtel, Hellmut 6140

Einige Bemerkungen zur Differentialgleichung $X'(t)=P(t)X(t)$ für Operatorfunktionen.

Wiss. Z. Humboldt-Univ. Berlin Math.-Natur. Reihe **13** (1964), 181.

From the author's summary: "$P(t)=A+V(t)$ ($t\in\Omega$, Ω die Gesamtheit der reellen Zahlen) sei eine Operatorfunktion abgeschlossener Operatoren eines Banachschen Raumes $\mathfrak{H}$, A sei abgeschlossen und unbeschränkt mit dem Definitionsbereich $\mathfrak{D}_A$, der dicht in $\mathfrak{H}$ liegt, es gebe Zahlen $M>0$, $\omega\geq 0$, so daß $\|R(\lambda,A)^n\|\leq M(|\lambda|-\omega)^{-n}$ gilt für alle $n=1,2,\cdots$, $V(t)$ sei beschränkt für jedes t und in t stark stetig, für $x\in\mathfrak{D}_A$ sei $V(t)x\in\mathfrak{D}_A$ für jedes t, $V(t)$ sei als Operator des Banachschen Raumes $\mathfrak{D}_A$ mit der Norm

$$[x]=\|x\|+\|Ax\|$$

beschränkt für jedes t, und für jedes $x\in\mathfrak{D}_A$ sei die Funktion $V(t)x$ stark stetig in t bezüglich [ ]. Dann gibt es eine in $\Omega\times\Omega$ stark stetige Operatorfunktion $X(t,t_0)$, die auf $\mathfrak{D}_A$ starke partielle Ableitungen nach t und t_0 besitzt mit den weiteren Eigenschaften $X(t_0,t_0)=I$ und $X(t,t_0)x\in\mathfrak{D}_A$, falls $x\in\mathfrak{D}_A$, die bezüglich t eine Lösung von $X'(t)=P(t)X(t)$ und bezüglich t_0 eine Lösung von $Y'(t)=-Y(t)P(t)$ ist. $X(t,t_0)$ ist auch die einzige stark stetige Lösung der beiden Differentialgleichungen mit den angegebenen Eigenschaften."

A. Stokes (Washington, D.C.)

Santagati, Giuseppe 6141

Problemi quasi lineari negli spazi di Banach. Unicità e dipendenza continua della soluzione dai dati.

Matematiche (Catania) **18** (1963), 15–39.

Let $x(t)$, $\mathfrak{x}(t)$ be functions defined on the closed interval $\Delta=[a,b]$ and taking values in the Banach spaces $\mathfrak{B}_1$, $\mathfrak{B}_2$, respectively. Let X and $\mathfrak{X}$ denote the spaces of functions $x(t)$ and $\mathfrak{x}(t)$ with norms $\|x\|_X=\sup_\Delta\|x(t)\|_{\mathfrak{B}_1}$, $\|\mathfrak{x}\|=\sup_\Delta\|\mathfrak{x}(t)\|_{\mathfrak{B}_2}$, respectively. $\tilde{\mathfrak{B}}_1$ is the space of endomorphisms A on $\mathfrak{B}_1$ with the usual norm $\|A\|_{\tilde{\mathfrak{B}}_1}=\sup_{\|u\|\le 1}\|Au\|_{\mathfrak{B}_1}$. Let $A(t)$ be a mapping from Δ into $\tilde{\mathfrak{B}}_1$, $g(t,u)$ a mapping from $\Delta\times\mathfrak{B}_1$ into $\mathfrak{B}_1$, $\mathscr{L}$ a linear continuous map of X into $\mathfrak{X}$ and $\mathscr{K}$ a continuous map of X into $\mathfrak{X}$. The author considers an equation (E) $\dot{x}=A(t)x+g(t,x)$ with auxiliary condition (C) $\mathscr{L}x=\mathscr{K}x$. A solution of (E), (C) is an element x in X which is absolutely continuous and satisfies (E), (C) for almost every value of t in Δ. A special case of this problem in which the auxiliary condition (C) is replaced by (C′) $\mathscr{L}x=\mathfrak{x}$, $\mathfrak{x}$ being an element of $\mathfrak{X}$, has been studied by Pulvirenti [Matematiche (Catania) **15** (1960), 98–107; MR **25** #5257] in connection with the existence question. In this paper, the author proves the uniqueness (Theorem 1) and the continuous dependence (Theorems 3–6) of the solution on the data $\mathscr{L}$, $A(t)$, $\mathscr{K}$, and $g(t,u)$, as well as the existence. In addition to the conditions on $A(t)$, $g(t,u)$ and $\mathscr{L}$, introduced by Pulvirenti, several more assumptions (Lipschitz continuity of g and $\mathscr{K}$, boundedness of $\|\mathscr{K}(X)\|_{\mathfrak{X}}$, etc.) are made.

J. S. Kim (College Park, Md.)

Valikov, K. V. 6142

Some criteria for stability of motion in Hilbert space. (Russian)

Uspehi Mat. Nauk **19** (1964), *no.* 4 (118), 179–184.

The author considers an equation of the form (1) $dx/dt=F(x,x,t)+f(x,t)$, where $F(x,y,t)$ and $f(x,t)$ are defined on a ball B centered at the origin in some Hilbert space H, $t\geq 0$, with (2) $F(x,0,t)\equiv 0$ and $\|f(x,t)\|\leq\gamma\|x\|$ for all $t\geq 0$, $x\in B$, $\gamma>0$. Further, assume F has a Gâteaux differential F defined on H, $\delta F(x,y,t;h)=S(x,y,t)h$, S a bounded linear operator for $x,y\in B$, $t\geq 0$. The author gives two conditions sufficient for the origin in (1) to be exponentially uniformly asymptotically stable for γ in (2) sufficiently small, namely: (i) $\lim_{x,y\to 0,t\to\infty}S(x,y,t)=S_0$ exists, where S_0 is a bounded operator with its spectrum in the left half-plane, or (ii) there exists a positive self-adjoint operator B such that the operator $G=BS(x,y,t)+S^*(x,y,t)B$ satisfies the condition $R(\lambda)\leq -\nu$, $\nu>0$, for all $x,y\in B$, $t\geq 0$. Here S^* denotes the adjoint of S.

A. Stokes (Washington, D.C.)

PARTIAL DIFFERENTIAL EQUATIONS

See also 5989, 6042, 6044, 6045, 6046, 6048, 6120, 6133, 6323, 6340, 6341, 6342, 6344, 6348, 6642, 6734, 6808, 6947, 6955, 6971.

Fuglede, Bent 6143

★Extremal length and closed extensions of partial differential operators.

Jul. Gjellerups Boghandel, Copenhagen, 1960. 82 *pp.*

This work is divided into two parts. In the first it is shown that three different L^p extensions of a first-order system of partial operators are actually the same extension. The second part is devoted to generalized derivation of exterior differential forms locally of class L^p. An essential feature of the paper is the idea of p-exceptional system of surfaces, which also appears in the author's paper [Acta Math. **98** (1957), 171–219; MR **20** #4187].

Let $1\leq p<\infty$, $\Omega\subset R^n$ be open, $D_\nu=\partial/\partial x_\nu$, and for each $u=(u_1,\cdots,u_s)$ of class $C^1(\Omega)$ let

$$Pu=\sum_{\nu=1}^{n}D_\nu(A_\nu u)+Au.$$

Let $\bar{P}$ and P^w denote, respectively, the L^p-closure and weak closure of P. Using a method of Deny and Lions [Ann. Inst. Fourier (Grenoble) **5** (1953/54), 305–370; MR **17**, 646] it is shown that the condition $\bar{P}u=v$ is purely local. Using Friedrich's mollifiers it is then shown that $\bar{P}=P^w$. A bounded open subset G of R^n is called a Green's set if (roughly speaking) the boundary Γ of G is regular enough that a Gauss-Green identity holds. The precise condition imposed on G is equivalent to the assumption that G is an open set of finite perimeter in the sense of De Giorgi [Ricerche Mat. **4** (1955), 95–113; MR **17**, 596]. A system $\mathbf{E}$ of Green's sets is called p-exceptional if there exists a non-negative Baire function $f\in L^p(R^n)$ such that $\int_\Gamma f\,d\sigma=+\infty$ for every $G\in\mathbf{E}$. The flux extension P^f of P has the property that if u,v are of class $L^p(\Omega)$, then $P^fu=v$ if and only if

$$\int_\Gamma\sum_\nu A_\nu ue_\nu\,d\sigma+\int_G Au\,dx=\int_G v\,dx$$

except for some p-exceptional system of Green's sets G with $G\cup\Gamma\subset\Omega$. Here $e(x)$ denotes the exterior unit normal at x. It is proved that $\bar{P}=P^f$.

In the second part, the L^p-closure $\bar{d}$ of the exterior differential d is considered, and the notion of p-exceptional system $\mathbf{E}$ of class C^1 singular chains is defined. Let u and v be differential forms locally of class $L^p(\Omega)$, of respective degrees $k-1$ and k. Then $\bar{d}u=v$ if and only if Stokes' formula holds for every k-chain not in some p-exceptional system. Periods of closed forms locally of class $L^p(\Omega)$ are then defined, and the appropriate version of De Rham's theorem is proved.

W. H. Fleming (Providence, R.I.)

Manninen, Jouko 6144

Über das Existenzgebiet von integralen partieller Differentialgleichungen erster Ordnung.

Ann. Acad. Sci. Fenn. Ser. A I No. 332 (1963), 16 *pp.*

The author gives a practical and efficient estimate for the largest sphere (or polycylinder) inside of which there exists the solution of the Cauchy problem for a system of implicit partial differential equations of first order with one dependent variable. Two rather interesting lemmas (or special cases) concern implicit functions and ordinary differential equations. The notation is that of F. and R. Nevanlinna [*Absolute Analysis*, Springer, Berlin, 1959; MR **22** #12176]; see also the author [Ann. Acad. Sci. Fenn. Ser. A I No. 283 (1960); MR **24** #A319]. The results are too complicated to be stated here.

G. Hufford (Boulder, Colo.)

Schlissel, Arthur 6145

The asymptotic behavior with respect to time of the generalized solution of a first order quasi-linear equation with particular initial values.

J. Math. Anal. Appl. **9** (1964), 356–359.

Let $u_0(x)$ be a real function on the real line such that $u_0(x) = u_+ + o(x^{-1})$ as $x \to \infty$ and such that $K_1 = \int_{-\infty}^0 (u_0(x) - u_-)\,dx$ and $K_2 = \int_0^\infty (u_0(x) - u_+)\,dx$ exist. Let $f''(u) > 0$, let $G(\tau) = \max_u (u\tau - f(u))$ be the conjugate function of f and let the maximum occur when $u = b(t)$. Let $y_0 = y_0(x, t)$ minimize $\int_0^y u_0(x)\,dx + tG((x-y)/t)$. Then $u = b((x-y_0)/t)$ is an appropriate solution of $u_t + f_x(u) = 0$ with initial data $u_0(x)$. It is shown that as t tends to ∞, u tends to u_- and u_+ to the left and to the right of an asymptotic shock line $(u_+ - u_-)x = (f(u_+) - f(u_-))t - K_1 - K_2$. The same result has been obtained by Il'in and Oleĭnik [Mat. Sb. (N.S.) **51** (**93**) (1960), 191–216; MR **22** #11222]. *L. Gårding* (Lund)

Wu, Zhuo-qun [Wu, Cho-chün] 6146

On the existence and uniqueness of the generalized solutions of the Cauchy problem for quasilinear equations of first order without convexity conditions.

Acta Math. Sinica **13** (1963), 515–530 (*Chinese*); *translated as Chinese Math.* **4** (1964), 561–577.

Existence and uniqueness questions for discontinuous solutions of the Cauchy problem

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \qquad 0 \le t \le T,$$

$$u|_{t=0} = u_0(x), \qquad -\infty < x < \infty,$$

are considered. The function $f(u)$ is assumed twice continuously differentiable and the solution $u(x, t)$ is required to satisfy

$$\frac{f(v) - f(u(t, x(t)-0))}{v - u(t, x(t)-0)} \ge \frac{f(u(t, x(t)+0)) - f(u(t, x(t)-0))}{u(t, x(t)+0) - u(t, x(t)-0)}$$

along any curve $x(t)$ of discontinuity. This condition is related to one suggested by Gel'fand [Uspehi Mat. Nauk **14** (1959), no. 2 (86), 87–158; MR **22** #1736; Amer. Math. Soc. Transl. (2) **29** (1963), 295–381; MR **27** #3921]. The author proves a uniqueness theorem which generalizes previous work of Oleĭnik [Uspehi Mat. Nauk **14** (1959), no. 2 (86), 165–170; MR **22** #8178; Amer. Math. Soc. Transl. (2) **33** (1963), 285–290]. Applying a method due to Douglis [Comm. Pure Appl. Math. **12** (1959), 87–112; MR **21** #3650], he proves existence in the large for weak solutions under assumptions on $f(u)$ and $u_0(x)$. The class of functions in which existence is proved is larger than the class for which uniqueness is proved. *M. Schechter* (New York)

Bouligand, G. 6147

Sur les surfaces intégrales d'équations aux dérivées partielles d'ordre 1 et leurs arêtes de rebroussement (dans un R^3).

J. Math. Pures Appl. (9) **43** (1964), 177–185.

The integrability condition for $p = A(x, y, z, u)$, $q = B(x, y, z, u)$ yields a linear partial differential equation for the auxiliary unknown $u(x, y, z)$. Under the usual type of restrictions, Cauchy conditions for $z(x, y)$ extend to Cauchy conditions for u, and the method of characteristics applies to give a solution that may be viewed in the form $[x(u, v), y, z, u]$ with parameters u and v. In (x, y, z)-space those points on this surface with indeterminate normal are projections of points on the surface in (x, y, z, u)-space with a tangent line parallel to the u-axis. In general, such points form the curves that are mentioned in the title of the paper. A first-order partial differential equation that depends on a parameter, and in a limiting case has a degenerate solution, is investigated by the above method. *A. Newlander, Jr.* (Denver, Colo.)

Kušnirčuk, I. F. [Kušnirčuk, Ĭ. F.] 6148

On the Cauchy problem for equations of higher order with multiple characteristics. (Russian)

Approximate methods of solving differential equations, pp. 54–59. *Izdat. Akad. Nauk Ukrain. SSR, Kiev*, 1963.

Consider the equation in two independent variables

$$\prod_1^m \left(\frac{\partial}{\partial x} + (c(x) + \gamma_s)\frac{\partial}{\partial w}\right)^{k_s} f(w, x) + Df(w, x) = h(w, x),$$

where $\sum k_s = n$ and D is a differential operator of order less than n. This is a differential equation of order n with m distinct characteristics. This paper shows that if and only if D is of order less than m, the above equation can be reduced to Bianchi's equation

$$(-1)^n \partial^n F/\partial t_1 \cdots \partial t_n + \cdots = H(t),$$

and hence [reference is made to M. K. Fage, Mat. Sb. (N.S.) **45** (**87**) (1958), 281–322; MR **21** #1456] the Cauchy problem can be solved. *G. Hufford* (Boulder, Colo.)

Wu, Zhueng-hai [Wu, Chung-hai] 6149

Theorems on differential inequalities for nonlinear Bianchi equations.

Acta Math. Sinica **13** (1963), 584–606 (*Chinese*); *translated as Chinese Math.* **4** (1964), 637–660.

The author derives comparison theorems for solutions of the Cauchy and characteristic problems for the non-linear Bianchi equation

$$u_{xyz} = F(x, y, z, u, u_x, u_y, u_z, u_{yz}, u_{zx}, u_{xy}).$$

For example, with $F \in C^3$, suppose that $v_{xyz} = \bar{F}(x, \cdots, u_{xy})$ and that u and v have identical C^2 Cauchy data on a non-characteristic initial manifold. If $\bar{F} \le F$, then $u \le v$ in a non-vanishing region whose size depends on inequalities on the partial derivatives of F alone with respect to $u, u_x, \cdots, u_{xy}$. Assuming stronger inequalities on these partial derivatives, the derivatives of u and v are compared. Similar results are given for the characteristic problem in which u is assigned on the boundary of an octant. The main ingredient of the proof is a study of properties of the Riemann function for the linearized equation. *L. Sarason* (Stanford, Calif.)

Zerner, Martin 6150

Solutions singulières d'équations aux dérivées partielles.

Bull. Soc. Math. France **91** (1963), 203–226.

Soit $a(x, \partial/\partial x)$ un opérateur linéaire à coefficients analytiques. Il s'agit de préciser les propriétés géométriques des points singuliers d'une fonction ou plutôt d'une distribution u vérifiant $a(x, \partial/\partial x)u(x) = 0$.

Désignons par Sg(u) l'ensemble des points au voisinage desquels u n'est pas une fonction indéfiniment différentiable. L'auteur définit d'abord la notion de la caractéristique au sens de Lie simple. Dans le cas où la partie principale g de a est réel, le théorème suivant est démontré. Soit V une caractéristique au sens de Lie simple et analytique. Tout point $x_0 \in V$ possède un voisinage W sur lequel une distribution u est définie qui vérifie $au=0$ et $\mathrm{Sg}(u)=V \cap W$. Dans le cas où g n'est pas réel, ce résultat est encore vrai, si l'on suppose une condition entre la partie réelle et la partie imaginaire de g. La démonstration est basée essentiellement sur la méthode d'Hadamard. Finalement, dans le cas où a est matriciel et constant, un résultat précis est démontré.

S. Mizohata (Kyoto)

Zitarosa, Antonio 6151

Sul problema di Nicoletti per le equazioni a derivate parziali.

Matematiche (Catania) **17** (1962), 85–120.

L'equazione che sta a base del problema di Nicoletti è costituita da una funzione incognita, derivata n volte rispetto ad x ed m rispetto ad y, eguagliata ad una funzione nota delle variabili indipendenti e di tutte le derivate di ordine più basso; sul contorno del dominio rettangolare sono assegnate le condizioni sulla funzione e sulle derivate non miste.

L'autore dimostra teoremi di esistenza, unicità e dipendenza della soluzione delle condizioni al contorno. Lo studio si basa su un teorema di J. Schauder relativo alla esistenza di elementi uniti in una trasformazione funzionale.

Il metodo qui proposto consente di ottenere nuovi teoremi rispetto a quelli stabiliti da F. Guglielmino mediante un metodo fondato sullo stesso citato teorema.

A. Pozzi (Naples)

Markovs'kiĭ, A. I. 6152

Well-posed boundary-value problems in L^2 for differential equations with constant coefficients in a half-space. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 861–865.

The problem is to find conditions on the Cauchy data for the solvability in L^2 of the equation $P(D)u = \sum_{k=0}^{m} P_{m-k}(i\partial/\partial x)(\partial^k/\partial t^k)u = 0$ for $t \geq 0$, $x \in \mathbf{R}^n$, where the P_j are polynomials with constant coefficients of degree $\leq p$, with $P_0 \equiv 1$ [cf. G. Dikopolov and G. Šilov, Izv. Akad. Nauk SSSR Ser. Mat. **24** (1960), 369–380; MR **22** #11219; Sibirsk. Mat. Ž. **1** (1960), 45–61; MR **23** #A3381]. Various conditions for solvability and correctness are given.

R. Carroll (Urbana, Ill.)

Cannon, J. R. 6153

Error estimates for some unstable continuation problems.

J. Soc. Indust. Appl. Math. **12** (1964), 270–284.

Supposto che $p(x)$, $\rho(x)$ $(0 \leq x \leq 1)$ siano funzioni positive e due volte differenziabili e che $q(x)$ $(0 \leq x \leq 1)$ sia una funzione continua e non negativa, si considera il problema

$$Lu = (p(x)u_x)_x + \rho(x)u_{yy} - q(x)u = 0$$

$$(0 < x < 1,\ y > 0),$$

$$u(0,y) = f_1(y), \qquad u(1,y) = f_2(y) \qquad (y \geq 0),$$

$$u(x,Y) = F(x) \qquad (0 \leq x \leq 1),$$

$$F(0) = f_1(Y), \qquad F(1) = f_2(Y) \qquad (0 < Y < \infty),$$

dove $F(x)$ è continua per $0 \leq x \leq 1$, e due volte differenziabile per $0 < x < 1$, mentre $f_1(y)$ e $f_2(y)$ sono continue e limitate per $0 \leq y < \infty$. Nel presente lavoro, supposta l'esistenza di una soluzione $u(x,y)$ del problema in questione, la quale è continua e limitata per $0 \leq x \leq 1$, $0 \leq y < \infty$, viene dato un metodo per approssimare la soluzione $u(x,y)$, quando le funzioni $F(x)$, $f_1(y)$, $f_2(y)$ sono note a meno di un dato errore, e inoltre si suppone che esista una costante $M_1 > 0$, tale che $|u(x,0)| \leq M_1$ $(0 \leq x \leq 1)$.

S. Cinquini (Pavia)

Geymonat, G. 6154

Su un problema relativo alle soluzioni delle equazioni lineari ellittiche.

Ann. Scuola Norm. Sup. Pisa (3) **18** (1964), 87–110.

Consider the boundary-value problem

$$Au = 0 \text{ in } \Omega, \qquad \gamma_j u = \phi_j \text{ on } \Gamma, \qquad j = 0, \cdots, m-1,$$

where Ω is a bounded domain in R^n with sufficiently regular boundary Γ, A is a linear, properly elliptic, partial differential operator of order $2m$ with sufficiently regular coefficients, and $\gamma_j = \partial^j/\partial\nu^j$ is the normal derivative of order j. Let S_j, $j=0, \cdots, m-1$, be the Neumann boundary operator associated with the γ_j. By this is meant that for $u, v \in C^{2m}(\bar\Omega)$

$$(Au, v) - (u, A^*v) = \sum_{j=0}^{m-1} \int_\Gamma \{S_j u \overline{\gamma_j v} - \gamma_j u \overline{T_j v}\}\, d\sigma$$

holds, where A^* is the formal adjoint of A. The operator $\phi = \{\phi_0, \cdots, \phi_{m-1}\} \to \mathcal{F}(\phi) = \{S_0 u, \cdots, S_{m-1} u\}$ is known to be continuous from

$$\prod_{j=0}^{m-1} W^{s-j-1/p,p}(\Gamma) \quad \text{to} \quad \prod_{j=0}^{m-1} W^{s-2m+j+1-1/p,p}(\Gamma)$$

for $1 < p < \infty$, $0 \leq s \leq m$, provided that $s - 1/p$ is not an integer, where $W^{t,p}(\Gamma)$ are the trace classes of Lions. The author now extends this, for $n=2$, to the case when $s-1/p$ is an integer. In this case the boundary values are not understood in the sense of completion by continuity of smooth functions, but rather as convergence in the mean on curves parallel to Γ. The method is based on the study of properties of an auxiliary function constructed by Agmon [Comm. Pure Appl. Math. **10** (1957), 179–239; MR **21** #5057] and Miranda [Ann. Mat. Pura Appl. (4) **46** (1958), 265–311; MR **23** #A1927].

M. Schechter (New York)

Kadlec, Jan 6155

On the maximum principle for second-order elliptic equations and the method of Wiener. (Russian)

Czechoslovak Math. J. **14** (89) (1964), 154–155.

A preliminary report giving results, many already known, about the Dirichlet problem for weak solutions of $(a^{ij}u_i')_j' = 0$.

P. C. Fife (Minneapolis, Minn.)

Kružkov, S. N.; Kupcov, L. P. 6156

Harnack's inequality for solutions of elliptic differential equations of second order. (Russian. English summary)

Vestnik Moskov. Univ. Ser. I Mat. Meh. **1964**, no. 3, 3–14.

In this note an inequality of the form $\max u \leq c(\min u + r^{\gamma})$ (where the maximum and minimum are taken over a sphere $|x| < r$) is derived for non-negative weak solutions of elliptic equations

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + \sum_i b_i(x)u_{x_i} + c(x)u + f(x) = 0.$$

The results are contained in the very general study of the reviewer [Acta Math. 111 (1964), 247–302; MR 30 #337] which appeared simultaneously with the present note.

J. Moser (New Rochelle, N.Y.)

Lithner, Lars 6157

A theorem of the Phragmén-Lindelöf type for second-order elliptic operators.

Ark. Mat. 5, 281–285 (1964).

On the region $|x| \geq r$ contained in E^n, let L be the uniformly elliptic operator given by

$$-\sum_{i,k} a_{ik}(x)\frac{\partial^2}{\partial x_i \partial x_k} + \sum_1^n b_k \frac{\partial}{\partial x_k} + a(x),$$

where a is positive and continuously differentiable, and a_{ik} and $b_k \to 0$ as $|x| \to \infty$. The author shows that if u is in $L_2(|x| \geq r)$ and satisfies $Lu = 0$, then for every positive ε, $ue^{(1-\varepsilon)\varphi}$ and $|\nabla u|e^{(1-\varepsilon)\varphi}$ belong to $L_2(|x| \geq r)$; here $\varphi(x) = \inf_\Gamma |\int_\Gamma (a(x)\sum_{i,k} a_{ik}'(x)\,dx_i\,dx_k)^{1/2}|$, Γ is a curve connecting 0 and x, and a_{ik}' is the matrix inverse to a_{ik}.

An example is given to show that ε may not be taken as zero. The proof, which is straightforward, is carried out in detail only for the case $L = -\Delta + a(x)$.

C. A. McCarthy (Minneapolis, Minn.)

Stampacchia, Guido 6158

Il principio di minimo nel calcolo delle variazioni.

Atti Convegno Lagrangiano (*Torino*, 1963) [*Atti Accad. Sci. Torino* 98 (1963/64), *suppl.*], *pp.* 152–171. *Accad. Sci., Torino*, 1964.

This lecture is an excellent summary of some recent applications of the methods of the calculus of variations to the study of solutions of elliptic partial differential equations.

H. F. Weinberger (Minneapolis, Minn.)

Vol'pert, A. I. 6159

On the reduction of boundary value problems for elliptical systems of equations of higher order to problems for systems of first order. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukrain. RSR 1960, 1162–1166.

Author's summary: "The author describes a method of reducing boundary-value problems for elliptical systems of differential equations of a higher order to the corresponding problems of systems of the first order, which is a modification of an earlier method of the author [Dokl. Akad. Nauk SSSR 114 (1957), 462–464; MR 20 #5960]. On the basis of this reduction the normal solvability of the boundary-value problem for a higher-order system is proved and a formula is derived for its index."

Dubinskii, Ju. A. 6160

Some integral inequalities and the solvability of degenerate quasi-linear elliptic systems of differential equations. (Russian)

Mat. Sb. (*N.S.*) 64 (106) (1964), 458–480.

The author presents a series of inequalities relating integrals of powers of an arbitrary function and its derivatives over a domain D and its boundary Γ in n dimensions, and which are obtained by particular integrations by parts, using the Sobolev embedding theorems and the Hölder-Young inequality. These inequalities are applied to prove the existence of a generalized solution of the first boundary-value problem for a quasilinear system of divergence form and of order $2m$, under a weakened ellipticity hypothesis which permits the associated energy integral to degenerate in certain ways. The underlying idea is to base the definition of generalized solution on the square summability of $D(D^{m-1}u)^{\alpha}$, $\alpha > 0$, rather than on that of $D^m u$, as has been done in the past. The type of degeneration permitted is illustrated by the single equation

$$\sum_1^n \frac{\partial}{\partial x_i}\left(|u|^{\alpha_i}\frac{\partial u}{\partial x_i}\right) = f(x), \qquad \alpha_i \geq 0,$$

which is accessible to the method.

The general inequalities presented have an independent interest and can be expected to find application also in other problems of analysis.

R. Finn (Stanford, Calif.)

Herzog, John O. 6161

Phragmén-Lindelöf theorems for second order quasi-linear elliptic partial differential equations.

Proc. Amer. Math. Soc. 15 (1964), 721–728.

The author derives Phragmén-Lindelöf type theorems for quasi-linear elliptic partial differential equations of the form

$$L[u] \equiv \sum a_{ij}(x,p)\partial^2 u/\partial x_i \partial x_j - f(x,u,p),$$

where the principal assumptions are: (1) The form a_{ij} is positive definite at every point x, p; (2) For suitable β, γ, $|f(x,u,p)| \leq \beta|p|^{2\gamma}$ and $\partial f/\partial u \geq 0$. One result is that if D is an unbounded domain contained in a half-space of n-dimensional Euclidean space, then, if $L[u] \geq 0$ and if the upper limit of $u(x)$ is nonpositive as x approaches any point on the boundary of D, and if $\limsup_{r\to\infty}(M(r)/r^{\eta}) \leq 0$ for some $\eta > 0$ (where $M(r) = \text{lub } u(x)$, for $|x| = r$, $x \in D$), one may conclude that $u(x) \leq 0$ throughout D. Further, one can take $\eta = 1$, if $f \equiv 0$ and $n = 2$, or if D is contained in the cone $x_n \geq r \sin\delta$, for some δ, $0 < \delta < \pi/2$.

This paper is an extension of previous papers and methods of proof by D. Gilbarg [J. Rational Mech. Anal. 1 (1952), 411–417; MR 14, 279], E. Hopf [ibid. 1 (1952), 419–424; MR 14, 279], J. B. Serrin [ibid. 3 (1954), 395–413; MR 16, 42], and A. Friedman [Pacific J. Math. 7 (1957), 1563–1575; MR 20 #7142]. Reference should have also been made to the extension of the Hopf maximum principle given by R. Redheffer [Monatsh. Math. 62 (1958), 76–83; MR 19, 1041; ibid. 66 (1962), 32–42; MR 26 #446].

R. Barrar (Santa Monica, Calif.)

Simoda, Seturo 6162

Traité sur la théorie des équations elliptiques et semi-linéaires. IV.

Mem. Osaka Univ. Lib. Arts Ed. Ser. B No. 12 (1963), 1–9.

Part III appeared in same Mem. No. 11 (1962), 1–22 [MR 28 #1372]. This portion of the author's series of papers contains an existence theorem for the Dirichlet

problem for a semi-linear elliptic differential equation of the form

$$(*)\qquad \sum_{i,j=1}^{n} a_{ij}(x)\partial_{ij}{}^2 u(x) = f(x, u, \nabla u).$$

There are several conditions on f, one of the principal ones being that there exist constants β and γ such that $|f(x, u, p)| \le \beta|p|^2+\gamma$ when (x, u, p) are suitably restricted. The author obtains an a priori bound for the gradient of a solution of (*). He then states an existence theorem. The proof is not given but is said to be essentially that contained in the work of M. Nagumo [Osaka Math. J. 6 (1954), 207–229; MR 16, 1116].

R. C. MacCamy (Pittsburgh, Pa.)

Talenti, Giorgio 6163

Intorno alle classi funzionali di Gevrey. (English summary)

Ann. Mat. Pura Appl. (4) **63** (1963), 151–173.

The classes of infinitely differentiable functions defined by inequalities of the form $|g^{(n)}(x)| \le LI^n\Gamma(mn+1)$, introduced by Gevrey in 1918, have again become fashionable, in part due to Hörmander's result [*Linear partial differential operators*, Chapter IV, § 4, Academic Press, New York, 1963; MR 28 #4221], according to which the solution of a hypo-elliptic partial differential equation belongs to such a class. For further recent applications see, e.g., Friberg's thesis [Medd. Lunds Univ. Mat. Sem. 17 (1963); MR 28 #349] and a lecture by Leray [Leray and Ohya Séminaire sur les Équations aux Dérivées Partielles, Fasc. I. Exposé 2°, pp. 20–71, Collège de France, Paris, 1964].

Let E be a subset of N-dimensional euclidean space R_N and $m=(m_1, \cdots, m_N)$ an N-tuple of real numbers. The author considers the class G^m of infinitely differentiable functions g on E such that for every compact subset K of E there exists $L>0$ which verifies

$$(*)\qquad |D^n g(x)| \le L^{|n|+1}n_1{}^{m_1n_1}\cdots n_N{}^{m_Nn_N}$$

for all N-tuples $n=(n_1, \cdots, n_N)$ of positive integers and all $x\in K$, where

$$|n| = n_1+\cdots+n_N \quad\text{and}\quad D^n = \partial^{|n|}/\partial x_1{}^{n_1}\cdots\partial x_N{}^{n_N}.$$

If the m_i are ≥ 0, then (*) is equivalent to $|D^n g(x)| \le LI^{|n|}\Gamma(\langle m, n\rangle+1)$ for all n and all $x\in K$, where $\langle m, n\rangle = m_1n_1+\cdots+m_Nn_N$. Given $L>0$, $l>0$, $h\ge 0$ and $m=(m_1, \cdots, m_N)$ with $m_i\ge 0$, the author also introduces the class $G(m, l; L, h)$ of functions which verify $|D^n g(x)| \le Ll^{|n|}\Gamma(\langle m, n\rangle+h+1)$ for all n and all $x\in E$. He proves that $g\in G(m, l; L, h)$ implies $D^n g\in G(m, l; Ll^{|n|}, h+\langle m, n\rangle)$ and $f\in G(m, l; L, h)$, $g\in G(m, l; M, k)$ with $m_i\ge 1$ imply $fg\in G(m, l; LM/(h+k+1), h+k+1)$. For the case $N=1$, $m\ge 0$, the author characterizes the functions of G^m by the behavior of their Fourier transforms and their Fourier coefficients.

Next he shows that if $p>q$, $1\le m<p/q$, $H(x, y, t)$ is a function defined in $a\le x\le b$, $|y|\le r$, $t(t-y)\le 0$, such that with respect to the variable x the function $(y-t)^{1-q}H$ belongs to G^m for every y and t, and $g(x, y)$ ($a\le x\le b$, $|y|\le r$) belongs to G^m with respect to x for every y, then the integro-differential equation $u(x, y)=g(x, y)+\int_0^y (H(x, y, t)\partial^p u(x, t)/\partial x^p)\,dt$ has a unique solution $u(x, y)$ which belongs to G^m with respect to x. The solution is given by a Neumann series. There is a substitute result for the case $m=p/q$.

Finally, the author proves that the partial differential operator $e^{i\lambda\pi/2}\,\partial^m/\partial x_1{}^m+\partial^n/\partial x_2{}^n$ is hypo-elliptic in the following two cases and only in these: (1) $m+\lambda-n$ is not an even integer, (2) m, n and $\frac{1}{2}(m+\lambda-n)$ are even integers.

J. Horváth (Vanves)

Talenti, Giorgio 6164

Un problema di Cauchy.

Ann. Scuola Norm. Sup. Pisa (3) **18** (1964), 165–186.

We keep the notation of the preceding review [#6163]. Let now E be the closure of an open subset of R_N, $N\ge 1$ an integer, $p=(p_1, \cdots, p_N)$, $p_i\ge 0$, $f(x, t)$ a continuous function for $x\in E$, $|t|\le T$, and $u_r(x)$ functions of class C^{n-r} ($r=0, \cdots, n-1$) for $x\in E$. Set $P_{n-i}(D)=\sum_{\langle p,h\rangle\ge n-i} a_{hi}D^h$ ($i=0, \cdots, n-1$), where the coefficients a_{hi} are constant. The author proves that the Cauchy problem

$$\frac{\partial^n u}{\partial t^n} = \sum_{i=0}^{n-1} P_{n-i}(D)\frac{\partial^i u}{\partial t^i}+f, \qquad \frac{\partial^r}{\partial t^r}u(x, 0) = u_r(x)$$

$$(r = 0, \cdots, n-1)$$

has one and only one solution $u(x, t)$ for which $u(x, t)-\sum_{r=0}^{n-1}(u_r(x)t^r/r!)$ belongs to G^p with respect to x, provided (1) the function

$$g(x, t) = f(x, t)+\sum_{i=0}^{n-1}\sum_{r=i}^{n-1} P_{n-i}(D)u_r(x)\frac{t^{r-i}}{(r-i)!}$$

belongs to G^p with respect to x, (2) if

$$l > \limsup_{|n|\to\infty}\left(\max_{\substack{x\in E\\ |t|\le T}}\frac{|D^n g(x, t)|}{\Gamma(\langle m, p\rangle+1)}\right)^{1/|n|},$$

then $\sum_{i=1}^{n}\sum_{\langle p,h\rangle\ge i}|a_{h,n-i}|l^{|h|}T^i<1$. An analogous result holds for variable coefficients a_{hi}. The method consists in transforming the Cauchy problem into the integro-differential equation

$$(1)\quad v(x, t) = g(x, t)+\sum_{i=1}^{n} P_i(D)\int_0^t \frac{(t-\tau)^{i-1}}{(i-1)!}\,v(x, \tau)\,d\tau$$

with the help of the substitution $v=\partial^n u/\partial t^n$.

J. Horváth (Vanves)

Talenti, Giorgio 6165

Studio di una equazione integrodifferenziale non lineare.

Matematiche (Catania) **18** (1963), 129–138.

Let $G^{p,r}$ ($p\ge 0$, $r>0$) be the Banach space of all functions $u(x, t)$ defined in $R=\{(x, t)|\ 0<x<1,\ |t|<1\}$ such that $\partial^i u/\partial x^i$ is continuous and $(1-|t|)^{pi+1}(\partial^i u/\partial x^i)$ is bounded in R for $i=0, 1, \cdots$, and for which the norm

$$\|u\| = \sup_i \frac{r^{-i}}{\Gamma(pi+1)}\sup_{(x,t)\in R}(1-|t|)^{pi+1}\left|\frac{\partial^i u}{\partial x^i}(x, t)\right|$$

is finite. The author shows that the equation

$$u(x, t) = g(x, t)+\left[a+\frac{b}{4}\int_0^1 dx\int_{-1}^{1}(1-t^2)u(x, t)\,dt\right]$$

$$\times\frac{\partial}{\partial t}\int_0^t \frac{(t-\tau)^s}{\Gamma(s+1)}\frac{\partial^m u}{\partial x^m}(x, \tau)\,d\tau,$$

which, for $b=0$, $s=n$, becomes a particular case of (1) of the preceding review [#6164], has a solution $u(x,t)$ belonging to $G^{s/m,r}$, provided $|a|^{1/m}r<1$, $g\in G^{s/m,r}$ and $4|b|\cdot\|g\|r^m<(1-|a|r^m)^2$. The proof is based on the Banach-Caccioppoli contraction principle and on the observation that $(\partial/\partial t)\int_0^t((t-\tau)^p/\Gamma(p+1))(\partial/\partial x)\,d\tau$ is a continuous operator from $G^{p,r}$ into itself with norm $\leq r$.

J. Horváth (Vanves)

Tsutsumi, Akira 6166

On the uniqueness of the Cauchy problem for semi-elliptic partial differential equations. I, II.

Proc. Japan Acad. **39** (1963), 781–786; *ibid.* **39** (1963), 787–790.

The author studies a subclass of semi-elliptic differential operators, that is to say, operators of the form

$$P(x,D)=P_0(x,D)+Q(x,D),$$

$$P_0(x,D)=\sum_{|p:m|=1}a_p(x)D^p,$$

$$Q(x,D)=\sum_{j=1}^{n}\sum_{|p:m|<1-1/m_j}a_p(x)D^p,$$

where $D=((1/i)\partial/\partial x_1,\cdots,(1/i)\partial/\partial x_n)$, $m=(m_1,\cdots,m_n)$, m_j are positive integers, $|p:m|=(p_1/m_1)+\cdots+(p_n/m_n)$. The semi-ellipticity of $P(x,D)$ is expressed by the fact that the polynomial in $\xi=(\xi_1,\cdots,\xi_n)$, $P_0(x,\xi)$, has no zeros $\xi\in R^n$ which are different from zero. Under some further assumptions, two weighted L^2 estimates are proved, and the uniqueness of the Cauchy problem with data on a non-characteristic surface is derived. The methods are adapted from those of L. Hörmander [Math. Scand. 7 (1959), 177–190; MR **22** #12306].

F. Treves (Lafayette, Ind.)

Tsutsumi, Akira 6167

On the uniqueness of the Cauchy problem for semi-elliptic partial differential equations. III.

Proc. Japan Acad. **40** (1964), 259–261.

This is an improvement of the two papers above [see #6166] to the effect that one of the conditions imposed upon the differential operator $P(x,D)$, used to prove the main theorem in those papers, is superfluous. This main theorem is shown to be valid for semi-elliptic operators of multi-order $(m_1,\cdots,m_n)$ with $m_1\geq m_j$ $(2\leq j\leq n)$, such that, for any non-zero real vector $\xi'=(\xi_2,\cdots,\xi_n)$, all the (complex) roots ζ_1 of $P(x^0,\zeta_1,\xi')$ are simple.

F. Treves (Lafayette, Ind.)

Zachmanoglou, E. C. 6168

An example of slow decay of the solution of the initial-boundary value problem for the wave equation in unbounded regions.

Bull. Amer. Math. Soc. **70** (1964), 633–636.

The solution of the equation $\Delta u-u_{tt}=0$ is considered in a domain $\mathcal{D}$ containing ∞ and bounded by $\mathcal{B}$, which consists of two planes intersecting at an angle α. When $\alpha=\pi/k$, $k=1,2,\cdots$, the solution of the initial-value problem satisfying $u=0$ or $\partial u/\partial n=0$ on $\mathcal{B}$ can be constructed by the reflection principle, and if the initial data have compact support, the solution is identically zero after a finite time. The author shows that if $\alpha\neq\pi/k$, the solution can be expressed as an integral over the initial data and decays pointwise for $\alpha=2\pi/(2k+1)$ like t^{-2k-3}. This demonstrates that one cannot expect exponentially decaying solutions in the exterior of star-shaped bodies if these bodies extend to infinity (proved for finite bodies by Lax, Morawetz and Phillips [same Bull. **68** (1962), 593–595; MR **26** #457]), and that the rate of decay is very sensitive to shape.

C. S. Morawetz (New York)

Leis, Rolf 6169

Zur Eindeutigkeit der Randwertaufgaben der Helmholtzschen Schwingungsgleichung.

Math. Z. **85** (1964), 141–153.

The purpose of this paper is to provide material for the extension of existence and uniqueness theorems for the Dirichlet and Neumann problems for the Helmholtz equation. The extension is from smooth boundaries to boundaries having a finite number of discontinuities in direction. The material is applicable to both interior and exterior problems. Specifically, the author derives the orders of magnitude of several integrals evaluated in the neighborhood of an angle where the boundary curve bends. These integrals are important in the existence and uniqueness theorems.

E. Pinney (Berkeley, Calif.)

Werner, Peter 6170

Über die Randwertprobleme der Helmholtzschen Schwingungsgleichung.

Math. Z. **85** (1964), 226–240.

In this paper the author establishes that the solution of the exterior Dirichlet problem for the Helmholtz equation can be approximated by a suitable interior problem. The precise statement of the theorem is as follows. Let F be a twice continuously differentiable closed surface separating E_3 into an interior domain G_i and an exterior domain G_a. On F suppose $g(x)$ is a given continuous function. Let K_r be a sphere, with center the origin and radius r, such that F is contained in K_r for $r>r_0$. Let G_r be the domain exterior to F and interior to K_r. Now if $U_r(x)$, where $r>r_0$, is a function with the properties (a) U_r is twice continuously differentiable in G_r and satisfies $(\Delta+k^2)U_r=0$ with k real; (b) U_r is continuous in G_r+F and on F satisfies $U_r=g$; (c) U_r is continuously differentiable in G_r+K_r and on K_r satisfies the boundary condition $(\partial/\partial r-ik)U_r=0$; then U_r exists and is unique. Further, if $k\neq0$, for $x\in G_a$, $U_r(x)\to U(x)$ as $r\to\infty$, where $U(x)$ is the solution of $(\Delta+k^2)U=0$ in G_a, $U(x)=g$ for x on F and $U(x)$ satisfies the Sommerfeld radiation condition. The convergence is proven to be uniform in any compact subdomain of G_a.

Extensions of the theorem are discussed. For instance, (b) can be replaced by the mixed boundary condition $\partial U_r/\partial n+hU_r=g$, where g and h are given continuous functions. The proof can be extended to Maxwell's equations with sinusoidal time dependence, provided Sommerfeld's radiation condition is replaced by the electromagnetic radiation condition.

R. A. Ross (Toronto, Ont.)

★Autovalori e autosoluzioni. 6171

2° Ciclo, Chieti, 1–9 agosto 1962. Centro Internazionale Matematico Estivo (C.I.M.E.).

Edizioni Cremonese, Rome, 1963. **234** *pp. (not consecutively numbered)* *L.* **2.500.**

This monograph is a report of the lecture series at the 1962 summer institute of the Centro Internazionale Matematico Estivo (C.I.M.E.). The topics included are as follows. S. Agmon, On eigenvalues, eigenfunctions and resolvents of general elliptic problems; A. M. Ostrowski, Il metodo del quoziente di Rayleigh; L. E. Payne, Isoperimetric inequalities for eigenvalues and their applications; L. De Vito, Calcolo degli autovalori e delle autosoluzioni per operatori non autoaggiunti; and Sul calcolo per difetto e per eccesso degli autovalori delle trasformazioni hermitiane compatte e delle relative molteplicità; J. B. Diaz, Upper and lower bounds for the torsional rigidity and the capacity, derived from the inequality of Schwarz; and M. Schiffer, Fredholm eigenvalues and conformal mapping.

W. H. Fleming (Providence, R.I.)

Ash, M. E. 6172

The basic estimate of the $\bar\partial$-Neumann problem in the non-Kählerian case.

Amer. J. Math. **86** (1964), 247–254.

Consider a finite domain $\Omega \Subset \mathbf{R}^n$. Let E_1, E_2, E_3 be unitary vector spaces and $\mathfrak{E}_j$ the E_j-valued C^∞ functions on $\bar\Omega$. For $\varphi, \psi \in \mathfrak{E}_j$, set $(\varphi, \psi) = \int_\Omega \langle\varphi, \psi\rangle$. Suppose that $D : \mathfrak{E}_1 \to \mathfrak{E}_2$, $D^1 : \mathfrak{E}_2 \to \mathfrak{E}_3$ are constant-coefficient differential operators. Then the adjoints D^*, D^{1*} of D, D^1 are defined, as is the Laplace operator $L = D^{1*}D^1 + DD^*$, on a subspace $\mathscr{L} \subset \mathfrak{E}_2$. Assume that $\mathfrak{E}_1 \xrightarrow{D} \mathfrak{E}_2 \xrightarrow{D^1} \mathfrak{E}_3$ is such that L is elliptic. Then integration by parts leads directly to the following formula: For $\varphi \in \mathscr{L}$,

$$(1)\quad (L\varphi, \varphi) = \|D^1\varphi\|^2 + \|D^*\varphi\|^2 = \int_\Omega Q\langle\nabla\varphi, \nabla\varphi\rangle + \int_{\partial\Omega} L\langle\varphi, \varphi\rangle,$$

where $Q\langle\nabla\varphi, \nabla\varphi\rangle$ is a quadratic form in φ and its first derivatives and $L\langle\varphi, \varphi\rangle$ is an ordinary quadratic form. If Ω is convex, then $L\langle\varphi, \varphi\rangle \geqq c|\varphi|^2$ $(c > 0)$. Furthermore, if φ has compact support, $\int_\Omega Q\langle\nabla\varphi, \nabla\varphi\rangle \geqq \text{const} \int_\Omega |\nabla\varphi|^2$. Furthermore, in certain favourable cases, $Q\langle\nabla\varphi, \nabla\varphi\rangle \geqq 0$ and (1) becomes

$$(2)\qquad (L\varphi, \varphi) \geqq \text{const} \int_\Omega L\langle\varphi, \varphi\rangle.$$

This happens, for instance, if $n = 2m$, $\mathbf{R}^n = \mathbf{C}^m$, $E_j = \Lambda^{q+j}\mathbf{C}^m$, $\mathfrak{E}_j$ are $C^\infty(0, q+j)$ forms on $\bar\Omega$, and $D = \bar\partial = D^1$. In this case, $Q\langle\nabla\varphi, \nabla\varphi\rangle = |\bar\nabla\varphi|^2 = \sum_{i=1}^m |\partial\varphi/\partial\bar z_i|^2$ and L is essentially the E. E. Levi form of Ω. Another good case is when $E_j = \Lambda^{q+j}\mathbf{R}^n$, $\mathfrak{E}_j$ are $C^\infty(0, q+j)$ forms on $\bar\Omega$, and $D = d = D^1$ is the exterior derivative. Then $Q\langle\nabla\varphi, \nabla\varphi\rangle = |\nabla\varphi|^2$ and L is the second fundamental form of the hypersurface $\partial\Omega \subset \mathbf{R}^n$.

Now let X be a manifold, $\Omega \Subset X$ a finite open domain, E_1, E_2, E_3 vector bundles over Ω, $\mathfrak{E}_j$ the C^∞ sections of E_j over $\bar\Omega$, and $D : \mathfrak{E}_1 \to \mathfrak{E}_2$, $D^1 : \mathfrak{E}_2 \to \mathfrak{E}_3$ differential operators. By putting metrics in the E_j and on X, we may define $(\varphi, \psi) = \int_\Omega \langle\varphi, \psi\rangle$ $(\varphi, \psi \in \mathfrak{E}_j)$. Then D^*, D^{1*}, and $L = D^*D + D^1D^{1*}$ are defined. Given any point $x_0 \in X$, the bundle operators induce an osculating constant-coefficient situation on the fibres $(E_j)_{x_0}$ and tangent space at x_0, and the essential content of this paper is to show how, in some special favorable cases, the constant-coefficient inequality (2) may be reduced to the analogous statement on X. The necessary computations, which are of a differential-geometric nature, are efficiently executed by using the "moving frame" calculus of E. Cartan.

In closing, there are a few historical comments which seem relevant. The genesis of the inequality (2), for the $\bar\partial$-operator and L the E. E. Levi form, is found in C. B. Morrey, Jr. [Ann. of Math. (2) **68** (1958), 159–201; MR **20** #5504]. The general inequality was obtained and extensive applications to several complex variables were given in J. Kohn's treatment of the so-called $\bar\partial$-Neumann problem [ibid. (2) **78** (1963), 112–148; MR **27** #2999]. (The basic formalism in this question and the ensuing boundary-value problem were set up by D. C. Spencer.) On the other hand, there is a formula in Riemannian geometry [cf. S. Bochner and K. Yano, *Curvature and Betti numbers*, Chapter III, Princeton Univ. Press, Princeton, N.J., 1953; MR **15**, 989] which, even for general manifolds, gives (1) in the case that E_j is the bundle of $q+j$ forms and $D = d = D^1$; then there must be added to the right-hand side of (1) a term $R(\varphi, \varphi) = \int_\Omega R\langle\varphi, \varphi\rangle$, where R is the Riemannian curvature on X. Here $L = \mathrm{II}$ is the second fundamental form of $\partial\Omega$ in X. This has been used by C. C. Hsiung [Math. Z. **82** (1963), 67–81; MR **27** #2943] to obtain vanishing theorems for ordinary cohomology on Ω.

Phillip A. Griffiths (Berkeley, Calif.)

Burnat, M. 6173

Über die Spektraldarstellung des Operators $-\Delta u + q(x_1, x_2, x_3)u$ mit dreifach periodischen $q(x_1, x_2, x_3)$ im Raume der fastperiodischen Funktionen.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 255–259.

In a previous paper [same Bull. **10** (1962), 247–253; MR **26** #6476] the author considered an eigenvalue problem motivated by a one-dimensional crystal and employs the Hilbert space B^2 of Besicovitch almost periodic functions to establish some theorems on the spectrum of the operator $-u'' + q(x)u$ (with periodic $q(x)$). In the present paper similar theorems are briefly stated for the three-dimensional case, and the reader is referred to another paper [Studia Math. **25** (1964), 33–64] for further details.

G. H. Meisters (Boulder, Colo.)

Burnat, M. 6174

Die wesentliche Selbstadjungiertheit des Operators $Lu = -\Delta u + q(x_1, x_2, x_3)u$ mit unendlich vielen Singularitäten in $q(x_1, x_2, x_3)$.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 261–264.

The author remarks that Nils Nilsson [Kungl. Fysiogr. Sällsk. i Lund Förh. **29** (1959), no. 1, 1–19; MR **21** #3652] and others have shown that the operator $-\Delta u + q(x)u$ is essentially self-adjoint if $q(x)$ possesses only a finite number of singular points. Then, motivated by the quantum theory of crystals, the author considers the case that $q(x)$ has infinitely many singular points and proves a similar result.

G. H. Meisters (Boulder, Colo.)

Frasca, Michele 6175

Un problema variazionale per operatori ellittici.

Matematiche (Catania) **18** (1963), 1–11.

Suppose the bounded domain A is of class C^2 and φ is of

class C^2 on the boundary ∂A. Let $\mathscr{L}_{\alpha\gamma\delta}$ be a class of elliptic operators L of the form

$$Lu \equiv au_{xx}+2bu_{xy}+cu_{yy}+a_1u_x+b_1u_y+c_1u,$$

where the coefficients are bounded and measurable on A and satisfy

$$a\lambda^2+2b\lambda\mu+c\mu^2 \geq \alpha(\lambda^2+\mu^2),$$
$$a+c = 1+\alpha,\ 0<\alpha\leq 1,$$
$$a_1^2+b_1^2 \leq \delta^2, \quad -\gamma\leq c_1\leq 0, \quad \gamma,\delta\geq 0.$$

If f is given, there is a unique solution $u\in C_\mu^1(A)\cap H_2^2(A)$ of $Lu=f$ for which $u=\varphi$ on ∂A. The author shows that if $(x_0,y_0)\in G$, there are unique functions u_1 and u_2 such that $u_1(x_0,y_0)\leq u(x_0,y_0)\leq u_2(x_0,y_0)$ for all solutions u for all $L\in\mathscr{L}_{\alpha\gamma\delta}$. The same functions work for every (x_0,y_0) in A and satisfy certain non-linear equations.

C. B. Morrey, Jr. (Berkeley, Calif.)

Giesecke, Burghart 6176

Zum Dirichletschen Prinzip für selbstadjungierte elliptische Differentialoperatoren.

Math. Z. **86** (1964), 54–62.

Green's theorem and the consequent Dirichlet and Thompson principles are proved in a slightly non-standard manner. The proof holds for an arbitrary, not necessarily bounded, domain.

A significant result is contained in the proof of Theorem 4. Namely, for an arbitrary bounded domain G, any function in $C^1(G)\cap C^0(\bar{G})\cap H^1(G)$ which vanishes on ∂G is in $H_0^1(G)$. In fact, the same result is proved for an unbounded domain with respect to the norm

$$\left\{\int_G [a^{ij}u_{x_i}u_{x_j}+qu^2]\,dx\right\}^{1/2}$$

when $q\in L^2(G)$ and the function approaches zero at infinity.

H. F. Weinberger (Minneapolis, Minn.)

Kreiss, Heinz-Otto 6177

Über sachgemässe Cauchyprobleme.

Math. Scand. **13** (1963), 109–128.

Given a linear partial differential equation written in matrix form, $\partial u/\partial t=P(x,t;\partial/\partial x)u$, we say that the Cauchy problem is well-posed if there exists a constant C such that, for all t_1,t_2, $0\leq t_1\leq t_2\leq T$, we have $\|u(x,t_2)\|_{L^2}\leq C\|u(x,t_1)\|_{L^2}$. The author considers this problem along the ideas of Petrowsky and Leray. Some precise results are obtained in the case where the coefficients are constant. Next, parabolic and hyperbolic equations are discussed in detail from this point of view. Finally, the author shows that the treatment of hyperbolic equations can be extended to the case where the matrix $\sum_{k=1}^n A_k(x,t)\omega_k$ has a simple structure and has only real eigenvalues. This last question is closely connected with the reviewer's paper [Mem. College Sci. Univ. Kyoto Ser. A Math. **32** (1959), 181–212; MR **23** #A420].

S. Mizohata (Kyoto)

Kumano-go, Hitoshi 6178

On the uniqueness for the solution of the Cauchy problem.

Osaka Math. J. **15** (1963), 151–172.

The uniqueness theorem of the solutions of the Cauchy problem for the differential equation

$$Lu = \sum_{j+|\alpha|\leq m} a_{j,\alpha}(t,x)\frac{\partial^{j+|\alpha|}}{\partial t^j\partial x^{|\alpha|}}u(t,x) = f(t,x)$$

has been investigated by many authors. However, these investigations do not cover some interesting cases. For example, the reviewer proved the following result [Mem. College Sci. Univ. Kyoto Ser. A Math. **31** (1958), 219–239; MR **21** #5081]: The parabolic equation

$$Lu = \left(\sum_{i,j=1}^n a_{ij}(t,x)\frac{\partial^2}{\partial x_i\partial x_j} + \sum b_i(t,x)\frac{\partial}{\partial x_i}+c(t,x)-\frac{\partial}{\partial t}\right)u(t,x) = f(t,x),$$

with data prescribed on a piece of time-like hypersurface, has the unique continuation property.

In this paper, the author proposes a systematic treatment of such equations. Let

$$Lu = L_0u + \sum_{j+m|\alpha:m|\leq m-1} b_{j,\alpha}(t,x)\frac{\partial^{j+|\alpha|}}{\partial t^j\partial x^{|\alpha|}}u(t,x) = f(t,x),$$

where $|\alpha:m| = \alpha_1/m_1+\alpha_2/m_2+\cdots+\alpha_\nu/m_\nu$, $m\geq m_i$, and

$$L_0u = \sum_{j+m|\alpha:m|=m} a_{j,\alpha}(t,x)\frac{\partial^{j+|\alpha|}}{\partial t^j\partial x^{|\alpha|}}u(t,x),$$

which is called the principal part of L. The author establishes a uniqueness theorem for this type of equation after extending the notion of singular integral operators. The results obtained here can be applied to a large class of operators.

S. Mizohata (Kyoto)

Matsumura, Hideyuki 6179

On an algebraic lemma of L. Hörmander. (Japanese)

Sûgaku **13** (1961/62), 159–160.

The author, after showing the invalidity of Lemma 3.2 of L. Hörmander's thesis [Acta Math. **94** (1955), 161–248; MR **17**, 853] by a counter-example: $R=k[x^2,x^3]$, gives another proof of the following generalized form of Lemma 3.3: If $f_1(X),\cdots,f_r(X)\in k[X_1,\cdots,X_n]$, and if $k(f_1,\cdots,f_r)$ is of degree of transcendency 1 over k, then there exists a polynomial W such that $k(f_1,\cdots,f_r)=k(W)$, $k[f_1,\cdots,f_r]\subset k[W]\subset k[X_1,\cdots,X_n]$. Using this lemma, the author shows that all defects concerning Lemmas 3.2 and 3.3 in Hörmander's paper are covered.

S. Mizohata (Kyoto)

Odeh, Farouk; Keller, Joseph B. 6180

Partial differential equations with periodic coefficients and Bloch waves in crystals.

J. Mathematical Phys. **5** (1964), 1499–1504.

Let V be a real continuous function on R_3, and (a_{ij}) a symmetric differentiable matrix such that the operator $-\sum_{i,j=1}^3(\partial/\partial x_i)(a_{ij}\,\partial/\partial x_j)+V$ is uniformly elliptic. Let H be its unique self-adjoint extension on $L_2(R_3)$. Require it to be invariant under the discrete group of translations of R_3 generated by three orthogonal vectors. Ω^* will be a closed unit cell in the reciprocal lattice defined by these three vectors.

The authors justify rigorously the following facts from a generalization of Floquet's theory, which has been

widely used in the theory of solids. There exists a complete set of eigenfunctions ("Bloch waves") of the form $\exp(2\pi ik\cdot x)\phi_n(x,k)$, where k varies over Ω^* and ϕ_n is a smooth function of x invariant under the discrete group. The functions $\exp(2\pi ik\cdot x)\phi_n(x,k_0)$ ("Kohn-Luttinger waves") are also complete for any fixed k_0 in Ω^*. The spectrum of H is a countable union of closed intervals. Then the analytic dependence on k of the eigenfunctions and corresponding eigenvalues is discussed.

J. M. Cook (Argonne, Ill.)

Schechter, Martin 6181

Systems of partial differential equations in a half-space. *Comm. Pure Appl. Math.* **17** (1964), 423–434.

Boundary problems for systems of partial differential equations with constant coefficients in a half-space $x_n>0$ of R^n are studied. An inequality of the type

$$\left\|\sum_j R_j(D)u_j\right\| \leq C\left(\sum_i\left\|\sum_j P_{ij}(D)u_j\right\|+\sum_j\|u_j\|\right),$$

where $\|\ \|$ is the L_2 norm, for functions $u\in C^\infty$ in $x_n\geq 0$ and with compact support also satisfying the boundary condition

$$\sum Q_{ij}u_j=0,\qquad x_n=0,$$

is established under proper assumptions on P_{ij} and Q_{ij}, the R_j being essentially "weaker" (in Hörmander's sense) than the P_{ij}. This extends earlier work in the case of one single equation [the reviewer, Math. Scand. **9** (1961), 337–351; MR **26** #4039; the author, Ann. Scuola Norm. Sup. Pisa (3) **18** (1964), 255–282]; the method of proof is similar.

J. Peetre (Lund)

Dionne, Philippe A. 6182

Problème de Cauchy hyperbolique et espaces fonctionnels de S. L. Sobolev. *Les Équations aux Dérivées Partielles (Paris, 1962)*, pp. 25–31. *Éditions du Centre National de la Recherche Scientifique, Paris*, 1963.

Comme on sait, pour la résolution des equations hyperboliques quasi-linéaires, l'estimation a priori joue un rôle essentiel. Il s'agit ici de résoudre, dans une bande et sous les hypothèses les plus faibles, le problème de Cauchy régulièrement hyperbolique de façon à obtenir les résultats locaux les plus forts possibles.

Le principal résultat obtenu ici est qu'il suffit de pouvoir majorer a priori les dérivées d'ordre $m+\frac{1}{2}(l+3)$ (J. Leray avait obtenu $m+l+3$) de la solution d'un problème non linéaire d'ordre m dans un domaine de dimension l pour avoir un théorème global d'existence.

Pour montrer ce résultat, l'auteur utilise des propriétés assez fines des espaces fonctionnels de Sobolev.

S. Mizohata (Kyoto)

Bruhat, Yvonne 6183

Un théorème d'unicité de solutions faibles d'équations hyperboliques. *C. R. Acad. Sci. Paris* **258** (1964), 3949–3951.

Un théorème d'unicité pour les solutions faibles de certaines équations hyperboliques linéaires et non linéaires est démontré. L'auteur utilise le principe de Holmgren et un résultat de Leray. Comme corollaire le résultat suivant est montré: Le problème de Cauchy sur une hypersurface spatiale pour l'équation

$$\square u+u^p=f,\qquad \square=\frac{\partial^2}{\partial x_0^2}-\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2},$$

où f est une distribution arbitraire, a une solution unique $u\in W_2^1\cap L_{2p-1}$.

S. Mizohata (Kyoto)

Chaillou, J. 6184

Opérateurs hyperboliques à partie principale tendant vers zéro. Comportement de la solution élémentaire. *Les Équations aux Dérivées Partielles (Paris, 1962)*, pp. 19–24. *Éditions du Centre National de la Recherche Scientifique, Paris*, 1963.

Soit $a_\varepsilon(\partial/\partial x,\partial/\partial y)=\varepsilon g+a_0$, où $a_0=g'+a_1$ est un opérateur à deux variables à coefficients réels et constants. Supposons que (1) g est hyperbolique et homogène d'ordre m, (2) g' est quelconque et homogène d'ordre $m-1$, (3) a_1 est d'ordre $m-2$. Supposons de plus que $g(t,1)$ et $g'(t,1)$ n'ont pas de zéro commun. Désignons par $E_\varepsilon(x,y)$ la solution élémentaire de a_ε. L'auteur donne une condition nécessaire et suffisante pour que $E_\varepsilon(x,y)$ tende vers une limite finie, lorsque $\varepsilon\to 0$, en dehors des demi-droites caractéristiques de g et de g'. La démonstration est assez délicate. D'abord, l'auteur montre une expression explicite de $E_\varepsilon(x,y)$ au moyen des résidus et puis utilise un résultat de M. Morse.

S. Mizohata (Kyoto)

Gloistehn, H. H. 6185

Monotoniesätze und Fehlerabschätzungen für Anfangswertaufgaben mit hyperbolischer Differentialgleichung. *Arch. Rational Mech. Anal.* **14** (1963), 384–404.

This paper deals with a quasilinear hyperbolic operator

$$Tu\equiv -h^2(x,y)u_{xx}+u_{yy}+F(x,y,u,u_x,u_y),$$

and a related first-order system defined by

$$(1)\qquad \begin{aligned} L_{11}u &\equiv hu_x+u_y+\alpha_1u=v_1,\\ L_{12}u &\equiv -hu_x+u_y+\alpha_2u=v_2,\end{aligned}$$

and

$$(2)\qquad \begin{aligned} hv_{1x}+v_{1y}+\beta_1v_1 &= Tu+G_1(x,y,u,v_1,v_2),\\ hv_{2x}+v_{2y}+\beta_2v_2 &= Tu+G_2(x,y,u,v_1,v_2).\end{aligned}$$

Here α_j and β_j $(j=1,2)$ are arbitrary functions, and G_1 and G_2 are defined by (2) and (1). The principal result of the paper is the following monotony theorem. Let B be a triangle in the (x,y)-plane, bounded by two characteristic arcs for T and a non-characteristic arc Γ. Assume that $G_j(x,y,u,v_1,v_2)$ is monotone nondecreasing in u, v_1 and v_2 $(j=1,2)$ and $\alpha_j(x,y)\geq 0$ on Γ $(j=1,2)$. Then $Tu<Tw$ in B, $\pm hu_x+u_y<\pm hw_x+w_y$ on Γ implies $u<w$, $L_{11}u<L_{11}w$ and $L_{12}u<L_{12}w$ in $B+\Gamma$. (The precise differentiability assumptions on u, v, h, $\alpha_j(x,y)$, $\beta_j(x,y)$ and $F(x,y,u,u_x,u_y)$ which are given by the author have been omitted for brevity.)

The monotony theorem is used to derive a number of error estimates for approximate solutions of initial-value problems for the operator T. The results generalize theorems of S. Agmon, L. Nirenberg and M. H. Protter [Comm. Pure Appl. Math. **6** (1953), 455–470; MR **15**, 432], M. H. Protter [Trans. Amer. Math. Soc. **71** (1951), 416–429; MR **14**, 281; ibid. **87** (1958), 119–129; MR **20**

#4079], H. F. Weinberger [Ann. of Math. (2) 64 (1956), 505–513; MR 19, 1058] and W. Walter [Arch. Rational Mech. Anal. 7 (1961), 249–272; MR 23 #A2640].

C. H. Wilcox (Madison, Wis.)

Kolodner, Ignace I. 6186

On the Carleman's model for the Boltzmann equation and its generalizations.

Ann. Mat. Pura Appl. (4) **63** (1963), 11–32.

T. Carleman has shown that the system

$$(1)\qquad \begin{aligned} u_{+,t}+u_{+,x} &= u_-^2-u_+^2,\\ u_{-,t}-u_{-,x} &= u_+^2-u_-^2 \end{aligned}$$

has many features in common with the integro-differential equation of Boltzmann [T. Carleman, *Problèmes mathématiques dans la théorie cinétique des gaz*, Almqvist and Wiksells, Uppsala, 1957; MR 20 #4935]. The pair (u_+, u_-) may be interpreted as the velocity distribution function of a fictive one-dimensional gas whose molecules travel with only two velocities, $+1$ and -1, and interact according to the collisional terms on the right-hand side of (1). With this interpretation, $\rho=u_++u_-$ is the number density and $c=(u_+-u_-)/\rho$ is the average mass velocity. They satisfy the moment equations

$$(2)\qquad \begin{aligned} &\rho_t+(\rho c)_x = 0, &&\text{the equation of continuity, and}\\ &(\rho c)_t+\rho_x = -2\rho^2 c, &&\text{the equation of momentum.} \end{aligned}$$

The system (2) is equivalent to (1). Moreover, the study of (2) can be reduced to the study of a single second-order equation by observing that if ϕ solves

$$(3)\qquad \phi_{tt}-\phi_{xx} = -2\phi_x\phi_t,$$

then $(\rho, c)=(\phi_x, -\phi_t/\phi_x)$ solves (2). The quantity

$$H(t) = \int_{-\infty}^{\infty} (u_+\log u_+ + u_-\log u_-)\,dx$$

plays the role of the total entropy of the gas, and the analogue of the H-theorem, $H'(t)\leq 0$, holds for solutions of (1) which satisfy suitable regularity assumptions.

This paper deals with the Cauchy problem for the system (1). The local existence and uniqueness theory can be established by well-known techniques [see Chapter V of R. Courant and D. Hilbert, *Methods of mathematical physics*, Vol. II, Interscience, New York, 1962; MR 25 #4216]. In this paper the author proves a global existence theorem for (1) in the strip $R\times(0,a)$ and the half-plane $R\times(0,\infty)$. The results are derived by exploiting the fact that the solution of a certain Riccati equation depends isotonically (with respect to a suitable partial order) on the forcing term and the initial data. Generalizations of the theory to systems having several dependent variables are discussed in a final section.

C. H. Wilcox (Madison, Wis.)

Lewis, Robert M. 6187

Asymptotic methods for the solution of dispersive hyperbolic equations.

Asymptotic Solutions of Differential Equations and Their Applications (Proc. Sympos., Math. Res. Center, U.S. Army, Univ. Wisconsin, Madison, Wis., 1964), pp. 53–107. *Wiley, New York*, 1964.

A heuristic method is described for asymptotic solution as $\lambda\to\infty$ of a symmetric hyperbolic system

$$\sum_0^n A^\nu \partial u/\partial x_\nu+\lambda Bu = f(t,x,\lambda), \qquad u(0,x) = u_0(x,\lambda),$$

together with boundary conditions, where B is anti-symmetric. The method is applicable to the problem of dispersion of electromagnetic waves in dielectrics, among others. It is akin to J. B. Keller's ray method in diffraction theory [B. R. Levy and J. B. Keller, Comm. Pure Appl. Math. 12 (1959), 159–209; MR 21 #1130]. It may be roughly described as follows. In the case of problems that can be solved exactly, the asymptotic expansion of the solution reveals that it consists of a sum of terms, each of which is an asymptotic power series in λ^{-1} multiplied by a phase function, the coefficients in the series being amplitude functions. The basic assumptions of the author's theory is that for more complex problems the solutions have the same asymptotic form. The phase functions are then found by the method of characteristics, and the amplitude functions are found by solving the transport equations that must be satisfied if the asymptotic solution is to formally satisfy the system of partial differential equations. In some cases, the initial conditions necessary to determine the proper rays (characteristics), phase functions, and amplitude functions are obtained from the solution of a canonical problem (a solvable one with the same local features as the given problem). That this may be done is the author's second main hypothesis.

The method is illustrated by applying it to a number of initial-boundary-value problems for the homogeneous and inhomogeneous wave equation

$$u_{tt}-c^2(x)\Delta u+\lambda^2 b^2(x)u = \lambda^2 f(x,t,\lambda),$$

involving oscillatory initial data and oscillatory sources.

N. D. Kazarinoff (Ann Arbor, Mich.)

Walk, Manfred 6188

Über das Cauchysche Anfangswertproblem schwach nichtlinearer hyperbolischer Differentialgleichungen.

Math. Nachr. **27** (1963), 9–28.

This paper deals with the following initial-value problem:

$$(1)\qquad \sum_{i,k=1}^{2} a_{ik}\left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, u\right)\frac{\partial^2 u}{\partial x_i\partial x_k}-\frac{\partial^2 u}{\partial t^2} = F(x_1,x_2,t), \qquad (x_1,x_2,t)\in\Omega,$$

$$(2)\qquad u(x_1,x_2,0) = \phi(x_1,x_2), \qquad \frac{\partial u(x_1,x_2,0)}{\partial t} = \psi(x_1,x_2), \qquad (x_1,x_2)\in\Omega_0.$$

Here Ω is the frustrum of a cone in (x_1,x_2,t)-space with its axis parallel to the t-axis and of height T. Ω_0 is the base of Ω (the intersection of Ω with the plane $t=0$). The existence of a solution for sufficiently small T was proved by J. Schauder [Fund. Math. 24 (1935), 213–246] with the help of his fixed-point theorem. The same result was proved by S. L. Sobolev [*Some applications of functional analysis in mathematical physics* (Russian), Izdat. Leningrad. Gos. Univ., Leningrad, 1950; MR 14, 565], using modern functional analysis.

In this paper the author proves the existence of a solution for arbitrary values of T, under the assumption

that the nonlinearity is not too large. More precisely, he assumes that

$$\mathfrak{A}_{ik}\left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, u\right) = \delta_{ik} + \lambda A_{ik}\left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, u\right),$$

where $|A_{ik}(\xi_1, \xi_2, \xi_3)| \leq R$ for all i, k, ξ_1, ξ_2 and ξ_3, and λ is not too large. Estimates for the allowable values of λ are given.

The proof makes use of the linearized problem

$$(3) \quad \sum_{i,k=1}^{2} \mathfrak{A}_{ik}\left(\frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2}, z\right) \frac{\partial^2 v}{\partial x_i \partial x_k} - \frac{\partial^2 v}{\partial t^2} = F(x_1, x_2, t), \qquad (x_1, x_2, t) \in \Omega,$$

with initial conditions (2). If z is continuously differentiable in the closure of Ω, this defines a unique solution v, and the original problem (1), (2) is equivalent to finding a fixed point for the mapping $z \to v = \Phi(z)$. The author shows that if λ is not too large, Φ has a fixed point in a subset of the Sobolev space $W_2^{(r+1)}(\Omega)$ with $r \geq 4$. The proof is based on the following estimate for the solution of the linear problem (2), (3):

$$\|v\|_{W_2^{(\nu)}(\Omega)} \leq P_1(\|F\|_{W_2^{(\nu-1)}} + \|\phi\|_{W_2^{(\nu)}} + \|\psi\|_{W_2^{(\nu-1)}}) \times \exp\{\lambda P_2(1 + \|z\|_{W_2^{(r+1)}}) \exp N\|z\|_{W_2^{(r+1)}}\}$$

for $\nu = 0, 1, 2, \cdots, r+1$, where P_1, P_2 and N depend on Ω and the A_{ik} only.

As an example, the results are applied to the equation for the torsional oscillations of an elastic, circularly cylindrical shaft:

$$\frac{\partial}{\partial x}\left[f(T^2)\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left[f(T^2)\frac{\partial u}{\partial y}\right] - \frac{\partial^2 u}{\partial t^2} = F(x, y, t),$$

$$T^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2.$$

C. H. Wilcox (Madison, Wis.)

Agmon, Shmuel 6189

Problèmes mixtes pour les équations hyperboliques d'ordre supérieur.

Les Équations aux Dérivées Partielles (Paris, 1962), pp. 13–18. *Éditions du Centre National de la Recherche Scientifique, Paris,* 1963.

Soit $A(D_x, D_t)$ un opérateur différentiel hyperbolique d'ordre m à coefficients constants. L'auteur considère les problèmes aux limites de la forme:

$$A(D_x, D_t)u(x, t) = f(x, t) \qquad (0 \leq t \leq T, x_n \geq 0),$$

$$D_t^j u|_{t=0} = 0 \qquad (x_n \geq 0, j = 0, 1, \cdots, m-1),$$

$$B_j(D_x, D_t)u|_{x_n=0} = 0 \qquad (0 \leq t \leq T, j = 1, 2, \cdots, l).$$

$\{B_j(D_x, D_t)\}$ est un système d'opérateurs différentiels à coefficients constants d'ordre $m_j < m$, sans termes d'ordre inférieur.

L'auteur énonce que, dans ce cas, on peut caractériser les problèmes bien posés, d'en tirer les majorations L^2 typiques et d'obtenir les résultats d'existence. La démonstration n'est pas donnée ici. Cette recherche est sans doute une contribution importante aux problèmes mixtes pour les équations hyperboliques. *S. Mizohata* (Kyoto)

Mel'nik, Z. O. 6190

A mixed problem for the general third- and fourth-order hyperbolic equations on the plane. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 1149–1151.

Author's summary: "Using the method of L. Gårding, the author proves the existence and the uniqueness of the solution for the mixed problem for general hyperbolic equations of the third and fourth order on a plane."

Blagoveščenskiĭ, Ju. N. 6191

The Cauchy problem for degenerate quasi-linear parabolic equations. (Russian. English summary)

Teor. Verojatnost. i Primenen. **9** (1964), 378–382.

The author considers the Cauchy problem for the quasi-linear parabolic equation

$$(*) \qquad \frac{\partial v}{\partial t} = \frac{b^2(t, x, v)}{2}\frac{\partial^2 v}{\partial x^2} + a(t, x, v)\frac{\partial v}{\partial x}$$

with given initial conditions $v(0, x) = \phi(x)$. This problem was studied by O. A. Ladyženskaja and N. N. Ural'ceva [Izv. Akad. Nauk SSSR Ser. Mat. **26** (1962), 5–52] under the assumption $b(t, x, v) \geq a_0 > 0$ for all (t, x, v). In the paper being reviewed $b(t, x, v)$ is allowed to be 0 and it is assumed only that the $a(t, x, v)$, $\phi(x)$, and $b(t, x, v)$ satisfy various smoothness conditions. The method of solution is to solve successively the equations

$$(**) \quad \frac{\partial v_n}{\partial t} = \frac{b^2(t, x, v_{n-1}(t, x))}{2}\frac{\partial^2 v_n}{\partial x^2} + a(t, x, v_{n-1}(t, x))\frac{\partial v_n}{\partial x}$$

with $v_n(0, x) = \phi(x)$, $v_0(t, x) = \phi(x)$. When v_{n-1} is known, the equation (**) is linear and is solved by constructing a diffusion process $\{x_n(t)\}$ satisfying a corresponding stochastic integral equation. Then $v_n(t) = \mathbf{M}[\phi(x_n(t))]$ solves (**) and $v_n(t)$ converges as $n \to \infty$ to a solution $v(t)$ of (*).

J. L. Snell (Hanover, N.H.)

Cannon, J. R. 6192

A priori estimate for continuation of the solution of the heat equation in the space variable.

Ann. Mat. Pura Appl. (4) **65** (1964), 377–387.

Suppose that $u(x, t)$ is a continuous function in the region

$$D = \{(x, t): x_1(t) \leq x \leq x_3(t), 0 \leq t \leq T\}$$

such that

$$(a) \qquad u_t = u_{xx}, \quad x_1(t) < x < x_3(t), \quad 0 < t \leq T,$$

$$(b) \qquad u(x, 0) = 0, \quad x_1(0) \leq x \leq x_3(0),$$

$$(c) \qquad u(x_2(t), t) = f(t), \quad 0 \leq t \leq T, \quad f(0) = 0,$$

$$(d) \qquad u(x_3(t), t) = g(t), \quad 0 \leq t \leq T, \quad g(0) = 0,$$

$$x_1(t) < x_2(t) < x_3(t),$$

$$(e) \qquad |u(x_1(t), t)| \leq M_0,$$

where $x_1(t)$, $x_3(t)$ are Lipschitz continuous and $x_2(t)$, $f(t)$, and $g(t)$ are continuous. If $v(x, t)$ is a bounded, continuous function on D satisfying (a), (b), and (e), then the author obtains an a priori estimate for $|u(x, t) - v(x, t)|$, $x_1(t) < x < x_2(t)$, in terms of $\varepsilon(t)$, where $\varepsilon(t)$ is

$$\max\left\{\sup_{0 \leq \tau \leq t} |v(x_2(\tau), \tau) - f(\tau)|, \sup_{0 \leq \tau \leq t} |v(x_3(\tau), \tau) - g(\tau)|\right\}.$$

A method is discussed for calculating $v(x,t)$ in such a way that $s(t)$ is as small as possible.

M. Lees (Pasadena, Calif.)

Komatsu, Hikosaburo 6193

Abstract analyticity in time and unique continuation property of solutions of a parabolic equation.

J. Fac. Sci. Univ. Tokyo Sect. I **9**, 1–11 (1961).

The analytic semi-group $e^{tA} = U(t)$ has been characterized by K. Yosida [Proc. Japan Acad. **35** (1959), 109–113; MR **21** #4298]. The author considers the case where A depends on t:

$$\frac{d}{dt}u(t) = A(t)u(t),$$

where the space is a fixed Banach space. Following the method of Tanabe [cf. Osaka Math. J. **11** (1959), 121–145; MR **22** #4964], the author obtains the following result. Assume $\mathcal{D}(A(t))$ is independent of t and moreover $B(t) = A(t)A_0^{-1}$ is analytic in t; if, for $0 \leq t \leq T$, a certain uniformity concerning the analytic semi-group property is satisfied, then $U(t,s)$ is analytic in (t,s) for $t > s$. Some applications are given.

S. Mizohata (Kyoto)

Lu, Wei-mian; Wang, Guang-fa; Chen, Pu-quan; Wang, Shao-shang; Sun, Jing-sen 6194

On the Cauchy problem for a parabolic Monge-Ampère equation.

Acta Math. Sinica **12** (1962), 273–277 (*Chinese*); *translated as Chinese Math.* **3** (1963), 302–306.

The authors consider the Monge-Ampère parabolic equation

$$L(u) = xr + 2ys - y^2 + rt - s^2 = 0$$

with initial conditions $u(x, 0) = u_0(x)$, $u_y(x, 0) = q_0(x)$. They transform it to a hyperbolic Monge-Ampère equation by the simple device of considering the equation $L(u) + a^2 = 0$, where a is a small constant. They then prove that as $a \to 0$ the solution of the hyperbolic equation approaches that of the parabolic equation. The same procedure is used in a more complicated case by K. Wang, T. Chen, and T. Mei [same Acta **14** (1964), 78–92; MR **29** #377].

R. Barrar (Santa Monica, Calif.)

Montaldo, Oscar 6195

Sulla non negatività delle funzioni paraboliche elementari. (English summary)

Boll. Un. Mat. Ital. (3) **19** (1964), 260–265.

Author's summary: "The elementary parabolic equation

$$(D_x^{2n} + (-1)^n D_y)u = 0$$

is considered and the non-negativity of its solutions is proved by finite differences."

W. G. Strang (Cambridge, Mass.)

Nirenberg, L. 6196

Elliptic partial differential equations and ordinary differential equations in Banach space.

Differential Equations and Their Applications (*Proc. Conf., Prague,* **1962**), *pp.* 121–122. *Publ. House Czechoslovak Acad. Sci., Prague; Academic Press, New York,* **1963**.

This is a report of joint work with S. Agmon [Agmon and Nirenberg, Comm. Pure Appl. Math. **16** (1963), 121–239; MR **27** #5142] on elliptic and parabolic differential equations in an infinite cylinder.

Čibrikova, L. I.; Pokazeev, V. I. 6197

The Tricomi problem for a multiply connected domain. (Russian)

Boundary-value problems in the theory of functions of a complex variable, pp. 73–79. *Izdat. Kazan. Univ., Kazan,* 1962.

The Tricomi problem for the equation in polar coordinates of mixed type

$$(1) \qquad r^2u_{rr} + ru_r + (\operatorname{sgn}(1-r))u_{\varphi\varphi} = 0$$

is considered in a multiply connected domain D of the (r, φ)-plane, divided by the unit circle $r = 1$ into two simply connected parts D_1 and D_2. D_1, the portion interior to the circle, is bounded by $m+1$ piecewise smooth curves $\sigma_1, \cdots, \sigma_m$, each of which consists of two circular arcs orthogonal to the unit circle, and intersects it at $2(m+1)$ points $t_0, t_0', \cdots, t_m, t_m'$. D_2 is outside the unit circle, between the circle and the pairs of characteristics $\gamma_0, \gamma_0', \cdots, \gamma_m, \gamma_m'$ of (1) passing through the points $t_0, t_0', \cdots, t_m, t_m'$, respectively.

A unique solution $u(r, \varphi)$ in D to the equation (1) is found which vanishes on the arcs $\sigma_1, \cdots, \sigma_m$ and assumes given values on the characteristics $\gamma_1, \cdots, \gamma_m$. This is an extension of work of A. V. Bicadze [*On the problem of equations of mixed type* (Russian), Trudy Mat. Inst. Steklov, Vol 41, Izdat. Akad. Nauk SSSR, Moscow, 1953; MR **16**, 43], V. G. Karmanov [Dokl. Akad. Nauk SSSR **95** (1954), 439–442; MR **16**, 369], and others [see L. Bers, *Mathematical aspects of subsonic and transonic gas dynamics*, Wiley, New York, 1958; MR **20** #2960].

The authors' work is based on a reduction of the problem in the ellipticity domain D_1 to a Riemann boundary problem for an automorphic function on the unit circle, for which a solution in terms of the Cauchy principal value is found. In D_2 the solution of Bicadze is used.

A. D. Solomon (New York)

Krikunov, Ju. M. 6198

On the Tricomi problem with derivatives in the boundary conditions. (Russian)

Kazan. Gos. Univ. Učen. Zap. **122** (1962), *kn.* 3, 30–53.

In this article the author gives a method of solving a boundary problem for the equations of Lavrentieff and Bicadze.

The article consists of five sections. The first three contain the lemmas and theorems used in the proof of problem (T_k). This problem is posed in the fourth section and the solution is given in the fifth.

The problem (T_k) is too long to write here, so it will be only sketched. The considered domain D is a connected domain bounded by a simple arc Γ lying in the half-plane $y \geq 0$, having the endpoints $A(0,0)$ and $B(1,0)$, as well as by the characteristic of (1) AC: $x + y = 0$, $y \leq 0$, and BC: $x - y = 1$, $y \geq 0$. Denote by L the arc Γ and the interval AB. L has continuous curvature. The domain bounded by L and the domain bounded by the interval AB and the characteristics AC and BC will be denoted by D_1 and D_2, respectively.

Problem (T_4). Find a function $u(x, y)$ having the properties: $u(x, y)$ satisfies equation (1), the corresponding partial derivatives satisfy some prescribed conditions, and

$$\sum_{k=0}^{n} \sum_{j=0}^{k} a_{kj}(t) \frac{\partial^k u(t)}{\partial x^{k-j}\partial y^j} = b(t), \qquad t \in \Gamma,$$

where the functions $a_{kj}(t)$ and $b(t)$ have the Hölder property on Γ, and $\sum_{j=0}^{n} i^j a_{nj}(t) \neq 0$. Finally, $u(x, y)$, which is continuous in $\bar{D}_2$, satisfies a boundary condition on AC. *L. Pintér* (Szeged)

Nazarov, G. I. 6199

Bergman functions in the theory of flow of a compressible fluid. (Russian)

Tomsk. Gos. Univ. Učen. Zap. No. 49 (1964), 13 pp.

Bergman's integral operator generates the stream functions ψ of compressible fluids. If ψ is considered in the (s, θ)-plane, where s is a function of the speed and θ the angle which the velocity vector forms with the x-axis of the physical plane, then ψ satisfies a linear partial differential equation of mixed type. In the slightly modified form obtained by von Mises and Schiffer [see *Advances in applied mechanics*, pp. 249–285, Academic Press, New York, 1948; MR **10**, 642] and in the subsonic region this operator has the form $\psi(s, \theta) = A + B\theta + \operatorname{Im} \sum_{n=0}^{\infty} R_n(u, \bar{u}) f_n(s)$, $u = s + i$, $\bar{u} = s - i$. Here $f_n(s)$ are certain fixed functions. They depend only upon the equation which ψ satisfies, while $R_n(u, \bar{u}) = \phi_n(u) + F_n(\bar{u})$ are arbitrary harmonic functions which are connected by the relations $\phi_n(u) = \int \phi_{n-1}(u)\, du$, $F_n(\bar{u}) = \int F_{n-1}(\bar{u})\, d\bar{u}$, $n = 1, 2, \cdots$. The function $R_0(u, \bar{u})$ depends upon the profile around which the flow takes place. The author gives an explicit formula for the functions $f_n(s)$, given by (2.21) of the reviewed paper. Similar formulas are obtained for the stream functions ψ in the supersonic region.

The above results are of great interest, particularly in the numerical determination of flow patterns of compressible fluids. *B. Chalmers* (Stanford, Calif.)

Prouse, Giovanni 6200

Esempi tipici per l'equazione delle onde.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **34** (1963), 143–150.

L'auteur étudie un problème mixte bien posé pour l'équation des ondes mise sous la forme $x''(t) = Ax(t) + f(t)$. Posons $f_\tau(t) = f(t + \tau)$. L'auteur construit f telle que la fonction $\tau \to f_\tau$ soit uniformément continue de $\mathbf{R}$ dans $L^2([-\frac{1}{2}, \frac{1}{2}] \times U)$, l'image de $\mathbf{R}$ par cette fonction soit relativement compacte, la fonction $\tau \to x_\tau'$ n'ait aucune de ces deux propriétés, mais l'intégrale d'énergie entre $\tau - \frac{1}{2}$ et $\tau + \frac{1}{2}$ reste bornée quand τ varie.

La méthode employée s'applique à plusieurs autres situations analogues. *M. Zerner* (Marseille)

Fujita, Hiroshi 6201

The unique existence of solutions for the initial-value problem of Navier-Stokes equations. (Japanese)

Sûgaku **14** (1962/63), 65–81.

The author discusses the Navier-Stokes initial-value problem:

$$(1) \quad \partial_t u = \Delta u - \nabla p + f - (u \cdot \nabla) u \qquad (x \in D,\ t > 0),$$

$$(2) \quad \operatorname{div} u = 0 \qquad (x \in D,\ t > 0),$$

$$(3) \quad u|_{\partial D} = 0 \qquad (t > 0),$$

$$(4) \quad u|_{t=0} = a \qquad (x \in D),$$

which can be written in operational form as

$$(5) \quad du/dt = -Au + Pf - P(u \cdot \nabla) u \qquad (t > 0),$$

$$(6) \quad \|u(t) - a\| \to 0 \qquad (t \downarrow 0).$$

The first part concerns the strong solution and has the same aim as his previous paper with T. Kato [Rend. Sem. Mat. Univ. Padova **32** (1962), 243–260; MR **26** #495]. But the method of treatment is different in that this time the author makes use of Odqvist's Green's tensor $G(x, y)$, $g(x)$, assuming the boundary ∂D to be of class C^3. Then by using the fact that the solution of the Stokes problem:

$$(7) \quad \Delta v - \nabla q = -f, \qquad \operatorname{div} v = 0, \qquad v|_{\partial D} = 0$$

can be represented as

$$(8) \quad v(x) = \int_D G(x, y) f(y)\, dy = \mathbf{G} f, \qquad q(x) = \int_D g(x, y) f(y)\, dy = \mathbf{g} f,$$

he shows that the following relations hold

$$(9) \quad \mathbf{G} P h = \mathbf{G} h \quad \text{for any } h \in \mathscr{H} \text{ (i.e., } L_2(D) \text{ space)},$$

$$(10) \quad A^{-1} = \mathbf{G} \quad \text{as the operator on } \mathscr{H}_\sigma \text{ such that } \mathscr{H} = \mathscr{H}_\sigma \oplus \mathscr{H}_\pi$$

and is able to obtain the following more precise results.

Theorem 1. Let $a \in D(A^\beta)$ for some $\frac{1}{2} < \beta < 1$ and $f \in C_s^\theta[0, T_1]$ (i.e., strong Hölder-continuity with exponent $0 < \theta < 1$), then to any γ satisfying $\frac{1}{2} < \gamma < 1$, $\beta - \gamma + \frac{1}{2} > 0$ there exists in some interval $[0, T]$ $(0 < T < T_1)$ a strong solution $u(t)$ of class $\mathscr{S}_t(\beta, \gamma)$, where $u \in \mathscr{S}(\beta, \gamma)$ means that the inequality

$$N(u, t, \beta, \gamma) \equiv \max(\|Au\|_{t,\gamma}, \|A^{1/2}u\|_{t,(\beta - \frac{1}{2})}) < \infty$$

holds for any $t \in [0, T]$. Moreover, such a solution is unique.

The last half concerns the regularity in the classical sense of the strong solution u, as was promised in the above cited paper. First, he shows that $u(t, x)$ is Hölder continuous in $\Omega_1 = I \times D$, $I = (0, T)$, and then by proving the integral representations

$$u(t, x) = \int_D G(x, y) w(t, y)\, dy, \qquad (t, x) \in \Omega = I \times D,$$

$$(11) \quad \partial_m u(t, x) = \int_D G_m(x, y) w(t, y)\, dy \qquad (m = 1, 2, 3),$$

$$p(t, x) = \int_D g(x, y) w(t, y)\, dy,$$

where $w = -du/dt - (u \cdot \nabla) u + f$, he deduces, under the additional assumptions:

$$(12) \quad f \in C_s^\theta(I; L_2(D)) \quad \text{with exponent } 1/2 < \theta < 1,$$

$$f \in C_s^\chi(I, L_q(D)) \quad \text{for some } 3 < q,\ 0 < \chi < 1,$$

that $\nabla u(t, x)$ and $p(t, x)$ are Hölder-continuous in Ω_1.

Finally, making use of Oseen's fundamental solution $E(t-s, x-y)$ of the Stokes initial-value problem: $\partial_t u = \Delta u - \nabla p$, div $u = 0$, which is defined by

$$(13)\quad E_{ij}(t-s, x-y) = \left(-\delta_{ij}\Delta_x + \frac{\partial^2}{\partial x_i \partial x_j}\right)\Phi(t-s, x-y),$$

where

$$\Phi(t, x) = \frac{1}{4\pi^{3/2}} \frac{1}{|x|\sqrt{t}} \int_0^{|x|} \exp\left(-\frac{\alpha^2}{4t}\right) d\alpha,$$

and using the above result in an ingenious way he proves the following (Theorem 2). Under the further assumption that $f(t, x)$ is Hölder-continuous in Ω, the strong solution in Theorem 1 is classical, i.e., (i) $u(t, x)$ is Hölder-continuous in Ω_1 and satisfies (3), (ii) $\nabla u(t, x)$, $\Delta u(t, x)$, $\partial_t u(t, x)$, $p(t, x)$ and $\nabla p(t, x)$ are Hölder-continuous in Ω and satisfy (1) and (2). *W. Sibagaki* (Fukuoka)

Golovkin, K. K.; Solonnikov, V. A. 6202

The first boundary-value problem for the non-stationary Navier-Stokes equations. (Russian)

Dokl. Akad. Nauk SSSR **140** (1961), 287–290.

The authors consider in $\Omega \times [0, T]$, where Ω is a bounded region of 3-dimensional space with the boundary surface S of Ljapunov type and $T > 0$, the boundary-value problem for the non-stationary Navier-Stokes equations. The velocity u is assumed to vanish on S. For a vector-function $v(x, t)$ defined for $x \in \bar{\Omega}$, $t > 0$, the following norm is introduced

$$|v(x, t)|_{\mathfrak{M}(\Omega, \beta)} = \max_{x \in \Omega} |v(x, t)| + \max_{x, x' \in \Omega} \frac{|v(x, t) - v(x', t)|}{|x - x'|^\beta}$$

for a certain $\beta > 0$. Theorem: if the exterior force f and the initial value a of u satisfy, for some $\beta > 0$, the conditions

$$\sup_{t > 0} |f(x, t)|_{\mathfrak{M}(\Omega, \beta)} < \infty,$$

$$\sup_{t, t' > 0} \max_{x \in \Omega} \frac{|f(x, t) - f(x, t')|}{|t - t'|^\beta} < \infty,$$

and $|a(x)|_{\mathfrak{M}(\Omega, \beta)} < \infty$, then for a certain $T > 0$ there exists in $\Omega \times [0, T]$ a classical solution of the problem considered. T depends upon $\sup_{t > 0} |f(x, t)|_{\mathfrak{M}(\Omega, \beta)}$ and $|a(x)|_{\mathfrak{M}(\Omega, \beta)}$ and is equal to ∞ if the latter are sufficiently small. The proof is based on some estimates, in terms of $|\ |_{\mathfrak{M}(\Omega, \beta)}$-norms, which are deduced by using the non-stationary hydrodynamical potentials. The latter are constructed by means of some special forms of fundamental solutions given by the authors. Moreover, the following theorem is formulated. The weak solution (in the sense of Kiselev and Ladyženskaja [Izv. Akad. Nauk SSSR Ser. Mat. **21** (1957), 655–680; MR **20** #6881]) u, p of the problem considered has the derivatives $u_{x_i x_j}$, u_t and p_x which belong to $L^{4/3}(\Omega \times [0, T])$. *A. Krzywicki* (Zbl **106**, 183)

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

See also 6128, 6061.

Carlitz, L. 6203

A partial difference equation related to the Fibonacci numbers.

Fibonacci Quart. **2** (1964), 185–196.

The author considers the difference equation

$$u_{m,n} - u_{m-1,n} - u_{m,n-1} - u_{m-2,n} + 3u_{m-1,n-1} - u_{m,n-2} = 0,$$

where $m \geq 2$, $n \geq 2$ are integers. He shows that the general solution can be expressed in terms of the Fibonacci numbers. Formulas are not simple and the work involves elaborate and complicated but elementary transformations. *T. Fort* (Columbia, S.C.)

Huta, A. 6204

Eine Bemerkung zur Zerlegung der natürlichen Zahlen. (Slovak and Russian summaries)

Acta Fac. Natur. Univ. Comenian. **9**, 57–62 (1964).

Author's summary: "Bezeichnen wir mit $R_n(x)$ die Anzahl aller verschiedenen Zerlegungen einer natürlichen Zahl x in n Glieder (Summanden) unter folgenden Voraussetzungen: (1°) die Glieder sind natürliche Zahlen; (2°) die Zerlegungen mit zwar gleichen Gliedern, jedoch in anderer Reihenfolge, werden als gleiche angesehen. Werden also Glieder einer Zerlegung mit x_j, $j = 1, 2, \cdots, n$ bezeichnet, so kann man im weiteren voraussetzen, daß $x_i \leq x_j$ für $i < j$. Dann gilt die Relation

$$(1)\qquad R_n(x + n) - R_n(x) = R_{n-1}(x + n - 1)."$$

The author proves this relation. He then solves it recursively, that is, assuming $R_{n-1}(x + n - 1)$, he solves for $R_n(x)$ which now is a solution of a linear difference equation. He appends a table completed by equation (1) where x ranges from 1 to 15 and n from 1 to 4. *T. Fort* (Columbia, S.C.

Panov, A. M. 6205a

Behaviour of the trajectories of a system of finite difference equations in the neighbourhood of a singular point. (Russian)

Učen. Zap. Ural. Gos. Univ. **1956**, vyp. 19, 89–99.

Panov, A. M. 6205b

Classification of singular points of difference equations in an n-dimensional space. (Russian)

Učen. Zap. Ural. Gos. Univ. **1959**, vyp. 23, 13–21.

Consider the system

$$(1)\qquad x_{n+1} = Ax_n + \varphi(x_n), \qquad n = 0, 1, 2, \cdots,$$

where A is a 2×2 matrix and $\varphi(x)$ is a 2-vector which is $o(|x|)$ as $|x| \to 0$. A singular or equilibrium point of (1) is a solution such that $x_n = x_0$ for $n = 1, 2, \cdots$. For the linear system (2) $x_{n+1} = Ax_n$, the solutions are expressible in terms of the characteristic roots of A and arbitrary periodic functions of period one. In the first paper under review, the author classifies the singular point 0 of (2) in terms of the characteristic roots of A. Some theorems are also given concerning the local preservation of the structure of the singular point when equation (2) is subjected to perturbations of the type φ as in (1).

The second paper considers similar questions for the case when A is an $n \times n$ matrix and the function φ also depends upon n. *J. K. Hale* (Providence, R.I.

Tauber, Selmo 620

Existence and uniqueness theorems for solutions of difference equations.

Amer. Math. Monthly **71** (1964), 859–862.

The author establishes by recurrence the existence of solutions of the vector equation $EX = F(X, t)$. Such proofs are usually given at the beginning of any course on difference equations. *T. Fort* (Columbia, S.C.)

Aczél, J. 6207

Remarks on probable inference.

Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **6** (1963), 3–11.

The author compares his work on the axiomatics of conditional probability [Magyar Tud. Akad. Mat. Kutató Int. Közl. **6** (1961), 111–121] with that of R. T. Cox [*The algebra of probable inference*, Johns Hopkins Press, Baltimore, Md., 1961; MR **24** #A563]. He points out a number of shortcomings in Cox's treatment of the subject. For example, he shows that the functional equations which Cox encountered and solved by assuming differentiability of the solution can be solved under considerably weaker hypotheses. From the point of view of functional equations, the interest of the paper lies in the equation $zf(f(y)/z) = yf(f(z)/y)$ for the one-place function f, which the author reduces to the pair of functional equations $F(xz, yz) = F(x, y)z$, $F(x, F(y, z)) = F(z, F(y, x))$ for the two-place function F given by $F(x, y) = xf(y/x)$. Using known results and techniques, this latter pair of equations is readily solved. *B. Schweizer* (Tucson, Ariz.)

Kabaila, V. 6208

Existence conditions for a solution of a system of equations. (Russian. Lithuanian and German summaries)

Litovsk. Mat. Sb. **4** (1964), 353–356.

Author's summary: "In der Arbeit untersucht man das Gleichungssystem (1) $f(\varepsilon z) = \lambda f(z)$, $f(1/z) = \mu f(z)$ wo λ und μ gegebene Konstanten sind und $\varepsilon = e^{2\pi i/n}$ (n – beliebige natürliche Zahl). In der Arbeit sind die notwendigen und hinreichenden Bedingungen für die Existenz einer meromorphen Lösung des Systems (1) erhalten und ist die allgemeine Lösung in der Klasse der meromorphen Funktionen angegeben. In Sonderfalle $\lambda = \mu = 1$ sind die Lösungen von (1) automorphe Funktionen mit Haupttransformationen $w = \varepsilon z$ und $w = 1/z$."

Laha, R. G.; Lukacs, E.; Rényi, A. 6209

A generalization of a theorem of E. Vincze. (Russian summary)

Magyar Tud. Akad. Mat. Kutató Int. Közl. **9** (1964), 237–239.

Vincze solved the functional equation $\phi(x) = \phi(ax)\phi(bx)$, $a, b > 0$; $a^2 + b^2 = 1$ [same Közl. 7 (1962), 357–361; MR **27** #2748]. The authors of this paper prove the following theorem in a straightforward manner. Let $\phi(x)$ be a complex-valued function of the real variable x and let $\{a_j\}$ be a sequence of non-negative real numbers such that $\sum a_j^2 = 1$ and $0 < a_1 < 1$. Suppose that there exist complex constants A and B such that

$$\lim_{x \to 0} \frac{\phi(x) - \phi(0) - Ax}{x^2} = B.$$

Assume further that $\phi(x)$ satisfies, for all real x, the functional equation $\phi(x) = \prod \phi(a_j x)$, where the infinite product converges. Then $A = 0$ and $\phi(x) = e^{Bx^2}$.

R. A. Rosenbaum (Middletown, Conn.)

Timan, A. F. 6210

A non-linear functional equation in the class of functions which are convex on a semi-axis. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 515–526.

On considère sur $x \geq 0$ une fonction continue convexe ω avec $\omega(0) = 0$, $\sup_x \omega(x) = \infty$, et l'équation fonctionnelle $\sup_{x \geq 0} \{f(x) - \omega(x) \cdot y\} = \varphi(y)$, où $\varphi(y)$ est une fonction continue concave pour $y \geq 0$. On indique la relation entre l'existence d'une solution convexe et bornée d'une telle équation et la théorie d'approximation.

S. Mandelbrojt (Paris)

Vincze, E. 6211

Über eine Verallgemeinerung der Cauchyschen Funktionalgleichung.

Funkcial. Ekvac. **6** (1964), 55–62.

Let $(Q_0, *)$ be a commutative semigroup, Q the field of complex numbers, and f a mapping from Q_0 to Q. For any positive integer n, f is a solution of $f(z * w)^n = (f(z) + f(w))^n$ if and only if f is a solution of $f(z * w) = f(z) + f(w)$, i.e., in the set of all mappings from Q_0 to Q, the Cauchy equation and its nth power are equivalent. The main tool in the proof is the author's "Determinantenmethode" [Publ. Math. Debrecen **9** (1962), 149–163; MR **26** #502]. The special case $n = 2$, in a somewhat more general setting, was considered by the author [Arch. Math. **15** (1964), 132–135; MR **29** #390].

B. Schweizer (Tucson, Ariz.)

SEQUENCES, SERIES, SUMMABILITY

See also 5786, 5787, 6234, 6244.

San Juan Llosá, R. 6212

The uniqueness problem in the theory of numerical divergent series and formal laws of calculus. I.

Collect. Math. **14** (1962), 235–255.

Let c be the space of convergent sequences $\{s_i\}$ with the norm $\|s\| = \sup_{0 \leq i < \infty} |s_i|$ and u the space of sequences $\{u_i\}$ such that $\sum_{i=0}^{\infty} u_i$ converges.

We have the following definition:

A sequence $\{\phi_n\} \in c$ of sequences $\{\phi_{n,i}\} \in c$ for $n = 0, 1, 2, \cdots$ and $i = 0, 1, 2, \cdots$ is complete in c when the only constant C and the only sequence $\{C_i\} \in u$ satisfying all the conditions,

$$C \lim_{i \to \infty} \phi_{n,i} + \sum_{i=0}^{\infty} \phi_{n,i} C_i = 0 \qquad (n = 0, 1, 2, \cdots)$$

are $C = 0$ and $C_i = 0$ for $i = 0, 1, 2, \cdots$.

Various theorems are proved about complete sequences in different spaces with parallel theorems for functions continuous on the positive real line.

G. M. Petersen (Swansea)

San Juan Llosá, R. 6213

The uniqueness problem in the theory of numerical divergent series and formal laws of calculus. II.

Collect. Math. **15** (1963), 23–70.

Let s_0 be the set of convergent sequences. The author discusses the problem: "To extend the isomorphism λ_0 by means of another isomorphism λ between a set $s_1 \supset s_0$

$(s_1 \neq s_0)$ and the same complex plane c with respect to certain formal laws, for instance the laws a, b, c, d, e, and f stated below in the weak form that will be used." Law f, for example, states: "The limit $\lambda \lim s_n$ is a linear continuous functional in the space s_1 with a topology."

A similar discussion is made concerning convergent series.

G. M. Petersen (Swansea)

Gigler, Johann 6214

The fundamental theorem of van der Corput on uniform distribution and its generalizations.

Compositio Math. **16**, 29–34 (1964).

The contents of this paper were published earlier in a slightly different form [J. Reine Angew. Math. **210** (1962), 141–147; MR **25** #4273].

Thron, W. J. 6215

On the convergence of the even part of certain continued fractions.

Math. Z. **85** (1964), 268–273.

Linear fractional transformations and nested circular regions are used to prove the following result. If the complex numbers c_n satisfy

$$|c_{2n-1} \pm ia| \le p, \qquad |c_{2n} \pm i(1+a)| \ge p,$$
$$|c_{2n}^2| \ge p^2 - |1+a|^2 + \varepsilon,$$

where $\varepsilon > 0$, a is a complex number, and p satisfies $p > |a|$, $p \ge |1+a|$, then the sequence of even approximants of the continued fraction

$$1 + \frac{c_1^2}{1} + \frac{c_2^2}{1} + \cdots$$

converges to a value c which satisfies $|c - (1+a)| \le p$.

W. T. Scott (Tempe, Ariz.)

Bosanquet, L. S. 6216

On the order of magnitude of fractional differences.

Calcutta Math. Soc. Golden Jubilee Commemoration Vol. (1958/59), *Part I, pp.* 161–172. *Calcutta Math. Soc., Calcutta*, 1963.

For a sequence (u_n) we write

$$\Delta u_n = u_n - u_{n+1}, \quad \Delta^0 u_n = u_n \quad \text{and} \quad \Delta^s = \Delta\Delta^{s-1}$$
$$(s = 1, 2, 3, \cdots).$$

For all real ρ, the fractional difference is defined by

$$\Delta^\rho u_n = \sum_{\nu=n}^{\infty} A_{\nu-n}^{-\rho-1} u_\nu,$$

whenever the series on the right is convergent, A_n^ρ being defined for $\mu \ge 0$ by the identity

$$(1-x)^{-\rho-1} = \sum_{n=0}^{\infty} A_n^\rho x^n \qquad (|x| < 1).$$

When ρ is a non-negative integer, this definition reduces to the first one. For a sequence (v_n) the fractional difference with base at $-\infty$ is defined by

$$\nabla^\rho v_m = \sum_{n=-\infty}^{m} A_{m-n}^{-\rho-1} v_n,$$

whenever the series on the right is convergent. In particular, $\nabla^1 v_m = v_m - v_{m-1}$, and when ρ is a non-negative integer, we have $\nabla^\rho v_m = (-1)^\rho \Delta^\rho v_{m-\rho}$.

An analogue of a result of Hardy and Littlewood [Proc. London Math. Soc. (2) 11 (1913), 411–478] and variants of some results of Mordell [J. London Math. Soc. 3 (1928), 119–121] and Kloosterman [ibid. **15** (1940), 91–96; MR **2**, 89] is (D): If $W(n)$, $V(n)$ are positive and either both non-decreasing or both non-increasing and if $u_n = O(W)$, $\Delta^r u_n = O(V)$ for $n \ge 0$, where r is an integer greater than 1, then $\Delta^1 u_n = O(W^{1-(1/r)} V^{1/r})$. In the present paper the author extends (D) to fractional differences. The main result is given in Theorem 1: If $0 < \sigma < k$, $W(n)$, $V(n)$ are positive and non-increasing in the range $-\infty < n < \infty$, and if $u_n = O(W)$, $\Delta^k u_n = O(V)$ for $-\infty < n < \infty$, then $\Delta^\sigma u_n = O(W^{1-(\sigma/k)} V^{\sigma/k})$ for $-\infty < n < \infty$.

When W, V are non-decreasing, an equivalent result is proved in Theorem 2 for differences with base at $-\infty$. Some variants of Theorems 1 and 2 with o are given. Finally, by using an iteration formula for fractional differences due to Andersen [Mat. Tidsskr. B **1950**, 110–122; MR **12**, 404] and Kuttner [Proc. London Math. Soc. (3) 7 (1957), 453–466; MR **20** #1131], a result for differences of order σ ($\sigma_1 < \sigma < \sigma_2$ and σ_1 any real number, positive or negative) is obtained.

S. K. Basu (Calcutta)

Harris, W. A., Jr.; Turrittin, H. L. 6217

Reciprocals of inverse factorial series.

Funkcial. Ekvac. **6** (1964), 37–46.

This paper contains five interesting theorems on factorial series. Two of these theorems will be quoted. The reader can consult the paper for the other theorems.

Theorem: If $G(z)$ is analytic in a neighborhood of the origin and $F(z)$ can be represented by an inverse factorial series convergent in a right half-plane, then $\{G(F(z)) - G(0)\}$ also has a factorial series representation in some right half-plane.

Theorem: If $F(z)$ is a convergent inverse factorial series of the type

$$F(z) = \sum_{s=m}^{\infty} \frac{s!\,a_s}{z(z+1)\cdots(z+s)}, \qquad a_m \neq 0,\ \mathrm{Re}(z) > \lambda,$$

then $f(z) = F(z)/z^{m+1}$ is representable as a convergent inverse factorial series in some half-plane.

T. Fort (Columbia, S.C.)

Iyer, A. V. V. 6218

The equivalence of two methods of absolute summability.

Proc. Japan Acad. **39** (1963), 429–431.

Let T be a summability method on sequences. If $T(s_n) = (t_n)$, then (s_n) is said to be absolutely T-summable or $|T|$-summable if $\sum |t_n - t_{n-1}|$ converges. Let $H(s_n) = (\sigma_n)$ and $H^*(s_n) = (\sigma_n^*)$, where

$$\sigma_n = \sum_{k=0}^{n} (n+1-k)^{-1} s_k \Big/ \sum_{k=0}^{n} (k+1)^{-1}$$

and

$$\sigma_n^* = \sum_{k=0}^{n} (n+1-k)^{-1} s_k / \log n.$$

Theorem: $|H|$ and $|H^*|$ are equivalent.

R. E. Williamson (Norwich, Vt.)

Mac Nerney, J. S. 6219

Characterization of regular Hausdorff moment sequences.

Proc. Amer. Math. Soc. **15** (1964), 366–368.

The following theorem was suggested to the author by a problem proposed by H. S. Shapiro [Amer. Math. Monthly **69** (1962), 1013]: In order that the infinite complex-number sequence μ, with $\mu_0 = 1$, be a regular Hausdorff moment sequence, it is necessary and sufficient that, for each continuous complex-valued function f on $[0, 1]$, the limit $L(f) = \lim_{n\to\infty} \sum_{k=0}^{n} (-1)^{n-k} f(k/n) C_{n,k} \Delta^{n-k}\mu_k$ exist, and in this case, $L(f) = f(1) \lim_{n\to\infty} \mu_n$. The solution of Shapiro's problem appears as the particular case $\mu_n = 1/2^n$. An equivalent condition is that

$$M(f) = \lim_{n\to\infty} \sum_{k=0}^{n} f(k/n) C_{n,k} \Delta^{n-k}\mu_k$$

exist, and in this case, $M(f) = \int_0^1 f\, d\phi$, where ϕ is a function of bounded variation having μ as its moment sequence.

H. S. Wall (Austin, Tex.)

Maddox, I. J. 6220

Some inclusion theorems.

Proc. Glasgow Math. Assoc. **6**, 161–168 (1964).

The author gives necessary and sufficient conditions for a sequence-to-sequence, or series-to-sequence, regular matrix to include the Riesz method $(R, \lambda, 1)$. Theorem 1: If $A = (a_{n\nu})$ is a sequence-to-sequence regular matrix, then $A \supset (R, \lambda, 1)$ if and only if

$$\sum_{\nu=0}^{\infty} \lambda_{\nu+1} |\Delta(a_{n\nu}/\Delta\lambda_\nu)| \leq M,$$

where M is a constant. This generalizes a result due to Lorentz [Trans. Roy. Soc. Canada Sect. III (3) **45** (1951), 19–32; MR **14**, 160]. Theorem 2: If A is a series-to-sequence matrix with $a_{n\nu} \to 1$ ($n \to \infty$; ν fixed), then $A \supset (R, \lambda, 1)$ if and only if (i) $|a_{n\nu}| \leq H_n \Lambda_\nu^{-1}$, where $\Lambda_\nu = \lambda_{\nu+1}/|\Delta\lambda_\nu|$ and H_n is a positive number depending on n, and

(ii) $$\sum \lambda_{\nu+1} |\Delta(\Delta a_{n\nu}/\Delta\lambda_\nu)| \leq M,$$

where M is a constant. Theorem 3: The series-to-sequence matrix A with $a_{n\nu} \to 1$ ($n \to \infty$; ν fixed) includes $(R, \lambda, 1)$, where $\Lambda_n = \lambda_{n+1}/(\lambda_{n+1} - \lambda_n)$ is unbounded if and only if the following conditions hold: (i) $|a_{n\nu}| \leq H_n \Lambda_\nu^{-1}$, where H_n is a constant depending on n, and (ii) there exists a sequence of functions $\{g_n(u)\}$ defined for $u \geq \lambda_0$ and such that $a_{n\nu} = \int_{\lambda_\nu}^{\infty} (u - \lambda_\nu)\, dg_n(u)$, with $\int_{\lambda_0}^{\infty} u |dg_n(u)| \leq M$, a constant.

M. R. Parameswaran (E. Lansing, Mich.)

Sherif, Soraya 6221

Tauberian classes and Tauberian theorems.

Quart. J. Math. Oxford Ser. (2) **15** (1964), 303–308.

Es werden neue Tauberсche Klassen eingeführt, von denen die umfangreichste C folgendermaßen definiert ist: Es gehört $s(e^y) \in C$, wenn es zu irgendeinem $\varepsilon > 0$ Konstanten $\mu = \mu(\varepsilon)$, $\lambda = \lambda(\varepsilon)$ mit $0 \leq \mu < 1 < \lambda$ und Funktionen $\theta = \theta(\varepsilon, x)$, $\alpha = \alpha(\varepsilon, x)$ mit $\alpha \to \infty$ für $x \to \infty$ derart gibt, daß $\mathrm{Re}[e^{i\theta} s(y)] \geq (1-\mu)|s(x)| - \varepsilon$ für $\alpha \leq y \leq \lambda\alpha$ und hinreichend große x gilt. Zahlreiche Sätze des Buches *Tauberian theorems* von H. R. Pitt [Oxford Univ. Press, London, 1958; MR **21** #5109] bleiben richtig, wenn man die dort auftretenden Tauberschen Klassen durch die neuen ersetzt. Beispielsweise gilt jetzt: Ist $s(y)$ (R, h)-limitierbar zum Wert 0 und ist $s(e^y) \in C$ mit $\liminf_{\varepsilon\to 0} [\varepsilon/(1-\mu(\varepsilon))] = 0$, so gilt $s(y) \to 0$.

L. Berg (Halle)

Wilansky, Albert 6222

Topological divisors of zero and Tauberian theorems.

Trans. Amer. Math. Soc. **113** (1964), 240–251.

Using mostly work of B. Yood [Ann. of Math. (2) **50** (1949), 486–503; MR **10**, 611], the author examines the algebra Γ of conservative matrices; the notation is the same as in the review of I. D. Berg's paper on the same subject [Proc. Amer. Math. Soc. **15** (1964), 648–652; MR **29** #2574]. The paper has a strongly eclectic cast, and the author's claim of "depth" for some of his results seems quite unjustified. The paper does give a useful summary of the present state of the theory of Γ.

Edwin Hewitt (Seattle, Wash.)

APPROXIMATIONS AND EXPANSIONS

See also 5968, 6030, 6031, 6070, 6075, 6611.

Privalov, A. A. 6223

Interpolation on countable sets. (Russian)

Uspehi Mat. Nauk **19** (1964), no. 4 (118), 197–200.

In this paper the author defines an interpolation matrix to be an infinite triangular matrix of real numbers such that the k numbers in the kth row are distinct. He proves the following theorem: There exists a closed countable set F in $[-1, 1]$ with one limit point such that for any interpolation matrix M defined in F the order of increase of norms of interpolation polynomials is asymptotically not less than $\ln n$.

The norms referred to are defined by

$$L_n(M, F) = \max_{x\in F} \sum_{k=1}^{n} |l_{k,n}(x)|,$$

where $l_{k,n}(x)$ is the kth Lagrangian interpolation polynomial of degree $n-1$. The theorem shows that there exists a countable set F with interpolation behavior as "bad" as the entire interval $[-1, 1]$.

P. C. Hammer (Madison, Wis.)

Aramă, Oleg 6224

Sur quelques polynomes de type Bernstein.

Mathematica (Cluj) **4** (**27**) (1962), 205–224.

In dieser Arbeit führt der Verfasser frühere Untersuchungen [Mathematica (Cluj) **2** (**25**) (1960), 25–40; MR **22** #A1986b] über die Approximation von Funktionen durch Polynome vom Bernsteinschen Typ fort. Solche Polynome sind

$$B_n^{(k)}(x, f) = f(0) + \frac{x}{1!} f'(0) + \cdots + \frac{x^{k-1}}{(k-1)!} f^{(k-1)}(0) + \int_0^x \frac{(x-u)^{k-1}}{(k-1)!} B_n\left(u; \frac{d^k f}{dx^k}\right) du \quad (n = 1, 2, \cdots),$$

worin $B_n(x, F)$ die Bernsteinpolynome der Funktion F darstellen. Und zwar werden aus Konvexitätseigenschaften der Ableitungen von F für die Folge der abgeleiteten Polynome $B_n^{(k)}(x, f)$ interessante Monotonieeigenschaften gefolgert, mit deren Hilfe für eine Funktion

$f \in C^{n}[0, 1]$ für jedes $x \in [0, 1]$ die folgende Aussage gewonnen wird:

$$f(x) - B_m^{(k)}(x, f) = -\frac{x^{k+1}}{m}\left(\frac{k+2}{2} - x\right)[y_{m,1}^{(k)}, \cdots, y_{m,k+2}^{(k)}; f] \quad (m = 1, 2, \cdots).$$

Dabei ist $[y_{m,1}^{(k)}, \cdots, y_{m,k+2}^{(k)}; f]$ die dividierte Differenz der entsprechenden Werte aus $(0, 1)$. Ähnliche Ergebnisse werden für die Ableitungen von $f(x) - B_m^{(k)}(x, f)$, wie auch für den Ausdruck $B_{m+1}^{(k)}(x; f) - B_m^{(k)}(x; f)$ und dessen Ableitungen bewiesen.

Schließlich verallgemeinert der Autor dann seine Ergebnisse auf zwei Veränderliche.

P. L. Butzer (Aachen)

Berg, Lothar 6225

Iterative Berechnung der Tschebyscheffschen Alternante für quadratische Polynome.

Z. Angew. Math. Mech. **44** (1964), 220–222.

Es sei $f(x)$ eine im reellen Intervall $[a, b]$ gegebene Funktion, für welche die dritte Ableitung $f'''(x)$ dort stetig und positiv ist. Gesucht ist das Polynom von höchstens zweitem Grad, welches $f(x)$ im Sinne von Tschebyscheff am besten approximiert. Die Alternante besteht aus den beiden Endpunkten a und b und aus zwei im Innern von $[a, b]$ gelegenen Punkten u und v. Zur Berechnung von u und v (aus denen leicht das beste Polynom bestimmt werden kann) gibt der Verfasser zwei iterative Methoden an, die von dem bisher üblichen Austauschverfahren wesentlich verschieden sind. Die Konvergenz beider Verfahren wird für den Fall bewiesen, daß das

$$\max_{a \le x \le b} f'''(x) < \frac{3}{2} \min_{a \le x \le b} f'''(x)$$

ist. Die erste Methode ist geometrisch interpretierbar und steht in Analogie zu dem (elementaren) Fall $n = 1$. An numerischen Beispielen zeigt sich die sehr gute Konvergenzgeschwindigkeit. Der Verfasser weist auf die prinzipielle Möglichkeit der Verallgemeinerung für $n > 2$ hin.

G. Meinardus (Clausthal-Zellerfeld)

Čuprigin, O. A. 6226

On the approximation of functions by interpolation polynomials with equally spaced nodes. (Russian)

Vesci Akad. Navuk BSSR Ser. Fiz.-Tèhn. Navuk **1964**, no. 3, 11–17.

Diese Arbeit ist eine Weiterführung einer Abhandlung von P. O. Runck [J. Reine Angew. Math. **208** (1961), 51–69; MR **26** #1994a]. Im Segment $[-1, +1]$ sei die äquidistante Knotenmatrix $x_i^{(n)} = -1 + 2in^{-1}$ gegeben, untersucht wird dann das Verhalten der Lebesgueschen Funktion $\lambda_n(x)$ in gewissen Teilintervallen. Ist $r > 0$ beliebig, und setzt man $t_n = r(n \log n)^{-1}$, so gibt es eine Konstante $c_r > 0$, für welche die Ungleichung $\lambda_n(x) < c_r$ unter den folgenden Bedingungen gilt: (a) bei geradem n für $x \in [-t_n, +t_n]$, (b) bei ungeradem n für $x \in [-n^{-1} - t_n, -n^{-1} + t_n]$ und $x \in [n^{-1} - t_n, n^{-1} + t_n]$, (c) bei jedem n für $x \in [-1, -1 + 2^{-n}r]$ und $x \in [1 - 2^{-n}r, 1]$.

Ist auch noch $k > 0$ beliebig, und setzt man

$$u_n = r\sqrt{(2n^{-1}(k \log n - \log \log n))}$$

so gilt in $[-u_n, +u_n]$ die Abschätzung $\lambda_n(x) = A(r)n^k$, wobei $A(r)$ nicht von n abhängt, und mit $v_n = 2^{-n}rn^{k+1}$ ist $\lambda_n(x) = B(r)n^k$ für $x \in [-1, -1 + v_n]$ und $x \in [1 - v_n, 1]$. Aus diesen Sätzen werden noch Folgerungen über die Güte der Approximation einer stetigen Funktion durch die äquidistante L-Interpolation in den betrachteten Teilintervallen (bei gegebenem Stetigkeitsmodul) gezogen.

K. Bögel (Ilmenau)

Ehlich, H.; Zeller, K. 6227

Schwankung von Polynomen zwischen Gitterpunkten.

Math. Z. **86** (1964), 41–44.

Die Verfasser geben verschiedene Abschätzungen für den Maximalbetrag des Fehlerpolynoms, die manche bekannte Ergebnisse enthalten bzw. verschärfen. Mit der Abschätzung der Ableitung bzw. steilster Abstieg beweisen sie die folgenden Sätze: (I) Sei P ein Polynom vom Grade $\le n$ $(n > 2)$. Die natürliche Zahl m befriedige

$$\rho = n^2(n^2 - 1)/6m^2 < 1.$$

Aus $|P(x_\mu)| \le 1$ $(x_\mu = -1 + 2\mu/m, \mu = 0, 1, \cdots, n)$ folgt dann $|P(x)| \le 1/1 - \rho$ $(-1 \le x \le 1)$. (II) Sei P ein Polynom; es gelte grad $P \le n < m$ $(n, m \ge 0)$. Aus $|P(x_\mu)| \le 1$ $(x_\mu = \cos((2\mu - 1)\pi/2m); \mu = 1, 2, \cdots, m)$ folgt dann $|P(x)| \le 1/\cos(n\pi/2m)$ $(-1 \le x \le 1)$. (III) Sei T ein trigonometrisches Polynom und α eine reelle Zahl; es gelte grad $T \le n < m$ $(n, m \ge 0)$. Aus $|T(y_\mu)| \le 1$ $(y_\mu = \alpha + (\pi\mu/m)$; $\mu = 0, 1, \cdots, 2m - 1)$ folgt dann $|T(y)| \le 1/\cos(n\pi/2m)$.

L. Leindler (Szeged)

Lavrov, S. S. 6228

Approximation of functions of several variables by using the method of least squares. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 547–550.

The author sets up and solves the normal equations for three different variants of the least squares method for fitting a second-degree polynomial to a function of n variables.

R. G. Langebartel (Urbana, Ill.)

Malik, M. A. 6229

On extremal properties of the derivatives of polynomials and rational functions.

Canad. Math. Bull. **7** (1964), 121–131.

Let $f(z)$ be a rational function which is the quotient of polynomials of degrees m and n such that $f(z)$ has no zeros or poles inside the unit circle and $|f(z)| \le 1$ on $[-1, 1]$. The author shows that for $0 < \varepsilon < 1$ and $-1 + \varepsilon \le x \le 1 - \varepsilon$, the inequalities $|f'(x)| \le \varepsilon^{-1}[2(m + n)]^{1/2}$ and $|f''(x)| \le \varepsilon^{-2}[2(m + n)]^{3/4}$ are valid. These results are an improvement over a result of Erdős [Ann. of Math. (2) 41 (1940), 310–313; MR 1, 323], who considered only the first derivative and $f(z)$ a polynomial of degree n.

D. S. Greenstein (Evanston, Ill.)

Mohr, Ernst 6230

Elementarer Beweis einer Ungleichung von W. A. Markoff.

Tensor (N.S.) **14** (1963), 71–85.

A well-known theorem of W. A. Markov states that among all polynomials $P_n(x)$ of degree not exceeding n and satisfying $-1 \le P_n(x) \le 1$ on $[-1, 1]$, the Chebyshev polynomial $T_n(x)$ and its negative maximize the uniform

norm on $[-1, 1]$ of the derivatives $P_n^{(r)}(x)$ $(r=1, \cdots, n)$. In the present paper, the author gives an elementary proof of this theorem which depends only on some simple properties of trigonometric cosine polynomials and trigonometric sine polynomials.

D. S. Greenstein (Evanston, Ill.)

Szegő, Gabor 6231

On a problem of the best approximation.

Abh. Math. Sem. Univ. Hamburg **27** (1964), 193–198.

A classical problem of Chebyshev is to minimize $\max|F(x)|$, $|x| \leq 1$, where

$$F(x) = (Ax^n + a_1x^{n-1} + \cdots + a_n)/q(x)$$

with given n and A and positive polynomial q of degree not exceeding n. The author finds that this problem can be solved easily by reducing it, via $x = \cos\theta$, to the corresponding problem for $G(\theta) = H(\theta)/h(\theta)$, where H and h are real trigonometric polynomials, the terms of highest order in H have prescribed coefficients, and h is positive. The solution is obtained by finding the particular G that assumes its maximum modulus $2n$ times with alternating signs. As an application the author shows that if $f(\theta)$ is continuous, positive and periodic, and $\mu_n(f)$ is the minimum of $\max f(\theta)|H(\theta)|$, where H runs through trigonometric polynomials of degree n with leading term $\cos n\theta$, then $\lim \mu_n(f)$ equals the geometric mean of f; there is a corresponding result for continuous functions f on $[0, 1]$.

R. P. Boas, Jr. (Evanston, Ill.)

Werner, Helmut 6232

On the local behavior of the rational Tschebyscheff operator.

Bull. Amer. Math. Soc. **70** (1964), 554–555.

Let $\mathfrak{R}(l, r)$ be the set of rational functions where the degrees of the numerator and denominator do not exceed l and r, respectively. If $R = p/q \in \mathfrak{R}(l, r)$ and p, q are relatively prime, then $d[R] = \min[l - \partial p, r - \partial q]$ is called the defect of R in $\mathfrak{R}(l, r)$. R is called degenerate if its defect is positive. Let $T[l, r; f]$ be the best (weighted) uniform approximation to $f \in C[a, b]$ from $\mathfrak{R}(l, r)$. Those f for which $T[l, r; f]$ is not degenerate are called normal. The author improves a recent result of Cheney and Loeb by stating Theorem 1: The operator $T[l, r; f]$ is continuous at f if and only if f is normal or belongs to $\mathfrak{R}(l, r)$.

A counterpart to this is given by Theorem 2. Given $f \in C[a, b]$. To every $\varepsilon > 0$, $\varepsilon_1 > 0$ one can find $\delta > 0$ such that $\|f - g\| < \delta$ implies that there is a finite number of intervals depending on g whose total length is less than ε_1 such that for all points of $[a, b]$ not lying in the said intervals the inequality $|T[l, r; f](x) - T[l, r; g](x)| < \varepsilon$ holds. Proofs will be given elsewhere.

T. J. Rivlin (Yorktown Heights, N.Y.)

Rice, John R. 6233

On the L_∞ Walsh arrays for $\Gamma(x)$ and Erfc(x).

Math. Comp. **18** (1964), 617–626.

The two functions may each be approximated in an appropriate range by a rational function of degree (n, m) (using an obvious notation). The author considers in each case the best L_∞ approximation of degree (n, m), that is, the approximation of that degree whose maximum difference from $f(x)$ in the range is the smallest. The optimum (n, m) pair for each $(n+m)$ value is noted, for $n+m = 1, 2, 3, \cdots, 15$. The striking practical result is observed that in the case of Erfc(x) in $[0, \infty]$, these optimum values follow a smooth path, while for $\Gamma(x)$ in $[2, 3]$ they are extremely erratic. This interesting phenomenon awaits explanation.

C. W. Clenshaw (Teddington)

Copson, E. T. 6234

★Asymptotic expansions.

Cambridge Tracts in Mathematics and Mathematical Physics, No. 55.

Cambridge University Press, New York, 1965. vii + 120 *pp.* \$6.00.

This monograph is an exposition of the most commonly employed methods for the determination of the asymptotic expansions of functions. These methods are amply illustrated by establishing asymptotic expansions for a number of special functions such as the gamma function, error function, Fresnel integrals, Bessel functions, the Legendre polynomials, and related functions. The chapter headings are: (1) Introduction, (2) Preliminaries, (3) Integration by Parts, (4) The Method of Stationary Phase, (5) The Method of Laplace, (6) Watson's Lemma, (7) The Method of Steepest Descents, (8) The Saddle-Point Method, (9) Airy's Integral, (10) Uniform Asymptotic Expansions. The emphasis throughout is on technique, especially in Chapters 7 and 8 where the subject matter is treated heuristically. The Laplace approximation in Chapter 5 is proved as is Watson's lemma in Chapter 6, this lemma providing a powerful tool for establishing the asymptotic expansions of functions representable as Laplace integrals.

This book will inevitably invite comparisons with the two well-known monographs on asymptotic expansions: [1] A. Erdélyi, *Asymptotic expansions* [Dover, New York, 1956; MR **17**, 1202] and [2] N. G. de Bruijn, *Asymptotic methods in analysis* [North-Holland, Amsterdam, 1958, MR **20** #6003; second edition, 1961]. The subject matter and approach of the present book are very much like those of [1] with the inclusion of more recent material and more illustrative examples worked out in detail. However, the fundamentals of asymptotic expansions and series are treated more extensively in [1] than in the present book. Furthermore, [1] includes the application of asymptotic methods to differential equations—an important subject not treated by Copson. On the other hand, although the present book and [2] treat some of the same topics (in particular, the Laplace and saddle-point methods), [2] is of a higher level of mathematical exposition and very likely to be of greater appeal to the mathematician; moreover, the reviewer feels that there is no better introduction to the subject of asymptotics than [2].

A. K. Dasse (Buffalo, N.Y.)

Shen, Xie-chong [Shen, Hsieh-chang] 6235

On the closure of $\{z^{\lambda_n} \log^j z\}$ on unbounded curves in the complex plane.

Acta Math. Sinica **13** (1963), 170–192 (*Chinese*); *translated as Chinese Math.* **4** (1963), 188–210.

L'auteur indique des conditions, assez compliquées, liant

une courbe $\mathscr{L}$ à une fonction $p(z)$ (de la variable complexe) pour que sur $\mathscr{L}$ ait lieu la relation

$$\inf_{Q} \sup_{z \in \mathscr{L}} \exp(-p(z)) \cdot |f(z) - Q(z)| = 0,$$

pour toute fonction $f(z)$ continue sur $\mathscr{L}$, l'infimum étant pris par rapport à toutes les combinaisons linéaires $Q(z)$ des expressions $z^{\lambda_n} \log^j z$. *S. Mandelbrojt* (Paris)

Shen, Xie-chang [Shen, Hsieh-chang] 6236

On the closure of $\{z^{\lambda_n} \log^j z\}$ in a domain of the complex plane.

Acta Math. Sinica **13** (1963), 405–418 (*Chinese*); *translated as Chinese Math.* 4 (1964), 440–453.

On traite un problème analogue à celui traité dans le travail précédent [#6235], la courbe étant remplacée par un domaine, l'approximation étant faite avec la norme L_2. *S. Mandelbrojt* (Paris)

FOURIER ANALYSIS

See also 5992, 6069, 6080, 6230, 6231, 6258, 6302, 6350, 6351.

Gosselin, R. P. 6237

On the approximation of L^p functions by trigonometric polynomials.

Fund. Math. **53** (1963/64), 121–134.

Let

$$I_{n,u}(x; f) = \frac{2}{2n+1} \sum_{j=0}^{2n} f(u + x_j) D_n(x - u - x_j),$$

$$x_j = \frac{2\pi j}{2n+1},$$

where D_n is the Dirichlet kernel. $I_{n,u}(x; f)$, which has been discussed by several authors, is a trigonometric polynomial which interpolates f at the fundamental points of interpolation translated by a parameter u.

$$\text{Condition } C_{\alpha,p}: \int_0^{2\pi} \int_0^{2\pi} \frac{|f(x+u) - f(x)|^p}{u^{1+\alpha}} \, du\, dx < \infty.$$

Theorem 1: If f satisfies $C_{\alpha,p}$ for $p > 1$ and $\alpha > (\sqrt{5} - 1)/2$, then for almost every (x, u), $I_{n,u}(x; f)$ converges to $f(x)$. If $p > 1$, but $\alpha < \frac{1}{2}$, then there is an f satisfying $C_{\alpha,p}$ for which $I_{n,u}(x; f)$ diverges for almost every (x, u).

Some applications are made to fractional integrals, and finally a concept, that of translation continuity, which seems natural in this context, is introduced.

The proofs are nicely arranged and use the concept of subadditive function in an interesting way.

R. O'Neil (Houston, Tex.)

Janjulenis, A. [Iasiulionis, A. I.] 6238

Application of matrices to algebraic manipulations of trigonometric series. (Russian. Lithuanian and German summaries)

Litovsk. Mat. Sb. **3** (1963), no. 2, 193–207.

The author develops matrix algorithms, suitable for high-speed machines, for computing multiplication of Fourier series and division of trigonometric polynomials.

R. R. Goldberg (Evanston, Ill.)

Xie, Ting-fan [Hsieh, Ting-fan] 6239

On lacunary Fourier series.

Acta Math. Sinica **14** (1964), 312–318 (*Chinese*); *translated as Chinese Math.* **5** (1964), 340–345.

Let $f(x)$ have the Fourier series $\sum_{k=1}^{\infty} (a_k \cos n_k x + b_k \sin n_k x)$, where, for each k, $n_{k+1}/n_k \geq \lambda > 1$; let

$$\omega = \omega(f, x_0; \delta) = \sup_{|h| \leq \delta} |f(x_0 + h) - f(x_0)|;$$

and let

$$\omega_2 = \omega_2(f, \delta) = \max_{|h| \leq \delta} \max_{x} |f(x+h) + f(x-h) - 2f(x)|.$$

The author proves that if r is a positive integer, if $\phi(\delta)$ is positive and monotone increasing on $(0, \frac{1}{2})$, if $\eta^r \int_\eta^{1/2} \delta^{-r-1} \phi(\delta)\, d\delta \leq K_r \phi(\eta)$ $(0 < \eta \leq \frac{1}{2})$, and if at some point x_0, $\omega = O(\phi(\delta))$ as $\delta \to 0$, then $a_k, b_k = O(\phi(n_k^{-1}))$. The case $\phi(\delta) = \delta^\alpha$ improves on a result of Tomić [J. London Math. Soc. **37** (1962), 117–120; MR **24** #A3465]. Corollaries are that $f(x) \in \operatorname{Lip} \alpha$ $(0 < \alpha < 1)$ if and only if $\omega = O(\delta^\alpha)$ for some x_0, and that $f(x)$ is quasi-smooth if and only if $\omega = O(\delta)$ for some x_0. A simpler proof is given of a result due to Freud [Math. Z. **78** (1962), 252–262; MR **25** #2365] which, together with the above result, shows that $f(x)$ is smooth if and only if it has a finite derivative at some point x_0.

It is also shown that, for any $\beta > 0$, $\omega_2 = O(\delta|\log \delta|^{-\beta})$ if and only if $a_k, b_k = O(n_k^{-1}|\log n_k|^{-\beta})$. As corollaries (1) $\omega_2 = O(\delta|\log \delta|^{-\beta})$ if and only if $\omega = O(\delta|\log \delta|^{-\beta})$ at some point x_0; (2) using a result of Weiss and Zygmund [Nederl. Akad. Wetensch. Proc. Ser. A **62** (1959), 52–58; MR **21** #5849], if there is a point x_0 such that $\omega = O(\delta|\log \delta|^{-\beta})$ $(\beta > \frac{1}{2})$, then $f(x)$ is the indefinite integral of a function $g(x)$ which belongs to L^p for every $p \geq 1$.

P. Heywood (Edinburgh)

Robertson, M. M. 6240

Integrability of trigonometric series. II.

Math. Z. **85** (1964), 62–67.

As in Part I [same Z. **83** (1964), 119–122; MR **28** #5290] the author considers functions $f(x) = \frac{1}{2}\lambda_0 + \sum_1^\infty \lambda_n \cos nx$, $g(x) = \sum_1^\infty \lambda_n \sin nx$ which are "well behaved of order k"; by this is meant that the sequence $\{\lambda_n\}$ can be partitioned into k disjoint subsequences $\Lambda_i = \{\lambda_{nk+i}\}$ $(i = 1, \cdots, k)$, each of which ultimately increases or decreases to 0. He shows that, if $\phi(x, k, \gamma) = \prod_0^{[k/2]} (x - 2l\pi/k)^{-\gamma}$ and if $f(x)$ $[g(x)]$ is well behaved of order k, then when $p > 1$ and $1 - p < \gamma < 1$, $\{f(x)\}^p \phi(x, k, \gamma) \in L(0, \pi)$ $[\{g(x)\}^p \phi(x, k, \gamma) \in L(0, \pi)]$ if and only if $\sum n^{p+\gamma-2} |\lambda_n|^p$ converges. The corresponding result for $p = 1$ was proved by the author in Part I (in the review of which the series $\sum n^{\gamma-1}|\lambda_n|$ appeared as $\sum n^{-\gamma}|\lambda_n|$). The case $k = 1$ of the present theorem was previously dealt with by Yung-Ming Chen [ibid. **66** (1956), 9–12; MR **18**, 303; ibid. **68** (1957), 227–244; MR **19**, 1176].

H. Burkill (Sheffield)

Ciesielski, Z. 6241

On the orthonormal Franklin system.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 461–464.

For terminology and notation, see the author [Studia Math. **23** (1963), 141–157; MR **28** #419]. Put $\Delta_h x(t) = x(t+h) - x(t)$, $\Delta_h^2 x(t) = x(t+2h) - 2x(t+h) + x(t)$. For $x \in L_p\langle 0, 1\rangle$, $1 \leq p \leq \infty$, put $D_1(x, p, h) = \|\Delta_h x\|_p$ $(0 < h < 1)$,

$D_2(x, p, h) = \|\Delta_h{}^2 x\|_p$ $(0 < 2h < 1)$, where $\Delta_h x(u)$ is defined as 0 for $1 - h < u \leq 1$ and $\Delta_h{}^2 x(u) = 0$ for $1 - 2h < u \leq 1$. Now if φ is linear in each interval $\langle t_{i-1}, t_i \rangle$, where $0 = t_0 < t_1 < \cdots < t_n = 1$, and $\delta = \min\{t_i - t_{i-1} : 1 \leq i \leq n\}$, then, for $1 \leq p \leq \infty$, $D_1(\varphi, p, h) \leq 4 \min(h/\delta, 1/2)\|\varphi\|_p$ $(0 < h < 1)$, $D_2(\varphi, p, h) \leq 8[\min(h/\delta, 2^{-p(p+1)})]^{-1+1/p}\|\varphi\|_p$ $(0 < 2h < 1)$. For $x \in L_p\langle 0, 1\rangle$ $(1 \leq p \leq \infty)$, define $E_n{}^{(p)}(x)$ as the infimum of $\|x - \varphi\|_p$ over all functions φ of the preceding type; moreover, put

$$\omega_1{}^{(p)}(\delta; x) = \sup\{D_1(x, p, h) : 0 < h \leq \delta\} \quad (0 < \delta < 1),$$

$$\omega_2{}^{(p)}(\delta; x) = \sup\{D_2(x, p, h) : 0 < h \leq \delta\} \quad (0 < 2\delta < 1).$$

If $n = 2^m + k$, $1 \leq k \leq 2^m$, $t_i = i2^{-m-1}$ $(0 \leq i \leq 2k)$, $t_i = (i-k)2^{-m}$ $(2k+1 \leq i \leq n)$, $x \in L_p\langle 0, 1\rangle$, then $E_n{}^{(p)}(x) \leq \|x - S_n(x)\|_p \leq 4E_n{}^{(p)}(x) \leq 20\omega_1{}^{(p)}(1/n; x)$ for $n = 1, 2, \cdots$; moreover, $1 \leq p < \infty$ and $E_n{}^{(p)}(x) = O(1/n)$ imply $\omega_2{}^{(p)}(\delta; x) = O(\delta)$; $\omega_2{}^{(\infty)}(\delta; x) = O(\delta)$ implies $E_n{}^{(\infty)}(x) = O(1/n)$; $1 \leq p \leq \infty$, $0 < \alpha < 1$, $E_n{}^{(p)}(x) = O(n^{-\alpha})$ imply $\omega_1{}^{(p)}(\delta; x) = O(\delta^\alpha)$; the same statements are true if O is replaced by o. If $x \in L_p\langle 0, 1\rangle$, $1 \leq p \leq \infty$, $x = \sum a_n f_n$, $0 < \alpha < 1$, then the following conditions are equivalent: (i) $E_n{}^{(p)}(x) = O(n^{-\alpha})$, (ii) $\|x - S_n(x)\|_p = O(n^{-\alpha})$, (iii) $2^{m(1/2-1/p)}T_m(p, a) = O(2^{-\alpha m})$, (iv) $\omega_1{}^{(p)}(\delta; x) = O(\delta^\alpha)$, where $a = (a_0, a_1, \cdots)$ and

$$T_m(p, a) = \left(\sum_{2^m+1}^{2^{m+1}} |a_k|^p\right)^{1/p} \quad (1 \leq p < \infty)$$

and $T_m(\infty, a) = \max\{|a_k| : 2^m + 1 \leq k \leq 2^{m+1}\}$. The same is true if O is replaced by o. If $x \in L_p\langle 0, 1\rangle$, $1 \leq p \leq \infty$, and $\sum_{n=1}^\infty (1/n)E_n{}^{(p)}(x) < \infty$, then $\sum_{n=0}^\infty |a_n f_n(t)|$ converges in the L_p norm and also almost everywhere. Let $L_p{}^{(\alpha)}$ be, for $1 \leq p \leq \infty$, $0 < \alpha < 1$, the space of all $x \in L_p\langle 0, 1\rangle$ such that $\omega_1{}^{(p)}(\delta; x) = O(\delta^\alpha)$ with the norm

$$\|x\|_p{}^{(\alpha)} = \max(\|x\|_p, \sup\{D_1(x, p, h)h^{-\alpha} : 0 < h < 1\}),$$

$L_p{}^{(\alpha,0)}$ the subspace of $L_p{}^{(\alpha)}$ of all x such that $\omega_1{}^{(p)}(\delta; x) = o(\delta^\alpha)$, m_p the space of all real sequences $a = (a_0, a_1, \cdots)$ such that $T_m(p, a) = O(1)$ with the norm $\|a\| = \sup(|a_0|, |a_1|, T_m(p, a)$ $(m = 0, 1, \cdots))$, $m_p{}^{(0)}$ the subspace of the last space consisting of all a such that $T_m(p, a) = o(1)$. Then $L_p{}^{(\alpha)}$, $L_p{}^{(\alpha,0)}$, m_p, $m_p{}^{(0)}$ are Banach spaces and the Fourier-Franklin expansion

$$x = \sum_{n=0}^\infty a_n f_n{}^{(\alpha,p)}, \qquad a_n = \|f_n\|_p{}^{(\alpha)} \int_0^1 x(u)f_n(u)\,du,$$

where $f_n{}^{(\alpha,p)} = f_n/\|f_n\|_p{}^{(\alpha)}$, establishes an isomorphism of $L_p{}^{(\alpha)}$ onto m_p, respectively, of $L_p{}^{(\alpha,0)}$ onto $m_p{}^{(0)}$; moreover, $\{f_n{}^{(\alpha,p)}\}$ is a Schauder basis for $L_p{}^{(\alpha)}$. If $x \in L_1\langle 0, 1\rangle$, then $|\sum_{k=0}^n f_k(t)f_k(s)| \leq c_1 n e^{-c_0 n|t-s|}$ for all $t, s \in \langle 0, 1\rangle$ and $n = 1, 2, \cdots$, where c_0 and c_1 are absolute constants; moreover, the Fourier-Franklin expansion of x converges at every Lebesgue point of x. No proofs are given.

Á. Császár (Budapest)

Gapoškin, V. F. 6242

On the convergence of orthogonal series. (Russian)

Dokl. Akad. Nauk SSSR **159** (1964), 243–246.

Let $\{\varphi_n\}$ be a real orthonormal system with respect to a measure μ. The Rademacher-Menshov theorem states that $\sum a_n\varphi_n(x)$ converges almost everywhere if $\sum a_n{}^2 \log^2 n$ converges. By generalising the fundamental lemma on which the proof depends, the author obtains the following general result. Let $|\varphi_n(x)| \leq M_n$ outside E_n, where for each E of finite μ-measure we have $\sum \mu(E_n \cap E) < \infty$. Then $\sum a_n\varphi_n(x)$ converges almost everywhere if

$$\sum a_n{}^2 \log^2\left(\sum_{k=1}^n |a_k| M_k + 1\right) < \infty.$$

A number of corollaries follow; in particular, if the φ_n are uniformly bounded, then $\sum a_k{}^2 \log^2(\sum_{k=1}^n |a_k| + 1) < \infty$ implies convergence almost everywhere. There are applications to the order of magnitude of partial sums and to (generalized) lacunary series.

R. P. Boas, Jr. (Evanston, Ill.)

Fomin, G. A. 6243

On linear methods of summability of Fourier series. (Russian)

Mat. Sb. (*N.S.*) **65** (**107**) (1964), 144–152.

Soit défini par la matrice triangulaire $\|\lambda_k{}^{(n)}\|$, $\lambda_0{}^{(n)} = 1$, $\lambda_{n+1}^{(n)} = 0$, $\Delta\lambda_k{}^{(n)} = \lambda_k{}^{(n)} - \lambda_{k+1}^{(n)}$, $k = 0, 1, \cdots, n-1$, un procédé régulier de sommation de la série de Fourier de $f(x) \in C[0, 2\pi]$ au moyen de l'opérateur

$$L_n(f; x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)U_n(t)\,dt,$$

$$U_n(t) = \frac{1}{2} + \sum_{k=1}^n \lambda_k{}^{(n)} \cos kt.$$

La plus générale condition suffisante, portant sur $\lambda_k{}^{(n)}$ telle que $L_n(f; x) \to f(x) \in C[0, 2\pi]$, $n \to \infty$, a été donnée par A. V. Efimov [Izv. Akad. Nauk SSSR Ser. Mat. **24** (1960), 743–756; MR **23** #A1204]. L'auteur donne ici les nouvelles conditions qui ne sont pas contenues dans cette condition. Théorème 2: $L_n(f; x) \to f(x) \in C[0, 2\pi]$, $n \to \infty$, si

$$n \sum_0^n (\Delta\lambda_k{}^{(n)})^2 = O(1).$$

Théorème 7: Soit $\phi(x)$ continue sur $[0, 1]$ et à variation bornée, $\phi(0) = 1$, $\phi(1) \neq 0$, on a alors

$$\|L_n\| = \pi^{-1}\int_{-\pi}^{\pi}\left|\frac{1}{2} + \sum_{k=1}^n \phi\left(\frac{k}{n+1}\right)\cos kt\right| dt > C\log n,$$

$$C > 0.$$

Théorème 9. Soit la matrice $\|\lambda_k{}^{(n)}\|$ telle qu'il existe $m_n < n$ avec $m_n \sum_0^{m_n} (\Delta\lambda_k{}^{(n)})^2 = O(1)$, $m_n \to \infty$, $\Delta\lambda_{m_n}{}^{(n)} = \lambda_{m_n}{}^{(n)}$, $n = 1, 2, \cdots$, et $\sum_{m_n+1}^n (\lambda_k{}^{(n)})^2 = O(1)$, alors $L_n(f; x) \to f(x) \in C[0, 2\pi]$, $n \to \infty$, $x \in [0, 2\pi]$.

M. Tomić (Belgrade)

Jakimovski, Amnon 6244

Analytic continuation and summability of series of Legendre polynomials.

Quart. J. Math. Oxford Ser. (2) **15** (1964), 289–302.

Let (*) $\sum a_n P_n(z)$ be the Neumann series associated with a function $f(z)$ holomorphic in a domain containing the closed interval $-1 \leq \mathrm{Re}(z) \leq 1$. The reviewer and J. P. King [J. Analyse Math. **10** (1962/63), 139–152; MR **28** #1421] discussed the application of the Taylor and Lototsky transforms to the sequence of partial sums of (*). In the present paper the author proves some very general theorems (in terms of the generating domain D of the matrix A) for the summability of the sequence of partial sums of (*).

V. F. Cowling (New Brunswick, N.J.)

Leindler, László 6245

Über die de la Vallée-Poussinschen Mittel allgemeiner Orthogonalreihen.

Publ. Math. Debrecen **10** (1963), 274–282.

Let $\{\varphi_n(x)\}$ be an orthonormal sequence of functions on a finite interval, and let $\sum c_n\varphi_n(x)$ be the Fourier expansion of a function f. The author considers the size of $V_n(x)$ and $V_n(x)-f(x)$ under various conditions, where $V_n(x)=[s_n(x)+\cdots+s_{2n-1}(x)]/n$ is the delayed mean of de La Vallée Poussin. Typical theorems are as follows. Let $\mu_n\le\mu_{n+1}$ and $\sum_{s=0}^{k}\mu_{2^s}{}^2=O(\mu_{2^k}{}^2)$ and assume $\sum c_n{}^2\mu_n{}^2<\infty$. Then, $V_n(x)-f(x)=o_x(1/\mu_n)$ a.e. Two interesting examples are $\mu_n=n^\gamma$, $\gamma>0$, and $\mu_n=q^n$, $q>1$.

Let $\{l_n\}$ be positive, monotone non-decreasing to infinity and let $l_{2^n+1}=l_{2^n+2}=\cdots=l_{2^{n+1}}$. Then, if $\sum c_n{}^2l_n{}^{-2}<\infty$ and $\sum c_nl_n{}^{-1}\varphi_n(x)$ is $(C,1)$ summable a.e. on a set E, $V_n(x)=o_x(l_{2n})$ a.e. on E, where $V_n(x)$ is the delayed mean of $\sum c_n\varphi_n(x)$.

In an earlier paper [Acta Sci. Math. (Szeged) **24** (1963), 129–138; MR **27** #2792], the author has found similar theorems for Riesz means. *R. A. Askey* (Madison, Wis.)

Rubinštein, A. I. 6246

On ω-lacunary series and functions of the classes H^ω. (Russian)

Mat. Sb. (*N.S.*) **65** (**107**) (1964), 239–271.

Let ω be a real-valued continuous function on some interval $[0,h_0]$ with the following properties: $\omega(0)=0$, $\omega(h)\uparrow$, $\omega(h_1+h_2)\le\omega(h_1)+\omega(h_2)$. For such a function let H^ω be the set of 2π-periodic complex-valued functions f on the real line for which there exists a real number A, depending only on f, such that $|f(x+h)-f(x)|\le A\omega(|h|)$ for all x and h. The author investigates Fourier series, Fourier coefficients and properties of functions in H^ω. In the following let ω always fulfill the additional condition $\lim_{t\to+0}t^{-1}\omega(t)=\infty$, and let it be convex upwards. An increasing sequence of natural numbers $\{n_k\}$ $(k=1,2,\cdots)$ is called ω-lacunary if

$$(1)\qquad \sum_{k=N}^{\infty}\omega(n_k{}^{-1})=O[\omega(n_N{}^{-1})]$$

and

$$(2)\qquad \sum_{k=1}^{N}n_k\omega(n_k{}^{-1})=O[n_N\omega(n_N{}^{-1})]$$

$(N=1,2,\cdots)$. A trigonometric series $\sum_{k=1}^{\infty}(a_k\cos n_kx+b_k\sin n_kx)$ is called ω-lacunary if $\{n_k\}$ is an ω-lacunary sequence.

The following statements are proved, and they are typical. (i) If an increasing sequence of natural numbers $\{n_k\}$ fulfills (1) or (2), then it can be decomposed into a finite number of ordinary lacunary sequences $\{n_k{}^{(i)}\}$, i.e., such that $n_{k+1}^{(i)}/n_k{}^{(i)}\ge\lambda_i>1$ $(i=1,2,\cdots,l)$. (ii) Let f be a continuous function with an ω-lacunary Fourier series. Then $f\in H^\omega$ if and only if the Fourier coefficients of f are of order $O[\omega(n^{-1})]$. (iii) Let $f\in H^\omega$ with Fourier coefficients a_k, b_k. Then $(a_{n_k}{}^2+b_{n_k}{}^2)^{1/2}\ge c\omega(n_k{}^{-1})$ (c an arbitrary constant >0) implies that $\{n_k\}$ is ω-lacunary. (iv) Let $f(x)\sim\sum a_n\cos nx$ with $f\in H^\omega$ and $a_n\ge0$. Then

$$(3)\qquad \sum_{k=n}^{\infty}a_k=O[\omega(n^{-1})]$$

and

$$n^{-1}\sum_{k=1}^{n}ka_k=O\left[n^{-1}\int_{n^{-1}}^{\delta_0}\frac{\omega(t)}{t^2}\,dt\right],$$

where $\delta_0>0$ is fixed. Conversely, if ω is such that $\delta\int_\delta^{\delta_0}t^{-2}\omega(t)\,dt=O[\omega(\delta)]$ $(\delta\to+0)$ and $a_k\ge0$ $(k=1,2,\cdots)$, then (3) implies $f\in H^\omega$ for $f(x)\sim\sum a_k\cos kx$.

The last theorem implies partly a statement of Lorentz [Math. Z. **51** (1948), 135–149; MR **10**, 33]: Let $\sum a_k\cos kx\sim f(x)$ with $a_k\downarrow0$. Then $f\in\operatorname{Lip}\alpha$ $(0<\alpha<1)$ if and only if $a_n=O(n^{-1-\alpha})$ (Lorentz proved such a statement also for odd functions). A generalization of a result of Loud [Proc. Amer. Math. Soc. **2** (1951), 358–360; MR **13**, 218] is the following: (v) There exists a pair of conjugate functions $f,\tilde f\in H^\omega$, i.e., functions f and $\tilde f$ belonging to Fourier series conjugate to each other, such that in $[0,2\pi]$ $\limsup_{h\to0}|(f(x+h)-f(x))/\omega(|h|)|\ge c$, and correspondingly for $\tilde f$. Here c is a certain positive constant. (vi) Let $\Lambda=\|\lambda_k{}^{(n)}\|$ be a triangular matrix. If there exists an increasing sequence of natural numbers $\{n_k\}$ such that

$$\sum_{k=1}^{N-1}|1-\lambda_{n_k}{}^{(n_N-1)}|\omega(n_k{}^{-1})=o[\omega(n_N{}^{-1})],$$

then there exists a pair of conjugate functions $f,\tilde f\in H^\omega$ (see above) such that almost everywhere in $[0,2\pi]$

$$\limsup_{n\to\infty}[\omega(n^{-1})]^{-1}\left|f(x)-\sum_{k=1}^{n}\lambda_k{}^{(n)}(a_k\cos kx+b_k\sin kx)\right|>c,$$

and correspondingly for $\tilde f$. Here c is a certain positive constant. Statements of similar type for various summability methods such as the (C,α) method, Abel's method and de La Vallée Poussin's method conclude the paper.

{In the literature under [9] "Lond" should be replaced by "Loud".} *G. Goes* (Chicago, Ill.)

Tomić, M. 6247

Sur un critère pour la convergence des séries de Fourier dans un point fixe.

Bull. Acad. Serbe Sci. Arts Cl. Sci. Math. Natur. Sci. Math. **31** (1963), no. 4, 23–27.

Let the Fourier series of $f(x)$ be designated by

$$S(f)=a_0+\sum_{n=1}^{\infty}(a_n\sin nx+b_n\cos nx).$$

A well-known theorem of Hardy and Littlewood states that if

$$(1)\qquad f(x_0\pm h)-f(x_0)=o(|\log h|^{-1}),\qquad h\to0,$$

and

$$(2)\qquad |a_n|=O(n^{-r}),\qquad |b_n|=O(n^{-r})$$

for some $r>0$, then $S(f)$ converges at x_0. A positive function $L(u)$, defined for $0<u<\varepsilon$, is said to be slowly increasing at 0 if for all δ, $L(u)u^\delta$ decreases and $L(u)u^{-\delta}$ increases as $u\to0$. The following theorems are proved. Theorem 1: If the function $f(x)$ satisfies at x_0 the condition

$$(3)\quad f(x_0+t)+f(x_0-t)-2f(x_0)=\varphi(x_0,t)=\varphi(t)\to0,\quad t\to0,$$

where $\varphi(t)$ is a slowly increasing function as defined above, and if

$$\sup|f(x+h)-f(x)|=\omega(\delta)\to0,\qquad \delta\to0,\quad x\in(0,2\pi),\quad 0\le h\le\delta,$$

then $s_n(x_0)\to f(x_0)$, $n\to\infty$.

Theorem 2: If $f(x)$ satisfies condition (3) and if $S(f)$ has infinitely many gaps (n_{k+1}, n_k) of Fabry type, i.e., $n_{k+1}-n_k\to\infty$, then $s_{n_k}(x_0)\to f(x_0)$.
G. M. Petersen (Swansea)

Tomić, M. 6248

Sur la sommation d'une classe des séries de Fourier.

Bull. Acad. Serbe Sci. Arts Cl. Sci. Math. Natur. Sci. Math. **31** (1963), *no.* 4, 31–36.

This paper is a French translation of a paper by the same author [Acad. Serbe. Sci. Arts Glas Cl. Sci. Math. Natur. **254** (1963), 25–34; MR **27** #2781]. The precise nature of the uniform convergence used in the hypothesis of Theorem 2 is not quite clear on reference to the proof and would seem to require a further reference to another work by the same author [ibid. **249** (1961), 225–232; MR **27** #2779].
G. M. Petersen (Swansea)

Griffith, James L. 6249

On the Gibbs' phenomenon in n-dimensional Fourier transforms.

J. Proc. Roy. Soc. New South Wales **97**, 163–173 (1964).

Author's summary: "It is shown that with the simple inversion formula, the Hankel transform exhibits a Gibbs' phenomenon at a point of discontinuity of the same magnitude as in the cases of the finite Fourier and one-dimensional Fourier transforms. In the case of the n-dimensional Fourier transform we must add reasonably heavy restrictions in order that the simple inversion formula should converge. When we do this we find that this transform will exhibit the Gibbs' phenomenon. In the simplest cases this magnitude will be equal to that in the two transforms mentioned above."
E. M. Stein (Chicago, Ill.)

Rajagopalan, M. 6250

Fourier transform in locally compact groups.

Acta Sci. Math. (Szeged) **25** (1964), 86–89.

I. E. Segal [same Acta **12** (1950), 157–161; MR **12**, 188; errata, MR **12**, 1002] proved that if the Fourier transformation on a commutative locally compact group G maps $L_1(G)$ onto $C_0(\hat G)$, then G is finite, and stated without proof that the analogue for noncommutative groups holds; this is proved by the author in the sense that if T_f is the operator $g\to f*g$ on $L_2(G)$, for an $f\in L_1(G)$, then $L_1(G)$ is complete under the norm $\|T_f\|$ if and only if G is finite. In the course of the proof it is shown that an extremally disconnected G is discrete and that a C^* algebra is weakly sequentially complete if and only if it is finite-dimensional. The author also proves a conjecture of Segal that if, for locally compact commutative G, the Fourier transform maps $L^p(G)$, $1<p<2$, onto the whole of $L^{p'}(\hat G)$, then G is finite.
J. L. B. Cooper (Pasadena, Calif.)

Baillette, Aimée 6251

Sur la transformée de Fourier-Carleman d'une fonction presque périodique de spectre donné.

C. R. Acad. Sci. Paris **258** (1964), 6049–6051.

Let $\Lambda=(\lambda_j)$ be an increasing sequence of positive real numbers such that $\lambda_j\to\infty$ and $\lim\sup(j/\lambda_j)<\infty$; put $C(w)=\prod_1^\infty(1-w^2/\lambda_j^2)$. Then Λ is called a (P) sequence if $1/C(w)$ is the Fourier-Carleman transform of an almost periodic function, all derivatives of which exist and are almost periodic [see J.-P. Kahane, Ann. Inst. Fourier (Grenoble) **5** (1953/54), 39–130; MR **17**, 732]. The proof of the following result is indicated. If

$$a_k=\frac{k\lambda_{k+1}-(k+1)\lambda_k}{\lambda_{k+1}-\lambda_k},$$

then (i) $\lim a_k=\infty$ implies that Λ is a (P) sequence, and (ii) $\lim\sup a_k<\infty$ implies that Λ is not a (P) sequence. A less simple condition, both necessary and sufficient for Λ to be a (P) sequence, is also given.
H. Burkill (Sheffield)

Bochner, S. 6252

Continuous mappings of almost automorphic and almost periodic functions.

Proc. Nat. Acad. Sci. U.S.A. **52** (1964), 907–910.

The author gives his own proof of a counterexample, communicated to him by H. Furstenberg. If θ is any non-rational real number, then

$$\varphi(n)=\operatorname{signum}(\cos 2\pi n\theta),\qquad -\infty<n<\infty,$$

is almost automorphic, but not almost periodic, on the additive group of integers. The proof contains ingredients of a general nature.
E. Følner (Copenhagen)

Hartman, S.; Ryll-Nardzewski, C. 6253

Almost periodic extensions of functions.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 427–429.

For this discussion, any point set S on the real line such that every bounded uniformly continuous function on S can be extended to a uniformly almost periodic function will be called an I-set. Previous results of Hartman [Colloq. Math. **8** (1961), 99–101; MR **23** #A3416] and Mycielski [ibid. **8** (1961), 95–97; MR **23** #A3415] exhibit countable sets which are not I-sets and countable sets which are I-sets. In the present paper, the authors announce a theorem affirming the existence of I-sets which are unbounded and the union of an infinite collection of disjoint intervals. They say that the proof depends mainly on the fact that certain sequences are I-sets if and only if their closures in the Bohr compactification of the real line are homeomorphic to the Stone-Čech compactification of the integers.
D. S. Greenstein (Evanston, Ill.)

Kachroo, I. C. 6254

On the Borel summability of double Fourier series.

Rend. Accad. Sci. Fis. Mat. Napoli (4) **30** (1963), 29–33.

The author proves that the double Fourier series of a Lebesgue integrable periodic function on $(-\pi,\pi;-\pi,\pi)$ is summable by Borel exponential means to S at (x,y) if

$$\int_0^u\int_0^v|\Phi(s,t)|\,ds\,dt=o(uv/\log u^{-1}\log v^{-1}),$$

$$\int_0^v dt\left|\int_0^u\Phi(s,t)\,ds\right|=o(u/\log u^{-1})$$

and

$$\int_0^u ds\left|\int_0^v\Phi(s,t)\,dt\right|=o(v/\log v^{-1})\quad\text{as } u,v\to+0,$$

where

$$\Phi(s, t) = \tfrac{1}{4}[f(x+u, y+v) + f(x+u, y-v) + f(x-u, y+v) + f(x-u, y-v) - 4S],$$

thus generalizing a theorem of Sahney [Boll. Un. Mat. Ital. (3) 16 (1961), 156–163; MR 25 #391].

J. Mitchell (Madison, Wis.)

Beurling, Arne 6255

A critical topology in harmonic analysis on semigroups.

Acta Math. 112 (1964), 215–228.

Let S denote a discrete Abelian cancellation semi-group with irreducible zero element, let $\le$ denote the natural partial ordering of S, and let ω denote a positive real-valued function defined on S such that $\omega(y) \le 2\omega(x)$ whenever $y \le 2x$ and such that $\sum_{x \in S} e^{-\lambda_0 \omega(x)} \le 1$ for some positive λ_0. For each positive real number λ, let $A(\lambda)$ be the space of all complex functions f on S such that $\|f\|_\lambda = \sup_{x \in S} |f(x)| e^{-\lambda\omega(x)} < \infty$, and let $A = \bigcup \{A(\lambda) : \lambda > 0\}$ with the inductive limit topology. The dual space A' of A is identified with the set of complex functions φ on S such that, for all $\lambda > 0$, $\|\varphi\|_\lambda' = \sum |\varphi(x)| e^{\lambda\omega(x)} < \infty$, by way of the bilinear form $(f, \varphi) = \sum f(x)\varphi(x)$. The set of characters on S that belong to A' is denoted by $\chi_{A'}$, and the Laplace transform $\hat{f}$ of an element f of A is defined on $\chi_{A'}$ by $\hat{f}(\xi) = (f, \xi)$. Given τ in S, the shift operator T_τ is defined on A by $T_\tau f(x) = f(x-\tau)$ $(\tau \le x)$, $T_\tau f(x) = 0$ $(\tau \not\le x)$; and the closed linear hull in A of the set $\{T_\tau f : \tau \in S\}$ is denoted by A_f. The closure theorem is said to hold in A if $A_f = A$ whenever the Laplace transform of f vanishes nowhere on $\chi_{A'}$. It is proved that if the closure theorem holds in A, then the series $\sum_{n=1}^\infty n^{-3/2}\omega(nx)$ diverges for each non-zero x in S. For a result in the opposite direction, restrictions on the form of the function ω are needed. Suppose that ω is of the form $\psi \circ \theta$, where θ is an additive real function on S such that $\theta(x) \to +\infty$ as $x \to \infty$, and $\psi(r)$ is a positive increasing function of r for $r \ge 0$, is a convex function of $\log r$, and $c \log r \le \psi(r) = o(r)$ as $r \to \infty$. Under these conditions, it is proved that if $\sum_{n=1}^\infty n^{-3/2}\omega(nx)$ diverges for each non-zero x in S, then the closure theorem holds in A.

F. F. Bonsall (Newcastle upon Tyne)

Semadeni, Z. 6256

Generalizations of Bohr's theorem on Fourier series with independent characters.

Studia Math. 23 (1963), 159–179.

The author gives a new proof of the following generalization of a theorem of H. Bohr to arbitrary abelian groups G. If $\sum a_n\chi_n$ is the Fourier series of a uniformly almost periodic function on G and if the characters χ_n are independent (in a suitable sense), then $\sum |a_n| < \infty$.

F. I. Mautner (Zürich)

INTEGRAL TRANSFORMS, OPERATIONAL CALCULUS

See also 6040, 6076.

Nixon, Floyd E. 6257

★Transformation de Laplace. Tables et exemples.

Traduit par Daniel P. Lachat.

Dunod, Paris, 1964. vii + 113 pp. 18 *F*.

Translation into French of *Handbook of Laplace transformation: Tables and examples* [Prentice-Hall, Englewood Cliffs, N.J., 1960; MR 22 #9814].

Churchill, R. V. 6258

Integral transforms associated with boundary conditions of third type.

Contributions to Differential Equations 3 (1964), 387–398.

This paper is concerned with boundary-value problems in $U(x, y)$ of the form

$$\mathscr{Y}_1[U_{xx}(x,y)] + \mathscr{Y}_2[U_{yy}(x,y)] = R(x,y) \qquad (x > 0,\ y_1 < y < y_2),$$

with $\mathscr{Y}_i$ $(i = 1, 2)$ linear differential operators of the type $\mathscr{Y}_i = \sum_{j=0}^{m} a_{ij}(y)\partial^j/\partial y^j$ on a bounded or unbounded interval $y_1 < y < y_2$, together with the requirement that U be bounded over the domain, satisfy at $x = 0$ a boundary condition of the "third type"

$$U_x(0, y) - hU(0, y) = Q(y) \qquad (h > 0),$$

and boundary conditions with respect to y of the form $B_i[U] = P_i(x)$ when $y = y_i$, where $B_i[U]$ is some linear combination with constant coefficients of $U(x, y_i)$ and the partial derivatives of U with respect to y are evaluated at $y = y_i$. The author presents two procedures for the solution of such boundary-value problems. The first method involves an integral transform of the solution of a corresponding problem in $V(x, y) = U_x(x, y) - hU(x, y)$, and the problem in V adapted to solution by the Fourier sine transformation with respect to x. The second method uses a modified Fourier integral transformation with respect to x designed for such problems, and for which the author derives a convolution formula; the second method is illustrated by the solution of a simple diffusion problem.

W. T. Reid (Norman, Okla.)

Mainra, V. P. 6259

On self-reciprocal functions.

Bull. Calcutta Math. Soc. 55 (1963), 41–49.

Consider the transform

$$\omega^{\lambda}_{\mu,\nu}(x) = \int_0^\infty \omega_{\mu,\nu}(xy) J_\lambda(y) y^{1/2}\,dy, \tag{1}$$

where

$$\omega_{\mu,\nu}(x) = (x)^{1/2} \int_0^\infty J_\mu(t) J_\nu(x/t) t^{-1}\,dt. \tag{2}$$

The author generalizes (1) to $m+n$ parameters and derives a number of formulas and relations therefrom. In addition, certain theorems on self-reciprocal functions belonging to a certain class are proved.

E. J. Scott (Urbana, Ill.)

Debnath, L. 6260

On Jacobi transform.

Bull. Calcutta Math. Soc. 55 (1963), 113–120.

The paper treats the operational calculus of the integral transformation $\int_{-1}^{1} p(x)P_n^{(\alpha,\beta)}(x)F(x)\,dx$ of functions F on the interval $(-1, 1)$, where $p(x) = (1-x)^\alpha(1+x)^\beta$, $\alpha > -1$, $\beta > -1$, and $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial of degree n.

The author presents the transformation of the basic differential form $[p(x)]^{-1}(d/dx)[(1-x)^{\alpha+1}(1+x)^{\beta+1}F'(x)]$, the transformation of the inverse of that form, the series representing the inverse of the Jacobi integral transformation, transforms of some particular functions, and special cases of the transformation. His treatment is different from, and more complete than, that given earlier by E. J. Scott [Quart. J. Math. Oxford Ser. (2) **4** (1953), 36–40; MR **14**, 869]. *R. V. Churchill* (Ann Arbor, Mich.)

Džrbašjan, M. M.; Akopjan, S. A. 6261

On the theory of integral transforms with Mittag-Leffler kernels. (Russian. Armenian summary)

Akad. Nauk Armjan. SSR Dokl. **38** (1964), 207–216.

L'auteur indique un noyau Ψ formé à partir de la fonction de Mittag-Leffler $E_\rho(z;\mu)=\sum_k z^k/\Gamma(\mu+k/\rho)$, tel que toute fonction f satisfaisant à $f(x)x^{\mu-1}\in L_2(0,\infty)$ peut être exprimée par une transformée intégrale avec un noyau Φ (également formé à partir d'une fonction E), de sa transformée avec le noyau Ψ. *S. Mandelbrojt* (Paris)

Sevost'janov, G. D. 6262

A remark on Mercer's paper, "On integral transform pairs arising from second-order differential equations". (Russian)

Sibirsk. Mat. Ž. **5** (1964), 1200–1202.

Mercer [Proc. Edinburgh Math. Soc. (2) **12** (1962), 63–68; MR **25** #5350] has shown that pairs of integral transforms can be associated with certain homogeneous linear differential equations. In this article, the author shows that Mercer's method can be extended to certain inhomogeneous differential equations. *P. G. Rooney* (Toronto, Ont.)

Norris, D. O. 6263

A topology for Mikusiński operators.

Studia Math. **24** (1964), 245–255.

In der Mikusińskischen Operatorenrechnung bereitet die Einführung einer geeigneten Topologie für Operatoren bekanntlich gewisse Schwierigkeiten. Hier wird ein neuer Weg zur Definition der Konvergenz von Operatorfolgen eingeschlagen, wobei zugleich eine Modifikation von Mikusińskis Definition zugrundegelegt wird. Mit C wird die Menge aller komplexwertigen stetigen Funktionen $a(t)$ $(-\infty<t<+\infty)$ bezeichnet, welche für $t\le\sigma(a)$ identisch verschwinden. Im Unterschied zu der Operatorenrechnung von Mikusiński werden also hier Funktionen nicht nur für nichtnegative Argumente betrachtet, aber die Bedingung des Verschwindens auf einem linken Halbstrahl läuft im wesentlichen auf dasselbe hinaus. Addition und Multiplikation mit Skalaren werden in C wie üblich definiert; das Produkt zweier Elemente a und b von C wird durch die Faltung definiert

$$(a*b)(t)=\int_{-\infty}^{+\infty}a(\tau)b(t-\tau)\,d\tau.$$

Man sieht sofort, daß diese Operationen nicht aus C hinausführen. Es ist auch klar, daß der Integrand des Faltungsintegrals nur in einem endlichen Intervall von Null verschieden ist. Es werden außerdem noch der Raum L der in jedem endlichen Intervall integrablen, links verschwindenden Funktionen und der Unterraum D aller unendlich oft differenzierbaren Funktionen von C eingeführt. D ist eine Subalgebra der Algebra C, ebenso $C*L\subset C$ und $D*L\subset D$. Der Verfasser betrachtet sodann allgemein das Problem der Einführung einer Topologie in einer Subalgebra eines Quotientenkörpers eines Integritätsbereichs, für den Fall, daß der Integritätsbereich bereits eine Topologie trägt, welche durch eine Familie von Seminormen definiert ist. Die daraus natürlicherweise folgende Theorie wird auf die Operatorenrechnung angewendet. Es sei Q der Quotientenkörper von C; in C selbst wird die Topologie der kompakten Konvergenz eingeführt, welche sich auch durch eine Familie von Seminormen definieren läßt (Suprema der Absolutbeträge in kompakten Intervallen). S bezeichne die Subalgebra aller a aus Q mit der Eigenschaft, daß $a*D\subset D$.

An einem Beispiel wird gezeigt, daß die hier erzeugte Topologie in S nicht mit der Topologie von C verträglich ist. Ist aber P eine Menge von allen Elementen $c\in C$ für die $\sigma(c)\ge\tau$, so ist die Konvergenz in S bezüglich P verträglich mit der Konvergenz in C. Es wird sodann der Vergleich mit Mikusińskis Konvergenzdefinition durchgeführt, wobei sich für alle praktisch wirklich auftretenden Operatorfolgen Übereinstimmung ergibt. Es zeigt sich ferner, daß Operatorfunktionen im Sinne von Mikusiński nur dann stetig sind, wenn sie in der S-Topologie stetig sind. Entsprechendes gilt für die Differenzierbarkeit. *D. Laugwitz* (Darmstadt)

Wuyts, P. 6264

The behaviour on the boundary of the region of convergence of the function represented by an integral of the form $\int_0^{+\infty}e^{-\lambda(t)}F(t)\,dt$ ($\lambda(t)$ complex). (Dutch)

Simon Stevin **37** (1963/64), 25–34.

Der Verfasser setzt frühere Untersuchungen fort [Simon Stevin **26** (1962/63), 72–77, MR **26** #2828]. Bezeichnungen und Voraussetzungen werden aus dieser früheren Arbeit übernommen. Es wird das Randverhalten des im Titel angegebenen Integrals untersucht. Die Ergebnisse sind analog zu entsprechenden Resultaten bei Laplace-Integralen und Dirichlet-Reihen. *D. Laugwitz* (Darmstadt)

INTEGRAL EQUATIONS

See also 6163, 6164, 6165, 6172.

Maroni, Pascal 6265

Sur les solutions de l'équation de Milne généralisée.

C. R. Acad. Sci. Paris **259** (1964), 501–503.

The integral equation considered is

$$\mathcal{F}(t)=\int_0^\infty K(|t-\theta|)\mathcal{F}(\theta)\,d\theta\qquad(t\ge0),$$

where $K(t)$ satisfies certain conditions we shall not reproduce. A special case is Milne's equation from radiative transfer theory. Similar generalizations of Milne's equation have been studied by a number of writers [cf. I. W. Busbridge, *The mathematics of radiative transfer*, Cambridge Univ. Press, New York, 1960].

Solutions are sought which are measurable, bounded on each $[a,b]\subset(0,\infty)$, and satisfy certain asymptotic conditions. A direct study of the integral operator shows that

such a solution is necessarily twice continuously differentiable on each $[a, b] \subset (0, \infty)$. Transform methods yield a solution unique up to a constant factor. Various identities are given. There are no proofs.

P. M. Anselone (Corvallis, Ore.)

Levin, A. Ju. 6266

The Fredholm equation with smooth kernel and boundary-value problems for a linear differential equation. (Russian)

Dokl. Akad. Nauk SSSR **159** (1964), 13–16.

Estimations are given for numerator and denominator of Fredholm's resolvent and for the growth of the moduli of eigenvalues for

$$x(t) = \lambda \int_a^b K(t, s)x(s)\, d\mu(s)$$

which are used for the study of boundary-value problems of linear differential equations and for the derivation of sufficient conditions for non-oscillation of the operator $x^{(n)} + q(t)x$.

M. Ráb (Brno)

Zahar-Itkin, M. H. 6267

On the growth of eigenvalues of a linear integral equation. (Russian)

Dokl. Akad. Nauk SSSR **158** (1964), 1022–1025.

Asymptotic estimates are given for the growth of proper values of the integral equation $y(x) = \lambda \int_0^1 K(x, t)y(t)\,dt$ for $K(x, t)$ smooth except at $(0, 0)$ and $(0, 1)$, and for $K(x, t)$ with Green's function type singularities on the line $x = t$.

L. B. Rall (Madison, Wis.)

Hayashi, Yoshio 6268

On some singular integral equations. I.

Proc. Japan Acad. **40** (1964), 323–328.

This note concerns solutions of singular integral equations of the type considered by Mushelišvili in his book *Singular integral equations* [Noordhoff, Groningen, 1953; MR **15**, 434]. The theorems are of two types. First a contraction mapping procedure is given for equations of first kind. It is shown that addition and subtraction of a suitable function modifies the reduced Fredholm operator (in Mushelišvili's terminology) so as to make it a contraction. Then the method of successive approximations is valid. The second result is that an equation of second kind can be reduced to one of first kind. Essentially no proofs are given, and this makes the results difficult to follow.

R. C. MacCamy (Pittsburgh, Pa.)

Kumano-go, Hitoshi 6269

On a definition of singular integral operators. I, II.

Proc. Japan Acad. **40** (1964), 368–373; *ibid.* **40** (1964), 374–378.

The author considers singular integral operators H, in the sense of Calderón and Zygmund [see A. P. Calderón and A. Zygmund, Amer. J. Math. **79** (1957), 901–921; MR **20** #7196]. Instead of requiring the symbols $\sigma(H)(x, \eta)$ to be homogeneous, it is assumed that $\sigma(H)(x, \eta)$ is analytic in η. Part I is devoted to the definition of singular integral operators H and to proving certain lemmas. In Part II the author derives the theorems of Calderón and Zygmund for singular integral operators as defined in Part I. H is shown to be a bounded operator in L_2. The operators $H_1 \circ H_2$ and H^* are defined, and the analogous theorems relating H^* with H^* and $H_1 \circ H_2$ with H_1H_2 are proved.

L. M. Sibner (Stanford, Calif.)

Widom, Harold 6270

Asymptotic behavior of the eigenvalues of certain integral equations. II.

Arch. Rational Mech. Anal. **17** (1964), 215–229.

The author continues and partly generalizes his work in Trans. Amer. Math. Soc. **109** (1963), 278–295 [MR **27** #5101]. Let $K(\xi)$ $(-\infty < \xi < \infty)$ be bounded, non-negative, and even, and suppose that $K(\xi) \to 0$ as $\xi \to \infty$. For $f \in L_2(-1, 1)$, let $F(\xi) = \int_{-1}^{1} e^{i\xi x} f(x)\,dx$, and let $T(K)f$ be the restriction to $(-1, 1)$ of the inverse Fourier transform of $K(\xi)F(\xi)$. $T(K)$ is positive definite (for $K \neq 0$ a.e.) and completely continuous. Order its eigenvalues $\lambda_0(K) \geq \lambda_1(K) \geq \cdots > 0$. Write $K(\xi) = e^{-\Gamma(\xi)}$. The author proves the following theorems. Theorem I: If Γ is continuous, increasing for large ξ, and $\Gamma(\xi)/\xi \to 0$ as $\xi \to \infty$, then $\lambda_n(K) \sim K(n\pi/2 + o(n))$ as $n \to \infty$. Theorem II: If $\Gamma(\xi) \sim \alpha\xi$ $(0 < \alpha < \infty)$, then $\log \lambda_n(K) \sim -n\pi[\mathbf{K}(\operatorname{sech} \pi/\alpha)/\mathbf{K}(\tanh \pi/\alpha)]$, where $\mathbf{K}$ is the complete elliptic integral of the first kind, $\mathbf{K}(r) = \int_0^{\pi/2} (1 - r^2 \sin^2\theta)^{-1/2}\,d\theta$. Theorem III: If $\Gamma(\xi)$ is convex for large ξ and $\Gamma(\xi)/\xi \to \infty$, then $\log \lambda_n(K) \sim -\Gamma(\sigma_n)$, where σ_n is the unique solution of the equation $\Gamma(\sigma) = 2n \log(n/\sigma)$ $(0 < \sigma < n)$. Corollaries: If $\log \Gamma(\xi) \sim \alpha \log \xi$ $(\alpha > 1)$, then $\log \lambda_n(K) \sim -(2 - 2/\alpha)n \log n$; If $\log \Gamma(\xi)/\log \xi \to \infty$, then $\log \lambda_n(K) \sim -2n \log n$; If there happens to be a function k belonging to $L_p(-\infty, \infty)$ $(1 \leq p \leq 2)$ whose Fourier transform is K, then $T(K)$ is the integral operator on $L_2(-1, 1)$ with kernel $k(x-y)$, and the $\lambda_n(K)$ are the eigenvalues of the equation

$$\int_{-1}^{1} k(x-y)f(y)\,dy = \lambda f(x).$$

J. Chover (Madison, Wis.)

Tranter, C. J. 6271

A note on dual equations with trigonometrical kernels.

Proc. Edinburgh Math. Soc. (2) **13** (1962/63), 267–268.

From the author's introduction: "Sneddon [Proc. Glasgow Math. Assoc. **5** (1962), 147–152; MR **25** #1412] has recently pointed out that the dual integral equations

$$(1) \qquad \int_0^\infty \xi^{-1}\psi(\xi)\cos(x\xi)\,d\xi = f(x), \qquad 0 \leq x \leq 1,$$

$$(2) \qquad \int_0^\infty \psi(\xi)\cos(x\xi)\,d\xi = 0, \qquad x > 1,$$

are not covered by Busbridge's solution [Proc. London Math. Soc. (2) **44** (1938), 115–129] of the more general equations with Bessel function kernels, and has given a solution to equations (1) and (2) in the form of a Neumann series. The purpose of this note is to show that a very simple formal solution can be obtained by using … well-known Bessel function integral representations. …"

H. Widom (Ithaca, N.Y.)

Nedjalkov, I.; Pendev, G. 6272

On the numerical solution of a class of non-linear integral equations of dispersion type. (Bulgarian. Russian and English summaries)

Bălgar. Akad. Nauk. Otd. Mat. Fiz. Nauk. Izv. Fiz. Inst. s Aneb **12** (1964), 143–151.

Authors' summary: "A method is presented in this work for a direct solution of non-linear singular integral equation systems of the type of the equation of F. E. Low and G. F. Chew, Castilejo, Dalitz and Dyson, G. F. Chew and S. Mandelstam, et al. The solution is obtained by the method of successive approximations, taking special measures to remove the instability of the iteration process. Resonance and adiabatic solutions are obtained for the Chew and Low equation. The adiabatic solution found in this work ceases to be analytic when the meaning of the link's constant is $f^2 = 0{,}07$."

FUNCTIONAL ANALYSIS

See also 5958, 5959, 5964, 5979, 6057, 6141, 6142, 6157, 6163, 6164, 6165, 6171, 6172, 6175, 6181, 6184, 6255, 6256, 6269.

Allahverdiev, Dž. È. 6273

On the completeness of a system of characteristic and adjoint elements of operators which are rational functions of a parameter. (Russian)

Dokl. Akad. Nauk SSSR **159** (1964), 951–954.

The problem is to determine whether a Hilbert space is the closed span of finite-dimensional invariant subspaces for a given completely continuous operator. The operators L considered are of the form $L = A + \sum_{i=1}^{n-1} \lambda^i A_i H^i + \lambda^n H^n$, where $A, A_1, \cdots, A_{n-1}$ are completely continuous operators, λ is a complex parameter, and H is a completely continuous, selfadjoint operator which has a sum of pth powers of its roots finite. The paper is of interest for the estimates of resolvents of such operators, from which the completeness results follow in a routine way.

L. de Branges (Lafayette, Ind.)

Edwards, D. A. 6274

On the homeomorphic affine embedding of a locally compact cone into a Banach dual space endowed with the vague topology.

Proc. London Math. Soc. (3) **14** (1964), 399–414.

The author constructs a norm on a real, locally convex, separated linear space generated by a locally compact cone. With respect to this norm, the space is a Banach space which is shown to possess numerous interesting properties. For example, it is shown that the cone can be embedded homeomorphically into the Banach space with the w^*-topology. A result of Schaeffer concerning the existence of eigenvalues for certain positive operators is proved using the construction and a theorem of Bonsall.

S. Goldberg (College Park, Md.)

Ellis, A. J. 6275

The duality of partially ordered normed linear spaces.

J. London Math. Soc. **39** (1964), 730–744.

As a sequel to a paper of Edwards [#6274], the author considers a real partially ordered vector space X with a locally convex topology τ and with an order unit e such that $[0, e]$ is τ-compact. It is proved that X has an order unit norm which makes it the dual space of the Banach space of all linear functionals on X which are τ-continuous on norm bounded sets. A result is given dual to the theorem of Grosberg and Krein which links generating and normal cones in dual Banach spaces. The last section of the paper considers when the annihilator of an order ideal in a partially ordered vector space with an order unit intersects the base in the order dual space in an extremal set. This generalizes a theorem of Bonsall.

S. Goldberg (College Park, Md.)

Gähler, Siegfried 6276

Lineare 2-normierte Räume.

Math. Nachr. **28** (1964), 1–43.

In a previous paper [same Nachr. **26** (1963), 115–148; MR **28** #5423] the author investigated the notion of a 2-metric, a real-valued function of point-triples on a set X, whose abstract properties were suggested by the area function for a triangle determined by a triple in euclidean space; associated with a given 2-metric was a natural topology. The author showed that a metric space has a topologically equivalent 2-metric, but gave an example of a 2-metric space which is not metrizable.

In his present paper the author studies a related concept in the category of linear spaces. If L is a linear space, a real function on $L \times L$ is called a 2-norm if (1) $\|a, b\| = 0$ exactly when the vectors a and b are linearly dependent; (2) $\|a, b\| = \|b, a\|$; (3) for each real number r, $\|a, rb\| = r\|a, b\|$; (4) $\|a, b+c\| \leq \|a, b\| + \|a, c\|$.

Here are some of the main results of the paper. Theorem 2: A linear space L of dimension greater than 1 with a 2-norm $\|a, b\|$ has a 2-metric $\sigma(a, b, c) = \|b-a, c-a\|$. Theorem 4: A linear 2-normed space L is a locally convex topological vector space.

On the other hand, the author gives an example of a 2-normed vector space of uncountable dimension which is not metrizable and hence does not have any norm.

R. H. Rosen (Ann Arbor, Mich.)

Hasumi, Morisuke; Seever, G. L. 6277

The extension and the lifting properties of Banach spaces.

Proc. Amer. Math. Soc. **15** (1964), 773–775.

A Banach space is said to have the extension property if for any Banach space F and any bounded linear map A from a subspace of F into E, there exists a linear extension A_e of A mapping all of F into E with $\|A_e\| = \|A\|$. E has the lifting property if for any closed subspace M of F, the space of bounded linear operators $[E, F]$ can be mapped onto the space of bounded linear operators $[E, F/M]$ under the map $\phi: T \to \eta T$, where η is the natural map from F onto F/M and $\|\phi\| = 1$. If E has the lifting property, then it is known that the conjugate space E' has the extension property. The authors prove that if E has the extension property, then E' has the lifting property if and only if E is finite-dimensional. *S. Goldberg* (College Park, Md.)

Cohen, Henry B. 6278

The k-norm extension property for Banach spaces.

Proc. Amer. Math. Soc. **15** (1964), 797–802.

Let k denote an infinite cardinal number. A k-family [set] is a family [set] whose cardinal number does not exceed k. If B is a set and C a family of subsets of B, C is said to have the binary intersection property [k-binary intersection property] if any non-empty subfamily [k-subfamily] of C whose members meet pairwise has a non-empty intersection. A normed linear space B is called k-separable if B has a dense k-subset. B has the norm extension property [k-norm extension property] if every bounded linear operator mapping a subspace of a normed linear space Y [k-separable normed linear space Y] can be extended to a linear operator, with the same norm, defined on all of Y. The author proves the following theorem. The following statements are equivalent for a real normed linear space B. (i) B has the k-norm extension property. (ii) If M is a k-separable subspace of B and Y is a normed linear space of the form $M \oplus \{x\}$, there is a linear operator $H: Y \to B$ of norm 1 such that H is the identity on M. (iii) The family of closed spheres of B has the k-binary intersection property. The theorem was proved by Nachbin [Trans. Amer. Math. Soc. 68 (1950), 28–46; MR 11, 369] with norm and binary replacing k-norm and k-binary, respectively, and $M = B$ in (i), (ii) and (iii). The Banach space $C^*(S)$ of bounded continuous functions on a topological space S is shown to have the k-norm extension property under certain conditions. For example, the compact Hausdorff spaces for which $C^*(S)$ has the k-norm extension property are characterized.

S. Goldberg (College Park, Md.)

de Branges, Louis; Rovnyak, James 6279

Correction to "The existence of invariant subspaces".

Bull. Amer. Math. Soc. 71 (1965), 396.

From the authors' correction: "The proof of the existence of invariant subspaces announced in this Bulletin 70 (1964), 718–721 [MR 28 #4329] is false. We now withdraw the announcement and make no statement either for or against the existence of invariant subspaces."

Karamata, J. 6280

Contribution à une théorie générale de la croissance des fonctions.

Bull. Acad. Serbe Sci. Arts Cl. Sci. Math. Natur. Sci. Math. 31 (1963), no. 4, 29–30.

Let $(A, <)$ be a directed set (in the sense of Moore-Smith limits), and consider the class Φ of functions f defined on A, with values in a Banach space F. A partial ordering of Φ is defined as follows. Let f and g be functions in Φ, and write $f \prec g$ in case there exists a scalar λ such that for every $\varepsilon > 0$ there exists an x_ε such that $\|f(x) - \lambda g(x)\| \leq \varepsilon g(x)$ whenever $x_\varepsilon < x$. The author fails to note that when $f \prec g$ with $\lambda \neq 0$, then $g \prec f$, and when $f \prec g$ with $\lambda = 0$, then $g \prec f$ is false unless g is eventually zero. A number of remarks concerning partially ordered sets are adjoined.

L. M. Graves (Chicago, Ill.)

Lozanovskiĭ, G. Ja. 6281

On topologically reflexive KB-spaces. (Russian)

Dokl. Akad. Nauk SSSR 158 (1964), 516–519.

Soit X un KB-espace, c'est-à-dire un espace linéaire réticulé complet, muni d'une norme satisfaisant les propriétés: $|x| \leq |y|$ entraîne $\|x\| \leq \|y\|$, $x_n \downarrow 0$ entraîne $\|x_n\| \to 0$, $0 \leq x_n \uparrow +\infty$ entraîne $\|x_n\| \to +\infty$.

L'auteur construit des sous espaces de X qui sont de KB-espaces réflexifs, et qui sont des espaces analogues au espaces L^p $(p > 1)$, ou aux espaces réflexifs de Orlicz.

Si X est un KB-espace à unité, alors pour chaque $x \in X$ il existe un sous espace linéaire E, normal (c'est-à-dire $|x| \leq |y|$ et $y \in E$ entraîne $x \in E$), et total (c'est-à-dire $x \perp E$ entraîne $x = 0$), qui contient x, et qui est un KB espace réflexif par rapport à une certaine nouvelle norme.

Si X est un espace linéaire réticulé complet quelconque, alors l'ensemble de tous les sous espaces linéaires, normaux et totals, qui sont des KB-espaces réflexifs, est un treillis distributif par rapport à l'inclusion.

R. Cristescu (Bucharest)

Mel'cer, M. M. 628

Weakly completely continuous spectra of locally conve spaces. (Russian)

Dokl. Akad. Nauk SSSR 157 (1964), 265–267.

The author announces without proof a number of resul about certain kinds of projective and inductive limits locally convex spaces. Let $\{E_\alpha, \pi_\alpha^\beta\}$ be an inverse spectr family of locally convex spaces E_α and continuous line maps $\pi_\alpha^\beta: E_\beta \to E_\alpha$, and let E be the projective limit. locally convex space E is said to be a W-space if it is th projective limit of $\{E_\alpha, \pi_\alpha^\beta\}$, where for each α there is $\beta > \alpha$ such that for each 0-neighborhood U in E_β the $\pi_\alpha^\beta(U)$ has weakly compact closure in E_α. Among oth things it is asserted that a W-space is complete and sem reflexive and that every W-space is the limit of a fami of Banach spaces. Moreover, closed subspaces of a space, quotient spaces, products, projective limits, a inductive limits of a sequence of W-spaces are also spaces if they are complete and Hausdorff. The author al considers inductive limits, where the maps $\pi_\beta^\alpha: E_\alpha \to$ are weakly completely continuous. This work extends th of D. A. Raĭkov [same Dokl. 113 (1957), 984–986; MR 754].

R. G. Bartle (Urbana, I

Pisanelli, Domingos 62

Sull'invertibilità degli operatori analitici negli spazi Banach.

Boll. Un. Mat. Ital. (3) 19 (1964), 110–113.

Let f be a function whose domain is an open subset o Banach space X and whose image space is a Banach spa Y. Suppose that, inside some open ball with center x_0 is analytic in the sense of Fréchet. [For definitions, see Hille, *Functional analysis and semi-groups*, Amer. Ma Soc., New York, 1948; MR 9, 594.] Then the followi theorem holds. If the Fréchet differential $\delta f(x_0, h)$ is a o one map from X onto Y, then there exists a uniq inverse analytic map g whose domain is an open ball in with centre y_0, where $y_0 = f(x_0)$. This theorem is a gener ization of the well-known theorem for Euclidian spaces invertibility of differentiable transformations with n vanishing Jacobian. The author leaves as an open questi the possibility of extending the above theorem to analy maps between locally convex spaces.

G. G. Gould (Syracuse, N.

Roelcke, W. 6

Über die Hebbarkeit der Unstetigkeiten gewisser Abl dungen topologischer Räume in Banach-Räume.

Math. Z. 79 (1962), 158–179.

Let X be a topological space and B a Banach space with conjugate space B^*. Let Ω be a family of subsets of X having empty interiors, such that if $N_k \in \Omega$ $(k=1, 2, \cdots)$, then $\bigcup N_k \in \Omega$, and such that if $M \subseteq N$ and $N \in \Omega$, then $M \in \Omega$. Let $p: X \to B^*$ be a mapping with the following property (here p_x is the image under p of $x \in X$): For all $u \in B$ there exists a complex-valued continuous function S_u on X such that the relation $p_x u = S_u(x)$ holds for all x except on some set $N_u \in \Omega$. The author seeks conditions that the discontinuities relative to the topology $\sigma(B^*, B)$ be "removable" in the following sense, namely, that there exist an "alteration" $q: X \to B^*$ of p such that q is (everywhere) continuous relative to $\sigma(B^*, B)$ and such that $q_x = p_x$ for all x except on some set in Ω.

The discontinuities of p are not in general removable, but are removable in several cases; for example, if X is locally compact, the discontinuities of p are removable.

S. Heckscher (Swarthmore, Pa.)

Gribanov, Ju. I. 6285

On the theory of the reduction method for infinite systems of linear equations. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 5 (42), 12–16.

A Banach space l of infinite sequences $X = (x_1, x_2, \cdots)$ is called a sequence space if (1) l contains the set h of all finitely non-zero sequences; (2) $\|X\| = 0$ implies $X = \{0\}$; (3) $\|X\| = \sup \|P_n X\|$, where P_n is the projection to the first n coordinates. Consider a system of linear equations (I) $X = AX + H$ in a sequence space l where A is a matrix operator in l with norm $\|A\| < 1$ and $X, H \in l$, and corresponding reduced systems $X_n = P_n A P_n X_n + P_n H$, $n = 1, 2, \cdots$. The assumptions on l and A imply the existence and the uniqueness of the solutions of these systems. The author says that the reduction method is norm-convergent in l to the solution of the system (I) if the sequence $\{X_n\}$ converges to X in l. Theorem 4: The reduction method is norm-convergent in l to the solution of the system (I) if and only if $X \in [l]$, where $[l]$ denotes the closure of h in l. In this case, (II) $\|X - X_n\| \leq \|X - P_n X\|(1 - \|A\|)^{-1}$. As a consequence, the reduction method is always norm-convergent for any sequence space l satisfying $l = [l]$ (e.g., $l = l_p$, $p \geq 1$). For $l = m$, the reduction method is norm-convergent if $A(m) \subseteq c_0$ and $H \in c_0$, because $[m] = c_0$. The sequence spaces considered here are more special and the restriction on A is much weaker than those studied earlier by the author [same Izv. **1962**, no. 1 (26), 28–40; MR **25** #1458]. The estimate (II) for approximation is less effective than those obtained in the previous paper because of the weaker assumption on A.

M. Hasumi (Mito)

Bahtin, I. A. 6286

On the existence of eigenvectors of positive linear operators which are not completely continuous. (Russian)

Mat. Sb. (*N.S.*) **64 (106)** (1964), 102–114.

The principal theorem proved here is this. Let A be a linear operator acting on a partially ordered Banach space. Suppose there exists a non-negative vector u_0 having the property that for each non-negative x there exist $\alpha, \beta > 0$ such that $\alpha u_0 \leq Ax \leq \beta u_0$ (A is said to be u_0-bounded), and suppose (*) that $\beta/\alpha \leq k$ for a fixed number k. Then A has a non-negative eigenvector. Furthermore, if every bounded monotonic sequence of vectors converges then condition (*) is unnecessary.

This generalizes a great many theorems, the oldest of which is O. Perron's classical statement [Math. Ann. **64** (1907), 1–76] that every non-singular matrix with non-negative elements has a non-negative eigenvector.

G. Hufford (Boulder, Colo.)

Orlicz, W. 6287

A note on modular spaces. VII.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 305–309.

Let ρ be a modular on a vector lattice X in which every bounded countable subset has a least upper bound. If one defines a metric on X by $d(x, y) = \inf\{\varepsilon \mid \rho[(x-y)/\varepsilon] \leq \varepsilon\}$, X has been shown by J. Musielak and W. Orlicz [Studia Math. **18** (1959), 49–65; MR **21** #298] to have all the properties of an F-space, in the sense of Banach [*Théorie des opérations linéaires*, Monogr. Mat., Tom 1, Warsaw, 1932], except possibly completeness. The present paper considers the hypothesis: If $x_1, x_2, \cdots \geq 0$ and $\sum \rho(x_n) < \infty$, then $\bigvee x_n$ exists. With this hypothesis, X is shown actually to be an F-space for the above metric. Furthermore, if we denote $d(x, 0)$ by $\|x\|$, the following theorem holds: Whenever $\|x_1\| + \|x_2\| + \cdots < \infty$, then $y = \bigvee_n(|x_1| + \cdots + |x_n|)$ exists, $z = \bigvee x_n$ exists, $\|y\| \leq \sum \|x_n\|$ and $\rho(z) \leq \sum \rho(x_n)$. The last section considers mappings from a suitable subset of X to another vector lattice. (The earlier papers in this series may be traced by starting with Part VI [Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 449–454; MR **28** #3316].)

H. Gordon (Philadelphia, Pa.)

Riedl, John 6288

Partially ordered locally convex vector spaces and extensions of positive continuous linear mappings.

Math. Ann. **157** (1964), 95–124.

The author generalizes to locally convex [or normed] partially ordered vector spaces the extension theorems and properties of locally convex [or normed] vector spaces, and establishes results that correspond to the known ones. The article also contains results about normal cones; about representation of spaces into, or onto, $C(X)$, about isomorphisms of spaces in terms of the lattice of closed linear subspaces and the incidence relation with the positive cone, and about isomorphisms of spaces in terms of the preordered semigroup of continuous linear operators.

L. Nachbin (Rochester, N.Y.)

Vekaler, A. I. 6289

The operations of partial multiplication in vector lattices. (Russian)

Dokl. Akad. Nauk SSSR **158** (1964), 759–762.

Wie es bekannt ist [vgl. z.B. B. Z. Vulih, *Introduction to the theory of partially ordered spaces* (Russian), Chapter V, Fizmatgiz, Moscow, 1961; MR **24** #A3494], gibt es zu jedem Archimedischen Vektorverband X einen extremalen bikompakten Raum P so, dass X zu einer Menge von stetigen Funktionen auf P isomorph ist, wobei angenommen wird, dass jede von diesen Funktionen auch die Werte $\pm\infty$ auf einer nirgends dichten Teilmenge von P erreichen darf. Die Multiplikation dieser Funktionen kann mit Hilfe des Isomorphismus in die Menge X übertragen

werden; so entsteht in X eine Multiplikation, welche im allgemeinen nicht zu jedem geordneten Paar von Elementen ein Produkt zuordnet. Diese Multiplikation heisst Darstellungsmultiplikation; sie erfüllt gewisse Axiome.

Es sei in X eine Multiplikation gegeben, welche die obenangeführten Axiome erfüllt. Der Verfasser gibt Bedingungen an, unter welchen diese Multiplikation mit einer Darstellungsmultiplikation identisch ist. Dies geschieht z.B. unter Voraussetzung, dass das Produkt für je zwei Elemente von X definiert ist. Eine Darstellungsmultiplikation in einem bedingt vollständigen Vektorverband X erklärt die Ordnung in X in gewissem Sinne eindeutig. *M. Novotný* (Brno)

Girardeau, Jean-Pierre 6290

Propriétés d'interpolation des espaces de Hilbert.

C. R. Acad. Sci. Paris **258** (1964), 44–46.

Let $(E_i, F_i)_{i=1,2}$ be two pairs of Banach spaces such that $F_i \subset E_i$, where this injection is continuous. Two Banach spaces H_1, H_2 such that $F_i \subset H_i \subset E_i$ (all injections being continuous) are said to have the property of interpolation if every continuous linear mapping of E_1 into E_2 whose restriction is continuous from F_1 to F_2 is also a continuous mapping from H_1 to H_2. In this paper the author gives certain conditions on the spaces (E_i, F_i) in order that the property of interpolation be valid. For example, he assumes the existence of a linear mapping $L_i : E_i' \to F_i$ with the following properties: $L_i(E_i') = F_i$ (algebraically) and $\langle z, u \rangle \geq 0$, where $z = L_i(u)$. The definition $(x \mid y)_i = \langle x, u \rangle$ for $y = L_i(u)$ yields a non-degenerate, positive, hermitian form on F_i. The completion of F_i with respect to this form is a Hilbert space H_i such that $H_i \subset E_i$, this injection is continuous, and F_i is dense in H_i [cf. L. Schwartz, Séminaire Bourbaki, 1961/62, deuxième édition, Fasc. 3, Exposé 238, Secrétariat mathématique, Paris, 1962; MR **28** #3561]. Under these conditions the author proves that the spaces H_1 and H_2 have the property of interpolation. As a corollary he derives a recent result of J.-L. Lions and J. Peetre [Inst. Hautes Études Sci. Publ. Math. No. 19 (1964), 5–68; MR **29** #2627]. *G. Maltese* (College Park, Md.)

Golovkin, K. K. 6291

The ε-entropy of certain compact sets of differentiable functions in spaces with monotone norm. (Russian)

Dokl. Akad. Nauk SSSR **158** (1964), 261–263.

Let $C^\infty(\Omega)$ be the space of all infinitely differentiable functions on an n-dimensional hypercube Ω. Let $\|\cdot\|$ be a monotone norm on $C^\infty(\Omega)$ which is invariant under translation, and denote by H the closure of the family $\{u \in C^\infty(\Omega) : \|u\| < \infty\}$. Having defined two norms $\|\cdot\|_k$ $(k = 1, 2)$ of type L_p [respectively, C], the author obtains a lower [upper] estimate for the ε-entropy of the sets $\{u \in H : \|u\| + \|u\|_k \leq M\}$; these are the "compact sets" in the title of this article. *G. L. Krabbe* (Lafayette, Ind.)

Hunt, Richard A. 6292

An extension of the Marcinkiewicz interpolation theorem to Lorentz spaces.

Bull. Amer. Math. Soc. **70** (1964), 803–807.

This note contains an extension of Marcinkiewicz's interpolation theorem to spaces of Lorentz type. If f is a measurable function on some measure space, denote by $f^*(t)$ its decreasing rearrangement on the real half-line $0 < t < \infty$. Set

$$\|f\|_{pq} = \left(\int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t}\right)^{1/q}, \qquad 0 < p < \infty,\ 0 < q < \infty,$$

$$= \sup_{0<t<\infty} t^{1/p} f^*(t), \qquad 0 < p \leq \infty,\ q = \infty.$$

(This is not a norm in general.) Theorem: If T is a quasi-linear operator, $|T(f+g)| \leq K(|Tf| + |Tg|)$, such that

$$\|Tf\|_{p_i' q_i'} \leq B_i \|f\|_{p_i q_i}, \qquad i = 0, 1,\ p_0 < p_1,\ p_0' \neq p_1',$$

then

$$\|Tf\|_{p_\theta' q} \leq B_\theta \|f\|_{p_\theta q}$$

for $0 < \theta < 1$, $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$, $1/p_\theta' = (1-\theta)/p_0' + \theta/p_1'$. The proof, which depends on Hardy's inequality, is short and elegant. (For previous work in the same direction, see Krein and Semenov [Dokl. Akad. Nauk SSSR **138** (1961), 763–766; MR **25** #4352], Calderón [Studia Math. (Ser. Specjalna) Zeszyt 1 (1963), 31–34; MR **26** #5409], Lions and the reviewer [Inst. Hautes Études Sci. Publ. Math. No. 19 (1964), 5–68; MR **29** #2627] and the reviewer [C. R. Acad. Sci. Paris **256** (1963), 1424–1426].) *J. Peetre* (Lund)

Hunt, Richard A.; Weiss, Guido 6293

The Marcinkiewicz interpolation theorem.

Proc. Amer. Math. Soc. **15** (1964), 996–998.

This paper consists of a new proof of the Marcinkiewicz interpolation theorem [see A. Zygmund, *Trigonometric series*, Vol. II, second edition, pp. 111–116, Cambridge Univ. Press, New York, 1959; MR **21** #6498]. [It should be noted that there is a misprint in the middle of the first page. Following formula (2) there appears the formula $f(x) > y$; it should read $|h(x)| > y$.] *S. R. Foguel* (Jerusalem)

Itô, Takashi 6294

On the continuity of lattice automorphisms on continuous function lattices.

Illinois J. Math. **8** (1964), 419–424.

Let $C(E)$ denote the lattice of all real-valued continuous functions on a compact Hausdorff space E. In general, lattice automorphisms are not continuous in the topology of uniform convergence. The author considers the following possibilities. Property (K): all lattice automorphisms of $C(E)$ are continuous; (K_0): all compact subspaces of E have property (K); (K_1): a lattice automorphism of $C(E)$ is continuous if and only if its inverse is continuous. Theorem 1: E has property (K_1) if and only if it is not the Stone-Čech compactification of any proper dense co-zero set. Theorem 2: E has property (K_0) if and only if it does not contain a copy of βN. The methods of proof follow those of Kaplansky [Amer. J. Math. **70** (1948), 626–634; MR **10**, 127]. *L. Gillman* (Rochester, N.Y.)

Jamison, Benton 6295

Asymptotic behavior of successive iterates of continuous functions under a Markov operator.

J. Math. Anal. Appl. **9** (1964), 203–214.

Let S be a compact metric state space and $P(x, E)$ a transition probability operator on S. It is assumed that the operator $Tf = \int_S f(y)P(\cdot, dy)$ maps the Banach space $C(S)$ of all complex-valued continuous functions under sup norm into itself. In this paper the asymptotic behavior of the sequence $\{T^n f\}$ is studied. T is called uniformly stable if the sequence $\{T^n f\}$ is equicontinuous for each $f \in C(S)$. It is called uniformly stable in the mean if in the above definition $\{T^n f\}$ is replaced by $\{(1/n) \sum_{k=1}^{n} T^k f\}$. Under the hypotheses of uniform mean stability the author shows that $\{(1/n) \sum_{k=1}^{n} T^k f\}$ converges uniformly to a constant if and only if there are not two disjoint nonvoid topologically and stochastically closed subsets in S. Furthermore, under the stronger assumption of uniform stability, several conditions are given which are necessary and sufficient for the uniform convergence of $\{T^n f\}$ to constants. Two applications of these results are made: one concerns convolutions of measures on compact separable groups, and the other a certain random mapping problem.

R. L. Adler (Yorktown Heights, N.Y.)

Kinokuniya, Yoshio 6296

Orthogonal projection of the space X of univoque functions.

Mem. Muroran Inst. Tech. **4**, no. 1, 297–307 (1962).

Mack, John 6297

The order dual of the space of Radon measures.

Trans. Amer. Math. Soc. **113** (1964), 219–239.

The author continues the study of the first and second duals of the space of continuous functions that was initiated by S. Kaplan [same Trans. **86** (1957), 70–90; MR **19**, 868; ibid. **93** (1959), 329–350; MR **22** #2888; ibid. **101** (1961), 34–51; MR **24** #A1598]. (The last paper will be referred to as III.)

This paper is concerned with the spaces L_k, L_b, and $\bigcup L(K)$ of measures on a locally compact space X that are Radon measures, that are finite, and that have compact support, respectively, and their order duals M_k, M_b, and M. Various relations among these spaces are studied, and among other things, it is shown that M_k and M_b appear as ideals in M, and that C (the set of all continuous real-valued functions on X) can be embedded in M.

The author further considers two questions raised in III by S. Kaplan: (1) if every atomic (Radon) measure on a space is finite, does it follow that every measure on the space is finite; and (2) for each $f \in M_k$ does there exist $g \in M_k$ so that the closed ideal generated by f is dominated by some multiple of g. Both questions are answered in the negative by exhibiting a locally compact space T for which each fails. The space is obtained using the following theorem. Let μ be a Radon measure on the σ-compact space X. Then there exists a countably compact space T containing X as a topological subspace, and a Radon measure v on T so that the restriction of v to X is μ if and only if the atomic part of μ (considered as a function on X) vanishes at infinity.

R. G. Douglas (Ann Arbor, Mich.)

Peetre, Jaak 6298

Sur le nombre de paramètres dans la définition de certains espaces d'interpolation.

Ricerche Mat. **12** (1963), 248–261.

On rappelle la définition des espaces de moyenne deux espaces de Banach A_0 et A_1 continûment dans un même espace vectoriel topologique localeme vexe $\mathscr{A}$: Pour $0 < \theta < 1$ et $1 \leq p_0, p_1 \leq +\infty$, $(A_0, A$ est l'espace décrit par $\int_0^{+\infty} u(t)\,dt/t$ lorsque u varie e assujettie aux conditions

$$\text{(i)} \qquad \int_0^{+\infty} \|t^{-\theta}u(t)\|_{A_0}^{p_0}\, dt/t < +\infty,$$

$$\text{(ii)} \qquad \int_0^{+\infty} \|t^{1-\theta}u(t)\|_{A_1}^{p_1}\, dt/t < +\infty.$$

L'auteur démontre l'identité

$$(A_0, A_1)_{\theta, p_0, p_1} = (A_0, A_1)_{\theta, p, p}$$

avec $1/p = (1-\theta)/p_0 + \theta/p_1$, ce qui montre que les de moyennes ne dépendent effectivement que d paramètres. L'auteur démontre ensuite l'identité

$$((A_0, A_1)_{\theta,1,1}, (A_0, A_1)_{\theta,\infty,\infty})_{\lambda,\rho,\rho} = (A_0, A_1)_{\theta,\rho,}$$

pour $1/\rho = 1 - \lambda$, qui complète un résultat antéri J.-L. Lions et J. Peetre [Inst. Hautes Études Sci Math. No. 19 (1964), 5–68; MR **29** #2627].

P. Grisvard (

Sendov, Bl.; Penkov, B.

On widths of the space of continuous functions.

C. R. Acad. Bulgare Sci. **17** (1964), 689–691.

Let C_Δ be the set of all continuous real functions on $\Delta = [a, b]$ and put, for $f, g \in C_\Delta$,

$$r(f, g) = \max\left\{\max_x \min_y d(x, y), \max_x \min_y d(y, x\right.$$

where $d(u, v) = \max(|u - v|, |f(u) - g(v)|)$ for $r(f, g)$ is a distance over C_Δ; put $r(f, X) = \inf\{r(f, g):$ and $\delta(C_\Delta, X) = \sup\{r(f, X) : f \in C_\Delta\}$ for $X \subset C_\Delta$. The ing theorems are stated without proofs. (1) For natural m and every ε such that $0 < \varepsilon < (b-a)/2m$ exists a $(3m+1)$-dimensional linear subspace L of that $\delta(C_\Delta, L) \leq (b-a)/2m + \varepsilon$. (2) If, for a linear su $L \subset C_\Delta$ and an $\varepsilon > 0$, $\delta(C_\Delta, L) \leq (b-a)/2m - \varepsilon$, th dimension of L is not lower than $3m-1$. Several laries follow, in particular, the asymptotic f $d_n(C_\Delta) \sim 3(b-a)/2n$ for $n \to \infty$, where $d_n(C_\Delta) = \inf\{\delta($ L is a linear subspace of C_Δ of dimension $\leq n\}$.

Á. Császár (Bud

Wang, Sheng-Wang

On the products of Orlicz spaces.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. **11** (1963), 19–22.

An even convex function $M(u)$ is an N-funct $\lim_{u\to\infty} M(u)/u = \infty$, $M(0) = 0$. Definition 1: If M, are N-functions and if there exist $\alpha, u_0 > 0$ such $M(\alpha u_1 u_2) \leq M_1(u_1) + M_2(u_2)$ for $u_1, u_2 \geq u_0$, we sh that the pair $[M_1(u), M_2(u)]$ is stronger than Definitions 2 and 3 define what is meant by saying th pair $[M_1(u), M_2(u)]$ is, respectively, completely stro essentially stronger than $M(u)$.

Two more definitions are given and thirteen the are stated without proof. Two references are give second of which is to the excellent article of T. Andô Ann. **140** (1960), 174–186; MR **22** #3965]. The

familiar with it will have no difficulty in supplying one- or two-line proofs to the theorems given here.
R. O'Neil (Houston, Tex.)

Wang, Sheng-Wang 6301

Convex functions of several variables and vector-valued Orlicz spaces.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 279–284.

A function of n variables $M(\mathbf{u})$, $\mathbf{u}=(u_1,\cdots,u_n)$, $u_1,\cdots,u_n\geq 0$, is said to be convex if for $a,b\geq 0$, $a+b=1$, $M(a\mathbf{u}_1+b\mathbf{u}_2)\leq aM(\mathbf{u}_1)+bM(\mathbf{u}_2)$. $M(\mathbf{u})$ is called non-decreasing if it is non-decreasing in each variable separately. The complement to a non-decreasing, convex $M(\mathbf{u})$ with $M(\mathbf{0})=0$ is given by

$$N(\mathbf{v}) = \sup_{\mathbf{u}\geq 0}\left[\sum_{i=0}^{n} u_i v_i - M(\mathbf{u})\right].$$

It is shown that if $N(\mathbf{v})$ is the complement of $M(\mathbf{u})$, then $M(\mathbf{u})$ is the complement of $N(\mathbf{v})$. (The statement and proof of this theorem are marred by the fact that the author fails to make the observation that a convex $M(\mathbf{u})$ is always expressible as the integral of a non-decreasing system.)

By means of a non-decreasing convex function $M(\mathbf{u})$ of n variables, a vector-valued Orlicz space may be defined, but this turns out to be merely the direct sum of n Orlicz spaces.
R. O'Neil (Houston, Tex.)

Rosenblatt, M. 6302

Almost periodic transition operators acting on the continuous functions on a compact space.

J. Math. Mech. **13** (1964), 837–847.

For a compact Hausdorff space X let T be a non-negative linear operator on $C(X)$ with $T1=1$. The author is concerned with consequences of the assumption that $\{T^n: n\geq 1\}$ be almost periodic on $C(X)$ in the sense that $\{T^n f, n\geq 1\}$ is conditionally compact, $f\in C(X)$.

From results of de Leeuw and the reviewer [Acta Math. **105** (1961), 63–97; MR **24** #A1632] one knows that $C(X)=C_0\oplus C_p$, where C_0 [respectively, C_p] is the closed invariant subspace of all f with $\|T^n f\|\to 0$ [spanned by eigenvectors of T corresponding to eigenvalues of unit modulus]; the corresponding projection of $C(X)$ onto C_p is provided by E, the identity of the least ideal K of the compact semigroup formed by the strong operator closure of $S=\{T^n: n\geq 1\}$. Call T irreducible if for any f in $C(X)$ and x in X, $T^n f(x)>0$ for some n. From an earlier result of the author [Teor. Verojatnost. i Primenen. **9** (1964), 205–222; MR **29** #1540] one then has C_p an algebra. This leads to the author's first theorem, which, roughly stated, asserts that if T is irreducible, then C_p is the algebra of all continuous functions on a quotient space X' of X, and $T|C_p$ is induced by a self-homeomorphism of X'.

K (above) is a compact topological group, and the author uses harmonic analysis on K (Theorem 2) to obtain an integral representation of $\lim_{n\to\infty}(1/n)\sum_1^n\lambda^{-j}T^j$, where λ is any eigenvalue of unit modulus.

Finally, when T is irreducible, there is a unique probability measure on X fixed under T^*; for the corresponding L^2 space the natural extension of T acts as an isometry on the closure of C_p and as a dissipative operator on the (orthogonal) closure of C_0. Extensions to other positive operators are given.

{The author's assertion (Lemma 2) that C_p is an algebra is false. Let $X=\{z\in\mathbf{C}: |z|=0,1\}$, $Tf(z)=f(z)$, $z\neq 0$, $Tf(0)=(1/2\pi)\int f(e^{i\theta})\,d\theta$. Then $T^2=T$ is a projection (thus $=E$) whose self-adjoint and separating range $EC(X)=C_p$ cannot be an algebra since $C_p\neq C(X)$. What can be proved is that $C_p|Y$ is an algebra, where Y, the "support of E", is the closure of the union of the supports of the measures $E(x,\cdot)$, where $Ef(x)=\int f(y)E(x,dy)$; one need only note that $C_p|Y$ is closed in $C(Y)$ and apply the argument of the author's earlier paper that the real elements of $C_p|Y$ form a lattice, hence a subalgebra of $C^R(Y)$ by Stone's proof of the Stone-Weierstrass theorem. When T is irreducible, $X=Y$, so that none of the results of this paper are invalidated.}
I. Glicksberg (Seattle, Wash.)

Anselone, P. M.; Korevaar, J. 6303

Translation invariant subspaces of finite dimension.

Proc. Amer. Math. Soc. **15** (1964), 747–752.

A very neat proof of the following theorem is given: Every finite-dimensional translation-invariant subspace W of the continuous functions on $(-\infty,\infty)$, or of the Schwartz distributions on $(-\infty,\infty)$, is the span of a set of exponential monomials of the form

$$t^{\mu-1}e^{\lambda t},\qquad \mu=1,\cdots,m(\lambda),\ \lambda=\lambda_1,\cdots,\lambda_k.$$

L. Bungart (Berkeley, Calif.)

Campos Ferreira, J.; Silva Oliveira, J. 6304

Problèmes aux conditions initiales, dans la théorie des distributions.

Univ. Lisboa Revista Fac. Ci. A (2) **10** (1963), 91–130.

Im Anschluß an Sebastião e Silvas axiomatische Einführung der Distributionen [dieselben Revista (2) **4** (1955), 79–186; corrigenda (2) **5** (1956), 169–170; MR **17**, 766] werden Anfangsaufgaben für lineare inhomogene Systeme von gewöhnlichen Differentialgleichungen untersucht. Seien $u=u(t)$, $f=f(t)$ vektorielle Distributionen, $A=A(t)$ eine quadratische Matrix. Untersucht wird das System $u'=Au+f$. Es wird gezeigt, daß jede Distribution u, welche in einem Intervall Lösung des Systems ist, sich eindeutig als Lösung für alle t fortsetzen läßt, falls $A(t)$ eine differenzierbare Funktion ist. Ist A von der Klasse C^p und f von der Ordnung höchstens $p+1$ und haben die Stammfunktionen Grenzwerte für $t\to a-0$, so ist die Anfangswertaufgabe für beliebige Vorgaben von $u(a-0)$ eindeutig durch eine Distribution $u(a-0)$ lösbar. Für den Wert einer Distribution werden dabei die üblichen Definitionen verwendet. Ist A unendlich oft differenzierbar, so ist die Anfangswertaufgabe für $a-0$ dann und nur dann für alle t eindeutig lösbar, wenn f eine Distribution ist, die eine Stammfunktion besitzt, welche einen Grenzwert für $t\to a-0$ hat. Es folgen Anwendungen auf Differentialgleichungen höherer Ordnung mit konstanten Koeffizienten und auf Systeme mit konstanten Koeffizienten, wobei Heaviside-Funktionen als Störfunktionen besondere Beachtung finden.
D. Laugwitz (Darmstadt)

Marcinkowska, H. 6305

On the duality of certain Hilbert spaces of tempered distributions.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 293–297.

For $f \in \mathscr{S}$ (the space of rapidly decreasing functions on N-dimensional Euclidean space E^N) the norms

$$\|f\|_{0,q} = \left(\int (1+|x|^2)^q |f(x)|^2\,dx\right)^{1/2},$$

$$\|f\|_{m,q} = \left(\sum_{|\alpha|\leq m} \|D^\alpha f\|_{0,q}^2\right)^{1/2},$$

where q is real and m is a non-negative integer, and

$$\|f\|_{-m,-q} = \sup\{|(\phi, f)| : \phi \in \mathscr{S} \text{ and } \|\phi\|_{m,q} = 1\}$$

are introduced. The completion of $\mathscr{S}$ in the norm $\|\cdot\|_{m,q}$, m an integer, q real, is denoted by $G_{p,q}$. A number of results are proved which are analogous to those usually proved for the spaces H^m. In particular, it is shown that the map $f \to (\cdot, f)$ maps $\bigcup_{p,q} G_{p,q}$ onto $\mathscr{S}'$. Some estimates involving these norms are also given, for example, an analogue of Ehrling's estimate.

R. S. Freeman (College Park, Md.)

Martineau, André 6306

Sur une propriété universelle de l'espace des distributions de M. Schwartz.

C. R. Acad. Sci. Paris **259** (1964), 3162–3164.

Author's summary: "La plupart des espaces nucléaires de l'analyse sont plongeables comme sous-espaces de $\mathscr{D}'$ ou d'un espace $\mathscr{D}'^N$, N étant un cardinal inférieur ou égal à $2^{\aleph_0}$."

A space E (presumably separated and locally convex) is termed s-nuclear if, for each convex balanced neighbourhood U of 0 in E, there exists a convex balanced neighbourhood V of 0 in E such that the natural map of E_V into E_U is expressible in the form $v \circ u$, where $v: E_V \to s$ and $u: s \to E_U$ are continuous, s being the space of rapidly decreasing sequences. (The terminology and notations are those of A. Grothendieck [Mem. Amer. Math. Soc. No. 16 (1955), see especially Chapter II, p. 60 et seq.; MR 17, 763].) The author announces some universal properties of $\mathscr{D}'$ (the space of distributions on the line). For example: the class of s-nuclear spaces is identical with the class of subspaces of $\mathscr{D}'^N$, and with the class of subspaces of s'^N (s' the space of slowly increasing sequences); the s-nuclear Fréchet spaces are exactly the Fréchet subspaces of $\mathscr{D}'$. Several standard spaces of real- and complex-analytic functions are stated to be subspaces of $\mathscr{D}'$. The subspaces of $\mathscr{D}'$ and those of $s'^{\aleph_0}$ form identical categories. Dual results: $\mathscr{E}(V)$ (the space of indefinitely differentiable functions on a σ-compact C^∞ manifold V) and $H(V)$ (the space of holomorphic functions on a Stein manifold V, or an open submanifold V of C^n) are each quotient spaces of s.

R. E. Edwards (Canberra)

Clark, Colin 6307

On Rellich's theorem concerning infinitely narrow tubes.

Canad. Math. Bull. **7** (1964), 435–440.

This brief paper contains a proof of a theorem of Rellich which states that the Laplacian operator with zero boundary conditions associated with a domain G has discrete spectrum if C is an "infinitely narrow tube". An infinitely narrow tube with a straight line l as axis is a set which is contained in some half-space and whose intersection with a solid cylinder of any positive radius about l contains all but a bounded part of the set. The proof makes use of a variational principle which is first applied to the corresponding operator associated [...] bounded subset of G, then to that associated w[...] unbounded remainder of G, and finally to the [...] operator.

C. C. Conley (Madiso[...]

Dias Agudo, F. R.

A note on extensions of linear operators.

Univ. Lisboa Revista Fac. Ci. A (2) **9** (1961/62), 2[...]

The author proves that: If $\hat{T}$ is a self-adjoint ex[...] of the closed symmetric operation T and λ is a[...] or complex) number in the resolvent set of $\hat{T}$, then

$$\text{domain}(T^*) = \text{domain}(\hat{T}) \oplus \text{null-space}(T^* - \ldots$$

It follows that if T is closed and symmetri[...] deficiency indices (m, m), then for any λ in the re[...] set of any self-adjoint extension of T, the e[...] $T^*f = \lambda f$ has exactly m linearly independent solutio[...]

C. C. Conley (Madiso[...]

Durszt, E.

On the numerical range of normal operators.

Acta Sci. Math. (Szeged) **25** (1964), 262–265.

A linear transformation T in Hilbert space is cons[...] which is bounded by 1 and whose numerical r[...] contained properly in the intersection of the nu[...] ranges of all its unitary dilations.

L. de Branges (Lafayett[...]

Gilbert, Richard C.

Extremal spectral functions of a symmetric oper[...]

Pacific J. Math. **14** (1964), 75–84.

From the author's summary: "M. A. Naimar[...] Akad. Nauk SSSR Ser. Mat. **11** (1947), 327–344; [...] 447] has shown that the finite-dimensional extens[...] a symmetric operator define extremal spectral fu[...] of the operator. Finite-dimensional extensions [...] however, only for a symmetric operator with [...] deficiency indices. In § 4 of this paper it is show[...] self-adjoint extensions defined by the addition of m[...] symmetric operators determine extremal spectra[...] tions for a symmetric operator with unequal def[...] indices. ... Section 2 is devoted to a description [...] self-adjoint extensions of a symmetric operato[...] Section 3 identifies some extremal spectral functio[...] symmetric operator with infinite deficiency indices [...] than the ones defined by finite-dimensional extens[...]

D. A. Edwards ([...]

Harazov, D. F.

On the separation of the eigenvalues of operato[...] discrete spectrum. (Russian)

Mathematica (Cluj) **4** (27) (1962), 253–260.

Theorem 1: Let A be a linear operator (of a typ[...] specified shortly) that is symmetrized by a strictly p[...] operator H, and let P be a real polynomial. S[...] $P(A) \neq P(0) \cdot$ iden. Then a necessary and su[...] condition that A possess (necessarily real) eigenv[...] satisfying both $P(\lambda) \leq 0$ and $P(\lambda) \geq 0$ is that ther[...] $f_0 \neq 0$ such that $(HP(A)f_0, f_0) = 0$. If, in addition, P[...] 0, then there exist eigenvalues satisfying both P[...] and $P(\lambda) > 0$.

The present note is based upon the author's earlier work [Uspehi Mat. Nauk 12 (1957), no. 4 (76), 201–207; MR 19, 873] and consequently employs, as the central hypothesis, the direct assumption that A has spectrum consisting only of eigenvalues of finite multiplicity which accumulate only at zero. (Indeed, the only other hypotheses on A are that it should be defined on, and leave invariant, a dense linear manifold contained in the domain of H. Observe that this insures that $P(A)$ is defined on the same domain and is also symmetrized by H. The main point is to show that, in fact, $P(A)$ enjoys all the properties of A.)

Theorem 1 can be used to obtain separation theorems for the eigenvalues of A, e.g., if $\alpha \neq \beta$ are real numbers, then, using $P(\lambda) = (\lambda - \alpha)(\lambda - \beta)$, we obtain Theorem 2: If there exists a vector $f_0 \neq 0$ such that $(H(A^2 - (\alpha+\beta)A + \alpha\beta)f_0, f_0) = 0$, then there is an eigenvalue of A between α and β. Similarly, let $f_0 \neq 0$ and define

$$\gamma_0 = (Hf_0, f_0), \quad \gamma_1 = (HAf_0, f_0), \quad \gamma_2 = (HA^2 f_0, f_0).$$

Then, for any α, $\alpha\gamma_0 \neq \gamma_1$, there is at least one eigenvalue of A between α and $(\alpha\gamma_1 - \gamma_2)/(\alpha\gamma_0 - \gamma_1)$ (Theorem 3).

A. Brown (Ann Arbor, Mich.)

Kaniel, S. 6312

Unbounded normal operators in Hilbert space.

Trans. Amer. Math. Soc. **113** (1964), 488–511.

The author introduces a class of operators in Hilbert space, called unbounded normal operators, which are extensions of symmetric operators. These operators are defined as follows. Let T_0 have domain $D(T_0)$, a dense subspace of Hilbert space. Let its adjoint T_0^* be densely defined and let T_1^* be the restriction of T_0^* to a dense subspace. Form the adjoint $T_1 = T_1^{**} \supset T_0$. Define H_λ to be the kernel of $T_1 - \lambda I$ and H_λ^* to be the kernel of $T_0^* - \lambda I$. (T_0, T_1^*) forms a normal pair if (a) $f \in D(T_1^* T_0)$ implies $f \in D(T_0 T_1^*)$ and $T_1^* T_0 f = T_0 T_1^* f$; (b) The closure of $D(T_1^* T_0)$ with respect to the T_0 norm is $D(T_0)$; (c) $D(T_0^*) \cap H_\lambda$ is dense in H_λ and $D(T_1) \cap H_\lambda^*$ is dense in H_λ^*. The operator T_0 is called normal if T_0 has a mate T_1^* such that (T_0, T_1^*) forms a normal pair. T_0 is completely normal if it is normal and has a bounded inverse defined on the entire space and which is also normal. The author mentions that many operators like constant coefficient partial differential operators satisfy conditions (b) and (c). Properties of normal operators are then given. For example, the following theorem is proved. Theorem: Let (T_0, T_1^*) form a normal pair. Suppose T_1^* is regular in a region $\bar{D}$, i.e., $T_1^* - \lambda I$ has a bounded inverse for each λ in $\bar{D}$. If T_1^* has finite deficiency index, then there exists an analytic function $G(z, w) = w^n + h_1(z)w^{n-1} + \cdots + h_n(z)$, $z \in D$, such that for any z_0, the zeros of $G(z_0, w)$ are the eigenvalues of T_0^* restricted to H_{z_0}. An interesting example is given of a normal operator which is regular in the plane and which is not a polynomial in a symmetric operator. The next part of the paper gives conditions under which an operator has a normal extension and, in particular, a complete normal extension. Most of the theorems assume that T_0 and T_1^* are regular in the plane. The remaining portion of the paper deals with properties of the spectrum of normal extensions. In particular, the distribution of the eigenvalues of complete normal extensions is investigated.

S. Goldberg (College Park, Md.)

Gohberg, I. C.; Krein, M. G. 6313

Factorization of operators in Hilbert space. (Russian)

Acta Sci. Math. (*Szeged*) **25** (1964), 90–123.

The essential part of the paper is a detailed exposition of a result announced earlier [Dokl. Akad. Nauk SSSR **147** (1962), 279–282; MR **26** #6777]. The paper also contains a generalization of this result to the case when the chain has discontinuities (a discontinuity of a chain $\mathfrak{P}$ is a pair (P^-, P^+) of members of $\mathfrak{P}$ such that no other member of the chain lies between P^- and P^+).

P. Saworotnow (Washington, D.C.)

Krein, M. G. 6314

A new application of the fixed-point principle in the theory of operators in a space with indefinite metric. (Russian)

Dokl. Akad. Nauk SSSR **154** (1964), 1023–1026.

Let P_+ be a projection on a Hilbert space $\mathscr{H}$, let $P_- = 1 - P_+$ and $J = P_+ - P_-$. The symmetry J defines an indefinite metric $[x, y] = (Jx, y)$. As is customary, we call an (invertible) operator U "J-unitary" if $U^*JU = J$ and a (densely defined) operator A "J-selfadjoint" if JH is selfadjoint. It was noted some time ago by the author [Uspehi Mat. Nauk **5** (1950), no. 2 (36), 180–190; MR **14**, 56] that the basic structure theorems for J-selfadjoint and J-unitary operators could be viewed as applications of fix-point theorems. (The original theorem in this connection goes back to Pontrjagin [Izv. Akad. Nauk SSSR Ser. Mat. 8 (1944), 243–280; MR **6**, 273]. The fix-point approach has been exploited by others, notably Iohvidov [Trudy Moskov. Mat. Obšč. **5** (1956), 367–432; MR 18, 320] and, more recently, Langer [Math. Ann. **152** (1963), 434–436; MR **28** #1492] and Fan [Bull. Amer. Math. Soc. **69** (1963), 773–777; MR **28** #1494].) In this note the Schauder-Tychonov fix-point theorem is used to prove the following. Let $\mathscr{P}_+$ denote the "positive set" $\{x : [x, x] \geq 0\}$ and let $\mathscr{M}_+$ denote the set of all maximal subspaces lying in $\mathscr{P}_+$. Let T be a bounded linear operator on $\mathscr{H}$ and suppose (1) $Tx \in \mathscr{P}_+$, $Tx \neq 0$ for all $x \in \mathscr{P}_+$, $x \neq 0$; (2) $\mathscr{P}_+ T \mathscr{P}_-$ is compact; and (3) for at least one $L_0 \in \mathscr{M}_+$ we have $T(L_0) \in \mathscr{M}_+$. Then (a) $T(\mathscr{M}_+) \subset \mathscr{M}_+$, and (b) for at least one $L_0 \in \mathscr{M}_+$ we have $T(L_0) = L_0$. This theorem is then used to deduce the appropriate spectral decomposition theorems for J-unitary and J-selfadjoint operators, the latter result subsuming a theorem recently announced by Langer [loc. cit.].

A. Brown (Ann Arbor, Mich.)

Krein, M. G.; Langer, G. K. [Langer, Heinz] 6315

On the theory of quadratic pencils of self-adjoint operators. (Russian)

Dokl. Akad. Nauk SSSR **154** (1964), 1258–1261.

A quadratic pencil of operators is a family of the form $L(\lambda) = \lambda^2 + \lambda B + C$, where B, C are operators. If $\lambda \neq 0$ and $L(\lambda)$ is invertible, then λ is a regular point; the irregular points constitute the spectrum of $L(\lambda)$. The present note is a brief report on a considerable body of theory concerning the spectral structure of such pencils, and this review will not attempt more than a hasty sampling of the stated results.

The minimal hypotheses on B and C are $B = B^*$ and C closed with $\mathscr{D}(B) \subset \mathscr{D}(C)$, and some results are obtained even for unbounded C. However, the more typical theorems concern the case $C \geq 0$ and compact, and we

suppose this condition satisfied from now on. (This forces the spectrum of $L(\lambda)$ to be symmetric in the real axis.) The most interesting feature of the investigation is the systematic use made of the relations between the spectral structure of $L(\lambda)$ and the quadratic operator equation

$$(*)\qquad Z^2+BZ+C=0.$$

Here is a slightly weakened version of a basic theorem. Let Λ, $\bar{\Lambda}$ be any partition of the non-real spectrum of $L(\lambda)$ into two disjoint sets, each the reflection of the other in the real axis. Then (*) possesses a bounded solution Z_Λ satisfying (1) $Z_\Lambda\mathscr{H}\subset\mathscr{D}(B)$, (2) $Z_\Lambda{}^*Z_\Lambda\leq C$, (3) the non-real spectrum of Z_Λ coincides with Λ. An application of this theorem yields the following. Suppose B is in the trace class and that

$$(Bx,x)^2<4(x,x)(Cx,x),\qquad x\neq 0.$$

Then $-\sum\Re\lambda_i\leq\operatorname{Tr}B$, where the sum is extended over all eigenvalues of $L(\lambda)$, counting multiplicities.

Suppose, on the other hand, that

$$(Bx,x)^2>4(x,x)(Cx,x),\qquad 0\neq x\in\mathscr{D}(B).$$

Then the spectrum of $L(\lambda)$ lies entirely on the negative real axis and the following is true: (*) has a unique solution Z_1 [Z_2] satisfying $Z_1{}^*Z_1\leq C$ [$Z_2{}^*Z_2\geq C$] and we have (a) $Z_2+Z_1{}^*=-B$, $Z_2{}^*Z_1=C$, (b) spec $L(\lambda)=$ spec $Z_1\cup$ spec Z_2, spec $Z_2\leq$ spec Z_1, (c) the eigenvectors x of Z_1 [Z_2] are eigenvectors of $L(\lambda)$ for eigenvalues λ such that $|\lambda|^2<(Cx,x)/(x,x)$ [$|\lambda|^2>(Cx,x)/(x,x)$], the case $|\lambda|^2=(Cx,x)/(x,x)$ being impossible.

Proofs are omitted or merely sketched, although enough is given to show that the above-mentioned basic theorem is an interesting application of the authors' work on "J-selfadjoint" operators [Heinz Langer, Math. Ann. **146** (1962), 60–85; MR **25** #1450; ibid. **152** (1963), 434–436; MR **28** #1492; and M. G. Kreĭn, #6314 above]. A detailed report is to appear in the "Proceedings of an International Symposium on the Application of Function Theory to the Mechanics of Continuous Media".

A. Brown (Ann Arbor, Mich.)

Lindenstrauss, Joram 6316

On the modulus of smoothness and divergent series in Banach spaces.

Michigan Math. J. **10** (1963), 241–252.

Given a Banach space X, we may introduce the notions of modulus of convexity $\delta_X(\varepsilon)$ and modulus of smoothness $\rho_X(\tau)$. Let $\delta_X(\varepsilon)=\frac{1}{2}\inf(2-\|x+y\|)$, $0\leq\varepsilon\leq 2$, where the infimum is taken over all x,y such that $\|x\|=\|y\|=1$ and $\|x-y\|=\varepsilon$. Let $\rho_X(\tau)=\frac{1}{2}\sup(\|x+y\|+\|x-y\|-2)$, $0\leq\tau$, where the supremum is taken over all x,y such that $\|x\|=1$, $\|y\|=\tau$. Let X^* denote the dual space of X.

Theorem 1: $\rho_{X^*}(\tau)=\sup_{0\leq\varepsilon\leq 2}(\frac{1}{2}\tau\varepsilon-\delta_X(\varepsilon))$. Corollary: $\rho_X(\tau)\geq(1+\tau^2)^{1/2}-1$ and if equality holds for all τ between 0 and 1, then X is an inner product space. Theorem 2: If $\{x_i\}_{i=1}^\infty$ is a sequence in X such that $\sum\pm x_i$ diverges for every choice of signs, then $\sum\rho_X(\|x_i\|)=\infty$.

It is shown that if $\delta_X(t)$ and $\rho_X(t)$ behave, for small t, as multiples of t, then under rather general conditions X is isomorphic to an inner product space.

Finally, an index of convergence and an index of divergence for a Banach space X are introduced and some properties are discussed.

R. O'Neil (Houston, Tex.)

Lindenstrauss, Joram

On the extension of operators with a finite-dimen[sional] range.

Illinois J. Math. **8** (1964), 488–499.

A number of interesting theorems on extensi[on of] operators are established. The statements being to[o long] for inclusion here, the following (part of Theo[rem …]) should serve as an example of the results obtaine[d.] X, Y, Z denote Banach spaces. If X has the pr[operty] that every linear operator $T:X\to Y$ has a norm-pres[erving] extension $T':Z\to Y$ whenever $Z\supset X$, dim $Z/X=$ … dim $Y=3$, then for every Y and $Z\supset X$ each co[mpact] operator $T:X\to Y$ has a norm-preserving ext[ension] $T':Z\to Y$.

B. Grünbaum (Jeru[salem])

Markus, A. S.

Eigenvalues and singular values of the sum and p[roduct] of linear operators. (Russian)

Uspehi Mat. Nauk **19** (1964), no. 4 (118), 93–123

This mainly expository article is an excellent survey … results on the eigenvalues and singular values … sum and product of matrices or completely cont[inuous] operators. Also included are some closely related res[ults on] doubly stochastic matrices and symmetric gauge fu[nctions] introduced by J. von Neumann. One finds here res[ults of] H. Wielandt [Proc. Amer. Math. Soc. **6** (1955), 10… MR **16**, 785], V. B. Lidskiĭ [Dokl. Akad. Nauk SS[SR] … (1950), 769–772; MR **12**, 581], A. A. Nudel'ma[n and] P. A. Švarcman [Uspehi Mat. Nauk **13** (1958), no. … 111–117; MR **21** #3436], L. Mirsky [Quart. J. … Oxford Ser. (2) **11** (1960), 50–59; MR **22** #5639], … [Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 652–655; … 600; ibid. **37** (1951), 760–766; MR **13**, 661], A. … [ibid. **36** (1950), 374–375; MR **12**, 505; Pacific J. M[ath.] … (1962), 225–241; MR **25** #3941], A. R. Amir-Moéz … Math. J. **23** (1956), 463–476; MR **18**, 105]. Some … theorems for the infinite-dimensional case are … although their analogues in the finite-dimensional ca[se are] known.

Ky Fan (Evansto[n])

Minty, George J.

Monotone (nonlinear) operators in Hilbert space.

Duke Math. J. **29** (1962), 341–346.

Let $\mathfrak{X}$ be a (real or complex) Hilbert space with … product $\langle x,y\rangle$. An operator F (not necessarily l[inear]) acting from $\mathfrak{D}\subset\mathfrak{X}$ into $\mathfrak{X}$, is called monoto[ne if] $\operatorname{Re}\langle x_1-x_2, F(x_1)-F(x_2)\rangle\geq 0$ for any $x_1,x_2\in\mathfrak{D}$ … fundamental result is that if F is a continuous mo[notone] operator, then the inverse operator $(I+F)^{-1}$, w[here I] is the identity operator, exists, is continuous … domain of definition, and is monotone. If, in addit[ion, F] is maximal, i.e., does not have an extension … preserves monotonicity, and if the set $\mathfrak{D}$ is open … $(I+F)^{-1}$ is defined everywhere on $\mathfrak{X}$.

The author gives some sufficient conditions th[at an] operator be monotone. In particular, if the set … convex and the derivative

$$\left[\frac{d}{dt}\operatorname{Re}\langle x_1-x, F(x+t(x_1-x))\rangle\right]_{t=0},$$

where t is real, exists and is non-negative for ar[bitrary] $x, x_1\in\mathfrak{D}$, then F is monotone.

On the basis of his results, the author is able to co…

the "graph" of F in the space $\mathfrak{X}\times\mathfrak{X}$ (endowed with the scalar product $\langle(x_1,y_1),(x_2,y_2)\rangle=\langle x_1,x_2\rangle+\langle y_1,y_2\rangle$) and to study in this space the relations $(x_1,y_1)M(x_2,y_2)$ and $(x_1,y_1)L(x_2,y_2)$ defined by

$$\operatorname{Re}\langle x_1-x_2,y_1-y_2\rangle \geq 0,\qquad \|x_1-x_2\| \geq \|y_1-y_2\|,$$

respectively. *I. Šragin* (RŽMat 1963 #6 B410)

Olubummo, A. 6320
A note on perturbation theory for semi-groups of operators.
Proc. Amer. Math. Soc. 15 (1964), 818–822.

A theorem of Miyadera [Tôhoku Math. J. (2) 11 (1959), 98–105; MR 21 #5903] gives a necessary and sufficient condition in order that a perturbation of a strongly continuous semigroup of positive contractions of an abstract L-space result in a strongly continuous semigroup of positive contractions dominating the unperturbed semigroup. The perturbation of the infinitesimal generator in this theorem is by means of a not necessarily bounded operator, whose domain contains that of the infinitesimal generator. The main result of the present paper can be considered as a generalization of the theorem of Miyadera to general Banach lattices, but it is actually more because the conditions characterizing the desired perturbing operators are somewhat simpler. These conditions become even simpler (the second main result) for bounded perturbations, still in the context of Banach lattices.

J. Gil de Lamadrid (Paris)

Saphar, Pierre 6321
Calcul fonctionnel et sous-espaces stables pour une application linéaire continue dans un espace de Banach.
C. R. Acad. Sci. Paris 258 (1964), 6055–6057.

For a continuous linear operator T on a (complex) Banach space E, $\operatorname{co}(T)$ denotes the largest subspace of E such that $T(\operatorname{co}(T))=\operatorname{co}(T)$. $S_1(n,p;T)$ is the open subset of those points z in the complex plane such that

$$\dim\ker(T-z)=n,\qquad \operatorname{codim}(T-z)(E)=p,$$

$$\ker(T-z)\subset\operatorname{co}(T-z).$$

For any connected component D of $S_1(n,p;T)$ there is a discrete set δ in D and a holomorphic $\mathscr{L}(E)$-valued function R on $D-\delta$ such that

$$(T-z)R(z)(T-z)=T-z,\qquad R(z)(T-z)R(z)=R(z),$$

$$R(z)-R(z')=(z-z')R(z)R(z'),$$

and $\operatorname{Im}R(z)$ and $\ker R(z)$ are topological complements of $\ker(T-z)$ and $\operatorname{Im}(T-z)$, respectively; the set δ can be chosen to avoid any compact subset of D. Using this function R, a functional calculus is established,

$$f\to-\frac{1}{2\pi i}\int_\gamma f(t)R(t)\,dt,$$

where f is holomorphic in a neighborhood of any compact and open subset of the complement of $D-\delta$. As an application, the existence of invariant subspaces tied to properties of the spectrum is obtained; furthermore, a discussion of the equation $(T-z)x=y$, $z\in D$, is given. No proofs are presented. *L. Bungart* (Berkeley, Calif.)

Sasser, D. W. 6322
Quasi-positive operators.
Pacific J. Math. 14 (1964), 1029–1037.

Let B denote a partially ordered real Banach space whose positive cone K is closed and fundamental in B. A (bounded) operator T in B is called quasi-compact if it has spectral radius $r(T)>0$ and if for some n and some operator V with $r(V)<r(T)^n$, T^n-V is compact. If, in addition, T is positive (i.e., if $T(K)\subset K$), then the peripheral spectrum $\sigma(T)\cap\{\lambda:|\lambda|=r(T)\}$ of T has a number of interesting properties [cf. Krein and Rutman, Amer. Math. Soc. Transl. No. 26 (1950); MR 12, 341; and the reviewer, Pacific J. Math. 10 (1960), 1009–1019; MR 22 #5893]; these are generalized here to the case of quasi-compact, quasi-positive operators in B. (T is called quasi-positive if for each $x\geq 0$ in B and each continuous linear form $x^*\geq 0$ on B, one has $x^*(T^kx)\geq 0$ whenever $k\geq n(x,x^*)$.) The idea of the proof is that Pringsheim's theorem on the singularities of a power series with non-negative scalar coefficients remains valid when all but a finite number of these coefficients are ≥ 0.

H. H. Schaefer (Tübingen)

Shinbrot, Marvin 6323
A fixed point theorem, and some applications.
Arch. Rational Mech. Anal. 17 (1964), 255–271.

Let $\mathfrak{H}$ be a separable Hilbert space; for $r>0$, let $\mathfrak{S}_r$ and $\mathfrak{B}_r$ denote the (open) sphere and (closed) ball of radius r with center at the origin. Let $A:\mathfrak{H}\to\mathfrak{H}$ be continuous with respect to the weak topology in $\mathfrak{H}$. The author proves (main theorem) that if there is an $r>0$ such that $\operatorname{Re}\langle Ax,x\rangle\leq\|x\|^2$ for all $x\in\mathfrak{S}_r$, then A has a fixed point in $\mathfrak{B}_r$. The main tool is the finite-dimensional special case, proved with (Weierstrass) polynomial approximations to A in a way reminiscent of existing analytical proofs of Brouwer's theorem. Much briefer proofs can be given by a degree argument (noted by the author) or Brouwer's theorem (not noted). Various corollaries are considered.

"Applications" are given to the proofs of other abstract theorems, e.g., let $A:\mathfrak{H}\to\mathfrak{H}$ be continuous in the weak topology and suppose $\operatorname{Re}\langle A(sx),x\rangle$ is a monotonically increasing function of the real variable s for all x and large enough s. Let L be either the identity or a positive linear operator with compact inverse. Then $(L+F)$ is "onto" $\mathfrak{H}$. More concrete applications are given to proofs of existence of solutions of quasilinear partial differential equations (in divergence form) satisfying certain "monotonicity conditions". In contrast to recent related work of F. Browder, the coefficients are not allowed to depend on the highest-order derivatives; however, the author's hypotheses are weaker in some other respects. A more concrete application is given to the stationary Navier-Stokes equation in n dimensions with $n\leq 4$.

G. J. Minty (New York)

Simpson, James E. 6324
Nilpotency and spectral operators.
Pacific J. Math. 14 (1964), 665–672.

Dunford's theory of spectral operators has been extended to operators on locally convex spaces, and this paper carries over some of the Banach-space theorems of the reviewer [same J. 9 (1959), 1223–1231; MR 21 #7442] to this more general setting.

Takesaki, Masamichi 6332

A complement to "On the unitary equivalence among the components of decompositions of representations of involutive Banach algebras and the associated diagonal algebras".

Tôhoku Math. J. (2) **16** (1964), 226–227.

The author shows the following. Let A_1 and A_2 be two maximal abelian subalgebras of a von Neumann algebra M. Let $e_1\,[e_2]$ be a non-zero projection in $A_1\,[A_2]$. If there exists a central projection z of M such that $e_1 \leq z$ and $e_2 \leq I-z$, then A_1e_1 and A_2e_2 are unrelated. Hence if A_1e_1 and A_2e_2 are similar, then e_1 and e_2 have the same central carriers. *B. Yood* (Eugene, Ore.)

Vala, Klaus 6333

On compact sets of compact operators.

Ann. Acad. Sci. Fenn. Ser. A I No. 351 (1964), 9 *pp.*

This note modifies an argument due to S. Kakutani [J. Math. Soc. Japan **3** (1951), 228–231; MR **13**, 355] to obtain a different formulation for Ascoli's theorem. Let A be a set and let F be a metric space; a mapping $f: A \to F$ is "precompact" if the set $f(A)$ is precompact (= totally bounded) in F. The collection $K(A, F)$ of all precompact maps of A into F is a metric space under $d(f, g) = \sup\{d(f(x), g(x)); x \in A\}$. The new form of Ascoli's theorem is: A subset $H \subseteq K(A, F)$ is precompact if and only if (1) $H(x) = \{f(x): f \in H\}$ is precompact in F, and (2) H has "equal variation" (in the sense that for $\varepsilon > 0$ there is a finite partition $\{A_j\}$ of A such that if $x, y \in A_j$, $f \in H$, then $d(f(x), f(y)) < \varepsilon$). As an application, the author proves a generalization of Schauder's theorem on the adjoint of a compact operator. A special case of his result is the following. Let E be a Banach space and let A, C be non-zero elements of $L(E)$, the normed algebra of bounded linear operators on E. Then the map $T \to ATC$ is a compact operator of $L(E)$ into $L(E)$ if and only if both A and C are compact operators. An extension of the theorem of Dini is also given. *R. G. Bartle* (Urbana, Ill.)

Varopoulos, N. Th. 6334

A note on the abstract Wiener-Pitt phenomenon.

Proc. Cambridge Philos. Soc. **61** (1965), 297–298.

Suppose that M is a semisimple commutative Banach algebra with unit and with an involution i defined on it. Suppose that $L \subset M$ is a closed star ideal such that L is a completely symmetric Banach algebra relative to $i\,|\,L$. Identify the maximal ideal space $\mathcal{M}_L$ with a subspace of $\mathcal{M}_M$. The author says that the Wiener-Pitt phenomenon occurs if and only if there is an $m \in M$ and a $\chi_0 \in \mathcal{M}_M$ such that $m(\chi_0) = 0$ and yet $\inf\{|m(\chi)| : \chi \in \mathcal{M}_L\} > 0$. The following theorem is proved; it adds to previous results by the author in this context: If the Wiener-Pitt phenomenon does not occur, then $\mathcal{M}_L$ is dense in $\mathcal{M}_M$. *K. H. Hofmann* (New Orleans, La.)

Xia, Dao-xing [Hsia, Tao-hsing] 6335

On locally bounded topological algebras.

Acta Math. Sinica **14** (1964), 238–251 (*Chinese*); *translated as Chinese Math.* **5** (1964), 261–276.

A locally bounded algebra is a topological algebra which, as a topological linear space, is a locally bounded space, i.e., possesses a basis of neighbourhoods of zero consisting of bounded sets. The topology of a locally bounded algebra
is always metrizable, and its metric may be given b
of a p-homogeneous norm $\|ax\| = |a|^p\|x\|$ for any
and vector x, where p is a fixed real number sa
$0 < p \leq 1$; other properties are the same as in the c
homogeneous norm. In the case when the algebra
tion is complete or its multiplication is jointly con
the p-homogeneous norm may be chosen in such
that it is submultiplicative; in this case, the al
called a p-normed algebra. The theory of locally b
algebras was given earlier by the reviewer [Studi
19 (1960), 333–356; MR **22** #A4033; ibid. **21** (1
203–206; MR **25** #4381; ibid. **21** (1961/62), 3
MR **25** #4380; Colloq. Math. **10** (1963), 57–60;
#6809]. It was shown [MR **25** #4380, loc. cit.] that i
$\lim \sqrt[n]{\|x^n\|}$, where $\|x\|$ is a submultiplicative p-homo
norm, then $\|x\|_s$ is a homogeneous pseudonorm,
follow certain basic facts concerning the algebra i
tion. In the paper under review the author ma
observation that for a p-normed algebra $(A, \|\ \|$
exists a maximal homogeneous pseudonorm $|\ |$ sa
$|x|^p \leq \|x\|$, called a support pseudonorm of $\|\ \|$. C
$\|x\|_s \leq |x|^p \leq \|x\|$, and the support pseudonorm i
defined as $|x| = \sup_{f \in F} |f(x)|$, where F is the famil
linear functionals defined on A and satisfying $|f(x)|$
thus $(\bar{A}, |\ |)$, the completion of A in $|\ |$, is a
algebra having "the same" maximal ideal sp
$(A, \|\ \|)$; observe that $\|x\|_s = (\lim \sqrt[n]{\|x^n\|})^p$. By
of this algebra the author obtains some pro
p-normed algebras. Some results are given also
normed algebras with involutions, called in the tra
"convolutions". *W. Żelazko* (W

Forelli, Frank

The isometries of H^p.

Canad. J. Math. **16** (1964), 721–728.

Fix $p \geq 1$. Consider the Hardy space H^p consistin
elements of L^p on the circle whose Fourier coef
vanish for negative indices. H^p is a Banach space
are the linear isometries of this Banach space ont
itself? If φ is a conformal self-mapping of the un
regarded as a homeomorphism of the circle, the
defined by $Tf = (d\varphi/dz)^{1/p} f(\varphi)$ evidently is such a
isometry of H^p onto itself. The author proves the i
ing result that, up to complex constants of mod
these are all the linear onto isometries, except in t
$p = 2$, when H^p is a Hilbert space, so that there ev
are many others. For $p = 1$ this result was proved
Amer. Math. Soc. **11** (1960), 694–698; MR **22** #12
deLeeuw, Rudin and the reviewer. However, th
$p = 1$ is very special, and the arguments based on e
points used for $p = 1$ are not available when $p >$
author proves and uses the following remarkable l
Let σ_1 and σ_2 be two probability measures on two
Let A be a subalgebra of $L^\infty(\sigma_1)$ that contains co
and let T be a linear transformation of A into $L^\infty(\sigma$
$T(1) = 1$. Suppose that $p \neq 2$ and $\int |Tf|^p d\sigma_2 = \int |f|^p$
all f in A. Then T is multiplicative, i.e., $T(fg) = Tf$

When T is an isometry of H^p into itself, $p \neq 2$, the
obtains the form for $T: Tf = Ff(\varphi)$, where now φ is
constant bounded analytic function of modulus 1
the circle and F is an H^p function satisfying a
condition. The case $0 < p < 1$, when H^p is no lo
Banach space, is also considered.

J. Wermer (Providence

Glicksberg, I. 6337

Maximal algebras and a theorem of Radó.

Pacific J. Math. 14 (1964), 919–941.

The theorem of Radó referred to in the title is the following. Let f be a continuous function on $D=\{|z|\le 1\}$, and let $K=\{z\in D; f(z)=0\}$. If f is analytic on $\{|z|<1\}-K$, then f is in fact analytic throughout the interior of D. The author gives a new proof, which is roughly as follows. If A_1 is the algebra of continuous functions on D which are analytic on the interior of D, it is known that A_1 is a maximal subalgebra of $C(\Gamma)$, $\Gamma=\{|z|=1\}$. Let B be the algebra of continuous functions on D which are analytic on $\{|z|<1\}-K$. Then B is a subalgebra of $C(\Gamma)$ which contains A_1. By proving that the Šilov boundary of B is Γ, the author verifies that B is a proper closed subalgebra of $C(\Gamma)$ and thus $B=A_1$, proving Radó's theorem.

Beyond this, the author provides a broad investigation of the implications of various types of maximality, applies his methods further in order to obtain a type of Schwarz lemma for Banach algebras, and studies integral closures of algebras of functions. The main tool is the following lemma. Let A be a uniform algebra with spectrum M and let X be a boundary for A. Let V be a relatively open subset of X. Suppose $g\in A$ peaks within X on a nonvoid subset of V, and let $\alpha=\sup\{|g(x)|; x\in X-V\}$. Then any $f\in A$ vanishing on V also vanishes on $\{m\in M; |g(m)|>\alpha\}$.

H. Rossi (Waltham, Mass.)

Krivine, Jean-Louis 6338

Quelques propriétés des préordres dans les anneaux commutatifs unitaires.

C. R. Acad. Sci. Paris 258 (1964), 3417–3418.

A pre-order in a commutative algebra A over the reals is a subset Ω closed under addition and multiplication, containing the non-negative reals but not -1. This differs from the concept in the author's paper [J. Analyse Math. 12 (1964), 307–326] in not assuming that Ω contains x^2 for all x in A. However, every maximal archimedean pre-order does contain all squares, and so the author's other results apply and can be improved. In particular, A is represented by continuous real-valued functions on the space of all maximal pre-orders containing a fixed archimedean Ω. If such a function is non-negative and not zero, then the corresponding element of A is $\ge 1/n$ in the pre-order Ω for some positive integer n. Suitable choices of Ω yield theorems such as the following. Every polynomial $H(X_1,\cdots,X_n)$ with real coefficients which is strictly positive on the unit cube is expressible as a polynomial in $X_1,\cdots,X_n$ and $1-X_1,\cdots,1-X_n$ with positive coefficients; and every power series which is absolutely convergent and strictly positive on $[-1,1]$ is a finite sum $\sum_i \exp\varphi_i(x)$, with φ_i a power series absolutely convergent on $[-1,1]$.

D. Zelinsky (Evanston, Ill.)

Barros Neto, J. [Barros-Neto, J.] 6339

Spaces of vector valued real analytic functions.

Trans. Amer. Math. Soc. 112 (1964), 381–391.

Given an open set K in a finite-dimensional vector space, a locally convex topological vector space E, and a notion of smooth mapping, one generally considers "weakly smooth" and "strongly smooth" mappings of K into E. The question here is of the equivalence of these notions. For "smooth" interpreted as C^∞, or complex analytic, Grothendieck [*Espaces vectoriels topologiques*, Inst. Mat. Pura e Apl., Univ. de São Paulo, São Paulo, 1954; MR 17, 1110] answered the question affirmatively when E is complete. The author is concerned with real analytic mappings; he shows that completeness is not enough, but having a Fréchet space as strong dual suffices. More precisely, we have the following definitions. Let U be an open set on the real line and E a complete topological vector space, with E' its dual. Let φ be a mapping of U into E. φ is weakly analytic at $t_0\in U$ if for all $f'\in E'$, $\langle\varphi(t), f'\rangle$ is an analytic function in a neighborhood of t_0. φ is strongly analytic if there exists $a_p\in F$ for $p\in Z$ such that $\varphi(t)=\sum a_p(t-t_0)^p$ uniformly in a neighborhood of t_0. The author introduces the notion of quasi-(DF) space which generalizes (DF) and proves the equivalence of weak and strong real analyticity in this case. He then considers natural topologies on these spaces of mappings, and proves that they coincide in case E has a Fréchet space as dual. Finally, he considers mappings of U into $L(E,F)$, where E, F are topological vector spaces. Here there are three notions of analyticity to consider; in case E is barrelled and F a complete quasi-(DF) space, these all coincide.

H. Rossi (Waltham, Mass.)

Barros-Neto, J. 6340

The Dirichlet problem for homogeneous elliptic operators in a half space.

Bull. Amer. Math. Soc. 70 (1964), 798–802.

In a recent series of papers [Ann. Scuola Norm. Sup. Pisa (3) 16 (1962), 1–44; MR 26 #4049; and articles there cited] Lions and Magenes have given a great number of precise results on the mapping by elliptic operators of various function spaces on a bounded region of Euclidean space. The present work considers the same questions in a half-space, where new difficulties arise. The following spaces are introduced. R_+^n is the half-space $x_n>0$ of R^n. If $m\ge 0$ is an integer, then $D^{k,m}(R_+^n)=\{u\in L^{q(m)}(R_+^n)$: $D^rD^pu\in L^{q(m-j)}(R_+^n)$, where $0\le|r|\le k$, $0\le j=|p|\le m$, $1/q(j)=\frac12-j/n\}$. $D_0^m(R_+^n)$ is the closure in $D^m(R_+^n)$ of $C_c^\infty(R_+^n)$; $D^{-m}(R_+^n)$ is the dual of $D_0^m(R_+^n)$. If $-n+1<2s<n-1$, then $D^s(R^{n-1})$ is the completion of $C_c^\infty(R^{n-1})$ in the norm $\|\phi\|_{D^s(R^{n-1})}=(\int_{R^{n-1}}|\xi|^{2s}|\phi^\wedge(\xi)|^2d\xi)^{1/2}$, with $\phi^\wedge$ the Fourier transform of ϕ.

Let $\gamma_j\phi(x_1,\cdots,x_{n-1})=(\partial^j\phi/\partial x_n^j)(x_1,\cdots,x_{n-1},0)$, and consider $Au=\sum_{|p|=|r|=m}D^ra_{rp}D^pu$, where a_{rp} is C^∞ and bounded on $\bar R_+^n$, and $\int_{R_+^n}(Av)\bar v\ge c\|v\|^2_{D^m(R_+^n)}$ for all v in $C_c^\infty(R_+^n)$. Theorem: The map $u\to(Au,\gamma_0u,\cdots,\gamma_{m-1}u)$ extends uniquely from $C_c^\infty(\bar R_+^n)$ to an isomorphism of $D^m(R_+^n)$ onto $D^{-m}(R_+^n)\times\prod_{j=0}^{m-1}D^{m-j-1/2}(R^{n-1})$; if f is in $D^{1-m}(R_+^n)\cap D^{-m}(R_+^n)$, then the unique solution u in $D_0^m(R_+^n)$ of $Au=f$ lies in $D^{1,m}(R_+^n)$. The isomorphism is extended to a class of spaces by interpolation.

R. T. Seeley (Waltham, Mass.)

Bittner, R. 6341

Algebraic and analytic properties of solutions of abstract differential equations.

Rozprawy Mat. 41 (1964), 63 pp.

The scope of this paper is much wider than is indicated by the title. The author considers a linear map S from a linear space C^1 into another linear space C^0, with $C^1\subset C^0$, with a right inverse T, i.e., T is a linear map from $C^0\to C^1$ such that $ST=$ identity, and $Tf=0$ if and only if $f=0$. The author calls S a derivative, T an integral, and proceeds to

develop a theory concerning such abstract differential equations. The first section is algebraic only, while the second section introduces spaces in which various types of convergence are defined, and the author studies abstract equations in such spaces. The development is clear, elegant, and natural. Rather than attempt to state the results, it would perhaps give a better idea of the significance of this paper if the applications considered by the author are listed: Ordinary or partial differential and difference equations, differential equations involving Mikusiński's operational calculus, asymptotic series, various difference formulas for interpolation, Schauder's and Banach-Caccioppoli's fixed-point theorems, and the Kojima-Schur and Toeplitz summation theorems.

A. Stokes (Washington, D.C.)

Chen, Wen-yuan [Chen, Weng-yuan] 6342

The index of solutions of nonlinear equations in Banach space.

Acta Math. Sinica **13** (1963), 315–322 (*Chinese*); *translated as Chinese Math.* **4** (1964), 341–350.

Let X, Y, Z be Banach spaces and $F: X \times Y \to Z$ be such that $F(x_0, y_0) = 0$. If the partial derivative $F_x'(x_0, y_0)$ has n-dimensional null space and its adjoint has m-dimensional null space, there is a one-one correspondence between the solutions of $F(x, y) = 0$ near (x_0, y_0) and the solutions of a finite-dimensional non-linear equation. The Brouwer index is considered for isolated solutions of this bifurcation equation. The equality of this Brouwer index and the Leray-Schauder index of a related mapping is established when both indices are defined. This note is closely related to papers of Jane Cronin [Trans. Amer. Math. Soc. **69** (1950), 208–231; MR **12**, 716; ibid. **76** (1954), 207–222; MR **16**, 47].

R. G. Bartle (Urbana, Ill.)

Gehtman, M. M.; Stankevič, I. V. 6343

On the spectrum of non-selfadjoint differential operators. (Russian)

Dokl. Akad. Nauk SSSR **158** (1964), 29–32.

Let A be a selfadjoint operator in a Hilbert space and let B have domain $D(B) \supset D(A)$. Suppose $\|Bf\| \leq a\|Af\| + b\|f\|$ for $0 \leq a < \frac{1}{2}$, $b > 0$, and $f \in D(A)$. Moreover, suppose G is a region in the complex plane and m is an integer such that $(B(A-z)^{-1})^m$ is completely continuous for $z \in G$ and that $I + B(A-z)^{-1}$ has a bounded inverse for $z \in G$. Then the operator $T = A + B$ with $D(T) = D(A)$ is closed and $\sigma(T) \cap G$ consists of isolated eigenvalues of finite multiplicity. Using this result, the authors state some results concerning the perturbation of a differential operator by adding a non-selfadjoint differential operator of lower order.

R. G. Bartle (Urbana, Ill.)

Grušin, V. V. 6344

Behaviour of solutions of differential equations near the boundary. (Russian)

Dokl. Akad. Nauk SSSR **158** (1964), 264–267.

A distribution $u(t, x)$ on $X = T \times R^n$, T an open interval in R, is said to be continuous in t if $t \to u(t, x) \in \mathcal{D}'(R^n)$ is continuous. Let S be T plus part of $\bar{T} - T$ and let $G^r(S)$, $r \geq 0$, be all $u \in \mathcal{D}'(X)$ such that all t-derivatives of order $\leq r$ are continuous in t when $t \in T$ and have continuous extensions to S, and let $G^{-r}(S)$ be all u which in T are sums of t-derivatives of order $\leq r$ of distributions in $G^0(S)$. I[t is] shown that if $P(t, x, \partial/\partial x)$ has infinitely differenti[able] coefficients and u belongs to some $G^p(S)$, then [...] $Pu \in G^r(S)$ if and only if $u \in G^{r+1}(S)$. Further[...] $P(D_t, D_x)u = 0$, where P is hypo-elliptic and normal [...] then $u \in G^\infty(S)$ if and only if $u(t, x) = O(d^{-a})$ for some [...] where x is bounded and d is the distance of t to ∂S.

L. Gårding (L[...]

Karlin, Samuel 6[...]

The existence of eigenvalues for integral operators.

Trans. Amer. Math. Soc. **113** (1964), 1–17.

A C^∞ kernel $K(x, s)$ $(x, s \in [a, b],\ -\infty < a < b < \infty)$ is ca[...] ETP (extended totally positive) if, for $p = 1, 2,$ [...] $K(\bar{x}, \bar{s})/u_p(\bar{x})u_p(\bar{s}) > 0$ for

$$\bar{x} = (x_1, x_2, \cdots, x_n), \qquad \bar{s} = (s_1, s_2, \cdots, s_n),$$

with

$$a \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq b, \quad a \leq s_1 \leq s_2 \leq \cdots \leq s_n \leq b$$

where

$$K(\bar{x}, \bar{s}) = \det\|K(x_i, s_j)\| \text{ and } u_p(\bar{x}) = \prod_{1 \leq i < j \leq p} (x_j - x_i)$$

the case of equality for the variables x_i and s_j being und[er]stood as a limiting case. Then (Theorem 3) the operato[r] defined by means of the kernel $K(x, s)$ possesses a cou[nt]able set of simple positive eigenvalues $\lambda_0 > \lambda_1 > \cdots$ $\lambda_n > \cdots$ decreasing to zero; there exists no other non-[...] spectrum of T; for the corresponding eigenfunctions φ_0[...] $\varphi_1(x), \cdots, \varphi_n(x), \cdots$, there holds an oscillation theo[rem] $\operatorname{sign}(\epsilon_p) \det\|\varphi_i(x_j)\|/u_p(\bar{x}) > 0$ $(\operatorname{sign}(\epsilon_p) = 1$ or $-1)$. The pr[oof] of this result is obtained by successive applications of [...] elaboration (Theorem 2) of the M. G. Krein-M. A. Rutm[an] generalization [Uspehi Mat. Nauk **3** (1948), no. 1 ([...] 3–95; MR **10**, 256] of the Frobenius-Perron-Jentz[sch] theorem.

K. Yosida (Tok[...]

Mitropol′skiĭ, Ju. A. 6[...]

A study of the integral manifold for a system of non-lin[ear] equations in Hilbert space which are close to equati[ons] with variable coefficients. (Russian. English su[m]mary)

Ukrain. Mat. Ž. **16** (1964), 334–338.

The author considers a system of differential equations [of] the form (1) $d\phi/dt = \omega(t) + P(t, \phi, h, \varepsilon)$, $dh/dt = H(t)h$ [+] $Q(t, \phi, h, \varepsilon)$, where ϕ is a scalar, h is an element of a Hilb[ert] space H, P and Q are continuous in their argumen[ts,] bounded in t, ϕ for all t, ϕ and Lipschitzian in h, w[ith] bounds and Lipschitz constants which approach zero [as ε] and h approach 0. The major assumption is that [the] operator equation $dU/dt = H(t)U(t)$, with $U(t_0) = E$, [the] identity operator, is uniformly exponentially asym[p]totically stable as $t \to +\infty$ in the operator norm. Using [the] method developed by Bogoljubov and the author [for] showing the existence of stable integral manifolds [for] systems in the form (1) (see Hale [*Oscillation in non-lin[ear] systems*, McGraw-Hill, New York, 1963; MR **27** #4[...] for a reference in English), it is then shown that (1) ha[s a] stable one-dimensional integral manifold. It would be [of] interest if the author gave some applications indicati[ng] how a system of the form (1) might occur and some inte[r]pretation of the results in terms of the applications.

A. Stokes (Washington, D.[C.)]

Sirčenko, Z. F. 6347

Extension of a theorem of N. N. Bogoljubov to the case of a Hilbert space. (Russian. English summary)

Ukrain. Mat. Ž. **16** (1964), 339–350.

The author considers an equation in Hilbert space in standard form (1) $dx/dt=\varepsilon X(t,x)$, where x, $X(t,x)$ are functions with values in a Hilbert space H, and ε is a small parameter. The author extends the method of averaging of Krylov and Bogoljubov to (1), making the usual assumptions that the averaged equation (2) $dy/dt=\varepsilon X_0(y)$ exists, has an equilibrium point y_0, and that the infinite matrix $\partial X_0(y)/\partial y$ at $y=y_0$ has no eigenvalue with zero real part. Following the proof given by Bogoljubov and Mitropol'skiĭ, the author obtains the standard results concerning the existence and stability of an almost periodic solution of (1). It would be of some interest to see some applications of these results, i.e., some source for equations of the form (1), and some interpretation of the above results.

A. Stokes (Washington, D.C.)

Trenogin, V. A. 6348

Existence and asymptotic behaviour of solutions of "solitary wave" type for differential equations in a Banach space. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 1033–1036.

The author deals with the boundary-value problem (1) $-d^2y/d\eta^2+Ay=F(\lambda,y)$, $-\infty<\eta<\infty$, $\lim_{\eta\to\pm\infty} y(\eta)=0$, where y is an element of a Banach space E, λ is a real parameter, A is a closed linear unbounded operator whose domain is dense in E, $F(\lambda,y)$ is a nonlinear operator in E, analytic in λ, y in the neighborhood of the point $(\lambda,0)$ for all λ, with $F(\lambda,0)=0$. The problem (1) always has the trivial solution. The object is to give conditions under which for some $\lambda=\lambda_0$ there exist non-trivial solutions as well. The basic methods are similar to those used by A. M. Ter-Krikorov and the author for a boundary-value problem for elliptic equations [Mat. Sb. (N.S.) **62** (104) (1963), 264–274; MR **28** #3266]. In preliminary lemmas, the differential equation $-d^2z/d\xi^2+Bz=h(\xi)$, subject to the boundary condition $\lim_{\xi\to\pm\infty} z=0$, is solved "explicitly" with the aid of the theory of semi-groups, under conditions on the closed linear operator B which permit the use of the Hille-Yoshida-Phillips theorem. The conditions of the principal theorem involve the concept of a generalized Jordan chain of length p of a linear operator B with respect to the analytic operator Φ and are too complicated to be given here.

D. H. Hyers (Los Angeles, Calif.)

Valikov, K. V. 6349

Characteristic indices of solutions of differential equations in a Banach space. (Russian)

Dokl. Akad. Nauk SSSR **158** (1964), 1010–1013.

The author considers an equation of the form (1) $dx/dt=Ax$, x in a Banach space X, and A a closed operator with domain $D(A)\subset X$ and values in X, satisfying auxiliary conditions so that A generates an analytic semi-group e^{At}, $t>0$. The author defines, following Lyapounov, the characteristic index $\mu(x)$ of a function $x(t)$, with values in X. The author's first two theorems relate the characteristic indices of solutions of (1) to the real part of the spectrum of A, thereby extending results of M. G. Krein when A is assumed bounded [Uspehi Mat. Nauk **3** (1948), no. 3 (25), 166–169; MR **10**, 128]. The author then considers the equation (2) $dx/dt=Ax+h(x,t)$, where h is defined for $t\geq 0$, $x\in D(A_0^\alpha)$, $0\leq\alpha<1$, $h(0,t)=0$, and $h(A_0^\alpha,x,t)$ is continuous for $x\in X$, $t\geq 0$. Here $A_0=z_0I-A$, where z_0 is such that the spectrum of A is contained in an angular sector opening to the left centered at z_0. The remaining five theorems then give conditions for the existence of solutions of (2) on finite intervals, then on infinite intervals, and finally, various relations are given between the real part of the spectrum of A and $\mu(A_0^\alpha x)$, where x is a solution of (2).

A. Stokes (Washington, D.C.)

Bass, Jean 6350

Espaces de Besicovitch, fonctions presque-périodiques, fonctions pseudo-aléatoires.

Bull. Soc. Math. France **91** (1963), 39–61.

Author's summary: "Les fonctions complexes $f(t)$ telles que la moyenne quadratique

$$\lim_{T\to\infty} T^{-1}\int_0^T |f(t)|^2\,dt$$

existe ne constituent pas dans leur ensemble un espace vectoriel. Mais on peut les grouper en espaces vectoriels incomplets E. Dans un espace E, toute suite de Cauchy converge vers une limite, et la limite obtenue est un prolongement de l'espace E. Une suite de Cauchy particulière permet de représenter la classe des translatées d'une fonction $f(t)$ de E par la formule

$$f(t+h)=\int_{-\infty}^{\infty}\exp(2i\pi\omega h)\,Y(t,d\omega),$$

où $Y(t,\Delta)$ est une fonction spectrale telle que la moyenne de $\bar{Y}(t,\Delta)Y(t,\Delta')$ soit nulle lorsque Δ et Δ' sont disjoints. Suivant les propriétés de $Y(t,\Delta)$, on classe les fonctions f en fonctions presque-périodiques, fonctions pseudo-aléatoires, fonctions singulières."

E. Følner (Copenhagen)

Bass, Jean 6351

Erratum: "Espaces de Besicovitch, fonctions presque-périodiques, fonctions pseudo-aléatoires".

Bull. Soc. Math. France **91** (1963), 434.

Correction of an obvious misprint on p. 40 of the author's paper [#6350].

CALCULUS OF VARIATIONS

See also 5829, 6158, 6402, 6446, 6972, 6973, 6976, 6977.

Bolonkin, O. O. 6352

Optimization of the parameters of variational problems. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukrain. RSR **1964**, 580–582.

In this paper the author considers Mayer's problem of the variational calculus regarding the extremum of the functional $I=G[c,x_1,y(x_1),x_2,y(x_2)]$, with differential constraints $\Phi_i\equiv \dot y_i-f_i(c,x,y,u)=0$, $i=1,2,\cdots,n$, and with boundary conditions $g_k[c,x_1,y(x_1),x_2,y(x_2)]=0$, $k=1,2,\cdots,p\leq 2n+1$. Here $y(x)=\{y_i(x)\}$ is a vector function, $c=\{c_1,c_2,\cdots,c_l\}$ an l-measured vector-constant subject to the constraints $\varphi_m(c)=0$, $m=1,\cdots,q<l$, and $\varphi_\xi^*(c)\leq 0$, $\xi=1,\cdots,\zeta$. Three theorems are established

and discussed on the necessary and sufficient conditions for the relative and absolute minimum of the above functional with the given constraints.

The proofs of these theorems are very weak, so that the treatment is difficult to comprehend.

D. P. Rašković (Belgrade)

Faedo, Sandro 6353

Semicontinuità e quasi-regolarità per gli integrali di Fubini-Tonelli.

Ann. Scuola Norm. Sup. Pisa (3) 18 (1964), 361–383.

A Fubini-Tonelli integral is a double integral of the form $I(y_1, y_2) = \int_a^b \int_c^d f[x, z, y_1(x), y_2(z); y_1'(x), y_2'(z)]\, dx dz$, and this double integral is here interpreted in a Weierstrass sense rather than as a Lebesgue integral. It is (for fixed f) a functional depending on a pair of non-parametric curves $y_1(x)$ $(a \leq x \leq b)$ and $y_2(z)$ $(c \leq z \leq d)$. Its study in no way reduces to that of the ordinary single variational integrals, and it was thought until now that it behaves quite differently from these. Indeed, Magenes [same Ann. (3) 2 (1948), 1–38; MR 12, 267; ibid. (3) 3 (1949), 95–131; MR 12, 267] established the validity of a strange semicontinuity condition, unrelated to convexity, and which has no parallel for single integrals. All this turns out, however, to have been a will-o'-the-wisp, for the author shows, surprisingly enough, that the Magenes condition is necessary only if we admit curves which reduce to single points. By excluding such degenerate non-parametric curves, the author is able to establish semicontinuity theorems which are natural analogues of those for single integrals. His necessary condition for semicontinuity is positive quasiregularity, and his sufficient condition consists of this, together with a uniform continuity condition for f, uniformity being in respect of the arguments y_1', y_2'.

L. C. Young (Madison, Wis.)

Fichera, Gaetano 6354

Semicontinuity of multiple integrals in ordinary form.

Arch. Rational Mech. Anal. 17 (1964), 339–352.

The author establishes a semicontinuity of considerable generality for non-parametric integrals of the form $\int f(x, Mu, Lu)\, dx$ taken over a bounded open subset A of Euclidean r-dimensional space. Semicontinuity is in regard to the weak topology in u, and here u takes values in a Banach space, while L, M are linear operators, not necessarily differential operators, which transform U into two vector-valued functions of x, with components in L^p for some $p > 1$. It is assumed that the integrand $f(x, y, z)$ is continuous in (x, y, z) and convex in z, and, moreover, that it possesses a convex minorant of the form $f_0(z)$ which is large compared with (Z) for large z. To these hypotheses, there is added a compactness condition in regard to M, to account for the absence of any convexity hypothesis of f in the vector y. The author's discussion is particularly well motivated, and he derives a corresponding existence theorem for an attained minimum.

L. C. Young (Madison, Wis.)

Hestenes, Magnus R. 6355

Variational theory and optimal control theory.

Computing Methods in Optimization Problems (*Proc. Conf., Univ. California, Los Angeles, Calif.*, 1964), pp. 1–22. *Academic Press, New York*, 1964.

This paper is concerned with a very general formul[…] of a calculus of variations problem (optimal co[…] problem) with variable endpoints, involving differe[…] integral and finite equations and inequalities as […] conditions. There is given an outline of a proof of […]ditions which must be satisfied by a solution. This […] involves an extension and modification of methods i[…]duced by McShane [Amer. J. Math. 61 (1939), 809–[…] MR 1, 78] and Pontrjagin [Uspehi Mat. Nauk 14 (1[…] no. 1 (85), 3–20; MR 22 #11189]. The form which […] conditions take in some special cases is discussed. The […] section states the condition involving the sign of […] second variation for the case when there are no […]equalities and no corners.

L. M. Graves (Chicago[…]

Sánchez, David A.

Calculus of variations for integrals depending […] convolution product.

Ann. Scuola Norm. Sup. Pisa (3) 18 (1964), 233–2[…]

Let $F(x, y, d, p)$ be a real-valued continuous functio[…] E^4, and denote by K the set of all real-valued func[…] $y(x)$, $-\infty < x < \infty$, satisfying: (1) $y(x)$ is absolutely […]tinuous in every finite interval; (2) $y'(x) \in L^1(-\infty,$ […] (3) $F[x, y(x), y'(x), p(x)] \in L^1(-\infty, \infty)$, where $p(x)$ is […] of the following convolution integrals: $\int_{-\infty}^{\infty} y'(x-t)y'($[…] $\int_{-\infty}^{\infty} |y'(x-t)|\,|y'(t)|\, dt$, $\int_{-\infty}^{\infty} y'(x-t)g(t)\, dt$ […] $\int_{-\infty}^{\infty} |y'(x-t)|\,|g(t)|\, dt$, where in the last two insta[…] g is a given function in $L^1(-\infty, \infty)$. Under the assu[…]tion that the class K is non-empty, direct methods o[…] calculus are applied to establish sufficient condition[…] $I[y] = \int_{-\infty}^{\infty} F[x, y(x), y'(x), p(x)]\, dx$ to be lower […] continuous in K with respect to uniform convergenc[…] $(-\infty, \infty)$; an example is given to show that, under […] hypotheses considered, the functional $I[y]$ need no[…] lower semi-continuous in K with respect to uniform […]vergence on every compact set in $(-\infty, \infty)$. In addi[…] for P a given compact set in E^2 and $\bar{K}$ a non-em[…] subclass of K such that the graph of every $y(x)$ i[…] contains a point of P, and $\bar{K}$ closed with respec[…] uniform convergence on every compact set in $(-\infty,$ […] the author establishes a theorem on the existence […] minimum of $I[y]$ on $\bar{K}$.

W. T. Reid (Norman, O[…]

Triscari, Dionisio 6[…]

Sulle singolarità delle frontiere orientate di mi[…] minima.

Ann. Scuola Norm. Sup. Pisa (3) 17 (1963), 349–37[…]

Triscari, Dionisio 6[…]

Sull'esistenza di cilindri con frontiera di misura mini[…]

Ann. Scuola Norm. Sup. Pisa (3) 17 (1963), 387–39[…]

Triscari, Dionisio 6[…]

Sulle singolarità delle frontiere orientate di mi[…] minima nello spazio euclideo a 4 dimensioni.

Matematiche (*Catania*) 18 (1963), 139–163.

Let $\mathcal{F}^*E$ denote the reduced boundary of a set $E \subset R$[…] finite perimeter $P(E)$, in the sense of De Giorgi [Rice[…] Mat. 4 (1955), 95–113; MR 17, 596]. Let $A \subset R^n$ […] bounded and open. Suppose that $(\mathcal{F}^*E) \cap A$ has m[…]mum $(n-1)$-area in the sense that $P(E) \leq P(E')$ for e[…] E' such that $E - A = E' - A$. De Giorgi [Seminari[…]

Mat. Scuola Norm. Sup. Pisa (1960–61), pp. 1–57, Editrice Tecnico Scientifica, Pisa, 1961] showed that any $x \in (\mathscr{F}^*E) \cap A$ has a neighborhood U such that $(\mathscr{F}^*E) \cap U$ is an analytic hypersurface. Let $S(E)$ be the set of all $x \notin \mathscr{F}^*E$ such that x is a limit point of $\mathscr{F}^*E$ and $(\mathscr{F}^*E) \cap A$ has minimum $(n-1)$-area for some open set A containing x. $S(E)$ is called the singular set. It is an open question whether $S(E)$ is always empty. In these three papers some partial results about it are obtained.

In the first paper it is shown that if $x = 0 \in S(E)$, then there exists a cone L with vertex 0 such that $\mathscr{F}^*L$ locally has minimum $(n-1)$-area and $0 \in S(L)$. The cone L is the limit, locally in measure, of E_{ρ_k} for a suitable sequence ρ_k tending to 0, where E_ρ is the ρ^{-1} homothetic image about 0 of E.

The second paper begins with a discussion of sets locally of finite perimeter. Then it is shown that if L is as above and 0 is an accumulation point of $S(L)$, then there exists a cylinder M such that $\mathscr{F}^*M$ locally has minimum $(n-1)$-area and $0 \in S(M)$.

In the third paper some results about cartesian products of sets of finite perimeter are proved. The results quoted above then imply that if $S(E)$ is not discrete for some $E \subset R^n$, then there exists a cone $N \subset R^{n-1}$ locally of minimum $(n-2)$-area such that $0 \in S(N)$. For $n \leq 3$ it is shown that $S(E)$ is always empty; the reviewer gave a different proof of this fact [Rend. Circ. Mat. Palermo (2) 11 (1962), 69–90; MR 28 #499]. Therefore, when $n = 4$, $S(E)$ is always a discrete set.

W. H. Fleming (Providence, R.I.)

GEOMETRY

See also 5748, 5872, 6512, 6668.

Guggenheimer, H. 6358

Ein Axiomensystem für die euklidische Geometrie.

Elem. Math. **19** (1964), 126–131.

Es wird ein Axiomensystem von einfacher algebraischer Struktur für die reelle Euklidische Geometrie aufgestellt. Die Axiome kennzeichnen im wesentlichen kollineare und nicht kollineare Lage von drei Punkten, die Vollständigkeit der Ebene und die Gültigkeit des Pythagoraischen Theorems. Obwohl in die Formeln dieses Axiomensystems nur gerade Potenzen der Entfernungen je zweier Punkte eingehen, können gerichtete Strecken in natürlicher Weise erklärt werden. Ein wesentlicher Teil der metrischen Euklidischen Geometrie (und der projektiven Geometrie) über einem reellen Zahlkörper beliebiger Charakteristik $p \neq 2$ kann so leicht entwickelt werden. Wird ein angeordneter Körper (der Charakteristik Null) zugrunde gelegt, so sind auch die üblichen Axiome der Anordnung und der Winkeladdition einfach zu beweisende Sätze. Für Körper der Charakteristik 2 decken sich Parallelismus und Orthogonalität, die Existenzsätze bereiten jedoch einige Schwierigkeiten.

H. R. Müller (Braunschweig)

Keedwell, A. D. 6359

A geometrical proof of an analogue of Hessenberg's theorem.

J. London Math. Soc. **39** (1964), 434–436.

Nach G. Hessenberg ist der Desarguesche Satz eine Folge des Satzes von Pappus. In Analogie dazu wird bewiesen, daß der schwächere Satz von Pappus, dessen Konfiguration aus fünf Punkten aufgebaut wird, die Gültigkeit der schwächeren Reidemeister-Bedingung nach sich zieht, die die Assoziativität aller Additionen zur Folge hat.

H. Brauner (Stuttgart)

Lumiste, Ju. [Lumiste, Ü. G.] 6360

On models of betweenness. (Russian. Estonian and English summaries)

Eesti NSV Tead. Akad. Toimetised Füüs.-Mat. Tehn.-tead. Seer. **13** (1964), 200–209.

An axiom system of geometry is given in terms of a ternary relation, called betweenness. It is then proved that in the two-dimensional case every model of this axiom system is a partial plane, and in case of dimension n greater than two, every model is isomorphic to a convex region of an n-dimensional linear space over an ordered division ring.

G. Grätzer (University Park, Pa.)

Du Val, Patrick 6361

★Homographies, quaternions and rotations.

Oxford Mathematical Monographs.

Clarendon Press, Oxford, 1964. xiv + 116 pp. $5.60.

Cet ouvrage comble une lacune de la littérature mathématique puisqu'il reprend et coordonne un grand nombre de propriétés dont certaines ne se trouvent guère que dans les mémoires originaux parfois anciens et souvent peu accessibles au lecteur moyen; de plus l'exposé ne suppose que peu de connaissances préalables et s'adresse ainsi à un public étendu. A partir de la théorie des homographies unidimensionnelles, permettant la représentation des rotations de la sphère, on passe aux groupes finis de rotation et ainsi à l'étude des polyèdres réguliers. Une étude parallèle des quaternions conduit à la théorie des polytopes, qui constitue la partie la plus importante de l'ouvrage. Enfin le dernier chapitre traite des groupes d'involution et constitue la partie la plus originale, apportant des vues nouvelles sur la théorie des singularités des surfaces algébriques à laquelle l'auteur a consacré d'importants travaux. On se doit de souligner la présentation de modèles qui fait de cet instrument de travail "un beau livre".

B. d'Orgeval (Dijon)

Martini, Giulia 6362

Ricerche locali sulle trasformazioni puntuali tra due piani nella geometria affine.

Atti Accad. Sci. Ist. Bologna Cl. Sci. Fis. Rend. (11) **9** (1961/62), no. 2, 117–127.

Etude des transformations ponctuelles entre deux plans affines en un couple de points régulier homologues dans les cas (a) directions caractéristiques distinctes, (b) une direction caractéristique double, (c) une direction caractéristique triple, sans que sur celle-ci la projectivité caractéristique soit une similitude. Dans chaque cas détermination des repères intrinsèques et interprétation géométrique des invariants du 2^e ordre; ils n'existent pas pour (c), ni pour (b) si sur la droite caractéristique simple, la projectivité caractéristique est une similitude; ces invariants se lient aux segments définis par l'origine et les points limites sur les axes caractéristiques.

B. d'Orgeval (Dijon)

Martini, Giulia 6363

Ricerche locali sulle trasformazioni puntuali tra due spazi nella geometria affine.

Atti Accad. Sci. Ist. Bologna Cl. Sci. Fis. Rend. (11) **10** (1962/63), *no.* 2, 209–223.

Etude en géométrie affine des transformations ponctuelles entre deux espaces linéaires S_r en un couple régulier de points homologues si (a) il y a au moins $r+1$ des 2^r-1 directions caractéristiques distinctes dont r ne sont pas dans un hyperplan, (b) si $r=3$, il y a un plan de droites caractéristiques et trois autres droites caractéristiques distinctes, (c) $r=3$, on a deux plans de droites caractéristiques et une autre droite caractéristique, (d) $r=3$, il y a un cône quadrique de droites caractéristiques et une autre droite caractéristique. Dans chaque cas détermination des repères intrinsèques et interprétation géométrique des invariants du 2ᵉ ordre, à partir des segments définis par l'origine et les points limites pour les cas (a), (b), (c); pour (d) l'interprétation est moins simple.

B. d'Orgeval (Dijon)

Martini, Giulia 6364

Sopra un caso particolare notevole che si presenta nella geometria affine delle trasformazioni puntuali.

Atti Accad. Sci. Ist. Bologna Cl. Sci. Fis. Rend. (11) **10** (1962/63), *no.* 2, 140–147.

Etude des transformations ponctuelles entre deux plans affines en un couple régulier de points homologues où les trois directions caractéristiques coïncident, la projectivité caractéristique étant sur elle une similitude. Pour définir le repère intrinsèque, il faut aller au voisinage du 3ᵉ ordre. Les invariants se lient à deux cubiques transformées des axes du repère intrinsèque par la transformation rationnelle approchant au 3ᵉ ordre la transformation donnée.

B. d'Orgeval (Dijon)

Ballanti, Pietro 6365

Trasformazioni puntuali fra piani nell'intorno di un punto unito.

Atti Accad. Sci. Ist. Bologna Cl. Sci. Fis. Rend. (11) **9** (1961/62), *no.* 2, 128–139.

Etude des transformations ponctuelles T entre deux plans superposés au voisinage d'un point uni 0, si T est osculable par une homologie de centre 0; on doit distinguer si cette homologie est générale, spéciale ou l'identité. Dans chaque cas l'auteur établit des équations canoniques de la transformation; pour obtenir le repère intrinsèque, il faut au 1ᵉʳ cas aller au 3ᵉ ordre, pour les autres au 4ᵉ.

B. d'Orgeval (Dijon)

Böhm, Wolfgang 6366

Über eine Verallgemeinerung des Satzes von Pascal.

Arch. Math. **15** (1964), 392–393.

Auf J. Plücker geht folgender Satz zurück: Sind drei Kegelschnitte k_i ($i=1, 2, 3$) einem Kegelschnitt k einbeschrieben, so haben die drei Tangentenvierseite der Paare k_1, k_2 je ein Gegeneckenpaar mit einem vollständigen Vierseit gemeinsam; sein Diagonaldreieck besteht aus den Schnittpunkten der drei Paare von Berührungstangenten von k_i mit k. Dieser Satz wird durch eine eineindeutige lineare Zuordnung der Kegelschnitte einer Ebene auf die Punkte eines projektiven fünfdimensionalen Raume[s aus] dem Satz von Desargues gefolgert. Außerdem wir[d ein] zum Satz von Plücker analoge Satz über Quadrik[en im] Raum formuliert.

H. Brauner (Stutt[gart])

Niče, Vilko [6367]

Über neue Eigenschaften der Büschel und der B[ündel] polarer Räume. (Serbo-Croatian summary)

Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hr[vatske] Ser. II **17** (1962), 189–204 (1963).

Die polaren Räume zweier quadratischen Flächen S_1 [und] S_2 bestimmen auf jeder Geraden l des Raumes [zwei] involutorische Punktreihen. Diejenigen Geraden l f[ür die] diese Punktreihen einen gemeinsamen Zentralpunkt h[aben,] bilden nach Majcen einen Komplex M dritter Ord[nung. Der] Verfasser hat früher [Rad Jugoslav. Akad. Znan. U[mjet.] Odjel Mat. Fiz. Tehn. Nauke **325** (1963), 107–125] g[ezeigt,] dass irgendwelche zwei quadratische Flächen [eines] Büschels denselben M bestimmen. Ein solcher Bü[schel] bestimmt auch einen tetraedralen Komplex R. [Der] Durchschnitt von M und R ist die Kongruenz K un[d die] Zentralpunkte der Strahlen von K bilden eine Fläc[he θ.] Die Strahlen von R kann man den Punkten des Ra[umes] eineindeutig zuordnen. Die Bildpunkte der Strahlen [von] K bilden eine zweite Fläche Γ. Verfasser untersucht die Flächen θ und Γ; sie sind beide sechster Ord[nung.] Er fügt einen neuen Beweis hinzu für die Ordnung v[on M] und betrachtet zuletzt die Komplexe M in einem B[ündel] quadratischer Flächen.

O. Bottema (D[elft])

Permutti, Rodolfo [6368]

Su certe classi di forme a hessiana indeterminata.

Ricerche Mat. **12** (1964), 97–105.

The author studies the properties, in complex proje[ctive] r-space (x_i), of forms which can be written as polyno[mials] in complex projective s-space (z_i), where

$$z_i = x_i, \qquad i \leq s,$$

$$z_{s+1} = \sum_{1}^{r-s+1} M_k(x_1, \cdots, x_s)x_{s+k}.$$

Both in algebra and in geometry, his results gener[alize] those of Gordan and Nöther [Math. Ann. **10** (1[876),] 547–568].

H. W. Guggenheimer (Minneapolis, M[inn.])

Segre, Beniamino [6369]

Galois spaces and non-Desarguesian geometries.

Rend. Mat. e Appl. (5) **23** (1964), 1–3.

This note is introductory to the author's paper [#[6370] below], giving also translations of the titles of the p[arts] and sections of the paper.

R. J. Crittenden (Providence, R[.I.])

Segre, B. [Segre, Beniamino] [6370]

Teoria di Galois, fibrazioni proiettive e geometrie [non] desarguesiane.

Ann. Mat. Pura Appl. (4) **64** (1964), 1–76.

L'autore ha suddiviso in tre parti questa sua volumi[nosa] e densa Memoria, complementare all'altra scritta [con]temporaneamente [Math. Ann. **154** (1964), 195–[256;] MR **29** #2697].

Nella 1ª parte (nn. 1–20), di carattere introduttivo e di contenuto prevalentemente algebrico, si occupa delle estensioni finite e separabili di un campo. Sia δ una tale estensione, propria, semplice, commutativa, algebrica e di grado n del campo γ, e sia $u_1, \cdots, u_n$ una sua base, cosicchè ogni elemento ξ di δ ha una espressione del tipo $\xi = x_1u_1 + \cdots + x_nu_n$, con le $x_1, \cdots, x_n$ variabili in γ. Sia inoltre δ^* il sovracorpo di δ e di γ, di decomposizione della equazione algebrica coi coefficienti in γ che definisce l'ampliamento da γ a δ. L'autore introduce uno spazio proiettivo ad $n-1$ dimensioni $S_{n-1} = \{(x_1, \cdots, x_n)\}$ su γ e fa vedere, sfruttando le costanti di struttura della base di δ su γ, come ad ogni cambiamento della base resti associata un'omografia dell' S_{n-1} in sè, mentre alla estensione stessa resta associata (in modo indipendente dalla base) una trasformazione cremoniana T, involutoria e di grado $n-1$ dell' S_{n-1}, le cui varietà fondamentali costituiscono la traccia di un simplesso rappresentabile (nell'ampliamento di S_{n-1} che si ottiene pensando le $x_1, \cdots, x_n$ variabili in δ^*) annullando la "norma" di ξ rispetto a γ. D'altra parte ad ogni estensione del tipo suddetto resta associata anche un'algebra A, commutativa, associativa e dotata di modulo, gli elementi della quale sono le stesse espressioni $\xi = x_1u_1 + \cdots + x_nu_n$, dove però ora le u_i si interpretino come puri simboli (legati dalle relazioni di struttura) e le x_i si pensino variabili in un sovracorpo χ di γ. Giovandosi ancora della considerazione di uno spazio proiettivo $S_{n-1} = \{(x_1, \cdots, x_n)\}$, l'autore affronta lo studio dei nullifici di A, allo scopo di precisarne ulteriormente la struttura. Si può supporre che χ contenga il campo di spezzamento δ^*. La A risulta priva di autonullifici, mentre il luogo dei punti ξ di S_{n-1} rappresentativi dei nullifici di A è lo stesso n-simplesso di S_{n-1} la cui restrizione (allo S_{n-1} su γ) è il luogo delle varietà fondamentali della trasformazione cremoniana T. Gli n vertici di questo simplesso sono le immagini dei nullifici di indice $n-1$ di A. Infine (sempre se $\chi \supseteq \delta^*$) l'autore riesce a caratterizzare compiutamente la struttura di A provando che essa è somma diretta di n algebre del 1° ordine, ciascuna "isomorfa" al campo χ. L'autore passa poi a considerare il gruppo Γ di Galois del campo δ rispetto al campo γ, nell'ipotesi (verificata automaticamente, se δ è finito) che sia $\delta = \delta^*$, ossia che l'estensione da γ a δ sia anche normale. Infine l'autore si occupa degli elementi di una estensione algebrica aventi norma assegnata c. Nel caso che il campo δ sia finito e l'ordine del suo sottocampo γ sia $q = p^h$, la norma c può essere un qualsiasi elemento non nullo di γ ed il numero degli elementi aventi quella norma (che si ottengono tutti da uno di essi con le trasformazioni del gruppo di Galois di δ rispetto a γ) è sempre uguale a $(q^n - 1)/(q - 1)$.

Nella 2ª parte della memoria l'autore si occupa dapprima delle fibrazioni (globali) mediante sottospazi S_{n-1} di uno spazio grafico irriducibile S_{s-1} $(1 < n < s)$. Nel caso che l'S_{s-1} sia finito la condizione che n divida s, cioè che sia $s = n \cdot r$, è necessaria e sufficiente per l'esistenza di una fibrazione siffatta. La stessa condizione è anche sufficiente, nel caso che lo spazio S_{s-1} sia lineare sopra un campo γ qualsiasi, supposto che esista un'algebra A su γ, primitiva, di ordine n e, se $r \neq 2$, anche associativa. Questo è appunto il caso che si presenta quando S_{s-1} è uno spazio grafico pascaliano, ossia lineare sopra un campo γ ed A è l'algebra relativa all'estensione (algebrica, separabile e di ordine $n \geq 2$) del campo γ ad un suo sovracampo δ. Si riesce allora a costruire (su γ) un "sistema grafico elementare" Σ di spazi S_{n-1} fibranti. Un sistema grafico elementare siffatto definisce in S_{s-1} una n-pla di sottospazi "direttori" S_{r-1}, linearmente indipendenti (e coniugati nella estensione da γ a δ^*), ed è costituito da quei sottospazi S_{n-1} di S_{s-1} che son definiti su γ e si appoggiano in un sol punto a ciascuno degli n spazi direttori S_{r-1}. Il sistema Σ costituisce un primo "modello" algebrico dell'S_{r-1} che si ottiene da uno spazio S'_{r-1} lineare su γ quando dal campo base iniziale γ si passa alla sua estensione algebrica δ. Un tale modello, interamente definito su γ, è utile per descrivere la geometria algebrica di S_{r-1} senza uscire dal campo γ iniziale, quando δ sia la chiusura algebrica di γ. Un secondo modello è fornito dalla immagine W di Σ sulla grassmanniana rappresentativa degli S_{n-1} di S_{s-1}. La W è una varietà di C. Segre (prodotto degli n spazi direttori del sistema grafico Σ), appartenente ad uno spazio S_{r^n-1} lineare su γ, della quale si conoscono anche l'ordine e la dimensione.

L'autore considera poi sui modelli Σ e W le immagini (definite in γ) G/G_0 e rispettivamente G' del gruppo delle omografie (definite in δ) dell' S_{r-1} in sè. Inoltre, su W, considera le sottovarietà immagini degli spazi subordinati all' S_{r-1} e le sottovarietà (di Veronese, generalizzate) immagini della totalità delle forme di ordine n di uno di quegli spazi. Il gruppo G consta di tutte e sole le omografie (su γ) di S_{s-1} che mutano in sè Σ, lasciando fisso ciascuno dei suoi spazi direttori. Il gruppo G' consta di tutte e sole le omografie (su γ) dell'ambiente che mutano in sè la W, lasciando fisso ciascuno dei suoi n sistemi di generatori. G_0 è il nucleo dell'omomorfismo che intercede tra G e G' ed è isomorfo al gruppo delle omografie che mutano in sè un S_{n-1} di Σ lasciando fissi i suoi n punti d'appoggio con gli spazi direttori. Tanto G quanto G' sono sottogruppi invarianti rispettivamente del gruppo $\bar{G}$ di tutte le omografie (su γ) di S_{s-1} che trasformano in sè Σ e del gruppo $\bar{G}'$ di tutte le omografie dell'ambiente (su γ) che trasformano in sè W. I due gruppi quozienti $\bar{G}/G$ e $\bar{G}'/G'$ risultano isomorfi tra loro ed al gruppo Γ di Galois di δ rispetto a γ. Si ottengono così delle rappresentazioni geometriche di Γ che consentono di studiarne sinteticamente la struttura.

Un terzo modello (definito su γ), dell' S_r che si ottiene da uno spazio S_r' lineare su γ quando il campo γ si amplia in quello δ, si ottiene partendo da una base $u_1, \cdots, u_n$ dell'ampliamento e considerando le coordinate proiettive non omogenee $\xi_i = x_{i1}u_1 + \cdots + x_{in}u_n$ $(i = 1, \cdots, r)$ di un punto dell' S_r. Allora, essendo $s = n \cdot r$, la s-pla di elementi x_{ij} si può interpretare come quella delle coordinate proiettive non omogenee di un punto di un S_s lineare su γ, cosicchè i punti propri dello S_r vengono a rappresentarsi coi punti propri dello S_s. D'altra parte le stesse ξ_i possono interpretarsi come coordinate proiettive omogenee di un punto dell' S_{r-1} improprio del riferimento di S_r, e per questo S_{r-1} (su δ) si può considerare un modello del primo tipo (definito su γ), rappresentandone biunivocamente i punti sugli spazi fibranti di un sistema grafico elementare di S_{n-1} appartenente all' S_{s-1} improprio dell' S_s già considerato. Questo terzo modello si può anche ottenere dal secondo mediante una "proiezione stereografica".

La 3ª parte della memoria inizia con la più generale definizione dei "sistemi grafici" $\Sigma_{n,r}$ di sottospazi S_{n-1} fibranti uno spazio grafico irriducibile S_{s-1}, con $s = n \cdot r$ ed $r, n > 1$. Di questi $\Sigma_{n,r}$ sono casi particolari i sistemi grafici "elementari", già considerati nella 2ª parte. Ogni $\Sigma_{n,r}$ si può pensare a sua volta come uno spazio grafico

astratto irriducibile R_{r-1}, di cui sono "punti" gli S_{s-1} del sistema. Questo R_{r-1} si può sempre immergere in un R_r, procedendo in modo analogo a quello tenuto nella costruzione dei modelli del 3° tipo considerati nella 2ª parte della Memoria. Per $r>2$ tanto lo S_{s-1} quanto lo R_{r-1} risultano desarguesiani, ossia lineari rispettivamente sopra dei corpi γ e $\delta\supset\gamma$, che risultano definiti a meno d'isomorfismi. La costruzione sintetica delle coordinate fa vedere che δ è una estensione di rango n di γ. L'autore si occupa poi dei casi in cui dapprima solo γ, e poi anche δ risultano commutativi, illustrando anche i collegamenti dell'argomento con la teoria delle algebre associative e primitive sopra un corpo γ.

Un sistema grafico $\Sigma_{n,r}$ di un S_{s-1} definito su di un campo γ si dice "generale" se—entro un S_{2n-1} di S_{s-1}—possono trovarsi quattro distinti S_{n-1} di Σ che ammettano n distinte rette trasversali comuni (definite in γ od in una sua estensione). Risulta allora che i soli sistemi grafici che siano pascaliani (cioè tali che δ sia commutativo) e generali sono quelli elementari. Nel caso che γ sia un campo finito, ossia in un S_{s-1} di Galois, possono aversi sistemi grafici $\Sigma_{n,r}$ generali che non siano elementari soltanto nel caso che risulti $r=2$, $s=2n$. Anzi, in un S_{2n-1} di Galois, un $\Sigma_{n,2}$ generale, ma non elementare, risulta certamente non desarguesiano (cioè non è desarguesiano il piano R_2). Lo stesso accade in un S_3 reale per ogni sistema grafico $\Sigma_{2,2}$ di rette che sia generale ma non elementare. Da ciò deriva un metodo per la costruzione di piani grafici R non desarguesiani, finiti o reali. Anche per precisare la portata di questo metodo, l'autore affronta e risolve il problema della determinazione e classificazione dei sistemi grafici $\Sigma_{n,2}$ e $\Sigma_{2,2}$ suddetti. Così nel § 10 vengono determinate e classificate le fibrazioni mediante rette di uno spazio tridimensionale S_3 sopra un campo γ e si forniscono due esempi di piani grafici finiti non desarguesiani, di ordini rispettivi 9 e 25. Nel § 11 l'autore si occupa delle fibrazioni di un S_{2n-1} di Galois mediante S_{n-1}. Nell'ipotesi sia valida la congettura (vera per $n=2, 3, 4$) che esista qualche collineazione non omografica di un S_{n-1} lineare su γ la quale sia priva di spazi uniti, l'autore dimostra l'esistenza di fibrazioni, ossia di sistemi grafici $\Sigma_{n,r}$ non elementari. Così ottiene dei piani grafici finiti non desarguesiani di ordine p^k, con p primo e k non primo.

Infine l'autore segnala alcuni problemi ancora da risolvere, sulla geometrizzazione delle estensioni algebriche, sulle fibrazioni proiettive, sugli elementi uniti delle collineazioni di uno spazio pascaliano, sulla classificazione dei piani grafici non desarguesiani ottenibili con i metodi indicati sopra e sul loro confronto con quelli già noti.

E. Morgantini (Padova)

Segre, Beniamino 6371

Sistemi polilaterali di spazi in un iperspazio.

Math. Notae **19** (1964), 1–10.

In questo lavoro l'autore considera alcune proprietà proiettive di certi notevoli insiemi finiti di sottospazi di un iperspazio. Tali sono gli $(n-1)$-lateri di uno spazio proiettivo S_{n-1} ad $n-1\geq 2$ dimensioni sopra un campo K (generalizzazione dei quadrilateri piani completi) ed i sistemi polilaterali di generatori della varietà di C. Segre prodotto di due spazi proiettivi (generalizzazione delle terne di generatrici di una stessa schiera di una quadrica rigata dell' S_3).

E. Morgantini (Padova)

Segre, Beniamino

Alcune questioni su insiemi finiti di punti in geo[...]
algebrica.

Univ. e Politec. Torino Rend. Sem. Mat. **20** (196[...]
67–85.

The bulk of this paper is a survey of the present st[...] knowledge on the following problem: $\{n|k_1, k_2, \cdots$ [...] or $\{n|k_1^{l_1}, \cdots, k_t^{l_t}\}$, where l_i of the k_j's have the va[...] denotes the complete linear system of plane cur[...] order n with k_1-ple, $\cdots$, k_s-ple base points. Its v[...] dimension is

$$d = \tfrac{1}{2}n(n+3) - \sum_{i=1}^{s} k_i(k_i+1),$$

and its effective dimension is $\delta = d+\sigma$ $(\sigma\geq 0)$. In [...] cases is $\delta > d \geq -1$, without any specialisation i[...] position of the base points? A number of cases are [...] in which $d=-1$, $\delta\geq 0$; in all these the system has [...] multiple fixed components, and it is surmised tha[...] may always be the case. The imposition of an addi[...] double base point in general position on any s[...] diminishes δ by 2 only (while diminishing d by 3) i[...] only if the system is compounded with a pencil. If [...] $(i=1, \cdots, s)$ and $d\geq -1$, $\sigma=0$ in all cases except [...] $(d=2, \delta=3)$. A number of further cases of a very dif[...] kind are given in which it can be proved that $\sigma=0$[...] instance, if $\sum_{i=1}^{s} k_i \leq 3n-1$, $\sigma=0$, not only for ge[...] positions of the base points, but in all cases in whic[...] system is irreducible.

The last part of the paper summarises some r[...] work on geometry over a Galois field, especially on po[...] definite forms, i.e., those whose value in every poi[...] the space is a non-zero square in the ground field, a[...] the existence of sets of hypersurfaces with no com[...] points. The simplest of these results is that a plane [...] with no points on it breaks up into three lines whic[...] conjugate in a cubic extension of the ground field.

P. Du Val (Lo[...]

Speranza, Francesco

Trasformazioni fra piani euclidei rigati.

Atti Accad. Sci. Ist. Bologna Cl. Sci. Fis. Rend. (1[...]
(1962/63), *no.* 2, 176–191.

Etude des transformations différentiables entre [...] plans réglés; sur chaque droite il existe un point lim[...] celui-ci est impropre pour toutes les droites on [...] classe particulière de ces transformations. Etude des [...] invariants liés au voisinage du 1er ordre, des transfo[...] tions que l'on peut approcher par une correspond[...] donnée. Différences importantes avec les transforma[...] ponctuelles; applications métriques. Détermination [...] transformations possédant deux groupes d'égalit[...] elles-mêmes et des groupes de transformations per[...] ables avec un groupe d'égalité.

B. d'Orgeval (D[...]

Efremovič, V. A.; Tihomirova, E. S.

Equimorphisms of hyperbolic spaces. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 1139–[...]

This paper contains detailed arguments for the re[...] with briefly outlined proofs given by the authors [[...] Akad. Nauk SSSR 152 (1963), 1051–1053; MR 27 #5[...] The following consequence of these results was [...] mentioned explicitly in the previous review: If H^n i[...] Klein model of hyperbolic geometry in E^n bounde[...]

the sphere S^{n-1}, then an equimorphism of H^n on itself can be continued to a topological map of $H^n \cup S^{n-1}$ on itself. There is no analogous theorem for E^n.

H. Busemann (Los Angeles, Calif.)

Jaglom, I. M.; Rozenfel'd, B. A.; Jasinskaja, E. U. 6375

Projective metrics. (Russian)

Uspehi Mat. Nauk **19** (1964), no. 5 (119), 51–113.

This is a very useful and well-written survey of our present state of knowledge regarding the Cayley-Klein metrics in n-dimensional real projective space P^n.

Since the subject is little known, but interesting, it seemed best for the purposes of this review to define the spaces precisely and then outline briefly the topics treated in the paper.

The classical non-euclidean geometries arise from an absolute locus $a_{ik}x^ix^k = 0$ $(a_{ik} = a_{ki})$ with $\det(a_{ik}) \neq 0$ $(i, k = 0, \cdots, n)$. If j $(0 \leq j < n)$ is the index of this quadratic form, then ${}^jS_n$ is the corresponding space, in particular, ${}^0S_n$ and ${}^1S_n$ are the ordinary elliptic and hyperbolic spaces. If the rank of (a_{ik}) is $r+1 < n+1$, then $a_{ik}x^ix^k = 0$ represents a quadratic cone. If the form has index j, then the cone has $(n-r+j-1)$-dimensional generators and an $(n-r-1)$-dimensional "apex plane".

The semi-non-euclidean space ${}^{l_0 \cdots l_r}S_n^{m_0 \cdots m_{r-1}}$ is the P^n with an absolute locus of the following type: first we have a quadratic cone Q_0 of index j_0 with an $(n-m_0-1)$-dimensional apex plane T_0, then in T_0 a quadratic cone Q_1 of index j_1 with an apex plane T_1 of dimension $n-m_1-1$, $\cdots$, finally in the apex plane T_r-1 of Q_r-1 a non-degenerate quadric Q_r of index j_r. If all $j_i = 0$ we obtain a semi-elliptic space. If $m_0 = 0$ we delete the points of T_0 from the space and get the semi-euclidean space ${}^{l_1 \cdots l_r}R_n^{m_1 \cdots m_{r-1}}$. For $r=1$, $m_0 = 0$ we have the euclidean space ${}^jR_n$, where the absolute locus consists of a pair of hyperplanes coinciding with the hyperplane at infinity and in this hyperplane a non-degenerate quadric of index j. For $j=0$ we get the ordinary euclidean space, ${}^1R_4$ is the Lorentz space; ${}^{0 \cdots 0}R^{12 \cdots n-1}$ is the flag space, where the absolute consists of a set of r-planes $(r = 0, 1, \cdots, n-1)$, each contained in the next. It is shown that the semi-non-euclidean and semi-euclidean spaces can be obtained from the ${}^jS_n$ or ${}^jR_n$ by limit processes.

A line is non-euclidean of the ath order if it lies in T_{a-1} and intersects Q_a in two points. It carries a natural elliptic or hyperbolic metric δ_a with space constant 1 according to whether these points are conjugate, imaginary, or real. A line is euclidean of order a with a natural distance d_a if it intersects T_a but not T_{a+1}.

Angles between lines are defined by means of elliptic, parabolic or hyperbolic metrics in plane pencils of lines. One obtains in this way 3^2 geometries in P^2, and generally 3^n metrics in P^n.

The first topic is spheres, i.e., loci equidistant from a point. In particular, ${}^{l_1 \cdots l_r}S_n^{m_1 \cdots m_{r-1}}$ is isometric to the sphere of radius 1 in ${}^{l_1 \cdots l_r}R_{n+1}^{m_1 \cdots m_{r-1}}$.

Motions are defined as collineations which take the absolute into itself and preserve the distances d_a (they automatically preserve the δ_a). For $m_0 \neq 0$, i.e., in the case of semi-non-euclidean spaces, it suffices to require that the δ_a and d_a are preserved. The structures of the groups of motions are involved, and there are open problems, but they all have dimension $n(n+1)/2$.

The paper then turns to hyperplanes, questions of duality and angles between hyperplanes, next to lower-dimensional planes, in particular, covariants and invariants of pairs of planes. As is well known, one "angle" is, in general, not sufficient to determine the relative position of two lower-dimensional planes even in ordinary space ${}^0R_n$. The theory is, of course, much more complicated in the more general cases.

Cycles, i.e., hypersurfaces which in the above-mentioned isometric representation of ${}^{l_1 \cdots l_r}S_n^{m_1 \cdots m_{r-1}}$ as unit sphere in ${}^{l_1 \cdots l_r}R_{n+1}^{m_1 \cdots m_{r-1}}$ appear as intersections of this sphere with hyperplanes, are discussed next. In analogy to stereographic projection, this leads in a natural way to a definition of conformal maps, in particular, to inversions as involutoric conformal maps.

The last chapter deals with relations to algebra which arise by introducing numbers $x+\varepsilon y$ $(\varepsilon^2 = -1, \varepsilon^2 = 0, \varepsilon^2 = 1)$ and the corresponding bicomplex numbers or quaternions and biquaternions. Analogues of the Gauss plane, the Riemann sphere and the Poincaré upper half-plane provide models for the different two-dimensional geometries, for example, the Gauss plane with $\varepsilon^2 = 0$ yields the flag space ${}^0R_2{}^1$ and with $\varepsilon^2 = 1$ it yields ${}^1R_2$. The extension of the usual applications of these numbers to line geometry in the three-dimensional spaces and certain four- and five-dimensional geometries are discussed. A general algebraic scheme for all dimensions is outlined in the concluding section.

H. Busemann (Los Angeles, Calif.)

Burau, Werner 6376

Untersuchungen zur Geometrie der projektiven Formen- und Invariantentheorie. IV.

Math. Nachr. **28** (1964), 57–65.

Teil III ist in Collect. Math. 14 (1962), 51–69 [MR 28 #106] erschienen. In diesem vierten Teil gibt Verfasser eine erneute Darstellung der Deutung von Invarianten, Kovarianten, Kontravarianten und Zwischenformen der klassischen Invariantentheorie durch Transformationsgruppen in den mit den Koeffizienten der Grundformen korrespondierenden Koeffizientenräumen. Für Einzelformen ist das die Gruppe der Autokollineationen der Veroneseschen Manigfaltigkeit korrespondierend mit der n-ten Potenz einer (nicht symbolischen) Linearform. Verfasser erläutert die fundamentalen Begriffe der Invariantentheorie mit dieser Darstellung für eine, zwei und mehrere Grundformen.

E. M. Bruins (Amsterdam)

Burniat, Pol 6377

Varietà algebriche V_3 con $P_g = P_a = 0$ e P_2 qualunque.

Univ. e Politec. Torino Rend. Sem. Mat. **20** (1960/61), 235–253.

There exist 3-dimensional varieties with genera $P_g = P_a = 0$ and arbitrary $P_2 \geq 3$ whose bicanonical systems are irreducible, and also such whose bicanonical systems are composed with linear systems of regular surfaces, necessarily of bigenus one, or of genera one.

M. Rosenlicht (Berkeley, Calif.)

Burniat, Pol 6378

Variétés algébriques V_3 à surfaces canoniques unies dans une involution d'ordre 3.

Univ. Nac. Tucumán Rev. Ser. A **14** (1962), 77–89.

S^0 is the algebraic threefold with rational function field defined by adjoining $\sqrt[3]{E}$ to that of three-dimensional projective space S, where $E=0$ is an irreducible member of a trisectable linear system $|E|=|3e|$, defined as complete with respect to certain base conditions. The canonical system on S^0 is the image of the system in S adjoint to $|2e|$. S^0 is shown to be regular, and its canonical and all pluricanonical systems irreducible, in the following two cases:

(i) $|e|$ of order $r+2$, with an r-ple base line and $q \leq 2r+2$ simple base points each with a simple base line in its neighbourhood (i.e., with fixed tangent plane). $P_g = P_a = 6r+1-q$.

(ii) $|e|$ of order $r+2$ with an r-ple base line a, a simple base line b skew to a, and $q \leq 2r+2$ further simple base lines meeting a and b. $P_g = P_a = 4r-q$.

Further, if e is of order $2r+1$, with a skew quadrilateral of r-ple base lines, S^0 is regular with $P_g = P_a = 2r$, the canonical and all pluricanonical systems being compounded with the pencil of images of quadrics containing the quadrilateral; these surfaces have all genera equal to 1. If a further simple base line (meeting two opposite sides of the quadrilateral) is imposed on e, one surface of the pencil, image of the quadric through this additional base line, becomes a fixed part of the canonical system, so that $P_g = P_a = 2r-1$. *P. Du Val* (London)

Demaria, Davide Carlo 6379

Sopra una proprietà delle varietà superficialmente regolari.

Univ. Nac. Tucumán Rev. Ser. A **14** (1962), 341–352.

In previous work the author has obtained conditions under which, on a non-singular algebraic variety V_d, a complete linear system $|T|$ of hypersurfaces cuts a complete linear system on a given non-singular hypersurface S. In the present work the author considers the case in which V_d is superficially regular. He proves that if the system $|S-T|$ contains an effective non-singular hypersurface, then the system cut by $|T|$ on S is complete.

He further proves that a non-singular hypersurface D of a totally regular variety V_d is itself totally regular if and only if the divisor $-D$ has all its indices of irregularity equal to zero. *L. Roth* (London)

Godeaux, Lucien 6380

Sur les surfaces représentant les couples de points de certaines courbes algébriques.

Univ. Nac. Tucumán Rev. Ser. A **14** (1962), 61–68.

Se L è una curva algebrica non iperellittica che contenga una involuzione γ del second'ordine priva di coincidenze, si considera la superficie F che rappresenta le coppie non ordinate di punti di L; essa possiede una involuzione del secondo ordine avente una curva unita, e la superficie Φ immagine di questa involuzione è anch'essa sostegno di una involuzione del secondo ordine, priva di coincidenze. Detta F' l'immagine di questa involuzione, F' risulta la superficie che rappresenta le coppie non ordinate di punti della curva immagine di γ. F' e Φ hanno la stessa irregolarità. *D. Gallarati* (Genoa)

Reich, Ludwig 6381

Beiträge zur algebraischen Geometrie. II.

Österreich. Akad. Wiss. Math.-Natur. Kl. S.-B. II **172** (1963), 135–157.

This is a continuation of a previous paper [same 172 (1963), 1–42; MR 29 #2700], and deals with a problems involving a quartic space curve of the kind, together with a set of lines and conics.

J. A. Todd (Cambridge, Eng

Tallini, Giuseppe

Calotte complete di $S_{4,q}$ contenenti due qua ellittiche quali sezioni iperpiane.

Rend. Mat. e Appl. (5) **23** (1964), 108–123.

In a Galois space $S_{r,q}$ (i.e., a projective r-dimensional over a $GF(q)$) a k-cap (k-arc if $r=2$) is a set of k poi three of which are on a line; moreover, a k-cap is s be complete when there is no $(k+1)$-cap containi While the theory of caps is well developed for r [cf. B. Segre, Ann. Mat. Pura Appl. (4) 48 (1959), MR 22 #7054; Acta Arith. 5 (1959), 315–332; M #4980; *Lectures on modern geometry*, Edizioni Crem Rome, 1961; MR 24 #A1045], very little is known it for $r \geq 4$.

The present author determines all the complete of $S_{4,q}$ which are cut along (non-ruled) quadric su by two distinct primes, S' and S'', of $S_{4,q}$. Several arise, according to the parity of q and the possible in which the two quadrics are intersected by the $S' \cap S''$. Thus the existence is proved of complete of $S_{4,q}$ satisfying the condition above, where k take following values: $2q^2+q+5$ (q even and >2), $2q$ $2q^2+1$ (q odd), $2q^2+2q+1$ (q odd and >3), 2 (q even and >2). *B. Segre* (R

Turri, Tullio

Superficie algebriche normali irreducibili invaria un'omografia ciclica.

Rend. Sem. Fac. Sci. Univ. Cagliari **33** (1963), 445

A special cyclic homography of a projective S_n in is referred to, and algebraic curves and surfaces inv under it are considered. The exposition is hazy and often misleading, and so are some hints to Gode work on the subject. *B. Segre* (R

Cofman, Judita

The validity of certain configuration-theorems i Hall planes of order p^{2n}.

Rend. Mat. e Appl. (5) **23** (1964), 22–27.

Sia π il piano di traslazione sopra il quasicorpo di Q, associato al campo di Galois K e al polinomio x^2- (a coefficienti in K e irriducibile su K): se p^n è l'ordi K, Q, e quindi π, ha ordine p^{2n}.

L'autore prova che, per ogni divisore f di p^n-1, esiste in π (almeno) una quaterna di punti a tre a tr allineati, O_1, D_1, A_1, B_1 che verifica la seguente condiz posto $A_{2i} = (D_1 \cup B_{2i-1}) \cap (O_1 \cup A_1)$ e

$$B_{2i+1} = \{[(O_1 \cup D_1) \cap (B_1 \cup A_1)] \cup A_{2i}\} \cap (O_1 \cup$$

($\cup$ = retta congiungente; $\cap$ = punto d'intersezione; 1, 2, $\cdots$) i punti A_1, B_1, B_{2f-1} sono allineati, ment non accade per i punti A_1, B_1, B_{2i-1} se i è minore di

Il risultato conseguito equivale alla validità, relativamente al quadrangolo O_1, D_1, A_1, B_1, proposizione configurazionale C_8^{2f-1}. Nel caso partic

in cui $p \neq 2$ e $f = 2$ si riottiene, così, un risultato, relativo ai piani di Hall, stabilito da H. Neumann [Arch. Math. **6** (1954), 36–40; MR **16**, 739]. *G. Panella* (Rome)

Reiman, Stefano 6385

Su una proprietà dei piani grafici finiti.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **35** (1963), 279–281.

For $n = p^{2m} + p^m + 1$ (p prime) a set of n points and one of n lines is given; two lines have at most one point in common. The author proves by means of the properties of an incidence matrix that for the number N of incidences of a point and a line the inequality $N \leq N_0 = n/p^m + 1$ holds. For $N = N_0$ the system is a finite projective geometry. *O. Bottema* (Delft)

d'Orgeval, B. 6386

A propos des courbes d'ordre $q+2$ qui contiennent tous les points d'un plan de Galois $S_{2,q}$.

Bull. Soc. Roy. Sci. Liège **33** (1964), 264–269.

Dans un plan de Galois $S_{2,q}$, entre les courbes irréductibles qui contiennent tous les points d'un $S_{2,q}$, celles d'ordre minimum sont des C_{q+2}, qui ont été introduites, étudiées et classifiées du point de vue projectif par le rapporteur [Rend. Mat. e Appl. (5) **20** (1961), 431–479; MR **25** #4413]. Chaque point de $S_{2,q}$ est simple pour une telle C_{q+2}; il y a donc une correspondance biunivoque entre les points et les droites de $S_{2,q}$ si l'on associe à chaque point de $S_{2,q}$ la tangente à C_{q+2} en ce point. Par dualité, dite C^{q+2} une enveloppe irréductible d'ordre minimum $q+2$ contenant toutes les droites de $S_{2,q}$, elle détermine une correspondance biunivoque entre les droites et les points de $S_{2,q}$ si l'on associe à chaque droite son point caractéristique (c'est-à-dire la notion duale de droite tangente en un point). L'auteur démontre que chaque C_{q+2} détermine de façon biunivoque une C^{q+2} telle que, pour chaque droite r de $S_{2,q}$, le point caractéristique coïncide avec le point de contact de r avec C^{q+2}. Il démontre aussi que la C^{q+2} est équivalente, à une corrélation près, à la courbe C_{q+2}. Après avoir fixé une polarité (qui néanmoins n'est pas intrinsèquement liée à la courbe C_{q+2}) et avoir dénoté par C'_{q+2} la courbe transformée de C^{q+2} par rapport à cette polarité, l'auteur examine la question de déterminer quand la relation entre C_{q+2} et C'_{q+2} est involutive.

G. Tallini (Rome)

Macková, Božena 6387

Classification of antireciprocity in the complex projective plane. (Slovak. German summary)

Mat.-Fyz. Časopis Sloven. Akad. Vied **14** (1964), 89–103.

Author's summary: "In der vorliegenden Arbeit wird die Klassifizierung der Antireziprozität in der komplexen projektiven Ebene durchgeführt. Die einzelnen Typen sind je nachdem bestimmt, welche Formation die Hauptelemente dieser Transformation bilden. Bei der Untersuchung der Hauptelemente geht man aus folgender Erkenntnis aus: Das Quadrat der Antireziprozität $\mathscr{T}$ ist die Homographie $\mathscr{H}$. Zu jedem invarianten Punkt M der Homographie $\mathscr{H}$ kann man eine solche Gerade m' finden, die auch in $\mathscr{H}$ invariant ist, dass (M, m') Hauptelemente in der Antireziprozität $\mathscr{T}$ sind. Und umgekehrt, wenn die Antireziprozität $\mathscr{T}$ ein Paar von involutorisch entsprechenden Elementen enthält, sind diese Elemente invariant in der Homographie $\mathscr{H}$." *Z. Nádeník* (Prague)

Seidel, J. J.; van Vollenhoven, J. 6388

Mutually congruent conics in a net.

Simon Stevin **37** (1963/64), 20–24.

Etant donnée une conique appartenant à un réseau de coniques, le but des auteurs est de trouver le nombre de coniques du même réseau qui lui sont congrues. Dans le cas général, ce nombre est au plus cinq, c'est-à-dire qu'à toute conique d'un réseau de coniques on peut au plus associer cinq autres du même réseau qui lui sont congrues. Cependant, il y a des réseaux pour lesquels ce nombre peut tendre vers l'infini. *F. Şemin* (Istanbul)

CONVEX SETS AND GEOMETRIC INEQUALITIES

See also 5747, 5791, 6450, 6487.

Böröczky, K. 6389

Über in einem Kreisring enthaltene Polygone.

Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **6** (1963), 113–115.

The author considers plane closed polygons whose sides lie in a fixed circular annulus and which satisfy the following condition: All except at most two of the vertices lie on the outer bounding circle, and if there are two exceptional ones they must be neighbouring vertices. He shows that the minimum area occurs for one or the other of two characterized positions.

H. T. Croft (Cambridge, England)

Kosmák, Ladislav 6390

A remark on Helly's theorem. (Czech. Russian and English summaries)

Spisy Přírod. Fak. Univ. Brno **1963**, 223–225.

The following statement is often called "Radon's theorem": "If V is a subset of affine n-space containing $n+2$ points, there exist sets $V_1, V_2 \subset V$ such that $V_1 \cap V_2 = \varnothing$ but $\operatorname{conv} V_1 \cap \operatorname{conv} V_2 \neq \varnothing$ (where $\operatorname{conv} A$ denotes the convex hull of A). The subsets V_i of V are uniquely determined if and only if every $n+1$ points of V are affinely independent" [R. Rado, J. London Math. Soc. **27** (1952), 320–328; MR **13**, 970; I. V. Proskurjakov, Uspehi Mat. Nauk **14** (1959), no. 1 (85), 219–222; MR **20** #6681]. In the present paper a new proof is given for the following addendum to Radon's theorem (first established by Proskurjakov, loc. cit.): If every $n+1$ points of V are affinely independent, two points of V belong to the same V_i if and only if they are separated by the $(n-1)$-dimensional affine variety determined by the remaining n points of V.

{Reviewer's remark: It is much easier to establish Proskurjakov's result algebraically. If $V = \{v_0, \cdots, v_{n+1}\}$, there exist reals $\alpha_0, \cdots, \alpha_{n+1}$, not all 0, such that $\sum \alpha_i = 0$, $\sum \alpha_i v_i = 0$. In case of unique V_i's the α_i's are determined up to a constant factor, and v_i's having α_i's of the same sign belong to the same V_i. Now, if $\{x \mid (x; u) = \beta\}$ is a hyperplane containing all points of V except v_j and v_k, then $\alpha_j[(v_j; u) - \beta] + \alpha_k[(v_k; u) - \beta] = 0$, which implies Proskurjakov's result.} *B. Grünbaum* (Jerusalem)

Ceder, Jack G. 6391

A property of planar convex bodies.

Israel J. Math. 1 (1963), 248–253.

The author proves that for every closed convex curve C in the plane and for any point O in the interior of C there exist points A_1, A_2, A_3 of C such that the vectors $\overrightarrow{OA_1}+\overrightarrow{OA_2}$, $\overrightarrow{OA_2}+\overrightarrow{OA_3}$, $\overrightarrow{OA_3}+\overrightarrow{OA_1}$ have terminal points on C. The six points thus determined on C are the vertices of a centrally symmetric hexagon inscribed in C. Denoting by V the set of all points of C which are vertices of such inscribed hexagons, the author shows that the complement N of V in C consists of isolated points provided C is rotund and smooth. The vertices of a triangle show that, for general C, $N\neq\varnothing$ is possible. As open problems the author mentions: (1) Is $V=C$ for rotund and smooth C? (2) Is N countable for all C?

{Reviewer's remark. The answer to (1) is affirmative. Simple continuity arguments show that, for smooth and rotund C, among all parallelograms inscribed into C and having a given point of C as vertex, there are at least two different parallelograms with coinciding centers. Their vertices determine a centrally symmetric hexagon inscribed in C, which has the given point as one of the vertices.} *B. Grünbaum* (Jerusalem)

Rényi, A.; Sulanke, R. 6392

Über die konvexe Hülle von n zufällig gewählten Punkten. II.

Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 2, 138–147 (1964).

If K is a bounded convex set in the plane and P_i, $1\leq i\leq n$, are n points in K chosen independently and with uniform distribution in K, the authors denote the convex hull of the n points by H_n. In a previous note [same Z. 2 (1963), 75–84; MR 27 #6190] the authors considered the number of sides in the polygon making up the boundary of H_n. In the present note they consider the area F_n of H_n and the length L_n of the boundary of H_n. It is clear that F_n and L_n are random variables whose expectations $E(F_n)$, $E(L_n)$ approach, as $n\to\infty$, the area F and length of the boundary L of the set K. They investigate the behaviour of the differences $F-E(F_n)$ and $L-E(L_n)$ as $n\to\infty$. Different asymptotic formulae are derived for two cases: (a) K has a continuously differentiable boundary with a bounded curvature; (b) K is a square.

S. J. Taylor (London)

Bleicher, M. N.; Fejes Tóth, L. 6393

Circle-packings and circle-coverings on a cylinder.

Michigan Math. J. 11 (1964), 337–341.

It has long been known that the closest packing and thinnest covering of the whole Euclidean plane by equal circles are provided by the incircles and circumcircles (respectively) of the hexagons in the regular tessellation {6, 3}. The authors consider how these arrangements have to be modified when the whole plane is replaced by the strip between two parallel lines, with orthogonally corresponding points on them identified. If the distance between the lines is p times the common radius of the circles, the regular arrangement will, in general, conflict with this identification, and some modification will be required. The authors have determined the cases when this necessary modification is most drastic. The con-clusions are as follows. The maximal packing dens… at least $\pi/4$, and this minimum is attained when $p=$… the minimal covering density is at most $\pi/2$, and… maximum is attained when $p=2$.

H. S. M. Coxeter (Boca Raton,…

Conway, J. H.; Croft, H. T.

Covering a sphere with congruent great-circle arcs…

Proc. Cambridge Philos. Soc. 60 (1964), 787–800.

The authors study the problem of expressing the su… of a sphere in Euclidean 3-space as the union of… overlapping congruent great-circle arcs. The term "… gruent" is taken to mean equal in length and topologi… equivalent, i.e., they consider three distinct cases: … the arcs contain neither endpoint, when they co… both endpoints, and when they contain precisely… endpoint. The three problems are denoted S1 (half o… S2 (closed), and S3 (open). Analogously, they denot… related plane problems involving line-segments by… P2, P3. The authors prove that P1, P2, S1 are pos… whereas P3, S3 are impossible. S2 remains an… question. The paper is almost entirely devoted to… proof of S3. *F. Supnick* (New Y…

Baranovskiĭ, E. P.

On packing n-dimensional Euclidean spaces by … spheres. I. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika 1964, no. 2… 14–24.

The author obtains an upper bound D_n for the densi… packings of equal n-dimensional spheres. D_n is expre… in terms of the volume of a certain simplex and, to… extent, is equivalent to the work of C. A. Rogers [… London Math. Soc. (3) 8 (1958), 609–620; MR 21 #… although there are differences in the methods used.… $n=3$ an explicit calculation is made giving

$$D_3 = 2^{1/2}(3\cos^{-1}(\tfrac{1}{3})-\pi) = 0.77963\ldots$$

in agreement with Rogers. No asymptotic evaluatio… large n is made as was done by H. E. Daniels in the p… of Rogers. *R. A. Rankin* (Glas…

DIFFERENTIAL GEOMETRY

See also 5961, 5962, 5984, 6065, 6134, 6172, 6274, 6275, 6503.

Šulikovskiĭ, V. I. [Шуликовский, В. И.]

★**Classical differential geometry in tensor form [К…сическая дифференциальная геометрия в тензор… изложении].**

Edited by A. P. Norden.

Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1… 540 pp. 1.44 r.

This book differs somewhat from the ordinary "clas… differential geometry" in so far as it is based on the th… of curve nets and congruences, which are analysed… the aid of vector and tensor fields. This is an approac… which the author, professor in Kazan, has himself… tributed in several papers. He has divided the book…

a part on two-dimensional and another part on three-dimensional space. The first chapter of the first part begins immediately with vector fields and nets on an X_2, where X_2 is a two-dimensional manifold, where the bivectors ε_{ij}, $i, j, s = 1, 2$; $\varepsilon_{12} = \varepsilon$ and ε^{ij}, $\varepsilon^{12} = 1/\varepsilon$ serve to pass from covariant to contravariant: $T_k\varepsilon^{sk} = T^s$, $T^s\varepsilon_{ks} = T_k$. A set of curves is then defined by $a_{ks}x^k x^s = 0$. Among the important concepts here is that of apolar nets a_{ks}, b_{ks} if $a^{ks}b_{ks} = 0$. The theory is then applied to Riemannian V_2 and affine A_2 (geodesic lines, curvature tensor, etc.) with special chapters on diagonal nets and other special nets. Among them are isothermal nets on V_2, in which $ds^2 = e^{-2\sigma}(du^2 + dv^2)$; if $e^{-2\sigma} = [U(u) + V(v)]^m$, then we have L_m-nets. Dubnov nets on an A_2 are such that their Čebyšev vector a_i satisfies the equation $a_{(i|s)} - a_i a_s = R_{(is)}$, where $R_{(is)}$ is the Ricci tensor. When $a_i = 0$ we have Čebyšev nets. There is a chapter on Lie groups and automorphisms of surfaces, and the chapter ends with so-called Π-families of nets, which are manifolds of nets with the same apolar net and equal Čebyšev vector.

The second part of the book is called "Surfaces and congruences of three-dimensional space" and begins with a fairly "classical" surface theory (in tensor form), though already here we meet the formulas of Lelieuvre and Moutard. A chapter is devoted to special classes of surfaces; among them are rotation, spiral, ruled, translation, minimal, evolute and W-surfaces, surfaces of Voss, and cyclides, with several aspects of these surfaces usually not found in other books. Then follows a chapter on line congruences, again with several special examples, and a chapter on congruences of spheres and circles. The book ends with infinitesimal bending and W-congruences, to which are added some special bendings of surfaces.

The material, thus presented from a central point of view, is based, to a great extent, on the work of Russian authors (Blank, Dubnov, Efimov, Finikov, Liber, Norden, Čahtauri, among others).

The book can be especially recommended to those who have to teach a course in "classical" differential geometry, and want a change in subject matter from the customary program.

The book, writes the author, has been written for scientific workers, aspirants and students of advanced courses in the universities and "pedinstitutes", who specialize in geometry. *D. J. Struik* (Belmont, Mass.)

Upadhyay, M. D. 6397

Director surfaces with orthogonal correspondence.

Tensor (*N.S.*) 15 (1964), 228–232.

Let S be the surface of reference of a congruence. A family of curves on S together with the lines of the congruence determines a family of ruled surfaces of the congruence, which in turn defines a family of lines of striction forming a surface $\bar{S}$. It seems that the author intends to find a condition for orthogonality between corresponding curves at corresponding points x^i on S and y^i on $\bar{S}$. The equation of $\bar{S}$ is given by (1.1): $y^i = x^i + t\lambda^i$, where t represents the distance from x^i to the central point y^i on the line of the congruence passing through x^i with direction cosines λ^i. It is evident that in general t is not a constant. Hence equation (3.2) $y^i{}_{,\alpha} = x^i{}_{,\alpha} + t\lambda^i{}_{,\alpha}$, cannot be obtained from (1.1). Since (3.2) plays an important role and is wrong, the paper needs revision.

T. K. Pan (Norman, Okla.)

Mihaĭlovskiĭ, V. I. 6398

Infinitesimal bendings of piecewise regular surfaces of revolution of negative curvature. (Russian)

Ukrain. Mat. Ž. 14 (1962), 422–426.

The author extends results proved by him previously for smooth surfaces of revolution [same Ž. 14 (1962), 18–29; MR 26 #4298] to surfaces of revolution $r = f(z)$ $(\alpha \leq z \leq \beta)$ of negative curvature, where $f(z)$ is piecewise of class C^3. The bending field is assumed to be continuous and of class C^2 where the surface is of class C^3. The results are the following. For each z_0 there is a countable dense set of z_1 such that the zone from z_0 to z_1 is not rigid. There is a countable dense set of $h_1 > 0$ such that the surface $r + h_1 = f(z)$ $(\alpha \leq z < \beta)$ is not rigid.

H. Busemann (Los Angeles, Calif.)

Pérez de Madrid, Aquilino 6399

Coordinatized differential of a curve. (Spanish)

Gac. Mat. (*Madrid*) (1) 16 (1964), 88–96.

Vincensini, Paul 6400

Sur les déformations équivalentes infinitésimales des surfaces.

Univ. Nac. Tucumán Rev. Ser. A 14 (1962), 177–188.

Cet article est consacré à l'étude des transformations infinitésimales équivalentes des surfaces (transformations infinitésimales qui conservent les aires). Pour ce faire, on considère des congruences rectilignes dont les rayons portent les directions des déplacements infinitésimaux des différents points de la surface S envisagée. Le cas où ces déplacements infinitésimaux sont tangents à S (auto-transformations) est pris en considération quelle que soit S. Dans une transformation infinitésimale équivalente, le champ des déplacements ne peut être normal à S, à moins que cette dernière soit minima. En vue de donner une idée sur ce qui a été obtenu par l'auteur, citons en particulier ce qui suit: Dans le cas où S est une sphère de centre O, la déformation infinitésimale équivalente la plus générale de S est fournie par la donnée d'une congruence rectiligne quelconque, dont la représentation sphérique est faite sur S. Si par le point représentatif M d'un rayon D, on mène le vecteur équipollent à $\overrightarrow{OI}$, I étant le milieu du segment focal de D, le déplacement infinitésimal de M est donné par $\delta\overrightarrow{M} = \varepsilon \cdot \overrightarrow{OI}$, où ε est une constante infinitésimale, dont les puissances supérieures à la première sont négligées. Pour les auto-transformations équivalentes de la sphère, l'enveloppée moyenne des congruences rectilignes (D) est le point fixe O. L'article se termine par l'application des transformations infinitésimales équivalentes aux surfaces harmoniques. *F. Şemin* (Istanbul)

Bishop, Richard L.; Crittenden, Richard J. 6401

★Geometry of manifolds.

Pure and Applied Mathematics, Vol. XV.

Academic Press, New York-London, 1964. ix + 273 pp. $10.50.

This book represents an excellent treatment of a wide section of modern differential geometry. As the authors acknowledge in the preface it owes much to the distinguished school of differential geometry at M.I.T. and,

in particular, to W. Ambrose and I. Singer, who in turn draw upon the pioneering work of Cartan, Chern, and Ehresmann. The style is elegant and at the same time considerate for the needs of a beginner. Many comments are given to related questions as well as a great number of well chosen problems with pertinent references whenever desirable. This makes it possible to cover in considerable detail the foundations of the field as there are Manifolds, Lie groups, Fibre bundles, Differential forms, Connexions, Affine connexions, Riemannian manifolds. The last 113 pages of the 257-page text are devoted to a study of the global structure of Riemannian manifolds: Geodesics and complete Riemannian manifolds, Riemannian curvature, Immersions and second fundamental form, Second variation of arc length. In this part, in particular, there are many subjects which appear for the first time in a textbook or even anywhere else. The authors seem to have been quite successful in avoiding serious errors without doing it the easy way by glossing over the details. In summa, the reviewer thinks that anybody who chooses to base his course on differential geometry at the graduate level on this book could not do better. *W. Klingenberg* (Mainz)

Hahn, Kyong T. 6402

Minimum problems of Plateau type in the Bergman metric space.

Pacific J. Math. 14 (1964), 943–955.

Let D be a bounded domain in the space C^n of n complex variables, let K_D be the Bergman kernel and ds_D the Bergman metric, $ds_D{}^2(z) = K_D(z, \bar{z})|dz|^2$, invariant under pseudo-conformal mappings. Then D can be metrized by d, where d is the geodesic distance determined by ds_D. Let $Q_0 = [0, 1] \times [0, 1]$. If $G: Q_0 \to C^2$ is piecewise of class C', then the invariant B-area of the surface S described by G is $B_D(S) = \iint_{Q_0} [K_D(G, \bar{G})]^{1/2} |\partial(G_1, G_2)/\partial(u_1, u_2)| \, du dv$. If Γ is a simple closed curve in D and if C' is the class of surfaces bounded by Γ each of which is given by a representation which, together with its derivative, is Lipschitzian, then C' contains a surface of minimal area if C' is not empty. If $n = 1$ and D is bounded by simple closed curves $b_1, \cdots, b_n$, where b_n is the outer boundary component, then there exists a curve of smallest length in D which is homotopic to b_1. There are also distortion theorems for both length and area and results concerning minimal closed surfaces. *E. Silverman* (Lafayette, Ind.)

Grincevičjus, K. I. [Grincevičius, K. I.] 6403

On a complex of correlative elements. (Russian. Lithuanian and German summaries)

Litovsk. Mat. Sb. 4 (1964), 329–335.

Author's summary: "Ein korrelatives Element wird von einer Gerade A_1A_2 und einer Klasse

$$e^{2\sigma}|\det\|u_{ij}\|| = \left\{\begin{vmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{vmatrix}\right\}^2$$

der Korrelationen

$$u_{ij}x^iy^j = 0, \qquad U = \det\|u_{ij}\| \neq 0,$$

gebildet. Im dreidimensionalen Punktraume hängt die Gesamtheit der korrelativen Elemente von fünf Parametern ab. Mit 'Komplex der korrelativen Elemente' verstehen wir eine vierdimensionale Gesamtheit derer
In dem Artikel wird die Differentialumgebung w
Ordnung eines Komplexes der korrelativen Elem
untersucht. Ein Strahlenkomplex ist ein beson
Komplex der korrelativer Elemente und kommt dah
vorliegenden Artikel nicht in Betracht."

Grincevičjus, K. I. [Grincevičius, K. I.]

A problem on pairs of complexes. (Russian. L
anian and German summaries)

Litovsk. Mat. Sb. 4 (1964), 37–40.

The paper deals with pairs of complexes possessi
property analogous to that of stratifiable pairs of
gruences. The author derives a Pfaffian system d
mining such pairs. The solution depends on one arbi
function of three variables. If one of the complex
given, the other complex of the pair depends on e
constants. *A. Goetz* (Wro

Oniščuk, N. M.

**Bianchi's problem in equi-affine geometry for a
consisting of a congruence and a surface.** (Russi

Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. 168 (1
150–160.

The author solves the following problem: To find a
M consisting of a congruence (K) and an interse
surface (P) such that in every tangent plane of (P) a
T can be chosen in such a way that the congruence
and (K) are stratifiable in both directions. The au
applies the semi-canonical frame introduced in a fo
paper [same Trudy 160 (1962), 107–114; MR 27 #5
A more detailed summary of the results would b
technical for a short review. *A. Goetz* (Wro

Malahovskiĭ, V. S.

**Congruences of second-order curves with undeterm
focal families.** (Russian)

Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. Geom
160 (1962), 5–14.

The starting point is the general theory of congru
of second order in projective three-space given by S
Tuganov [Dokl. Akad. Nauk SSSR 100 (1955), 1
MR 16, 1050]. In the present paper the special ca
studied in which two contiguous curves of the
gruence intersect each other under arbitrary disp
ment in two different or coinciding points. For
congruences, named C_0, the foci of the conics and
focal families are not determined, so that Tuga
canonical reference system becomes illusory and
necessary to construct a new canonical reference sy
It is shown how this can be done, and how such a sy
can be geometrically characterized. Among the theo
that hold for congruences C_0 we mention that a C_0
two families of principal focal lines and two prin
focal surfaces; if these do not coincide, the congruen
determined by an arbitrary function of two vari
(focal points are the points of intersection of two
tiguous conics). An example of a congruence C
euclidean space is the congruence of great circles
sphere. Two special types of congruences C_0 are
investigated. *D. J. Struik* (Belmont, M

Malahovskiĭ, V. S. 6407

Manifolds of algebraic elements in an n-dimensional projective space. (Russian)

Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. **168** (1963), 28–42.

An $(n-2)$-dimensional non-degenerate hypersurface of degree k (k even and positive) of the hyperplane P_{n-1} of the n-dimensional projective space P_n is said to be an algebraic element A^k_{n-2}. Let $(h, m, n)^k$ denote an m-dimensional manifold of the algebraic elements $A^k_{n-2} \subset P_n$, the set of all hyperplanes of P_n containing algebraic elements of the given manifold being h-parametric ($h \leq m$, $h \leq n$, $1 \leq m < C^n_{n+k-1} + n - 1$, $n \geq 3$).

The projective differential geometry of the manifolds $(h, h, n)^k$ and $(n, n, n)^k$ is developed. Special theorems concerning the manifolds $(n, n, n)^2$ are obtained.

A. Urban (Prague)

Malahovskiĭ, V. S. 6408

Non-degenerate congruences of second-order curves in a three-dimensional projective space. (Russian)

Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. **168** (1963), 43–53.

Using the notation of focal surfaces of congruences of conics, the author develops the projective theory of non-degenerate congruences of conics (i.e., congruences with at least two non-degenerate focal surfaces satisfying some additional conditions) in a three-dimensional projective space. The author establishes the interior fundamental object of the congruence (in the sense of G. F. Laptev), constructs a canonical frame, introduces some relative and absolute invariants and derives some results concerning special classes of congruences of conics (I-, I_1- and C_3-congruences). *A. Urban* (Prague)

Malahovskiĭ, V. S. 6409

Congruences of second-order curves with one focal surface which degenerates into a point. (Russian)

Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. **168** (1963), 54–60.

This paper is devoted to A-congruences of conics in a three-dimensional projective space, i.e., to non-degenerate congruences of conics with at least two non-degenerate focal surfaces [see #6408 above], one further focal surface of which degenerates into a point. Special classes of A-congruences are introduced and studied.

A. Urban (Prague)

Malahovskiĭ, V. S. 6410

Congruences of second-order curves whose planes form a one-parameter family. (Russian)

Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. **168** (1963), 61–65.

The present paper deals with congruences of conics in a three-dimensional projective space whose planes form a one-parameter family. The author studies the non-parabolic case (the characteristic lines of the planes of conics are not tangent to the conics of the congruence) under the following restrictive assumption. The intersecting points of the conic C of the congruence with the characteristic line of its plane are not conjugated with respect to the conic C^* determined by the four focal points of the conic C and the pole of the characteristic line with respect to the same conic C. The congruences under investigation are called T-congruences of conics.

A. Urban (Prague)

Šul'man, T. A. 6411

Invariant nets on hypersurfaces in a four-dimensional projective space. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 5 (42), 137–142.

The present paper deals with asymptotic and conjugate curves and nets of the first and second order on hypersurfaces in a four-dimensional projective space. By asymptotic and conjugate curves of the first order are meant the usual ones. Asymptotic [conjugate] curves of the second order are defined—if the conventional notation and choice of the moving coordinate frame is applied—by the conditions $(A_0A_1A_2A_3d^2A_0)=(A_0A_1A_2A_3d^3A_0)=0$ $[(A_0A_1A_2A_3d_2d_1A_0)=(A_0A_1A_2A_3d_3d_2d_1A_0)=0]$.

A. Urban (Prague)

Ancochea, G. 6412

Sur la représentation des éléments différentiels de l'espace projectif.

Ann. Mat. Pura Appl. (4) **65** (1964), 341–359.

L'auteur présente un exposé d'ensemble de la représentation des éléments différentiels de l'espace projectif en la traitant dans le cadre de la théorie des espaces algébriques homogènes. Il donne d'abord la définition d'un élément différentiel de l'espace projectif $P(n, K)$ de dimension n sur le corps K en s'appuyant sur le concept de point proche introduit par A. Weil; la représentation des éléments différentiels se fait au moyen de variétés algébriques. Quelques exemples traités dans le cas du corps des réels permettent d'illustrer les particularités qui peuvent se présenter dans les cas généraux. L'auteur retrouve, simplement, tous les résultats déjà connus et il en établit de nouveaux, relatifs, en particulier, aux éléments d'ordre trois du plan et aux éléments superficiels d'ordre deux de l'espace à trois dimensions.

M. Decuyper (Lille)

Bompiani, Enrico 6413

Reti e tritessuti di curve nel piano proiettivo.

Rend. Circ. Mat. Palermo (2) **12** (1963), 299–329.

L'auteur commence par montrer comment l'exposition de la théorie des réseaux plans, telle qu'elle a été faite par E. Čech au Chapitre X de l'ouvrage qu'il a rédigé en collaboration avec G. Fubini [*Introduction à la géométrie projective différentielle des surfaces*, Gauthier-Villars, Paris, 1931], et divers travaux récents s'y rapportant, gagnent à être présentés à la faveur de certaines de ses propres recherches [l'auteur, Atti Accad. Naz. Lincei Rend. (6) **22** (1935), 483–491; ibid. (6) **24** (1936), 323–332]. Non seulement la plupart des résultats connus reçoivent ainsi une exposition nouvelle, mais d'importants résultats inédits se trouvent établis, conférant au sujet étudié une ampleur de développement particulièrement suggestive. Dès le début l'introduction de la notion de courbure projective d'un E^2 (élément différentiel curviligne du 2e ordre) par rapport à un réseau plan conduit à la définition du système des courbes du plan à courbure projective nulle

relativement au réseau, et à la recherche des connexions symétriques conférant à ces dernières courbes le caractère auto-parallèle. Cette recherche conduit, en particulier, à un type de connexions complètement individualisées par le réseau envisagé, moyennant lesquelles peuvent être assignées les significations géométriques des coefficients des dérivées du 1^er ordre dans le système différentiel du 2^e ordre (mis sous forme normale) vérifié par les coordonnées homogènes du point courant du réseau. L'auteur reprend ensuite l'étude des correspondances entre deux plans π, $\bar\pi$. Il montre qu'étant donnée une transformation ponctuelle $\bar x = Tx$ de π en $\bar\pi$, il existe ∞^2 homographies Ω équivalentes à T quant à la transformation des voisinages du 1^er ordre des points x et $\bar x$, et que pour toutes ces homographies il existe trois pinceaux de E_2^2 (éléments curvilignes différentiels du 2^e ordre de centre x et de même tangente en x) tels que $TE_2^2 = \Omega E_2^2$, les tangentes aux trois pinceaux étant celles du 3-tissu caractéristique (dans π ou $\bar\pi$) de la correspondance. Il montre comment cette étude peut être approfondie et prolongée par celle de la correspondance entre les courbes de deux réseaux donnés dans π et $\bar\pi$, correspondance à laquelle est subordonnée une correspondance entre les plans (les ouverts) eux-mêmes. Si, indépendamment de toute correspondance entre π et $\bar\pi$, un réseau est donné dans π, divers 3-tissus peuvent lui être associés. Un premier procédé pour en obtenir consiste à normaliser les coordonnées projectives dans π, à considérer la correspondance entre π et un plan affine déduite de la normalisation, et à prendre les 3-tissus (dépendant de deux fonctions arbitraires d'un argument) caractéristiques de cette correspondance. Un autre procédé est basé sur la considération de la correspondance entre un voisinage d'un point x de π et son transformé dans une transformation de Laplace. Ce type de correspondance ne semble pas encore avoir été étudié; l'auteur pousse l'étude jusqu'aux voisinages du 2^e ordre, et introduit ainsi l'intéressante notion de 3-tissus $\mathscr{L}$-caractéristiques d'un réseau (définis par les transformés de Laplace du réseau et par suite intrinsèquement liés à ce dernier). Le mémoire se termine par la considération de certains invariants d'un réseau ou d'un 3-tissu, en relation avec la déformation projective du réseau ou du 3-tissu, et des significations géométriques sont données pour ces divers éléments. L'auteur montre en particulier que, pour un 3-tissu, il existe un invariant infinitésimal homogène, du 1^er ordre par rapport aux différentielles 1^ères, se présentant comme le rapport d'une forme quadratique et d'une forme (monôme) linéaire, et pouvant être pris comme définition de l'élément linéaire du 3-tissu.

P. Vincensini (Marseille)

Černylenko, V. M. 6414

Spaces with a special complex of geodesic curves. (Russian)

Trudy Sem. Vektor. Tenzor. Anal. **11** (1961), 253–268.

An affine manifold A_n is called almost-projective if it is possible to map it on a euclidean E_n in such a way that a complex of geodesic lines (i.e., a family of $2n-3$ parameters) remains geodesic. This complex is called the basic complex. This complex must either be linear or quadratic, i.e., the lines which pass through a point must either lie in a plane or form a quadratic cone (compare the author's paper [Naučn. Zap. Dnepropetrovsk. Univ. **55** (1961), 105–110]). In the first case the A_n is called of the first type, in the second case, of the second A necessary and sufficient condition that the A_n be second type is that there exists a coordinate systen that in it Γ^i_{jk} can be written

$$\Gamma^i_{jk} = \xi^i a_{jk} + \delta^i_j\varphi_k + \delta^i_k\varphi_j,$$

where $a_{jk} = a_{kj}$ determines the quadratic cone. Wh space is also Riemannian, $\partial g_{ij}/\partial u^k = \xi_i a_{jk} + \xi_j a_{ik} + 2$ $\varphi_i g_{jk} + \varphi_j g_{ik}$, where g_{ij} is the fundamental tensor, ξ_i For a_{ij} to be $= g_{ij}$ it is necessary and sufficient that be conformally euclidean. Conditions are derived f g_{ij} in the case that the basic complex is mapped complex of tangents to a non-degenerate quadric of t

D. J. Struik (Belmont,

Hasin, G. B.

Successive Laplace transformations with stratifie gruences of the axes. (Russian)

Dokl. Akad. Nauk SSSR **145** (1962), 1235–1238.

Let M_0 be a net of conjugate lines in projective space and M_1 its first Laplace transform. Then, acc to S. P. Finikov [*Theory of congruences* (Russian), GITTL, Moscow, 1950; MR **12**, 744], the line conn the first transforms of the net is called its second and the line in which the tangent planes to th transforms of the net intersect is called the first a this net. Conditions are set up for a net M_0 such th congruence of the first axes is stratified with the gruences of the first axes of M_1 and, at the same tim congruences of the second axes of M_0 and M_1 are stra It is found that if the congruences of the first a M_0 and M_1 are stratified, and the congruences second axes are stratified on one side only, the congruences of the second axes of the nets M_0 an are also stratified on the other side. The solution and depends on four arbitrary functions of one vari

D. J. Struik (Belmont,

Mihăilescu, Tiberiu

Problèmes variationnels projectifs dans la théori réseaux plans.

Bull. Sci. Math. (2) **86** (1962), 1ère partie, 5–48.

La première partie de ce mémoire contient la théor réseaux plans, développée par la méthode du mobile, de la manière selon laquelle l'auteur l'a appl dans le traité [*Projective differential geometry* (Roma Acad. R. P. Romîne, Bucharest, 1958; MR **22** #29 l'étude des surfaces et des réseaux. Il retrouve ain résultats connus et obtient des nouveaux résultats, su en ce qui concerne quelques classes de réseaux plans

On considère le repère mobile (A_0, A_1, A_2, I), de manière que le sommet A_0 décrive le réseau respectif les deux arêtes $[A_0, A_1]$, $[A_0, A_2]$ soient les tangente courbes t_1, t_2 du réseau. Les deux formes de Pfaff ω_0 sont linéairement indépendantes par rapport aux rentielles dt_1, dt_2, pouvant s'écrire

$$\omega_{12} = a_2\omega_{01} + a_2'\omega_{02},$$

$$\omega_{21} = b_2'\omega_{01} + b_2\omega_{02}.$$

La famille de ces repères (R_0) possède seulement

invariants infinitésimaux, qui sont des fonctions des deux invariants fondamentaux de Bompiani

$$\varphi_1 = \frac{a_2{\omega_{01}}^3}{\omega_{02}}, \qquad \varphi_2 = \frac{b_2{\omega_{02}}^3}{\omega_{01}}.$$

Moyennant ces deux invariants on définit ensuite les lignes de Darboux, les lignes de Segre, ainsi que les réseaux simplement réglés et les réseaux doublement réglés. A l'aide des deux transformés de Laplace A_1, A_2 du réseau A_0, on étudie certains couples de coniques et les réseaux de Königs, pour lesquels les deux coniques coïncident.

Dans la deuxième partie du mémoire sont envisagés des problèmes variationnels, concernant les réseaux plans et appartenant au type étudié dans le traité cité ci-dessus. On considère l'invariant infinitésimal

$$\Phi = \frac{\gamma_1{\varphi_1}^{2n+1} + \gamma_2{\varphi_2}^{2n+1}}{{\varphi_1}^n{\varphi_2}^n}$$

et puis l'intégrale à extrémités fixes $I = \int_{t_1}^{t_2} \Phi$.

P. Drăgilă (Timişoara)

Norden, A. P. 6417

Cartesian composition spaces. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1963**, no. 4 (35), 117–128.

Given r basic manifolds X_{n_a}, $a, b = 1, \cdots, r$, and in each of them a point M_a is selected. The composition of the X_{n_a} is such a space X_N whose points M are in 1-1 correspondence with all possible sets (M_a), $N = n_1 + n_2 + \cdots + n_r$. If all points of the set (M_a) except M_b are fixed, then the corresponding point M of X_N describes a submanifold of n_b dimensions in 1-1 correspondence with the basic X_{n_b}, and this submanifold is called the position of X_{n_b}. Two different positions of one basic manifold are called analogue, the positions of two different basic manifolds are called transversal; two analogue positions have no point in common, two transversal ones have a unique common point.

A system of curvilinear coordinates in the space of composition X_N is adapted with respect to the composition when it is defined by a system of curvilinear coordinates in every basic manifold. The place of a point in the position of an X_{n_a} determines n_a adapted coordinates u^i and $N - n_a$ adapted coordinates $u^{\bar{i}}$, called the internal and external coordinates with respect to the given position. The i and $\bar{i}$ are in opposition. The composition is called Cartesian if the tangent elements of an arbitrary family of analogue positions form a field of absolute parallelism. Then an arbitrary position will be an entirely geodesic X_N. Now a symmetrical connection $\Gamma^\gamma_{\alpha\beta}$, $\alpha, \beta, \gamma = 1, \cdots, N$, is introduced. A necessary and sufficient condition that a given composition is Cartesian is that the Γ vanish if and only if two indices are in opposition. An example is the composition of two X_1 to a set of X_2. If this set is Cartesian, then the position of the basic X_1 coincides with the isotropic lines of a Weyl space W_2 ($g_{11} = g_{22} = 0$, $g_{12} = 1$, $\nabla_\gamma g_{\alpha\beta} = 2\omega_\gamma g_{\alpha\beta}$, $\omega_1 = -\frac{1}{2}\Gamma^1_{11}$, $\omega_2 = -\frac{1}{2}\Gamma^2_{22}$). Another example is a Γ_{n+m}, allowing a Cartesian composition of an X_n and an X_m, with fully orthogonal position. Thus we can write $ds^2 = g_{ij}(u^k)du^i du^j + g_{\bar{i}\bar{j}}(u^{\bar{k}})du^{\bar{i}} du^{\bar{j}}$. It is also possible to apply these notions to surfaces and hypersurfaces in projective space. A special study is devoted to the autocomposition of a hyperquadric, which leads to tensors of the type of Širokov and Raševskiĭ [P. K. Raševskiĭ, Trudy Sem. Vektor. Tenzor. Anal. **6** (1948), 225–248; MR **15**, 62].

D. J. Struik (Belmont, Mass.)

Skopec, Z. A. 6418

Mapping of a 4-dimensional space onto a 2-dimensional plane by means of a norm-curve. (Russian)

Trudy Sem. Vektor. Tenzor. Anal. **12** (1963), 443–450.

Generalization of the investigation of the mapping of projective three-space on a plane by means of a norm-curve of the third degree [Izv. Vysš. Učebn. Zaved. Matematika **1961**, no. 6 (25), 97–107; MR **26** #646] to the mapping of a four-dimensional space on three- and two-dimensional space by means of norm-curves of the fourth and third degree.

D. J. Struik (Belmont, Mass.)

Marcus, F. 6419

Le congruenze di rette a falde focali in corrispondenza proiettivo-simile.

Univ. e Politec. Torino Rend. Sem. Mat. **22** (1962/63), 181–208 (1964).

Studio delle congruenze E di rette dell' S_3 che realizzano sulle loro falde focali una corrispondenza tale che risulti $\sigma = k\bar{\sigma}$ ove k è una costante reale ($k^2 \neq 1$), e σ, $\bar{\sigma}$ sono gli elementi lineari di Fubini. Le falde focali delle E dipendono in generale da funzioni ellittiche, e vengono elencate quelle particolari E che si determinano con funzioni elementari. Per le congruenze E (che rientrano tra le congruenze R) vengono indicate numerose proprietà, che non è possibile qui elencare.

D. Gallarati (Genoa)

Marcus, F. 6420

Sur les déformations infinitésimales projectives similaires des surfaces.

Univ. Nac. Tucumán Rev. Ser. A **14** (1962), 353–367.

En prolongement d'un travail antérieur sur les translations projectives d'une surface en elle-même [Acad. R. P. Romîne Fil. Iaşi Stud. Cerc. Şti. Mat. **12** (1961), 69–94; MR **25** #505], l'auteur traite ici du problème de la recherche effective (effectuée sans doute pour la première fois) de surfaces admettant des déformations infinitésimales projectives similaires (p.s.), la similarité d'une correspondance projective entre deux surfaces étant caractérisée par la proportionnalité, avec rapport constant, des éléments linéaires projectifs de Fubini des deux surfaces. Les transformations simplement projectives d'une surface conservant les asymptotiques, l'auteur est naturellement conduit à envisager deux classes de surfaces admettant un groupe G_1 de déformations (p.s.) en elles-mêmes, dites respectivement d'espèce A^* ou B^* suivant que les trajectoires du groupe constituent ou non un système d'asymptotiques. Par l'emploi du système complètement intégrable déterminant, à une projectivité près, une surface (x) rapportée à des coordonnées homogènes normalisées au sens de Wilczyński, il est tout d'abord montré que les déformations des surfaces admettant un groupe G_1 ou G_2 de déformations (p.s.) conservent la normalisation de Wilczyński à des infiniment petits du $2^{\text{ème}}$ ordre près, puis sont établies un certain nombre de propriétés où interviennent les translations projectivo-similaires (introduites par l'auteur comme généralisation des translations projectives), les lignes de Darboux, les surfaces de coïncidence, les pangéodésiques, les lignes de

Segre et les congruences W. Vient ensuite la détermination des surfaces admettant ∞^1 déformations (p.s.). Pour les surfaces d'espèce A^*, après la mise en équations du problème, l'auteur explicite quelques familles de telles surfaces dépendant de trois constantes arbitraires. Et pour les surfaces d'espèce B^*, dont la recherche s'avère un problème nettement plus difficile, il parvient, en imposant aux trajectoires d'appartenir à certaines familles spéciales de courbes, à en déterminer des classes remarquables. Ainsi, si l'on impose aux trajectoires d'être des pangéodésiques et d'appartenir à une congruence W, on obtient des surfaces découvertes par A. Terracini et étudiées par cet auteur et par E. Cartan, et la considération du cas où les trajectoires sont simplement pangéodésiques permet l'explicitation d'une famille à trois paramètres de surfaces B^*. Le passage au domaine complexe conduit à une classe de surfaces B^* admettant un groupe G_2 de déformations infinitésimales du type (p.s.), à savoir les surfaces de coïncidence, et l'auteur attire l'attention sur le problème de savoir s'il existe d'autres surfaces B^* admettant un tel groupe G_2 de déformations (p.s.). *P. Vincensini* (Marseille)

Mokriščev, K. K. 6421

Some special curves in a space of constant curvature. (Russian)

Comment. Math. Univ. Carolinae **5** (1964), 79–84.

Dans ce travail sont étudiées dans l'espace à trois dimensions à courbure constante $1/k^2$ les courbes analogues aux courbes de Bertrand, de Mannheim, de Cesàro et autres. On en donne, avec une indication seulement de la démonstration, les propriétés principales qui permettent de retrouver les propriétés connues dans le cas de la géométrie euclidienne. *M. Decuyper* (Lille)

Blank, Ja. P.; Motornyi, L. T. 6422

Translation surfaces of an elliptic space which carry two translation nets. (Russian)

Uspehi Mat. Nauk **19** (1964), no. 1 (115), 139–142.

The authors, returning to the subject of their paper [Učen. Zap. Har'kov. Gos. Univ. **28**; Zap. Naučno-Issled. Inst. Mat. Meh. i Har'kov. Mat. Obšč. (4) **20** (1950), 61–76; MR **14**, 405] determine the equations of all translation surfaces with two translation nets in elliptic space $(x^0:x^1:x^2:x^3)$ in quaternion form: $x=A(v^1)B(v^2)$, $x=C(v^3)D(v^4)$, where $x=\sum x_he_h$, $A(v^1)=\sum A_h(v^1)e_h$, $B(v^2)=\sum B_h(v^2)e_h$, etc., and where $h=0,1,2,3$. The A_h, B_h, C_h, D_h are of the form

$$A_0=\cos\frac{1-a_1}{2}v^1\sin\frac{\alpha+\beta}{2},\quad B_0=\cos\frac{1-a_1}{2}v^2\cos\frac{\alpha-\beta}{2},$$

$$C_0=\cos\frac{1+a_1}{2}v^3\sin\frac{\alpha-\beta}{2},\quad D_0=\cos\frac{1+a_1}{2}v^4\cos\frac{\alpha+\beta}{2},$$

etc.,

where

$$\cos\alpha=\sqrt{\left(\frac{b_1}{2a_1^2}+k\right)},\qquad \cos\beta=\sqrt{\left(\frac{b_1}{2a_1^2}-k\right)},$$

$$k=\frac{b_1^2}{4a_1^4}-\frac{a^2}{a_1^2},$$

$a_1^2=b+2c+1$, $b_1^2=a+b-c^2$ (a,b,c arbitrary constants). The tangents to the translation curves of one family along the translation curves of the other fam[ily ...] Clifford parallels. *D. J. Struik* (Belmont,

Santaló, L. A.

A relation between the mean curvatures of [...] convex bodies in spaces of constant curvature. (S[panish; ...] English summary)

Rev. Un. Mat. Argentina **21**, 131–137 (1963).

Let Q be a closed convex hypersurface of class C^{∞} [...] n-dimensional space of constant curvature K. [...] $(i=1,2,\cdots,n-1)$ be the principal curvature ra[dii ...] point P of Q and dP the volume element at P. Th[e] ith mean curvature of Q is

$$M_i=\frac{1}{\binom{n-1}{i}}\int_Q\left(\sum\frac{1}{\rho_{h_1}\rho_{h_2}\cdots\rho_{h_i}}\right)dP,\quad i=1,2,\cdots$$

where the sum is extended to all the $\binom{n-1}{i}$ combi[nations] of class i of the indices $1,2,\cdots,n-1$. In particul[ar ...] is the volume of Q. Let $Q(\lambda)$ be the hypersurface [...] to Q at distance λ and $M_i(\lambda)$ the ith mean curva[ture of] $Q(\lambda)$. If $M_i'(\lambda)=dM_i(\lambda)/d\lambda$, the author prove[s the] identities

$$M_i'(\lambda)=-iKM_{i-1}(\lambda)+(n-i-1)M_{i+1}(\lambda).$$

From these identities he deduces some consequen[ces ...] convex curves and surfaces $(n=2,3)$. Among th[em he] obtains a proof of the inequality

$$M_1^2+KF^2-4\pi F\geq 0\qquad (M_0=F)$$

conjectured by Blaschke [*Über eine geometrische* [...] *von Euklid bis heute*, Teubner, Leipzig, 1938] an[d] proved by Knothe [Univ. Lisboa Revista Fac. Ci. [...] Mat. (2) **2** (1952), 336–348; MR **15**, 819] in a differen[t ...] *G. Tallini* (

Stanilov, G.

On the bi-axial theory of a congruence of [...] (Russian)

Comment. Math. Univ. Carolinae **4** (1963), 117–1[...]

This is a short announcement of results publis[hed in] extended form in Czechoslovak Math. J. **15** (**90**) ([...] 64–73. The bi-axial space B_3 is defined by two ske[w lines] j, k. Frames are used with $A_1\in j$, $A_2\in k$ and the [...] point E on the line (A_2+A_3, A_1+A_4). By associat[ing A_3] and A_4 with the line of the congruence and using Ca[rtan's] method, the author finds principal forms and inv[ariant] forms. The null-manifolds of the latter are disc[ussed.] A canonical frame and invariants b, α, β, γ, x_1, [...] found. As an example, a geometrical interpretat[ion of] the invariant b is given, namely, $b=-\delta/(\delta+1)^2$, $\delta=(A_3,A_4,F_1,F_2)$, F_1, F_2 being the foci of the [line of] the congruence, and the vertices A_3, A_4 of the can[onical] frame parabolic points of the line. Some properties [...] can be expressed in terms of this invariant ar[e] discussed. *A. Goetz* (Wr[...]

Stanilov, G.

Geometrische Deutungen der differentialgeomet[rischen] Invarianten eines Geradenkomplexes des euklid[ischen] Raumes.

C. R. Acad. Bulgare Sci. **17** (1964), 331–334.

Die Infinitesimalverschiebungen des kanonischen Koordinatensystems $Ae_1e_2e_3$, welches mit der Geraden eines Geradenkomplexes im euklidischen Raum verbunden ist, werden wie bekannt durch die Formeln $dA=\omega_i e_i$, $de_i=\omega_{ik}e_k$, $\omega_{ik}=-\omega_{ki}$, $\omega_2=a\omega_3$ $(i, k=1, 2, 3)$ gegeben. In der Arbeit werden die Invariante a und die weiteren neun Invariante, die als Koeffizienten in den Gleichungen $\omega_{12}=x_1\omega_1+x_2\omega_{31}+x_3\omega_{32}$, $da=x_3\omega_1+x_4\omega_{31}+x_5\omega_{32}$, $\omega_3=(ax_1-x_2)\omega_1+(ax_2-x_5)\omega_{31}+(ax_3-x_6)\omega_{32}$ (welche man aus den vorherstehenden Gleichungen ableiten kann) vorkommen, mit Hilfe passend gewählten geometrischen Objekte, die mit der Geraden des Komplexes verbunden werden, geometrisch interpretiert. *A. Urban* (Prague)

Mas'kin, N. M. 6426

Infinitesimal dually conformal transformations of line complexes. (Russian)

Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. **168** (1963), 167–182.

Infinitesimal transformations of a geometric object in three-dimensional affine space are defined in terms of a moving frame, and it is shown that they form a group. The method is applied to infinitesimal transformations of line complexes in E^3. Conditions are given under which a complex K is transformed into its first differential neighborhood, i.e., a line L of K goes into a line common to all complexes touching K at L. Using the well-known representation of lines in terms of dual numbers suggests an obvious definition of dually conformal maps. The complexes in E^3 which admit groups of infinitesimal dually conformal transformations into themselves are discussed in some detail; also, explicit examples are given.

H. Busemann (Los Angeles, Calif.)

Rezničenko, R. A. 6427

On an equi-affinely invariant class of pairs of congruences. (Russian)

Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. **168** (1963), 161–166.

The following was shown by E. T. Ivlev in a paper on a pair of congruences in the equi-affine geometry of three-dimensional space [same Trudy **168** (1963), 103–119; MR **30** #2416b]. Each pair of corresponding rays of an arbitrary equiaffine pair of line congruences lies on two pairs of ruled surfaces belonging to the pair, such that each pair contains one ruled surface whose asymptotic plane is parallel to the corresponding ray of the other ruled surface. These pairs of ruled surfaces are called A-pairs. In the construction of a canonical frame [loc. cit.] these pairs of congruences were excluded, when both A-pairs coincide. The present paper deals with this case. It is shown, for example, that these pairs of congruences still depend on five arbitrary functions, that the singular points of the lines of an A-pair are quasiflecnodal points, and that both surfaces of an A-pair are cylindroids. Some special cases are discussed. *H. Busemann* (Los Angeles, Calif.)

Vasil'ev, A. M. 6428

Families of linear elements which are bent by completely geodesic families. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, no. 3 (40), 28–35.

In vorhergehenden Arbeiten der Jahre 1959 bis 1963 [siehe z.B. Mat. Sb. (N.S.) **60** (102) (1963), 411–434; MR **27** #686] hatte Verfasser die Klasse homogener Räume und die Invarianten ihrer affinen Zusammenhänge untersucht; insbesondere vollständig geodätische Untermannigfaltigkeiten dieser Räume. Die Ergebnisse fanden Anwendung im konkreten Falle des homogenen Raumes m_5, der Gesamtheit aller Linienelemente des euklidischen Raumes. Nunmehr handelt es sich um die Entwicklung einer allgemeinen Methode zur Bestimmung homogener Untermannigfaltigkeiten, die von Untermannigfaltigkeiten gegebener Klasse eingehüllt werden, nämlich von Linienelementvereinen, die vollständig aus geodätischen Untermannigfaltigkeiten bestehen. Alle diese Linienelementvereine ergeben sich auf gleiche Weise und gestatten eine einfache geometrische Charakterisierung. Ihre analytische Behandlung wird unter Verwendung kanonischer Begleitsimplices auf die Integration Pfaffscher Systeme zurückgeführt. Zunächst werden notwendige und hinreichende Bedingungen für die Berührung der Klasse $\mathscr{F}$ angegeben. Weiterhin werden zunächst die vierparametrigen Elementvereine behandelt. Sie zerfallen in drei Klassen. Dann folgt die Theorie dreidimensionaler Elementvereine, welche bereits sechs Fälle unterscheidet. Im letzten Abschnitt wird noch auf zweiparametrige Fälle eingegangen (mit 9 weiteren Unterfällen).

M. Pinl (Muenster)

Tola, José [Tola Pasquel, José] 6429

The metric tensor, inertia tensor, and stress tensor. (Spanish)

Univ. Nac. Ingen. Inst. Mat. Puras Apl. Notas Mat. **1** (1962/63), 1–34.

These are notes presumably written for students of the Univ. Nac. Ingen. and give a classical presentation of vectors and second-order tensors. Nineteen pages are devoted to vectors, and the remaining part to examples of metric, inertia and stress tensors. *J. Kestens* (Brussels)

Pach, Klára; Frey, Tamás 6430

★**Vector and tensor analysis.**

Terra, Budapest, 1964. 596 pp. $14.00.

Although this book is intended for undergraduate engineers, it has many features that will appeal to other students. First, the authors introduce scalars, vectors, and the various products of these quantities via the usual Gibbs approach. This is followed by a detailed discussion of differentiation and the properties of the gradient. These first three chapters contain many applications to analytic and differential geometry and kinematics. Chapter 4 treats the various integral forms of Stokes's theorem and concludes the standard type of introduction to vector analysis. However, the next three chapters, which deal with vector and tensor fields, furnish several interesting developments. For instance, the introduction of a norm for A_{ik}, a second-order tensor (the norm is defined as the trace of the product of the matrix A_{ik} and its transpose $A_{ik}^*=A_{ki}$) and the discussion of some inequalities involving this norm are given. Further, applications to hydrodynamics and potential theory are furnished.

The exposition is in "theorem-proof" form and is clear except in a few sections. The only defect in the text is a

tendency to oversimplify some general theory. An example of this is the "embedding problem" which is discussed on page 564. However, the text is interesting and should be of considerable value to engineering and other students. *N. Coburn* (Ann Arbor, Mich.)

Tevan, György 6431

Die Darstellung der Tensoren mit ihren Tensorkomitanten niedrigerer Stufe in Räumen definiter und indefiniter Metrik.

Publ. Math. Debrecen **10** (1963), 40–52.

The author's summary: "In diesem Artikel werden wir von den, für die Tensoren 2-ter Stufe des Raumes definiter Metrik gultigen, bekannten Sätzen ausgehend, den beliebigen Tensor p-ter Stufe, als lineare Kombination der dyadischen (tensoriellen) Produkte der Komitantentensoren $(p-q)$-ter und q-ter Stufe darstellen (Satz 2). Danach werden wir durch Anwendung des allgemeinen Satzes auf den Fall $q=1$, den Tensor p-ter Stufe vollständig bestimmende Komitantenvektoren and Invarianten angeben (Satz 3). Endlich verallgemeinern wir unsere Resultate auf Räume indefiniter Metrik (Sätze 4–7)."

T. K. Pan (Norman, Okla.)

Vagner, V. V. 6432

The foundations of differential geometry and modern algebra. (Russian)

Proc. Fourth All-Union Math. Congr. (Leningrad, 1961), Vol. I, pp. 17–29. *Izdat. Akad. Nauk SSSR, Leningrad,* 1963.

From time to time a paper appears in which it is shown how far present-day differential geometry deviates from the principles laid down in the Erlangen programs. "For contemporary differential geometry", writes the author, "the concept of group is quite insufficient for the examination, from an algebraic point of view, of the basic concepts of the corresponding geometrical theories. Moreover, algebraic problems arising in investigations concerning the foundations of contemporary differential geometry require the study of special algebraic systems which at present are not very seriously discussed". The author indicates some of the reasons for criticism: the local character of usual differential geometry, the use of functions whose existence is only confirmed by the classical existence theorems which are also local in character, the difficulties met in understanding the operations of multiplication as binary relations in the partial transformations of spaces under classical Lie groups. An elementary example is found in the set of all projective transformations of n-dimensional arithmetic space, where every transformation has a singular hyperplane. The author now endeavors to give an outline of a modern approach to differential geometry, an approach first undertaken by Veblen and Whitehead in their monograph of 1932 [*The foundations of differential geometry*, Cambridge Univ. Press, London, 1932]. The investigation of binary relations owes much to J. Riguet [Bull. Soc. Math. France **76** (1948), 114–155; MR **10**, 502], that of generalized groups to previous work by the author [e.g., Ukrain. Mat. Ž. **8** (1956), 235–253; MR **20** #3924; see also Dokl. Akad. Nauk SSSR **84** (1952), 1119–1122; MR 14, 12].

D. J. Struik (Belmont, Mass.)

Gaeta, Federico 6433

Some characterizations of the complete integrability of a given Pfaffian system by means of the Lie derivative.

Bol. Soc. Mat. São Paulo **15** (1960), 37–46 (1964).

Let M be a connected, analytic, real differentiable manifold, $E^h(x)$ an h-dimensional subspace of the tangent space at x, and $\hat{E}^{n-h}(x)$ a subspace of the cotangent space at x orthogonal to $E^h(x)$. Let $\Pi: x\to E^h(x)$ be an h-dimensional field (a distribution) defined over M_n, and $V(x)$ a local analytic h-vector representing Π, and let $\hat{V}^{n-h}(x)$ be defined in a similar fashion. Then it is proved that Π is completely integrable if and only if either of the following conditions is true: (1) $L_v(V)=f_v(x)V$ for any v contained in Π, where L_v is the Lie derivative and $f_v(x)$ is a local analytic function depending on v; (2) $L_v(\hat{V})=f_v(x)\hat{V}$.

C. B. Allendoerfer (Seattle, Wash.)

Reátegui, José [Reátegui C., José] 6434

Cohomology and integrable forms. (Spanish)

Univ. Nac. Ingen. Inst. Mat. Puras Apl. Notas Mat. **2** (1964), 85–92.

Esposizione elementare di alcuni risultati noti sulla coomologia di forme differenziali e forme di Pfaff completamente integrabili. Applicazioni ad R^n. Alcuni esempi assai semplici, ma significativi. *D. Gallarati* (Genoa)

Johnson, H. H. 6435

A new type of vector field and invariant differential systems.

Proc. Amer. Math. Soc. **15** (1964), 675–678.

In another paper [same Proc. **15** (1964), 432–437, MR **29** #526], the author defined and studied k-vector fields associated to mappings from a manifold N to a manifold M. In this paper, he continues the study and shows the analogy with classical continuous groups. Defining a system Σ of partial differential equations as a finitely generated ideal in $C^\infty(J^k)$, where $J^k=J^k(N, M)$ is the manifold of k-jets $j_x^k(f)$ of maps $f: N\to M$, and using the notions of prolongations of k-vector fields and of systems of partial differential equations, the author shows that in the real analytic case, if θ is a k-vector field leaving Σ invariant, then the integral curves of θ which pass through one solution of Σ yield solutions for all parameter values. *E. T. Kobayashi* (Evanston, Ill.)

Reeb, G. 6436

Formes de Pfaff polynomes complètement intégrables.

Ann. Inst. Fourier (Grenoble) **14** (1964), *fasc.* 1, 37–42.

L'auteur considère dans une variété réelle de la forme $V\times\mathbf{R}$ (ou complexe $V\times\mathbf{C}$) une forme de Pfaff complètement intégrable polynomiale $\omega(x,t)=\omega_0+\omega_1 t+\cdots+\omega_r t^r-dt$, où les ω_i proviennent de formes sur V. Il construit alors explicitement les variétés intégrales de ω dans certains cas. *A. Haefliger* (Geneva)

Hashimoto, Shintaro 6437

On the differentiable manifold M^n admitting tensor fields (F, G) of type (1, 1) satisfying $F^3+F=0$, $G^3+G=0$, $FG=-GF$ and $F^2=G^2$.

Tensor (N.S.) **15** (1964), 269–274.

The following result is proved: A necessary and sufficient

condition for the structural group of the tangent bundle of a differentiable manifold M^n to be reducible to the group $\mathrm{sp}(r)\times O(n-4r)$ is that M^n admits tensor fields F, G of type (1, 1) and rank $4r$ satisfying the relations of the title. *A. M. Rodrigues* (São Paulo)

Ogawa, Yosuke; Okumura, Masafumi 6438

On almost contact metric structures. (Japanese)

Sûgaku **16** (1964), 41–45.

This paper is a summary and supplement to the results of Y. Hatakeyama, S. Sasaki, S. Tanno and the authors [M. Okumura, Tôhoku Math. J. (2) **14** (1962), 135–145; MR **26** #708; Y. Hatakeyama, Y. Ogawa, and S. Tanno, ibid. (2) **15** (1963), 42–48; MR **26** #4295; Y. Hatakeyama, ibid. (2) **15** (1963), 176–181; MR **27** #705; S. Sasaki and Y. Hatakeyama, J. Math. Soc. Japan **14** (1962), 249–271; MR **25** #4458], concerning the properties of almost contact, normal contact and K-contact metric structures.

A. Morimoto (La Jolla, Calif.)

Nguyen-Van-Hai 6439

Un type de connexion linéaire invariante sur un espace homogène.

C. R. Acad. Sci. Paris **259** (1964), 2065–2068.

The author continues his investigation of invariant affine connections on homogeneous spaces [same C. R. **258** (1964), 3952–3955; MR **29** #1599; ibid. **259** (1964), 49–52; MR **29** #3995]. He constructs an invariant connection (called a connection of special type) on a homogeneous space satisfying certain conditions which are met by any reductive homogeneous space in the sense of Nomizu. Assuming that a homogeneous space with invariant connection of special type admits also an invariant pseudo-Riemannian metric, he studies the relationship between the invariant connection of special type and the Levi-Civita connection defined by the metric.

S. Kobayashi (Berkeley, Calif.)

Yano, K.; Ledger, A. J. 6440

Linear connections on tangent bundles.

J. London Math. Soc. **39** (1964), 495–500.

From the authors' introduction: "It is shown in a recent paper by Dombrowski [J. Reine Angew. Math. **210** (1962), 73–88; MR **25** #4463] that if M is a C^∞ manifold with a linear connection ∇, then the tangent bundle $T(M)$ of M admits an almost complex structure determined by ∇. Furthermore, a Riemannian metric on M determines a Kahlerian metric on $T(M)$. We show in this paper that ∇ on M determines a linear torsion free connection on $T(M)$. In order to do this we first consider certain vector fields on $T(M)$, in addition to the horizontal and vertical lifts of vector fields of M. In particular, we show that $T(M)$ admits a non-trivial vector field which is independent of any connection or vector field on M."

A. M. Rodrigues (São Paulo)

Arregui, J. 6441

The infinitesimal connection of a differentiable manifold X as a representation of the differential determined by the topological space X. (Spanish)

Rev. Mat. Hisp.-Amer. (4) **23** (1963), 57–74.

The author's intention is to give a specific illustration of material found in papers of Botella Raduán [same Rev. (4) **20** (1960), 183–211; MR **23** #A2885; Actas de la Primera Reunión Anual de Matematicos Españoles, pp. 110–117, Universidad de Madrid, Madrid, 1961; MR **24** #A3638]. He first gives explicit coordinate representations of the group of projective collineations leaving the vertices of a Euclidean simplex fixed and the interior invariant. Then for a differentiable manifold X he considers coordinate systems having a Euclidean simplex as range, and thus induces local groups of homeomorphisms on X, as discussed in the first of the references cited. These local groups are tied together rather artificially, giving a presheaf of groups, the sheaf of which illustrates Botella Raduán's differential.

A linear connexion is assumed given and is used to prolong the infinitesimal transformations given by the local group to the bundle of bases. The prolongation is nothing more than the horizontal space of the connexion. The author calls it the geometric representation of the differential (sheaf) but the reviewer fails to see how it represents anything but the connexion itself.

R. L. Bishop (Urbana, Ill.)

Tashiro, Yoshihiro 6442

An example of non-complete affine connection on a torus. (Japanese)

Sûgaku **15** (1963/64), 221.

The author gives an example of a compact affinely connected manifold which is not complete. Let T^2 be a torus with coordinate system (x, y) (mod 2π), and consider an affine connection whose geodesics are given by

$$x''+x'^2\sin^2 x-y'^2\cos^2 x = 0 \quad\text{and}\quad y''+x'^2+y'^2 = 0.$$

Then the geodesic given by $x=t$ and $y=\log\cos t$ ($|t|<\pi/2$) is not extendible to $t=\pi/2$. A simpler example of non-complete affine connection on a circle was constructed by Auslander and Markus [Ann. of Math. (2) **62** (1955), 139–151; MR 17, 298]. Contrary to the statement made by the author, there exists also a simply connected, compact, affinely connected manifold which is not complete [see Kobayashi and Nomizu, *Foundations of differential geometry*, Vol. I, p. 292, Interscience, New York, 1963; MR **27** #2945]. *S. Kobayashi* (Berkeley, Calif.)

Vedernikov, V. I. 6443

A pseudo-linear connection. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* **2 (39)**, 41–52.

Verfasser geht aus von einem zentralaffinen Raum E_{n+1} mit ausgezeichnetem E_n; dieser E_{n+1} wird aud eine Vektorbasis $e_0, e_1, \cdots, e_n$ bezogen, wobei $e_1, \cdots, e_n$ den E_n aufspannen. Nach dem Muster von Cartan werden dann die infinitesimalen Änderungen dieses System durch $de_0=\omega_0{}^0e_0+\omega_0{}^ie_i$, $de_i=\omega_i{}^ke_k$ beschrieben, wobei die ω ein System Pfaffscher Formen bedeuten, die die Strukturgleichungen

$$d\omega_0{}^0 = 0, \qquad d\omega_0{}^i-(\omega_0{}^0\wedge\omega_0{}^i+\omega_0{}^k\wedge\omega_k{}^i) = 0$$

erfüllen. Nach dem Vorbild von E. Cartan verallgemeinert sich dann dies System der Strukturgleichungen zu einem nicht holonomen System, indem man die rechten Seiten nicht gleich 0 setzt, sondern gleich gewissen

äußeren Differentialformen 2. Grades Ω_0^0, Ω_0^i und Ω_i^k. Die ω_i^j definieren eine Übertragung, deren Torsion durch die ω_0^i bestimmt ist. Es wird in der Arbeit meist Torsionsfreiheit, d. h. $\omega_0^1 = \cdots = \omega_0^n = 0$ vorausgesetzt. Die übrigen Formen werden zu einer Matrix $\omega_0 = \begin{pmatrix}\omega_0^0 & 0\\ 0 & \omega_k^j\end{pmatrix}$ zusammengefaßt, und es wird untersucht, wie diese Matrix sich bei der Untergruppe G aller zentralen Affinitäten verändert, die außer E_n noch die Richtung des Vektors e_0 unverändert lassen. Die Gruppe G wird durch Matrizen der Art $A_1 = \begin{pmatrix}\sigma & 0\\ 0 & \sigma a_i^k\end{pmatrix}$ beschrieben, und ω_0 geht bei den infinitesimalen Transformationen von G, die auch die statische Gruppe der Richtung e_0 heißt, in die Formenmatrix $\tilde{\omega}_0$ über, die sich durch

$$\tilde{\omega}_0 = (dA_1)A_1^{-1} + A_1\omega_0 A_1^{-1}$$

berechnet. Die Formen der Matrix $\tilde{\omega}_0$ definieren den sogenannten pseudolinearen Zusammenhang. Dieser läßt sich auffassen als ein affiner Zusammenhang in dem gefaserten Raum $P(B, G)$; dabei ist G eine n-dimensionale Mannigfaltigkeit mit den Parametern $u^1, \cdots, u^n$, und die Gruppe G tritt als Faser auf. Bei Festlegung der Funktion $\sigma(u^1, \cdots, u^n)$ definieren die $\tilde{\omega}_i^k$ einen linearen Zusammenhang, und der pseudolineare Zusammenhang erweist sich somit als eine Gesamtheit linearer Zusammenhänge. Es werden dann Pseudotensoren, ein kovariantes Differential und damit eine Parallelverschiebung bei pseudolinearen Zusammenhängen erklärt. Als besonderer Fall wird der sogenannte Weylsche Zusammenhang behandelt, bei dem eine Bilinearform $g_{ij}u^iw^j$ sich als invariant bei pseudoparalleler Verschiebung der Vektoren u^i, w^j erweist.

W. Burau (Hamburg)

Vedernikov, V. I. 6444

On conjugate connections. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, no. 4 (41), 19–29.

Verfasser betrachtet als Grundbereich die Gesamtheit der ∞^{2n} Paare (A, ξ), wobei A ein Punkt und ξ eine Hyperebene des projektiven Raumes Π_n bedeuten. Die Gesamtheit dieser Paare wird durch $(n+1)^2$ Grundpaare (A_i, ξ^j) je der Punkte $A_0, \cdots, A_n$ und Seiten $\xi^0, \cdots, \xi^n$ eines Grundsimplex aufgespannt. Dann rechnet der Verfasser die Ableitungsformeln

$$dA_i = \omega_i^0 A_0 + \omega_i^\beta A_\beta, \qquad d\xi^j = \omega^*{}_0{}^j\xi^0 + \omega^*{}_\beta{}^j\xi^\beta$$

aus, die sich bei infinitesimaler Änderung des Grundsimplex ergeben. Unter der statischen Gruppe des Paares (A_0, ξ^0) wird die Gesamtheit derjenigen Kollineationen verstanden, die (A_0, ξ^0) unverändert lassen. Dann ergibt sich durch einfache Rechnung: Bezeichnet man mit $A = (a_\alpha{}^\beta)$ eine reguläre (n, n)-Matrix, ihre Inverse $A^{-1} = (b_\alpha{}^\beta)$, und ist σ eine Konstante, so gehen die Pfaffschen Formen ω_i^j bei den Transformationen der statischen Gruppe von (A_0, ξ^0) über in $\tilde{\omega}_i^j$ mit $\tilde{\omega}_0{}^\alpha = \omega_0{}^\beta b_\beta{}^\alpha$, $\tilde{\omega}_0{}^0 = \omega_0{}^0 + d\log\sigma$, $\tilde{\omega} = dA\cdot A^{-1} + A\omega A^{-1} + d\log\sigma\cdot E$. Darin bedeuten $\tilde{\omega}$ die Formenmatrix $\tilde{\omega}_\alpha{}^\beta$ $(\alpha, \beta = 1, \cdots, n)$ und E die Einheitsmatrix. In dem mit Π_n verbundenen Dualraum $\Pi_n{}^*$ ergeben sich analog Formen $\tilde{\omega}^*{}_\alpha{}^\beta$. Die $\omega_\alpha{}^\beta$ und die $\tilde{\omega}^*{}_\alpha{}^\beta$ definieren je einen pseudolinearen Zusammenhang [siehe #6443] und diese beiden pseudolinearen Zusammenhänge heißen konjugiert zueinander. Mit ihnen sind durch die Formen der Matrizen $\tilde{\theta} = \tilde{\omega} - \tilde{\omega}_0{}^0E$ und $\tilde{\theta}^* = \tilde{\omega}^* - \tilde{\omega}_0{}^0E$ lineare Zusammenhänge verbunden, die gleichfalls zueinander konjugiert heißen. Durch die Pfaffschen Gleichungen $\omega_\alpha{}^0 = b_{\alpha\beta}\omega_0{}^\beta$ wird eine Abbildung der Punkte von Π_n in die Menge der Hyperebenen definiert. Im besonderen Fall der Symmetrie von $b_{\alpha\beta}$ ist der durch θ definierte Zusammenhang projektiv euklidisch.

W. Burau (Hamburg)

Fava, Franco 6445

Ulteriori contributi allo studio delle varietà riemanniane a connessione costante.

Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **98** (1963/64), 60–85.

This paper contains some further results in the author's researches on Riemannian manifolds with constant coefficients of connection. It depends upon certain results obtained in a previous note by the author [same Atti **96** (1961/62), 451–462; MR **25** #5477] and also modifies certain of the results given in the note mentioned. The main part of the research concerns Riemannian manifolds V_n "of class C_p" whose fundamental tensor $g_{ij}(x)$ can be written in the form $\phi_\alpha a_{ij}{}^{(\alpha)}$ $(\alpha = 1, 2, \cdots, p)$, where the ϕ_α are functions of x and the $a_{ij}{}^{(\alpha)}$ are constants in a certain system of coordinates called coordinates C. In this paper the non-Euclidean V_n of type C_2 are taken particularly into consideration, and an exhaustive classification is given of spaces of this class.

E. T. Davies (Southampton)

Pignedoli, Antonio 6446

Sulle "famiglie naturali di curve" di Kasner.

Univ. Nac. Tucumán Rev. Ser. A **14** (1962), 221–251.

E. Kasner [Trans. Amer. Math. Soc. **10** (1909), 201–219] hatte "natürliche Kurvenfamilien" in die Dynamik als Extremalen von Variationsproblemen der Form $\int F\,ds$ = min mit beliebigen Funktionen F des Ortes eingeführt. Der Verfasser untersucht diese Kurvenfamilien in eingehender Weise und stützt sich hierbei auf die Differentialgleichung und ihre Integrationsmöglichkeiten. Im besonderen wird auf die speziellen Kurvenklassen von Brachistochronen, geodätischen Linien und Kettenlinien der Mannigfaltigkeit V_n das Augenmerk gerichtet, die auch schon Kasner betrachtet hat.

H. R. Müller (Braunschweig)

Sen, R. N.; Mittra, S. S. 6447

On a sequence of conformal Riemannian spaces.

Bull. Calcutta Math. Soc. **54** (1962), 107–121.

Let g_{ij} be the metric tensor of a Riemannian space, and let λ_i, μ_i be two covariant vector fields which are nowhere zero and nowhere orthogonal. Denote the product of the lengths of these vectors by l and the angle between them by θ. Then, as a specialisation of a previous paper by R. N. Sen [Proc. Nat. Inst. Sci. India Part A **26** (1960), suppl. II, 14–20; MR **25** #522], the authors consider the sequence of metric tensors

$$g_{ij}{}^{(2p-1)} = l^2\cos^{2p}\theta\, g_{ij}, \qquad g_{ij}{}^{(2p)} = \sec^{2p}\theta\, g_{ij}.$$

Assuming that l is constant, relations are found between the curvature tensors of the metrics in this sequence, and these are applied to obtain necessary conditions for pairs of spaces in the sequence to be Einstein spaces.

F. Brickell (Southampton)

Singal, M. K. 6448

Normal curvature of a vector field in a hypersurface V_n imbedded in a Riemannian V_{n+1}.

Rend. Circ. Mat. Palermo (2) **12** (1963), 275–290.

Wakakuwa, Hidekiyo 6449

Remarks on 4-dimensional differentiable manifolds.

Tôhoku Math. J. (2) **16** (1964), 154–172.

The author classifies Riemannian 4-manifolds X_4 according to restricted homogeneous holonomy, taking the viewpoint of linear Lie groups. Roughly speaking, much of the paper covers material considered from the viewpoint of structure equations by Ishihara [J. Math. Soc. Japan 7 (1955), 345–370; MR 18, 599] and based on Chern's report [Bull. Amer. Math. Soc. 51 (1945), 964–971; MR 7, 216]. The classification of possible restricted homogeneous holonomy groups for X_4 is a straightforward exercise in Lie theory. Then some examples are given, the various almost-complex structures present are considered, and the vanishing of Nijenhuis tensors of the latter is discussed. The reviewer finds the paper useful for testing conjectures in dimension 4.

J. A. Wolf (Berkeley, Calif.)

Diskant, V. I. 6450

Stability in a theorem of Liebmann's. (Russian)

Dokl. Akad. Nauk SSSR **158** (1964), 1257–1259.

A stability theorem of A. I. Fet [same Dokl. **153** (1963), 537–539; MR **28** #527] is strengthened so as to give the precise order of magnitude: If the Gauss curvature K of an n-dimensional convex surface satisfies $1-\varepsilon \leq K \leq 1+\varepsilon$ $(0<\varepsilon<\frac{1}{2})$, then the radius r of the maximal inscribed sphere and the radius R of the minimal circumscribed sphere are bounded by $r \geq 1-C\varepsilon$, $R \leq 1+C\varepsilon$, where C depends only on n. *H. Busemann* (Los Angeles, Calif.)

Radziszewski, Konstanty 6451

Sur la courbure intégrale d'une classe de courbes. (Polish and Russian summaries)

Ann. Univ. Mariae Curie-Skłodowska Sect. A **16** (1962), 19–40 (1964).

Cet article constitue un apport intéressant à la géométrie externe des lignes géodésiques d'une surface convexe. Il se relie principalement à H. Busemann et W. Feller [Acta Math. **66** (1936), 1–47] et à A. D. Alexandrov [*Intrinsic geometry of convex surfaces* (Russian), OGIZ, Moscow, 1948; MR **10**, 619] dont plusieurs résultats sont utilisés. Toutes les courbes et les surfaces considérées sont situées dans l'espace euclidien à trois dimensions. $\langle A*B\rangle$ désigne une courbe d'extrémités A et B, $\mathfrak{K}$ la classe des courbes $\langle A*B\rangle$ vérifiant les conditions suivantes: $\langle A*B\rangle$ admet en tout point M les vecteurs (unitaires) demi-tangents $\mathfrak{t}^+=\mathfrak{t}^+(M)$ et $\mathfrak{t}^-=\mathfrak{t}^-(M)$ tels que, pour $M\to M_0{}_+$, $\lim \mathfrak{t}^+(M)=\lim \mathfrak{t}^-(M)=\mathfrak{t}^+(M_0)$, que, pour $M\to M_0{}_-$, $\lim \mathfrak{t}^+(M)=\lim \mathfrak{t}^-(M)=\mathfrak{t}^-(M_0)$ et que $\mathfrak{t}^+(M_0)\neq -\mathfrak{t}^-(M_0)$. Les courbes de $\mathfrak{K}$ sont rectifiables, on peut prendre comme paramètre la longueur d'arc s comptée à partir de A: vecteur $\overline{AM}=\mathfrak{r}(s)$. Par "tangente au sens strict" $\mathfrak{t}(M)$ est entendue la limite unique des droites $M'M''$ où M' et M'' sont deux points distincts de $\langle A*B\rangle$ tendant vers M, par "plan osculateur au sens de Menger" $\sigma(M)$ la limite unique des plans passant par trois points M', M'' et M''' de $\langle A*B\rangle$ non alignés et tendant vers M, par "normale orientée" de $\sigma(M)$ la limite unique dans les mêmes circonstances du quotient du produit vectoriel $\overline{M'M''}\times\overline{M''M'''}$ par $|\overline{M'M''}\times\overline{M''M'''}|$, les points étant rangés dans l'ordre des paramètres croissants. Si $\langle A*B\rangle$ inclut un segment de droite I, on prendra comme plan osculateur en un de ses points M un plan passant par I "convenant le mieux au problème considéré", mais le même en tous les points de I. Pour une courbe $\langle A*B\rangle$ de $\mathfrak{K}$ on donne aux vecteurs $\mathfrak{t}^+(M)$ et $\mathfrak{t}^-(M)$ une origine fixe O. Aux points où $\mathfrak{t}^+\neq\mathfrak{t}^-$, on joint les extrémités par un arc de grand cercle de longueur inférieure à π. On obtient ainsi une courbe (continue) dite "indicatrice sphérique" de la courbe $\langle A*B\rangle$ et notée $\langle\langle A*B\rangle\rangle$. La longueur de $\langle\langle A*B\rangle\rangle$ est appelée "courbure intégrale" de $\langle A*B\rangle$ et désignée par $k(\langle A*B\rangle)$. S désigne la surface fermée limitant un solide convexe ayant des points intérieurs. S est dite "lisse" si elle admet en chacun de ses points un plan tangent continu. Théorème 1: Une géodésique sur une surface convexe lisse S admet en tout point $M\in\langle A*B\rangle$ un plan osculateur au sens de Menger continu, orienté, perpendiculaire au plan tangent à la surface au point M. Théorème 2: L'indicatrice sphérique d'une géodésique $\langle A*B\rangle$ sur une surface convexe lisse est une courbe $\langle\langle A*B\rangle\rangle$ admettant une tangente au sens strict. Théorème 3: Si la suite de surfaces convexes lisses S_n tend vers une surface convexe lisse S, si la suite des géodésiques $C_n=\langle A_n*B_n\rangle$ de S_n tend vers la géodésique $C=\langle A*B\rangle$ de S et si ces courbes sont rapportées à un même paramètre x, $x_0\leq x\leq x_1$, $C_n: M_n(x)$, $C: M(x)$, tel que, pour tout x, $M_n(x)\to M(x)$, alors la courbure intégrale $k_n(x)$ de $\langle A_n*M_n(x)\rangle$ tend uniformément vers la courbure intégrale $k(x)$ de $\langle A*M(x)\rangle$. Théorème 4 exprime une propriété analogue pour une suite de lignes brisées (polygonales) inscrites dans une géodésique $\langle A*B\rangle$ sur une surface convexe lisse et convergeant vers $\langle A*B\rangle$. Théorème 5 donne des conditions pour qu'une suite de courbes $\langle A_n*B_n\rangle\in\mathfrak{K}$ convergeant vers une courbe $\langle A*B\rangle\in\mathfrak{K}$, l'on ait

$$\lim k(\langle A_n*B_n\rangle) = k(\langle A*B\rangle).$$

Ces résultats visent à un théorème principal qui se trouve dans l'article suivant de l'auteur [Ann. Univ. Mariae Curie-Skłodowska Sect. A **16** (1962), 41–51; MR **29** #1605] et qui exprime la déviation linéaire $\mathfrak{r}(s)-s$ d'une géodésique d'une surface convexe lisse comme une intégrale faisant intervenir la courbure totale $k(x)=k(\langle A*M(x)\rangle)$. Les résultats obtenus pour les géodésiques des surfaces convexes lisses sont valables pour des courbes plus générales ainsi que le signalent des remarques. (Remarque du rapporteur. Le théorème de la Thèse du rapporteur cité p. 35 [*Les méthodes directes en géométrie différentielle*, Actualités Sci. Indust., No. 886, Hermann, Paris, 1941; MR 7, 67] recèle une erreur concernant la suffisance de la condition ainsi que l'a signalé E. J. van der Waag [Nederl. Akad. Wetensch. Proc. Ser. A **54** (1951), 390–403; MR **13**, 771] mais la conclusion qu'en tire l'auteur est correcte.) *Chr. Y. Pauc* (Nantes)

Avez, André 6452

Formule de Gauss-Bonnet-Chern en métrique de signature quelconque.

Rev. Un. Mat. Argentina **21**, 191–197 (1963).

This paper extends to compact riemannian manifolds with a metric of arbitrary signature the Gauss-Bonnet-Chern

formula. The same result was recently obtained independently by a rather different method by Chern [An. Acad. Brasil. Ci. **35** (1963), 17–26; MR **27** #5196].

Applications are made to Einstein spaces of four dimensions, and a result due to Milnor in the case of elliptic signature is generalised to apply to arbitrary signature.
T. J. Willmore (Liverpool)

Morimoto, Akihiko; Tanno, Shukichi 6453

Transformation groups of almost contact structures. (Japanese)

Sûgaku **16** (1964), 46–54.

This is one of the articles in the special issue of Sûgaku on the theories of almost complex structures and almost contact structures; substantially, it is a brief survey of the works of the first author [J. Math. Soc. Japan **15** (1963), 420–436; MR **27** #5275; Tôhoku Math. J. (2) **16** (1964), 90–104; MR **29** #549], M. Okumura [ibid. (2) **14** (1962), 398–412; MR **26** #4294], and the second author [ibid. (2) **15** (1963), 140–147; MR **27** #703; ibid. (2) **15** (1963), 322–331; MR **28** #4485].
T. Nagano (Berkeley, Calif.)

Ichijyo, Yoshihiro [Ichijô, Yoshihiro] 6454

On almost contact metric manifolds admitting parallel fields of null planes.

Tôhoku Math. J. (2) **16** (1964), 123–129.

This paper is concerned with local properties of manifolds which admit almost-contact metric structures whose associated field of complex null planes is parallel. It is shown that such manifolds have several properties in common with Kähler manifolds.
T. J. Willmore (Liverpool)

Tachibana, Shun-ichi 6455

Infinitesimal transformations of almost Hermitian spaces. (Japanese)

Sûgaku **16** (1964), 18–27.

This paper is a relevant exposition of recent developments in the field of infinitesimal transformations in almost Hermitian manifolds and in locally product manifolds. The author gives 20 theorems, including the following, with concise proofs, and mentions those manifolds to which these theorems have been generalized. (1) An almost analytic function in a compact almost complex manifold is a constant. (2) An infinitesimal isometry in a compact Kählerian manifold is an automorphism. (3) A harmonic vector in a compact Kählerian manifold is locally a gradient of an analytic function. (4) An infinitesimal conformal or projective transformation in a compact Kählerian manifold is isometric. (5) An infinitesimal conformal or projective transformation in a compact locally product manifold is isometric. (6) An infinitesimal holomorphically projective transformation in a compact manifold is analytic. (7) An infinitesimal analytic transformation preserving the differential form $4\varphi_\lambda{}^a R_{a\mu} dx^\lambda \wedge dx^\mu$ in a compact Kählerian manifold is an automorphism.
Y. Tashiro (Okayama)

Bruschi, Maria Luisa 6456

Un théorème de décomposition pour les variétés cosymplectiques homogènes.

C. R. Acad. Sci. Paris **257** (1963), 4120–4121.

Using the terminology and results of Lichnerowicz [Rend. Mat. e Appl. (5) **22** (1963), 197–244; MR **28** #564], it is shown that if G is a compact Lie group and G/K is a homogeneous co-symplectic space such that the associated horizontal algebra is semi-simple, then $G/K \simeq (G/H) \times S^1$, where G/H is an S-space.
J. W. Gray (Urbana, Ill.)

Hatakeyama, Yoji 6457

Complex and almost complex structures. (Japanese)

Sûgaku **16** (1964), 1–9.

Expository article on complex, almost complex, almost Hermitian, and almost symplectic structures.
K. Nomizu (Providence, R.I.)

Adler, A. 6458

Classifying spaces for Kähler metrics. III. The first Chern form and isometric imbeddings in Euclidean spaces.

Math. Ann. **156** (1964), 378–392.

Part II appeared in same Ann. **154** (1964), 257–266 [MR **28** #5408]. In a previous paper [ibid. **152** (1963), 164–184; MR **28** #1569] the author constructed a Riemannian manifold $B^*_{U(n)}$ with the property that every compact $2n$-dimensional Kahler manifold M is isometric to a submanifold $g(M)$ of $B^*_{U(n)}$ defined by means of an isometric embedding of M in some large Euclidean space. The map $g: M \to B^*_{U(n)}$ induces various differential operators and vector fields on M in terms of which the author obtains a necessary and sufficient condition for the first Chern form of M to be harmonic.
R. L. E. Schwarzenberger (Liverpool)

Luccioni, Raúl E. 6459

On the existence of measure for singular hyperquadrics in projective spaces. (Spanish)

Univ. Nac. Tucumán Rev. Ser. A **14** (1962), 269–276.

It is known that sets of projectively equivalent non-singular hyperquadrics have an invariant measure with respect to the projective group [the reviewer, Publ. Math. Debrecen **7** (1960), 226–237; MR **22** #12499; M. I. Stoka, Acad. R. P. Romîne Bul. Şti. Secţ. Şti. Mat. Fiz. **7** (1955), 903–937; MR **18**, 63]. In this paper the author proves that such an invariant measure does not exist for sets of singular hyperquadrics. However, there exists an invariant measure for sets of pairs of hyperquadrics whose vertices (locus of singular points) are linear spaces L_r, L_s which do not intersect each other and have complementary dimensions ($r+s=n-1$) and for sets of pairs of singular hyperquadrics whose vertex is of dimension r and linear spaces of dimension $n-r-1$ without common point with the vertex of the hyperquadric.
L. A. Santaló (Buenos Aires)

Stoka, Marius I. 6460

Géométrie intégrale dans un espace E_n.

Univ. Nac. Tucumán Rev. Ser. A **14** (1962), 25–59.

A Lie group of transformations is said to be measurable if it possesses one and only one integral invariant, up to a constant factor. The author gives the conditions of

measurability of a given r-parameter group G, operating in euclidean space E_n in terms of the infinitesimal transformations and the constants of structure of the group, and deduces some consequences and applications. Let F_q be a family of q-dimensional varieties in E_n and G the maximum group of transformations which leaves F_q invariant; the relations between the measurability of G and that of its subgroups are investigated. Most of the results, which are given clearly and systematically here, have been previously published by the author [Acad. R. P. Romîne Stud. Cerc. Mat. **9** (1958), 547–558; MR **21** #2285; Boll. Un. Mat. Ital. (3) **13** (1958), 470–485; MR **21** #2284; ibid. (3) **14** (1959), 467–476; MR **22** #4041; Rev. Math. Pures Appl. **4** (1959), 123–156; MR **22** #5005a; Rend. Circ. Mat. Palermo (2) 8 (1959), 192–205; MR **22** #5005b]. *L. A. Santaló* (Buenos Aires)

GENERAL TOPOLOGY

See also 5953, 5956, 6276, 6294, 6333, 6500.

Tondeur, Philippe 6461

Ein Beispiel zur allgemeinen Topologie: Die Topologie einer Äquivalenzrelation.

Ann. Acad. Sci. Fenn. Ser. A I No. 344 (1964), 7 pp.

Let ρ be a relation on the set X. The operator $\varphi(A) = \{p \mid p\rho q$ for some $q \in A\}$ $(A \subset X)$ is a Kuratowski closure operator on X if and only if ρ is reflexive and transitive. If ρ is, in addition, symmetric, then the resulting topology (denote it τ_ρ) is normal, locally compact, locally connected and first countable. τ_ρ is compact if and only if X/ρ is finite. Let ρ and σ be equivalence relations on the sets X and Y, respectively. A function $f: (X, \tau_\rho) \to (Y, \tau_\sigma)$ is continuous if and only if f maps ρ-equivalent points into σ-equivalent points. There are further results. *J. Mayer* (Albuquerque, N.M.)

Lynn, I. L. 6462

Linearly orderable spaces.

Trans. Amer. Math. Soc. **112** (1964), 189–218.

A topological space is linearly orderable if a linear ordering can be introduced into the space so that the original topology coincides with the interval topology. In an earlier note [Proc. Amer. Math. Soc. **13** (1962), 454–456; MR **25** #1536] the author gave a sufficient condition for a topological space to be linearly orderable. In the present paper sufficient conditions for a subspace of the real line to be linearly orderable are given. The following is the main theorem of the paper. Let X be a subspace of the real line R, and let $\eta(X)$ be the set of end points of the open ends of those components of $R \sim X$ which are half-open intervals. If no open subset of X is compact and $\eta(X)^- \cap X$ is countable (the closure being taken in R), then X is linearly orderable. The proof of the main theorem is quite long and occupies most of the paper. From this theorem, it follows that if a subspace X of R satisfies any of the following conditions, X is linearly orderable: (1) no open subset of X is compact and X has only countably many components, (2) X is a union of intervals containing no isolated closed (in R) interval, and (3) X is a union of open or half-open intervals. *I. Namioka* (Seattle, Wash.)

Day, Mahlon Marsh 6463

Correction to my paper "Fixed-point theorems for compact convex sets".

Illinois J. Math. **8** (1964), 713.

A left-invariant mean on $C(S)$, the space of all bounded continuous functions on the topological semigroup S, exists if and only if for each compact convex set K in each l.c.s. and for each homomorphism τ of X into the semigroup of affine continuous self-maps of K, with $[\tau s](y)$ continuous for some $y \in K$, there is a $p \in K$ such that $\sigma(p) = p$ for all $\sigma \in \tau(S)$.

This result replaces an incorrect remark made in a previous paper [same J. **5** (1961), 585–590; MR **25** #1547]. *M. Edelstein* (Halifax, N.S.)

Grace, E. E.; Heath, R. W. 6464

Separability and metrizability in pointwise paracompact Moore spaces.

Duke Math. J. **31** (1964), 603–610.

Alexandrov [Uspehi Mat. Nauk **15** (1960), no. 2 (92), 25–95; MR **22** #9947] introduced the notion of a "uniform base" in order to state (in part) the following metrization theorem: A regular space is metrizable if it has a uniform base and is paracompact or collectionwise normal. But, in order that a regular space have a uniform base, it is necessary and sufficient that it be a pointwise paracompact Moore space. Bing had already shown [Canad. J. Math. **3** (1951), 175–186; MR **13**, 264] that a Moore space is metrizable if it is paracompact or collectionwise normal. Thus, in Alexandrov's theorem, pointwise paracompactness is an unnecessary part of the notion of a uniform base. In the paper under review, metrization theorems are obtained in which pointwise paracompactness is necessary.

Suppose that S is a connected, pointwise paracompact Moore space (or equivalently, a connected, regular (T_1) space with a uniform base). Then S is separable (and hence metrizable) if (1) no point separates S and (2) for $p, q \in S$ $(p \neq q)$ there exists a closed, connected, separable set arbitrarily close to p which separates p from q or if (1) no point separates S, (2) S is locally connected and (3) for $p, q \in S$ $(p \neq q)$ there exists a closed separable set which separates p from q.

Furthermore, in a locally connected, pointwise paracompact Moore space, every monotonic collection (i.e., inclusion chain) of peripherally separable, connected, open sets contains a countable subcollection running upward through it. Similarly, and somewhat more generally, every connected, locally connected, locally peripherally separable, pointwise paracompact topological space is a Lindelöf space. Other results of a related nature are obtained. *F. Burton Jones* (Riverside, Calif.)

McCandless, Byron H. 6465

Perfect normality, local neighborhood extension spaces, and contractibility of absolute retracts.

Math. Z. **85** (1964), 385–391.

Let P denote the class of perfectly normal spaces. The author reviews the relation of P to the classes of collectionwise normal, of paracompact, and of Lindelöf spaces; he characterizes the metric absolute G_δ's by the perfect normality of certain spaces. Though the extension of certain results of Hanner [Ark. Mat. **2** (1952), 315–360; MR **14**, 396; errata, MR **14**, 1278] to the class P is not

straightforward, the author succeeds in proving (1) an AR(P) is contractible, and (2) a countable union of open NES(P) is also an NES(P). {Reviewer's remark: Theorem 7 is contained in a paper of E. Michael [Pacific J. Math. 3 (1953), 789–806, p. 793; MR 15, 547].}

J. Dugundji (Los Angeles, Calif.)

Lavallee, Lorraine D. 6466

Mosaics of metric continua and of quasi-Peano spaces.
Pacific J. Math. 14 (1964), 1327–1333.

Let X be a mosaic of compact metric spaces X_a, $a \in A$ [see Davison, Trans. Amer. Math. Soc. 91 (1959), 525–546; MR 21 #6575]. The author calls X an m-continuum space if each X_a is connected, and a quasi-curve space if instead each X_a is locally connected. Necessary and sufficient conditions (in terms of convergent sequences in X) are given for a topological space X to be an m-continuum space, and it is shown that X is then the mosaic of all its metrizable subcontinua. Other properties of m-continuum spaces are derived; for example, the components are open and are strongly (continuumwise) connected. Quasi-curve spaces are shown to coincide with curve spaces (for which the X_a's can be taken to be Peano spaces). The paper concludes by mentioning some unsolved problems. *A. H. Stone* (Rochester, N.Y.)

Chiang Chin-he [Jiang, Jia-he] 6467

The essential component of the set of fixed points of multivalued mappings and its application to the theory of games.
Acta Math. Sinica 14 (1964), 451–460 (*Chinese*); *translated as Chinese Math.—Acta* 5 (1964), 485–495.

Kinoshita showed [Osaka Math. J. 4 (1952), 19–22; MR 14, 193] that the set of fixed points of a continuous transformation of the Hilbert cube has an essential component. The author generalizes this to multivalued upper semi-continuous transformations with convex values, for compact convex sets in Banach spaces, with the natural application to sets of equilibrium points in non-cooperative games. *J. R. Isbell* (New Orleans, La.)

Juhász, I. 6468

On extremal values of mappings. I.
Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 6 (1963), 39–42.

The author calls a topological space R a $\sigma(k)$-space if the set of local extreme values of a function into any partially ordered set is always of cardinal $\leq k$. Theorem: If the weight of R (least cardinal of a base) is k, then R is a $\sigma(k)$-space; and if R is a $\sigma(k)$-space, then the least cardinal of a dense set is $\leq k$. It follows that a pseudometric space is a $\sigma(k)$-space if and only if its weight is $\leq k$. The author conjectures that the conclusion holds as well for arbitrary completely regular spaces; in any event, he proves that each completely regular $\sigma(k)$-space has weight $\leq 2^k$. In contrast, he also shows that a $\sigma(k)$-space can have arbitrarily large weight, even if it is T_1.

L. Gillman (Rochester, N.Y.)

Pasynkov, B. 6469

Partial topological products. (Russian)
Dokl. Akad. Nauk SSSR 154 (1964), 767–770.

Consider a space X, a family $\{G_\alpha\}$ of open subsets and a family $\{Z_\alpha\}$ of spaces. A "partial product" $P(X, \{Z_\alpha\}, \{G_\alpha\})$ is defined. For the case of compact spaces Z_α, the definition is equivalent to that of a "local product" introduced by the author [same Dokl. 150 (1963), 40–43; MR 28 #576]; if all G_α are equal to X, then $P(X, \{Z_\alpha\}, \{G_\alpha\})$ coincides with the cartesian product $X \times \prod Z_\alpha$. Seven theorems are given (without proofs); only some simpler results may be stated here.

For any cardinal number n, let $\mathcal{T}^n$ [D^n] denote the cartesian product of n segments [two-point spaces]. Let $\{G_\alpha\}$ be an open base in $\mathcal{T}^n$ and let Z_α be spaces. Then (1) if n is finite and every Z_α is a discrete space of power τ, then $P(\mathcal{T}^n, \{Z_\alpha\}, \{G_\alpha\})$ is universal for all metrizable spaces of dimension $\leq n$ and of weight $\leq \tau$; (2) if every Z_α is discrete of power τ, then $P(\mathcal{T}^{\aleph_0}, \{Z_\alpha\}, \{G_\alpha\})$ is universal for all metrizable spaces of weight $\leq \tau$; (3) if $n \leq \tau$ and Z_α coincides with D^τ, then $P(\mathcal{T}^n, \{Z_\alpha\}, \{G_\alpha\})$ is universal in the class of all those completely regular spaces which admit of a separating mapping of weight τ [cf. A. Zarelua, ibid. 144 (1962), 713–716; MR 26 #5541]; (4) if R is metrizable of weight τ and $\dim R > 0$, then there exists a space S_R and a mapping f of S_R onto R such that S_R is metrizable, $w(S_R) = \tau$, $\dim S_R = 1$, and f is closed, open, compact, 0-dimensional. *M. Katětov* (Prague)

Bothe, H. G. 6470

Ein eindimensionales Kompaktum im E^3, das sich nicht lagetreu in die Mengersche Universalkurve einbetten lässt.
Fund. Math. 54 (1964), 251–258.

The Menger universal curve $U \subset E^3$ [K. Menger, *Kurventheorie*, Teubner, Leipzig, 1932] is a 1-dimensional curve (i.e., continuous image of [0, 1]) in which every 1-dimensional metric space can be embedded. The author describes a 1-dimensional curve $X \subset E^3$ for which there is no homeomorphism $h: E^3 \to E^3$ such that $h(X) \subset U$. Like U, X is constructed by punching holes through a cube, except that knotted holes are used. Apart from classical spadework, the proof is elementary and proceeds by showing that a certain simple closed curve cannot be freed from X via a small homeomorphism of E^3.

R. F. Williams (Chicago, Ill.)

Cook, H. 6471

Concerning connected and dense subsets of indecomposable continua.
Fund. Math. 53 (1963), 21–23.

In this paper it is shown that if the continuum hypothesis is true, every compact metric indecomposable continuum M contains a connected point set K, containing no perfect set, such that each component of M contains one and only one point of K. An early result of P. M. Swingle [Bull. Amer. Math. Soc. 37 (1931), 254–258] is used. The paper is well-organized, with the proofs concise.

L. K. Barrett (Knoxville, Tenn.)

Hudson, Anne L. 6472

Note on pointwise periodic semigroups.
Proc. Amer. Math. Soc. 15 (1964), 700–702.

The author calls an element x of a semigroup S periodic if there is a natural number n such that $x^{n+1} = x$. The least

such n is called the period of x and is denoted by $p(x)$. (Since the main result of the paper concerns the notion of the period of an element, it should be remarked that Poole [Amer. J. Math. 59 (1937), 23–32] and others have called an element x periodic if there are positive integers n and ρ such that $x^{n+\rho}=x^{\rho}$. In this case the least n is the period and ρ the index or preperiod.) The author establishes as a corollary the following. Let S be a pointwise periodic semigroup on an n-cell. Then $p(x)\leq 2$ for all x in S. This answers a problem raised by A. D. Wallace [Bull. Amer. Math. Soc. 68 (1962), 447–448]. The result established is that if S is a compact semigroup which is the union of totally disconnected (compact) subgroups and contains a closed set B such that $S-B$ is a connected manifold dense in S and $\dim B<\dim S$, then $p(x)\leq 2$ for all x in S. This is done by invoking results of the reviewer and Anderson [Math. Ann. 147 (1962), 248–268; MR 26 #4324] to first get $\dim S=\dim E$ and then by using a result of Newman on periodic homeomorphisms [Quart. J. Math. Oxford Ser. 2 (1931), 1–8].

R. P. Hunter (University Park, Pa.)

Haddock, A. Glen 6473

Some theorems related to a theorem of E. Helly.

Proc. Amer. Math. Soc. 14 (1963), 636–637.

The author generalizes a result of J. Molnár [Mat. Lapok 8 (1957), 108–114; MR 20 #6689] with his Theorem 2: If $\{C_a\}$ is any collection of compact simply connected sets in the plane such that the intersection of any two is nonempty and the union of any three fails to separate the plane, then the intersection of the collection is nonempty.

R. H. Rosen (Ann Arbor, Mich.)

Lickorish, W. B. R. 6474

On the homeomorphisms of a non-orientable surface.

Proc. Cambridge Philos. Soc. 61 (1965), 61–64.

This short paper answers a question asked by the author in an earlier paper [same Proc. 59 (1963), 307–317; MR 26 #3029]. Theorem. Let X be any closed, connected, non-orientable 2-manifold other than the projective plane. There is a Y-homeomorphism $y:X\to X$ such that if $f:X\to X$ is any homeomorphism, f is isotopic to Hy^{ϵ}, where H is a product of twists and $\epsilon=0$ or 1.

A twist is a homeomorphism previously called a c-homeomorphism, and this and Y-homeomorphisms are defined in the earlier paper cited above.

H. R. Gluck (Cambridge, Mass.)

Černavskiĭ, A. V. 6475

Isotopy of elements and spheres in n-dimensional space with $k<\frac{1}{2}n-1$. (Russian)

Dokl. Akad. Nauk SSSR 158 (1964), 62–65.

Let E^n be Euclidean n-space. The author establishes the following results. Theorem 1: If $k<\frac{1}{2}n-1$, then every homeomorphism $E^n\to E^n$ is expressible as a composition h_2h_1, where h_1 is stable (in the sense of Morton, Brown and Gluck) and h_2 is the identity on a k-dimensional hyperplane. Theorem 2: If $k<\frac{1}{2}n-1$, then every locally flat embedding of a k-sphere in E^n is isotopic to the embedding of the boundary of a $(k+1)$-simplex, by an isotopy which is constant outside a bounded open set. Theorem 3: If $k<\frac{1}{2}n-1$, then any two locally flat embeddings of a k-simplex in E^n which coincide on the boundary of the simplex are isotopic, by an isotopy which is the identity on the image of the boundary and outside a bounded open set. The author observes that it follows from a classical result of Alexander that Theorem 1 is equivalent to Theorem 1′: If $k<\frac{1}{2}n-1$, then every locally flat embedding of a k-simplex in E^n is isotopic to the standard semilinear embedding by an isotopy which is the identity outside a bounded open set. Theorem 1 is proved by a small generalization of the Stallings-Zeeman engulfing lemma, and Theorems 2 and 3 are then deduced from Theorem 1′ by applying results of Stallings [Ann. of Math. (2) 77 (1963), 490–503; MR 26 #6946].

The author observes that, if $k\leq\frac{1}{2}n-2$, then his results may be deduced from results of Gluck [Bull. Amer. Math. Soc. 69 (1963), 824–831; MR 27 #5241b] and Greathouse [ibid. 69 (1963), 820–823; MR 27 #5241a]. On the other hand, his results lead in turn to the extension of the results of Gluck and Greathouse to the case $k<\frac{1}{2}n-1$.

P. J. Hilton (Ithaca, N.Y.)

King, L. R.; Roberts, J. H.; Rosenstein, G. M., Jr. 6476

Concerning some problems raised by A. Lelek.

Fund. Math. 54 (1964), 325–334.

Let C denote a collection of disjoint subsets of Euclidean n-space E^n, let C^* denote the union of the elements of C, and let $A(C)$ denote the set of all points p such that $p=\lim C_i$, where $C_1,C_2,\cdots$ are elements of C. Lelek [Fund. Math. 51 (1962/63), 87–109; MR 26 #1862] defined a "fixation" of C as a set $Z\subset E^n$ which intersects each element of C, and raised some questions about conditions under which C (I) has a 0-dimensional compact fixation, or (II) has an arc as a fixation, or (III) has, for each ϵ, a fixation which is the union of a finite number of disjoint closed sets of diameter less than ϵ. The authors construct two examples in E^2 to give negative answers to the following three of Lelek's questions. (1) Does (I) hold if C^* is bounded and the elements of C are connected sets whose diameters have a positive lower bound? (2) Does (III) imply (I) if C^* is bounded and the elements of C are connected? (3) Does (II) imply (III) if $\dim A(C)\leq 0$ and the elements of C are connected?

C. E. Burgess (Salt Lake City, Utah)

Mazur, Barry 6477

The method of infinite repetition in pure topology. I.

Ann. of Math. (2) 80 (1964), 201–226.

The object of this paper, as stated by the author, is to present the formal aspects of a method of infinite repetition in topology which, in various guises, has already been capitalized on by the author himself [Bull. Amer. Math. Soc. 65 (1959), 59–65; MR 22 #8469], M. Brown [ibid. 66 (1960), 74–76; MR 22 #8470b], M. Morse [ibid. 66 (1960), 113–115; MR 22 #8470a] and J. R. Stallings [ibid. 66 (1960), 485–488; MR 23 #A2214]. Essentially, the method is embedded in a theory of canonical neighborhoods for arbitrary subspaces of arbitrary topological spaces, and standard applications of the method appear as straightforward corollaries in the formal theory. The author plans to apply the theory to obtain his stable homeomorphism theorem [ibid. 67 (1961), 377–384; MR 24 #A557] in a sequel to this paper.

H. R. Gluck (Cambridge, Mass.)

McMillan, D. R., Jr. 6478

The singular points of a topological embedding.
Duke Math. J. **31** (1964), 711–716.

Let Y be a subset of the space X and let $x \in X$. Following the terminology introduced by Eilenberg and Wilder [Amer. J. Math. **64** (1942), 613–622; MR **4**, 87] the author calls Y locally p-connected at x if for each neighborhood U of x there is a neighborhood V of x such that $V \subseteq U$ and each map of S^p in $V-Y$ is null-homotopic in $U-Y$.

In this paper the author strengthens some of his previous work in characterizing open domains of S^n ($n \neq 4$) which are topologically equivalent to E^n by a study of their connectivity—in the above sense—at their boundaries [see Ann. of Math. (2) **79** (1964), 327–337; MR **28** #4528]. He seems to prefer stating his results in terms of embeddings of S^{n-1} in S^n, as may be witnessed by his Theorem 2. Let Σ be a topological $(n-1)$-sphere in S^n, where $n \neq 4$; let K be a component of $S^n - \Sigma$; and suppose that K is locally 1-connected at each point of $\Sigma - X$, where X is a compact subset of Σ. In either of the following cases K is homeomorphic to E^n: (I) If X is countable; (II) If $\Sigma - X$ is 1-connected and for each open set U of S^n such that $\Sigma \subseteq U$, there is an open set V such that $\Sigma \subseteq V \subseteq U$ and each loop in $V - X$ is null-homotopic in $U - X$.

This result is interesting and useful, but the restriction that the boundary of K is an $(n-1)$-sphere seems somewhat artificial since few properties of the sphere are needed to prove the theorem. Apparently if Σ were only an ANR of the homotopy type of S^{n-1} such that $\Sigma - X$ were 1-connected, the result would still hold.

R. H. Rosen (Ann Arbor, Mich.)

Poénaru, Valentin 6479

Sur quelques propriétés des variétés simplement connexes à trois dimensions.
Rend. Mat. e Appl. (5) **20** (1961), 235–269.

From the author's introduction: "Le présent travail contient les démonstrations des deux propositions suivantes. Théorème 1: Soit V_3 une variété simplement connexe, compacte à trois dimensions, $B_3 \subset V_3$ une cellule à trois dimensions contenue dans V_3. Il existe alors une application localement topologique: $f\colon (V_3 - \operatorname{int} B_3) \to E_3$. Théorème 2: Soit V_3 une variété simplement connexe, compacte à trois dimensions. Il existe alors un nombre positif, entier, $n(V_3)$ tel que si l'on enlève de V_3, $n(V_3)$ cellules à trois dimensions, deux à deux disjointes $B_3{}^1$, $B_3{}^2, \cdots, B_3{}^{n(V_3)}$ on ait:

$$B_2 \times \left(V_3 - \bigcup_1^{n(V_3)} \operatorname{int} B_3{}^i\right) = B_2 \times \left(S_3 - \bigcup_1^{n(V_3)} \operatorname{int} B_3{}^i\right),$$

où S_3 est la sphère à trois dimensions.

"Le théorème 2 a comme corollaire immédiat: Soit V_3 une variété simplement connexe, compacte à trois dimensions. Il existe un entier positif $n(V_3)$ tel que si $P^1, P^2, \cdots, P^{n(V_3)}$ sont des points distincts de V_3:

$$E_2 \times \left(V_3 - \bigcup_1^{n(V_3)} P^i\right) = E_2 \times \left(S_3 - \bigcup_1^{n(V_3)} P^i\right)."$$

Phillip A. Griffiths (Berkeley, Calif.)

ALGEBRAIC TOPOLOGY

See also 5746, 5747, 5874, 5943, 6475.

Andrásfai, B. 6480

Graphentheoretische Extremalprobleme.
Acta Math. Acad. Sci. Hungar. **15** (1964), 413–438.

The graphs of this paper are without loops or multiple joins. The valency of a vertex p of a graph G is denoted by $\phi_G(p)$, and the least valency in G by $\phi_0(G)$.

The paper is mainly concerned with the following problem. Suppose the number n of vertices of G and the maximum number τ of independent vertices of G are given. Suppose further that k is an integer such that no odd polygon of G has fewer than $2k+1$ vertices. What is the greatest possible value of $\phi_0(G)$, and for what graphs is it attained?

The case $\tau/n \geq \frac{1}{2}$ is solved easily; $\phi_0(G) \leq n - \tau$, and the equality holds for the complete bipartite graph with τ vertices in one class and $n-\tau$ in the other. The main result of the paper deals with the interval $k/(2k+1) \leq \tau/n < \frac{1}{2}$ and is as follows. Write $n = 2\tau + \delta$. Then if k is even, $(3k+2)\phi_0(G) \leq 6\tau + 4\delta$, and if k is odd, $(k+1)\phi_0(G) \leq 2\tau + 2\delta$. Moreover, the stated upper bound for $\phi_0(G)$ can be attained whenever it is an integer.

The last section of the paper is an interesting survey of some related results and conjectures.

W. T. Tutte (Waterloo, Ont.)

Andreatta, Antonio 6481

Alcuni sviluppi sulla teoria relativa dei singrammi finiti.
Ann. Mat. Pura Appl. (4) **65** (1964), 1–25.

The author investigates the embedding of a graph on a surface, which may be orientable or non-orientable. It is shown that each such embedded graph is obtainable from a cubic graph with a Hamilton circuit lying on the same surface. This leads to a new proof of the Kuratowski criterion for planar graph and also a criterion for a cubic graph with Hamilton circuit to be embeddable in the projective plane.

O. Ore (New Haven, Conn.)

Dirac, G. A. 6482

Valency-variety and chromatic number of abstract graphs.
Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe **13** (1964), 59–63.

In this paper the symbol $\langle k \rangle$ denotes a complete k-graph, and $\Phi(c)$ denotes the join of a pentagon to a $\langle c-3 \rangle$. The valency-variety of a finite graph Γ is the number of different valencies occurring in it. The author studies the relation between the valency-variety w, the total number n of vertices and the chromatic number k of Γ. He finds that $k \leq n - [\frac{1}{2}w]$, equality being possible whenever $n \geq 2$ and $1 \leq w \leq n-1$. If Γ contains no $\langle n - [\frac{1}{2}w] \rangle$, then $k \leq n - [\frac{1}{2}w] - 1$, and the equality can hold whenever $n \geq 4$ and $2 \leq w \leq n-2$, and whenever n is even, $n \geq 4$ and $w = n-1$. If w is odd and Γ contains no $\langle n - [\frac{1}{2}w] - 1 \rangle$ or $\Phi(n - [\frac{1}{2}w] - 1)$, then $k \leq n - [\frac{1}{2}w] - 2$.

The author also considers a regular graph Γ ($w = 1$) in which the valency v satisfies $3 \leq v \leq n-2$. It is known that $k \leq v+1$ in such a graph, with equality only when Γ contains a $\langle v+1 \rangle$. The author shows that if $[\frac{1}{2}n] \leq v \leq n-3$

and $n \equiv v+1 \bmod 2$, then $k \le \min(v, \frac{1}{2}n)$. For $[\frac{1}{2}n] \le v \le n-4$ and $n \equiv v \bmod 2$ he finds that

$$k \le \min(v, (2n-2v-3)n/(3n-3v-4)).$$

If $v = n-2$, then $k = \frac{1}{2}n$. *W. T. Tutte* (Waterloo, Ont.)

Foster, B. L. 6483

Short proof of a theorem of Rado on graphs.

Proc. Amer. Math. Soc. **15** (1964), 865–866.

Radó's theorem is stated here as follows. Given a locally finite graph (G, Γ), a finite set of integers K, and a mapping T of subsets of K into subsets of K; if each finite subgraph (A, Γ_A) admits a function ϕ_A such that $\phi_A(x) \in T(\phi_A(\Gamma_A x))$ for all $x \in A$, then (G, Γ) admits a function ϕ such that $\phi(x) \subset T(\phi(\Gamma x))$ for all $x \in G$. Here a (directed) graph is defined by a (not necessarily single-valued) mapping Γ of a set G into itself.

The author states that the proof of this theorem in Berge's textbook [C. Berge, *Théorie des graphes et ses applications*, deuxième édition, Dunod, Paris, 1963; MR **27** #5246] suffers from inaccuracies. He offers instead a short proof based on König's "Unendlichkeitslemma".

W. T. Tutte (Waterloo, Ont.)

Gallai, T. 6484

Elementare Relationen bezüglich der Glieder und trennenden Punkte von Graphen. (Russian summary)

Magyar Tud. Akad. Mat. Kutató Int. Közl. **9** (1964), 235–236.

Harary [Amer. Math. Monthly **66** (1959), 405–407; MR **21** #2986] proved that if a connected graph has g blocks and t articulation points, the ith of which is contained in g_i blocks, then $g = 1 + \sum_{i=1}^{t} (g_i - 1)$. In the present note it is shown that this result and its dual follow almost immediately from the fact that in any tree the number of points exceeds the number of edges by one.

J. W. Moon (Edmonton, Alta.)

Grünbaum, Branko 6485

A simple proof of a theorem of Motzkin.

Nederl. Akad. Wetensch. Proc. Ser. A **67** = *Indag. Math.* **26** (1964), 382–384.

Let G be a cubic planar graph, each of whose connected components is 2-connected, such that the number of sides of each face of G is a multiple of 3. It is shown that this implies that the parity of the number of faces of G is different from the parity of the number of connected components of G. This contains as a special case a result announced by Motzkin [Notices Amer. Math. Soc. **11** (1964), 242]. *J. W. Moon* (Edmonton, Alta.)

Halin, R. 6486

Über simpliziale Zerfällungen beliebiger (endlicher oder unendlicher) Graphen.

Math. Ann. **156** (1964), 216–225.

In another paper [same Ann. **147** (1962), 126–142; MR **26** #756] the author has defined the simplicial decomposition of a finite graph. In the present paper he extends the idea to infinite graphs. He finds that any graph which has no infinite complete subgraph has a unique simplicial decomposition into "prime" subgraphs.

W. T. Tutte (Waterloo, Ont.)

Klee, Victor 6487

A property of d-polyhedral graphs.

J. Math. Mech. **13** (1964), 1039–1042.

A graph is d-polyhedral if it is isomorphic to the 1-dimensional skeleton of a d-dimensional convex polytope. It is known that any d-polyhedral graph is d-connected. The author extends this result and applies it.

Given a graph G and two disjoint sets of its nodes, U and V, U is totally separated by V in G if every path joining a pair of nodes of U intersects some node of V. For $n \ge 0$, $\sigma_n(G)$ is the largest integer m for which there is some set of m nodes in G that is totally separated by some set of n nodes in G. Finally, for $n \ge d+1$, let $\mu(d, n)$ be the maximum number of $(d-1)$-faces of a d-polytope with n vertices. The result is this: $\max_G\{\sigma_n(G) | G$ is d-polyhedral$\} = 1$ if $n \le d-1$, $= 2$ if $n = d$, and $= \mu(d, n)$ if $n \ge d+1$. The result is used to show that for every d there exist d-polyhedral graphs that are not e-polyhedral for any $e \ne d$. *M. L. Balinski* (Princeton, N.J.)

Parthasarathy, K. R. 6488

Enumeration of paths in digraphs.

Psychometrika **29** (1964), 153–165.

Author's summary: "The solution of the problem of enumeration of the n-paths in a digraph has so far been attempted through an indirect approach of enumerating the redundant chains. The approach has yielded an algorithm for determination of the general formula for the matrix of redundant n-chains and also a partial recurrence formula for the same. This paper presents a direct approach to the problem. It gives a recurrence relation expressing the matrix of n-paths of a digraph in terms of the matrices of $(n-1)$-paths of its first-order subgraphs. The result is exploited to give an algorithm for computing the matrix of n-paths. The algorithm is illustrated with a 6×6 matrix." *G. Sabidussi* (Hamilton, Ont.)

Sabidussi, Gert 6489

Infinite Euler graphs.

Canad. J. Math. **16** (1964), 821–838.

The author studies infinite Euler graphs, that is, graphs whose local degrees are even or infinite. It is shown that a graph is an Euler graph if and only if it has an edge-disjoint covering by means of finite or infinite circuits, generalising a result by Veblen for finite graphs. The paper also contains criteria for an Euler graph to be coverable by finite circuits. *O. Ore* (New Haven, Conn.)

Ivanovskiĭ, L. N. 6490

Cohomologies of a Steenrod algebra. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 1284–1287.

The first three theorems in this paper are concerned with the spectral sequence for extensions of Hopf algebras, similar to some work of Adams [Ann. of Math. (2) **72** (1960), 20–104; MR **25** #4530]. Next, the author announces some results on the cohomology of the Steenrod algebra, as well as applications to stable homotopy groups of spheres. Barratt, Liulevicius, May, and others have made similar calculations, although their work is not yet in print. *D. W. Kahn* (Minneapolis, Minn.)

Hsiang, Wu-yi [Hsiang, Wu-Yi] 6491

On the unknottedness of the fixed point set of differentiable circle group actions on spheres—P. A. Smith conjecture.

Bull. Amer. Math. Soc. **70** (1964), 678–680.

The principal result can be stated without reference directly to knottedness. The assertion is that a differentiable action of S^1 on S^n with an $(n-2)$-dimensional fixed-point set F is equivalent to an orthogonal action if and only if F is an $(n-2)$-sphere for $n > 5$. This implies that if F is S^{n-2}, then this $(n-2)$-sphere is unknotted. An essentially complete argument is presented.

P. E. Conner (Charlottesville, Va.)

Eckmann, B.; Ganea, T.; Hilton, P. J. 6492

Generalized means.

Studies in mathematical analysis and related topics, pp. 82–92. Stanford Univ. Press, Stanford, Calif., 1962.

A mapping $f: X^n \to X$, where X^n denotes the n-fold Cartesian product, is called an n-mean if it is symmetric in the variables and satisfies $f(x, x, x, \cdots, x) = x$. Such means have been studied by Aumann and Carathéodory [cf. Aumann, Math. Ann. **119** (1944), 210–215; MR **6**, 277] and by Eckmann [Comment. Math. Helv. **28** (1954), 329–340; MR **16**, 503]. Eckmann showed, among other things, that an n-mean of a space is carried over, in an appropriate way, to its homotopy groups. The present paper places the study in a category-theoretic framework. The authors make an appropriate definition of an object with n-mean within the context of a category in which direct products exist. Among the categories considered are $\mathscr{G}$—that of groups and homomorphisms, $\mathscr{A}$—that of abelian groups and homomorphisms, $\mathscr{T}_h$—that of based spaces (of the based homotopy type of polyhedra) and based homotopy classes. Having made appropriate definitions and having shown that a covariant D-functor preserves means, the authors obtain results having as corollaries the following. If $A \in \mathscr{T}_h$ admits an n-mean, then so does A^X, ΩA, the loop space on A, and $\tilde{A}$, the universal covering space of A. If $A \in \mathscr{T}_h$ admits an n-co-mean, then A^X admits an n-mean (for any X).

After considering notions of primitivity and their connections with means and co-means, the authors note that the only objects with an n-co-mean in $\mathscr{G}$, where $n \geq 2$, are the trivial groups. Assuming the spaces considered to be connected and to have the based homotopy type of polyhedra, they show that if $A \in \mathscr{T}$ admits an n-mean, then it admits the structure of a homotopy commutative H-space. It follows that if $n \geq 2$ and $A \in \mathscr{T}$ has the homotopy type of a polyhedron, then A admits a homotopy n-mean if and only if it is contractible. (As was shown by Eckmann, an indecomposable continuum such as the n-adic solenoid may admit an n-mean. Another example of a continuum with n-mean would be any compact connected semi-lattice.)

There are a number of results on n-co-means, contractibility and H'-spaces.

A notion of means in groups, not necessarily abelian, was considered by Scott [Amer. J. Math. **74** (1952), 667–675; MR **13**, 910] and B. H. Neumann [J. London Math. Soc. **38** (1963), 226–227; MR **28** #137].

R. P. Hunter (State College, Pa.)

Brooks, Edgar 6493

Homotopy groups of compact Abelian groups.

Proc. Amer. Math. Soc. **15** (1964), 878–881.

The author computes the homotopy groups of an arbitrary compact abelian group. Let G be such a group. Then $\pi_n(G) = 0$, $n \geq 2$, and $\pi_1(G)$ is isomorphic to $\mathrm{Hom}(G^*, Z)$. Here G^* denotes the Pontrjagin dual of G ($G^* = \mathrm{Hom}(G, T)$, T the group of complex numbers of length 1), and Z is the additive group of integers.

P. G. Kumpel, Jr. (Stony Brook, N.Y.)

Toda, Hirosi 6494

A survey of homotopy theory. (Japanese)

Sûgaku **15** (1963/64), 141–155.

This survey is a skillful treatise on homotopy theory designed to present some recent developments of the subject, and it consists of a well-organized offering of materials and results.

To begin with, the author comments briefly on three main streams of methods in homotopy theory to be called, say, constructive, cohomological and differentio-topological, respectively, along which the contents of this survey are arranged.

The first half contains some basic notions, such as the homotopy set functors, the space functors, together with their properties and relations. The author gives various examples of homotopy sets admitting some natural group structures, among which are included the K-groups in the sense of Grothendieck, Atiyah and Hirzebruch, and also several exact sequences of importance concerning the above groups.

The second half deals with various types of operations. The operations of cohomotopy type $\Psi: \pi(\ , Y) \to \pi(\ , Y')$ are of special importance, and give, as examples, the usual cohomology operations, the generalized Hopf invariants, the Chern character and the operations in the K-theory. The operation of the second kind is defined as the secondary composition (the so-called Toda bracket). An example is called a functional Chern character, which gives rise to an interesting invariant

$$CH^{n+k}: \pi_{2n+2k-1}(S^{2n}) \to Q/Z.$$

This invariant is closely related to a work of Adams and Dyer.

The last section is for applications and contains the following: a brief indication of the composition method to calculate the homotopy groups of spheres, some topics on the invariant CH^{n+k}, some results on the unstable homotopy groups of unitary groups and on the stable homotopy groups of Moore spaces.

N. Shimada (Kyoto)

Mimura, Mamoru; Toda, Hirosi 6495a

Homotopy groups of SU(3), SU(4) and Sp(2).

J. Math. Kyoto Univ. **3** (1963/64), 217–250.

Mimura, Mamoru; Toda, Hirosi 6495b

Homotopy groups of symplectic groups.

J. Math. Kyoto Univ. **3** (1963/64), 251–273.

These are two papers of a series planned by the authors on the homotopy of the simple groups.

In the first paper they compile π_i, $G = SU(3)$, $SU(4)$ or $Sp(2)$ up to $i = 23$.

In the second one they study the almost stable groups $\pi_i(\mathrm{Sp}(n))$, $i=4n+2, 4n+3, 4n+4$. Their result here is that

$$\begin{aligned}\pi_{4n+2} &= \mathbf{Z}_{2(2n+1)!}, && n \text{ odd},\\ &= \mathbf{Z}_{(2n+1)!}, && n \text{ even},\\ \pi_{4n+3} &= \mathbf{Z}_2,\\ \pi_{4n+4} &= \mathbf{Z}_2, && n \text{ odd},\\ &= \mathbf{Z}_2+\mathbf{Z}_2, && n \text{ even}.\end{aligned}$$

R. Bott (Cambridge, Mass.)

Porter, Gerald J. 6496

Homotopical nilpotence of S^3.

Proc. Amer. Math. Soc. **15** (1964), 681–682.

The homotopical nilpotence of S^3 (the group of unit quaternions) is shown to be 4. According to Samelson, one has to prove that $[[j,j],j]\neq 0$, $[[[j,j],j],j]=0$, where j is a generator of $\pi_4(Q)$, and $Q=B_{S^3}$, the infinite quaternionic projective space. This is deduced from results of Hilton on iterated Whitehead products in S^4.

A. Dold (Heidelberg)

TOPOLOGY AND GEOMETRY OF DIFFERENTIABLE MANIFOLDS

See also 5962, 5984, 6063, 6068, 6069, 6306, 6458, 6475, 6494, 6495a–b.

Anderson, Peter G. 6497

Cobordism classes of squares of orientable manifolds.

Bull. Amer. Math. Soc. **70** (1964), 818–819.

Theorem: For every orientable manifold N there exists a spin manifold N which is cobordant to $M\times M$ mod 2, i.e., $[N]=[M\times M]\in\mathfrak{N}$. No full proof is given, but there are indications as follows: By a certain algebraic-geometric procedure applied to projective spaces, a family of manifolds V is constructed. With ground field $\mathbf{R}$ one gets manifolds $V_{\mathbf{R}}$, with $\mathbf{C}$ one gets $V_{\mathbf{C}}$, and comparison of Stiefel-Whitney numbers shows $[V_{\mathbf{C}}]=[V_{\mathbf{R}}\times V_{\mathbf{R}}]$. The $V_{\mathbf{C}}$ turn out to be spin manifolds, and among the $V_{\mathbf{R}}$ generators are found for $\mathrm{Im}(\Omega\to\mathfrak{N})$.

A. Dold (Heidelberg)

Sotomayor, Jorge 6498

On the abundance of regular points for a differentiable map and submanifolds defined implicitly. (Spanish)

Univ. Nac. Ingen. Inst. Mat. Puras Apl. Notas Mat. **1** (1962/63), 152–155.

Golo, V. L. 6499

Smooth structures on manifolds with boundary. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 22–25.

Continuing the work of W. Browder [Colloq. Algebraic Topology, 1962, Mat. Inst., Aarhus Univ., Aarhus, 1962; MR 26 #3565] and S. P. Novikov [Dokl. Akad. Nauk SSSR 143 (1962), 1046–1049; MR 25 #1628; Izv. Akad. Nauk SSSR Ser. Mat. 28 (1964), 365–474; MR 28 #5445], the author studies problems of equivalence for manifolds with boundary. Let $W=W^n$ be a smooth n-manifold (always 1-connected) with boundary ∂W (also 1-connected). Let T_W and $T_{\partial W}$ be the Thom spaces. Define $A\in\pi_{N+n}(T_W, T_{\partial W})$ by setting $\alpha\in A$ if there is a smooth map $f:(D^{N+n}, S^{N+n-1})\to(T_W, T_{\partial W})$, N large, representing α, which is t-regular on W and a diffeomorphism on $f^{-1}(\partial W)$. Let π^+ be the homotopy classes of maps $F:(W,\partial W)\to(W,\partial W)$ which are represented by maps which, when restricted to the boundary, are diffeomorphisms of degree 1 such that $F^*\nu_W=\nu_W$ (ν_W is the normal bundle). Also define $\pi_{SO}=\pi(W, SO_N)$. The author then announces the following results. π_{SO} operates on A with factor space B. π^+ operates on B. (1) If $n\not\equiv 2 \pmod 4$, B/π^+ may be put into 1-1 correspondence with the set of smooth manifolds, which are homotopically equivalent to $(W,\partial W)$ by a map which respects normal bundles and is a diffeomorphism of degree 1 on ∂W, divided into equivalence classes via diffeomorphisms of degree 1. (2) If $n\equiv 2 \pmod 4$, B/π^+ contains a subset in 1-1 correspondence with the set of such manifolds equivalent under diffeomorphism. The author also announces the following theorem: If W with boundary ∂W is such that

$$H_{4k}(W)\otimes Q=0 \quad \text{for } k=1,2,\cdots,$$

then the set of classes of smooth manifolds with boundary ∂W, which are equivalent under diffeomorphism and which satisfy the condition that the pair of the manifold and boundary are homotopy equivalent to $(W,\partial W)$ by a map which is a diffeomorphism of degree 1 on the boundary, is finite.

The methods indicated include Morse theory, results of Smale, and Bott periodicity for $B_{SO(N)}$. An interesting fact arises as a lemma. While a generalization of the Hirzebruch index theorem to manifolds with boundary appears to be unknown, the author generalizes the proposition that cobordant manifolds have the same index to manifolds with boundary. The author excludes manifolds W^n with $n<7$ or $n=14$ at the beginning.

D. W. Kahn (Minneapolis, Minn.)

Wall, C. T. C. 6500

Cobordism of combinatorial n-manifolds for $n\leq 8$.

Proc. Cambridge Philos. Soc. **60** (1964), 807–811.

L'auteur compare les groupes de cobordisme orienté Ω_n^d et Ω_n^c [non orienté $\mathfrak{N}_n^d$ et $\mathfrak{N}_n^c$] des variétés différentiables et combinatoires de dimension n. Il remarque tout d'abord que les théorèmes de triangulation de J. H. C. Whitehead permettent de définir des homomorphismes $T_n:\mathfrak{N}_n^d\to\mathfrak{N}_n^c$ et $U_n:\Omega_n^d\to\Omega_n^c$ qui sont en fait injectifs, puisque les nombres de Stiefel-Whitney et de Pontrjagin déterminent complètement Ω_n^d et $\mathfrak{N}_n^c$ et peuvent aussi se définir dans le cas combinatoire.

L'auteur démontre alors que: (a) pour $n\leq 7$, T_n et U_n sont des isomorphismes, (b) $\Omega_8^c=\mathbf{Z}+\mathbf{Z}+\mathbf{Z}_4$ et le rang de $\mathfrak{N}_8^c$ est 6; les nombres de Stiefel et de Pontrjagin ne déterminent pas ces deux groupes. L'affirmation (a) découle du fait que les théorèmes de lissage des variétés combinatoires donnent des obstructions dans les groupes Γ_i des structures différentiables sur les sphères S^i et que $\Gamma_i=0$ pour $i<7$. Un des points importants de la démonstration de (b) est l'existence d'un homomorphisme C_n de Γ_{n-1} dans Ω_n^c/Ω_n^d défini comme suite. Toute sphère

d'homotopie de dimension $n-1$ borde une variété différentiable; on obtient une variété combinatoire en lui collant un n-disque le long de son bord. Les théorèmes de lissage montrent que C_8 est un isomorphisme. Pour déterminer $R_8{}^c$ l'auteur utilise des suites exactes qu'il a établies dans un travail précédent [Ann. of Math. (2) 77 (1963), 1–15; MR 26 #3072]. *A. Haefliger* (Geneva)

Reinhart, Bruce L. 6501

Cobordism and foliations.

Ann. Inst. Fourier (Grenoble) **14** (1964), *fasc.* 1, 49–52.

L'auteur discute certaines relations de cobordisme associées aux feuilletages de codimension un. Il obtient en particulier une relation entre des classes caractéristiques des feuilles compactes d'un feuilletage de codimension un sur une variété V, lorsque leur nombre dépasse le premier nombre de Betti de V.

A. Haefliger (Geneva)

Ise, Mikio 6502

Generalized automorphic forms and certain holomorphic vector bundles.

Amer. J. Math. **86** (1964), 70–108.

Let $X=G/K$ be a bounded symmetric domain, where G is a semi-simple Lie group whose complexification G_c is simply connected and K is a maximal compact subgroup of G. Let Γ be a discrete subgroup of G which acts freely on X and has a compact fundamental set, and ρ an irreducible representation of K. This paper studies the dimension of the space $H(\Gamma,\rho)$ of holomorphic cross-sections of the complex vector bundle E_ρ on X associated to ρ, which are invariant under Γ. This space may be identified with a space of automorphic forms, and also with the 0th cohomology group $H^0(Y,\mathbf{E}_\rho)$ of $Y=\Gamma\backslash X$ with coefficients in the sheaf $\mathbf{E}_\rho$ of germs of holomorphic cross-sections of the bundle on Y canonically associated to ρ. To study it, the author introduces the bundle $E_\rho{}^*$ associated to ρ on $X^*=G^*/K$, where G^* is a maximal compact subgroup of G_c. Generalizing Hirzebruch's proportionality argument [Internat. Sympos. on Algebraic Topology, pp. 129–144, Univ. Nac. Autónoma de México, Mexico City, 1958; MR 21 #2058], he first shows that $\chi(X,\mathbf{E}_\rho)=\chi(Y)\cdot\chi(X^*,\mathbf{E}_\rho{}^*)$, where $\chi(Y)$ is the arithmetic genus of Y and the other terms are the usual Euler characteristics. The groups $H^i(X^*,\mathbf{E}_\rho{}^*)$ are known from Bott's work on homogeneous vector bundles [Ann. of Math. (2) 66 (1957), 203–248; MR 19, 681], and $\chi(Y)$ is discussed by Hirzebruch [loc. cit.; also Seminars on Analytic Functions, Vol. 2, pp. 92–104, Institute for Advanced Study, Princeton Univ. Press, Princeton, N.J., 1957]; there remains therefore the problem of giving a sufficient condition under which the groups $H^i(Y,\mathbf{E}_\rho)$ vanish for $i\geq 1$. The quotient G/T, where T is a maximal torus of K, can be viewed in a natural manner as the total space of a complex analytic fibre bundle over X with fibre K/T. A spectral sequence argument and Bott's results [loc. cit.] imply that $H^i(Y,\mathbf{E}_\rho)\simeq H^i(\Gamma\backslash G/T,\mathbf{F}_\sigma)$, where $\mathbf{F}_\sigma$ is the line bundle associated to the highest weight σ of ρ (for a suitable ordering of the roots). After a discussion of some differential forms, it is then shown that $\mathbf{F}_\sigma$ is positive if σ satisfies certain conditions. [Reviewer's remark: By a completely different method, Langlands [Amer. J. Math. 85 (1963), 99–125; MR 27 #6386] has obtained a formula for $\dim H(\Gamma,\rho)$ under a slightly more restrictive condition for ρ, but allowing elements of Γ to have fixed points.]

A. Borel (Princeton, N.J.)

Andreotti, A.; Vesentini, E. 6503

On deformations of discontinuous groups.

Acta Math. **112** (1964), 249–298.

Let V and M be C^∞ manifolds and let ψ be a C^∞ map of V onto M. The triple (V,ψ,M) is called a C^∞ family of complex manifolds if (1) ψ is of maximal rank at each point, (2) for every point $x\in V$ there exists a neighborhood W [respectively, U] of x [$\psi(x)$] in V [M], an open set S in $\mathbf{C}^n$ and a diffeomorphism $\varphi: U\times S\to W$ such that (α) $\mathrm{pr}_U=\psi\circ\varphi$, ($\beta$) if $\varphi_i: U_i\times S_i\to W_i$ $(i=1,2)$ are any two such diffeomorphisms, then $\varphi_2{}^{-1}\circ\varphi_1$ is an isomorphism of $\varphi_1{}^{-1}(W_1\cap W_2)$ and $\varphi_2{}^{-1}(W_1\cap W_2)$ endowed with the structural sheaves of germs of C^∞ functions holomorphic on the fibers of the projection pr_{U_i} $(i=1,2)$. It follows then that for every $t\in M$, $\psi^{-1}(t)=X_t$ has a natural structure of a complex manifold. Such a family (V,ψ,M) is called a C^∞ deformation of a complex manifold X_0 if we are given $m_0\in M$ and an isomorphism $j: X_0\to\psi^{-1}(m_0)$. Analogously, we define a holomorphic deformation. Following Kodaira and Spencer [Ann. of Math. (2) 67 (1958), 328–466; MR 22 #3009], we define equivalent deformations, locally equivalent deformations, trivial deformations, etc. A C^∞ deformation (V,ψ,M) of X_0 is called rigid at infinity if we can find a compact set $K_0\subset X_0$ and an isomorphism g of $(X_0-K_0)\times M$ onto an open subset of V such that (1) $\psi\circ g=\mathrm{pr}_M$, (2) $\psi|(V-\mathrm{Im}(g))$ is a proper map. Let (V,ψ,M) be a C^∞ family of complex manifolds, and let $\pi:\tilde V\to V$ be the universal covering manifold of V. Then $(\tilde V,\psi\circ\pi,M)$ is a C^∞ family of complex manifolds. The authors say that (V,ψ,M) is a family of uniformizable structures on a complex manifold D_0 if we can find an isomorphism $\sigma:\tilde V\to D_0\times M$ such that $\psi\circ\pi=\mathrm{pr}_M\circ\sigma$. The authors prove the following rigidity theorem for a C^∞ deformation of a complex manifold satisfying the pseudo-concavity introduced by Andreotti and Grauert [Bull. Soc. Math. France 90 (1962), 193–259; MR 27 #343]. Theorem: Let M be the unit open ball in $\mathbf{R}^m$ and (V,ψ,M) a C^∞ deformation of a complex manifold $X_0=\psi^{-1}(0)$ such that (a) (V,ψ,M) is a family of uniformizable structures on a bounded symmetric domain D whose irreducible components are of $\dim_{\mathbf{C}}>1$, (b) (V,ψ,M) is rigid at infinity, (c) X_0 is connected and strongly q-pseudoconcave with $0\leq q\leq\dim_{\mathbf{C}}X_0-2$, and (d) the fundamental group $\pi_1(X_0)$ is finitely generated. Then (V,ψ,M) is trivial. The proof is based on (I) the W-ellipticity (which will not be explained here) of the tangent bundle of D/Γ, where Γ is a properly discontinuous group acting on D without fixed points (Theorem 4), and (II) the fact that if D/Γ is strongly pseudoconcave, then there exists a relatively compact open set $B\subset D/\Gamma$ whose inverse image in D has D as an envelope of holomorphy (Theorem 1). As a corollary of the rigidity theorem, using an unpublished result of A. Borel, the authors assert that there exist, for D as above, only trivial families of discontinuous groups Γ containing the arithmetic group Γ_1 and keeping the part at infinity of D/Γ rigid. On the other hand, the authors prove that any holomorphic family of complex manifolds uniformizable on a bounded (not necessarily symmetric) domain in $\mathbf{C}^n$

is locally trivial (Theorem 2), i.e., holomorphic deformations and C^∞ deformations behave quite differently.

A. Morimoto (La Jolla, Calif.)

PROBABILITY

See also 5767, 5807, 5839, 5978, 5989, 6044, 6045, 6191, 6207, 6295, 6591, 6741, 6955, 6960, 6962, 6963a–b, 6964, 6984.

Morgenstern, Dietrich 6504

★**Einführung in die Wahrscheinlichkeitsrechnung und mathematische Statistik.**

Die Grundlehren der Mathematischen Wissenschaften, Band 124.

Springer-Verlag, Berlin-Göttingen-Heidelberg, 1964. x + 224 *pp.* *DM* 34.50.

Das vorliegende Buch ist aus Vorlesungen entstanden, die der Autor an mehreren deutschen Universitäten gehalten hat. Teilweise ist der Vorlesungscharakter noch erhalten geblieben: Das Buch ist lebhaft und pädagogisch geschickt geschrieben. Es werden nur verhältnismässig geringe Vorkenntnisse verlangt "sodass die Darstellung auch Ingenieuren und Interessenten aus den Wirtschaftswissenschaften zugänglich sein sollte. Die Darstellung ist mathematisch vollständig, indem alle Sätze bewiesen werden, auch wenn die Regularitätsvoraussetzungen weggelassen werden" (aus dem Vorwort). Dieser Zielsetzung kommt vielleicht Gnedenkos Buch [*Lehrbuch der Wahrscheinlichkeitsrechnung*, Akademie-Verlag, Berlin, 1962; siehe #6505] am nachsten, doch ist die mathematische Statistik hier viel stärker betont als bei Gnedenko. Im ersten Teil beschränkt sich der Autor auf diskrete Verteilungen und erst im zweiten Teil beschäftigt er sich mit stetigen Verteilungen, welche Verteilungsdichten besitzen. Im Grossen und Ganzen enthält das Buch Standard-Material, doch müssen einige hübsche und originelle Ideen hervorgehoben werden. So wird der Normierungsfaktor im Satz von Laplace ohne Benützung der Stirling'schen Formel mittels der Čebyšev'schen Ungleichung bestimmt. Weiters sei der kurze Beweis des Satzes von Cramér und Wold ohne Benützung der Fourier-Transformierten erwähnt. Dieser Satz wird übrigens mit Recht nach Radon benannt. Geschickt wird dieser Satz zur Definition der mehrdimensionalen Normalverteilung und der Herleitung ihrer wichtigsten Eigenschaften herangezogen. Schliesslich erwähnen wir noch die Beweisanordnung für den Satz von Kolmogoroff über die Verschärfung des Hauptsatzes der Statistik.

Leider enthält das Buch eine Unzahl von kleineren und grösseren Druckfehlern. Der Referent hat nur wenige druckfehlerfreie Seiten gefunden. Manche dieser Druckfehler sind sinnstörend, zum Beispiel bei der Formulierung der Ungleichung von Kolmogoroff (S. 103) und bei der Definition der Konsistenz (S. 62). In der Terminologie ist der Autor inkonsequent. So wird auf Seite 158 zwischen positiv definit und eigentlich positiv definit unterschieden, auf Seite 175 wird aber dann positiv semidefinit benützt.

Auch wenn man auf Angabe der Regularitätsbedingungen verzichtet, müsste man bei den Sätzen von Gnedenko und Kolmogoroff auf die Bedeutung der Stetigkeit der Verteilungsfunction hinweisen. Die Konsistenzaussagen über die Maximum-Likelihood Schätzungen sollten angesichts der bekannten Beispiele von Le Cam, Bahadur und anderen vorsichtiger formuliert werden. Die Effizienz Aussagen berücksichtigen nicht das Phänomen der Super-Effizienz. In der Form der Literaturzitate ist ebenfalls kein System zu erkennen.

Trotz dieser Kritik soll jedoch diesem Werk das Interesse nicht abgesprochen werden. Der Leserkreis, an den es sich hauptsächlich wendet, wird vieles von der Lektüre profitieren, wozu auch die vielen Übungsaufgaben beitragen werden.

L. Schmetterer (Vienna)

Gnedenko, B. W. [Gnedenko, B. V.] 6505

★**Lehrbuch der Wahrscheinlichkeitsrechnung.**

Vom Autor neubearbeitete und autorisierte Ausgabe in deutscher Sprache herausgegeben von Hans-Joachim Rossberg. Dritte erweiterte Auflage.

Akademie-Verlag, Berlin, 1962. xiii + 393 *pp.* *DM* 29.50.

Two mistakes in this third German edition should be corrected in any future edition since they are rather subtle at the level of the book. One is in the proof of Satz 3 on p. 255, where the conclusion (5) cannot be drawn on the basis of the uniform convergence of $\varphi^{(k)}$ and the choice of the "Hauptwert" alone without a further argument. One of the ways of rectifying this is given in Loève [*Probability theory*, p. 297, Van Nostrand, Toronto, 1955; MR 16, 598]. The other mistake occurs on p. 279, where neither the statement nor the proof of a well-known proposition makes sense as given. It would be an instructive exercise for the reader to analyze the fallacy there. {The same mistakes are in the second English translation [Chelsea, New York, 1962] of the original Russian version which appeared in 1961 [MR 25 #2622].}

K. L. Chung (Stanford, Calif.)

Gnedenko, B. V. 6506

On the law of signs. (Russian)

C. R. Acad. Bulgare Sci. 17 (1964), 793–796.

Simple remarks concerning a result by the reviewer and Feller in the fluctuation theory of coin-tossing [Proc. Nat. Acad. Sci. U.S.A. 35 (1949), 605–608; MR 11, 444].

K. L. Chung (Stanford, Calif.)

Dempster, A. P. 6507

On direct probabilities.

J. Roy. Statist. Soc. Ser. B 25 (1963), 100–110.

Blackwell, D. [Blackwell, David]; Deuel, P.; Freedman, D. [Freedman, David] 6508

The last return to equilibrium in a coin-tossing game.

Ann. Math. Statist. 35 (1964), 1344.

Let X_n, $n = 1, 2, 3, \cdots$, be independent and identically distributed random variables assuming the values ± 1 with probability $\frac{1}{2}$ each. Let $S_n = X_1 + \cdots + X_n$ and let $P(m, n)$ be the probability that $S_{2i} = 0$ for at least one i satisfying $m \leq i < m + n$. The authors prove that $P(m, n) + P(n, m) = 1$ for $m \geq 1$ and $n \geq 1$.

J. E. Cigler (Groningen)

Blackwell, David; Freedman, David 6509

A remark on the coin tossing game.

Ann. Math. Statist. 35 (1964), 1345–1347.

Let X_n, $n=1, 2, 3, \cdots$, be independent and identically distributed random variables assuming the values ± 1 with probability $\frac{1}{2}$ each. Let $S_n = X_1 + X_2 + \cdots + X_n$ and let $\tau(N, c)$ denote the least $n \geq N$ with $|S_n| > c\sqrt{n}$, $0 < c < \infty$.

With a simple but interesting method the authors prove the following two theorems. Theorem 1: The mean waiting time for $|S_n|$ to exceed $\sqrt{n}$ is infinite, that is, $E(\tau(1, 1)) = \infty$. Theorem 2: If $0 < c < 1$, the mean waiting time for $|S_n|$ to exceed $c\sqrt{n}$ is finite, that is, $E(\tau(N, c)) < \infty$ for all N. *J. E. Cigler* (Groningen)

Miles, R. E. 6510

Random polygons determined by random lines in a plane. II.

Proc. Nat. Acad. Sci. U.S.A. **52** (1964), 1157–1160.

The author describes methods of proof for the theorems announced in a previous paper [same Proc. **52** (1964), 901–907; MR **29** #5266]. An application to paper making is also mentioned. *P. A. P. Moran* (Canberra)

Tortorici, Paolo 6511

Problemi riguardanti una successione di suddivisioni casuali.

Rend. Mat. e Appl. (5) **23** (1964), 147–155.

The paper refers to the theory of random partitions of the interval [0, 1] obtained by the successive introduction of random points (a point to any moment $t = 1, 2, \cdots$), and especially to some problems treated by B. de Finetti. The author deals particularly with the probability of conservation of the minimum or of the maximum interval by one or more successive introductions of random points. *O. Onicescu* (Bucharest)

Richards, Paul I. 6512

Averages for polygons formed by random lines.

Proc. Nat. Acad. Sci. U.S.A. **52** (1964), 1160–1164.

New results are given for averages over the set of all polygons formed by a uniform isotropic Poisson field of straight lines in a plane. The average of a function of the distance between two points independently and uniformly chosen inside such a polygon is found, and by specialisation of the function many particular results are obtained. *P. A. P. Moran* (Canberra)

Castoldi, Luigi 6513

About a generalized stochastic problem of Laplace.

Rend. Sem. Fac. Sci. Univ. Cagliari **33** (1963), 112–116.

The urn scheme of Laplace which leads to the law of succession is discussed in somewhat more generality and detail than in Feller's book [*Introduction to probability theory and its applications*, Vol. I, second edition, pp. 113–114, Wiley, New York, 1957; MR **19**, 466]. *J. W. Lamperti* (Hanover, N.H.)

Castoldi, Luigi 6514

A continuous analogon of Poisson's distribution. (Italian summary)

Rend. Sem. Fac. Sci. Univ. Cagliari **33** (1963), 245–249.

The author proposes, as a "continuous analogue of Poisson's distribution", the distribution with density $e^{-\lambda}\lambda^x/\Gamma(x+1)$ for $x > 0$ and the remaining mass concentrated at the origin. *J. W. Lamperti* (Hanover, N.H.)

Čencov, N. N. 6515

Geometry of the "manifold" of a probability distribution. (Russian)

Dokl. Akad. Nauk SSSR **158** (1964), 543–546.

Let H denote the set of probability measures P on a measurable space $(\Omega, \mathcal{S})$, which are absolutely continuous with respect to a measure μ. The author defines a differential geometry in H, in which the geodesic line passing through two "points" P_0 and P_1 is the set of measures P_t $(0 \leq t \leq 1)$ such that

$$(1) \qquad \frac{dP_t}{d\mu} = \frac{1}{b(t)} \left(\frac{dP_0}{d\mu}\right)^{1-t} \left(\frac{dP_1}{d\mu}\right)^t,$$

where $b(t)$ is the normalizing factor

$$(2) \qquad b(t) = \int_\Omega \left(\frac{dP_0}{d\mu}\right)^{1-t} \left(\frac{dP_1}{d\mu}\right)^t d\mu.$$

The measure $I_\alpha(P_0, P_1)$ of order α of the information gain (obtained by replacing the measure P_0 by the measure P_1) is given [see A. Rényi, Proc. Fourth Berkeley Sympos. Math. Statist. and Prob., Vol. I, pp. 547–561, Univ California Press, Berkeley, Calif., 1961; MR **24** #A2410] by

$$(3) \qquad I_\alpha(P_0, P_1) = \frac{\log b(\alpha)}{\alpha - 1} \quad \text{for } 0 < \alpha < 1.$$

Shannon's measure of information is the limiting case for $\alpha \to 1$ of (3):

$$(4) \qquad \lim_{\alpha \to 1} I_\alpha(P_0, P_1) = \int_\Omega \left(\log \frac{dP_1}{dP_0}\right) dP_1 = I_1(P_0, P_1).$$

and Fisher's information is obtained as the limit of $2I_1(P_0, P_t)/t^2$ if P_t tends to P_0 along the geodesic line mentioned.

Thus, all these quantities of information can be interpreted geometrically. *A. Rényi* (Budapest)

Kolmogorov, A. N. 6516

Approximation of distributions of sums of independent terms by infinitely divisible distributions. (Russian)

Trudy Moskov. Mat. Obšč. **12** (1963), 437–451.

Let $F_k(x)$ be the distribution functions of the independent random variables X_k, $k = 1, 2, \cdots, n$, and let $H_n(x)$ be the distribution function of $X = X_1 + X_2 + \cdots + X_n$. Let $E(x)$ be the distribution function of the identically zero random variable, and let $\mathcal{D}$ be the collection of all infinitely divisible distribution functions $\{D(x)\}$.

If the X_k are equi-distributed, $F_k(x) = F(x)$, $k = 1, 2, \cdots, n$, the author proves that there exists a positive constant c_1 such that for at least one element $D \in \mathcal{D}$ one has $|H_n(x) - D(x)| < c_1 n^{-1/3}$ for all x. Let $\rho(F', F'') = \sup_x |F'(x) - F''(x)|$ for two distribution functions F' and F'', and in terms of this metric let $\psi(n) = \sup_F \rho(H_n, \mathcal{D})$. The above theorem implies that $\psi(n) = O(n^{-1/3})$, and this improves an earlier work of the author [Teor. Verojatnost. i Primenen. 1 (1956), 426–436; MR **19**, 586], in which it was proved that $\psi(n) = O(n^{-1/5})$, and of Prohorov [ibid. 5 (1960), 103–113; MR **24** #A2425], who showed that $\psi(n) = O(n^{-1/3} \log^2 n)$. In the other direction Meškin

[ibid. 6 (1961), 257–275; MR 24 #A2963] showed that $\phi(n) \geq c_2 n^{-2/3} \log^{-4} n$.

In the general, non-equidistributed case there is a positive constant c_3 such that whenever the $F_k(x)$ satisfy $E(x-l)-\varepsilon < F_k(x) < E(x+l)+\varepsilon$ for all x, for some $\varepsilon > 0$, $L \geq 2l > 0$, and for $k = 1, 2, \cdots, n$, there exists an element $D \in \mathscr{D}$ such that for all x

$$D(x-L) - \delta \leq H_n(x) \leq D(x+L) + \delta,$$

where $\delta = c_3 \max((l/L)(\log L/l)^{1/2}, \varepsilon^{1/3})$.

D. A. Darling (Ann Arbor, Mich.)

Kubik, L. 6517

On the classes of probability distributions closed under decompositions.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 12 (1964), 397–403.

According to the author, a subclass $\mathscr{T}$ of $\mathscr{Y}$ of probability distribution functions (non-degenerate) is said to be closed under decompositions in the class $\mathscr{Y}$ if the relation that the distributions of random variables X_1 and X_2 belong to $\mathscr{Y}$, the distribution of X belongs to $\mathscr{T}$ and $X = X_1 + X_2$ (X_1 and X_2 being independent) implies that both distributions of X_1 and X_2 belong to $\mathscr{T}$. This is the generalization of the ordinary closure under decompositions in which $\mathscr{Y}$ is the class of all distribution functions. The author is interested in the case in which $\mathscr{Y}$ is the class of infinitely divisible distributions (i.d.d.) and $\mathscr{T}$ is a subclass of $\mathscr{Y}$, which has the property that for any two distributions of $\mathscr{T}$ which have $G_1(u)$ and $G_2(u)$ for $G(u)$ in the Lévy-Khintchine formula for the logarithm of the characteristic function

$$i\gamma t + \int_{-\infty}^{\infty} \left(e^{itu} - 1 - \frac{itu}{1+u^2}\right) \frac{1+u^2}{u^2}\, dG(u),$$

one has that $G_1(u) = aG_2(u)$ for all u, for some $a > 0$. In this case $\mathscr{T}$ is called a class of distributions of the same sort. The author proves that a class $\mathscr{T}$ of distributions of the same sort closed under decompositions in $\mathscr{Y}$ of i.d.d. is the class of normal distributions or the class of Poisson distributions with characteristic function $\exp(ita + \lambda(e^{itb} - 1))$ with the same value of $b \neq 0$. He also shows that the class $\mathscr{Y}$ of the same sort introduced by him [Studia Math. 21 (1961/62), 245–252; MR 26 #1916] is closed under decompositions in a subclass $\mathscr{S}$ of the class of i.d.d. He also discusses the same problem about the class of stable distributions. *T. Kawata* (Washington, D.C.)

Ladohin, V. I. 6518

On non-positive distributions. (Russian)

Kazan. Gos. Univ. Učen. Zap. 122 (1962), kn. 4, 53–64.

Let Ω be the space of real-valued functions $\omega = \xi(\cdot)$ on $T = [0, \infty)$, H the family of Borel cylinder subsets of Ω, and B the smallest σ-algebra containing H. Suppose that there is given a system of set functions $Q_{t_1 t_2 \cdots t_n}$, $t_1, t_2, \cdots, t_n \in T$, $1 \leq n < \infty$, such that $Q_{t_1 t_2 \cdots t_n}$ is σ-additive and of bounded variation over Borel subsets of R^n, where $Q_{t_1 \cdots t_n}(R^n) = 1$, and the $Q_{t_1 t_2 \cdots t_n}$ jointly satisfy Kolmogorov's consistency conditions. Then a finitely additive set function Q over H is determined. Krylov [Dokl. Akad. Nauk SSSR 132 (1960), 1254–1257; MR 22 #9722] constructed such a Q from the fundamental solution of a parabolic equation of order $q > 2$ and deduced a formula of the Feynman-Kac type. The present author shows that although Q cannot be extended to be σ-additive over B, an analogy to the Markov process theory enables one to deduce a known expansion theorem connected with the Feynman-Kac theorem. *G. Maruyama* (Fukuoka)

Statuljavičjus, V. A. [Statulevičius, V. A.] 6519

Limit theorems and their sharpening for additive random functions and sums of weakly dependent random variables. (Russian)

Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (*Liblice*, 1962), pp. 683–687. *Publ. House Czech. Acad. Sci.*, *Prague*, 1964.

Let Ω be a space on which a probability measure P and a collection of σ-algebras $F^{(s,t)}$ are defined, satisfying $F^{(s',t')} \subseteq F^{(s,t)}$ for $s \leq s' \leq t' \leq t$.

The author states some rather sharp theorems on the limit distribution of

$$Z_n(\Delta) = \frac{H_n(\Delta) - E(H_n(\Delta))}{D(H_n(\Delta))},$$

where $H_n(\Delta)$ is a sequence of functions additively dependent on the interval $\Delta = (s, t)$ and satisfying some measurability and ergodicity conditions.

J. K. Cigler (Groningen)

Volodin, I. N. 6520

The joint distribution of the maximum and minimum of a trajectory of a Wiener process. (Russian)

Kazan. Gos. Univ. Učen. Zap. 122 (1962), kn. 4, 39–52.

Let $X(t)$, $0 \leq t \leq 1$, be the Wiener process, $X(0) = 0$, $EX(0) = 0$, $EX(t)X(s) = \frac{1}{2}\min(t, s)$, and let η_1, η_2 be positive numbers. The author expresses the probability distribution of the portion of time the process spends outside the interval $[-\eta_1, \eta_2]$ and the probability distribution of $\max X(t)$, $\min X(t)$, and of their difference as an infinite series of integrals of transcendental functions. He gives also several moments of these distributions.

Petr Mandl (Prague)

Cohn, Harry 6521

Le problème limite central pour les systèmes aléatoires à liaisons complètes.

C. R. Acad. Sci. Paris 259 (1964), 3423–3426.

Author's summary: "L'auteur établit plusieurs propriétés relatives au comportement asymptotique des lois des sommes normées de variables aléatoires attachées à un système à liaisons complètes. Le théorème (1.1) montre que le comportement asymptotique de ces sommes est identique, sous certaines conditions, au comportement asymptotique des sommes qui correspondent au processus stationnaire attaché au système à liaisons complètes donné. Le théorème (2.1) étend ce résultat au cas des variables dépendantes d'un type plus général."

S. M. Berman (New York)

Iosifescu, Marius 6522

Sur la loi forte des grands nombres pour les systèmes aléatoires homogènes à liaisons complètes à un ensemble quelconque d'états.

C. R. Acad. Sci. Paris 258 (1964), 4421–4423.

For each s in S, $P(s)$ is a p-measure on X and, for each

x in X, $u(x)$ is a mapping of S into S. $P_m^{(n)}$ denotes the distribution of $x_{n+1}, \cdots, x_{n+m}$ when the distribution of x_i, given $x_1, \cdots, x_{i-1}$, is $P(s_i)$, with $s_{i+1} = u(x_i)$ for $i = 1, 2, \cdots$. If there are distributions P_m with

$$\sum n \sup_{s_1, m} \| P_m^{(n)} - P_m \| < \infty,$$

then certain analogues of the theorems of de Moivre-Laplace, Borel-Cantelli, and V. I. Glivenko hold. Proofs will be published in Rev. Roumaine Math. Pures Appl.

James Hannan (E. Lansing, Mich.)

Keilson, J.; Wishart, D. M. G. 6523

A central limit theorem for processes defined on a finite Markov chain.

Proc. Cambridge Philos. Soc. **60** (1964), 547–567.

Let $\mathbf{B}(x) = \{B_{rs}(x)\}$ be for each x an $m \times m$ substochastic matrix, such that $\mathbf{B}(+\infty)$ is stochastic, irreducible, aperiodic, and such that each element of it is positive and monotone non-decreasing in $-\infty < x < +\infty$. The authors consider a vector process $\{R(n), X(n)\}$ ($n = 1, 2, \cdots$; $R(n) = 1, 2, \cdots, m$; $-\infty < X(n) < +\infty$) whose evolution is determined as follows. Let $\mathbf{F}_n(x)$ be a row vector of m components whose rth component is $\Pr\{R(n) = r, X(n) < x\}$ and determine transition probabilities by $\mathbf{F}_{n+1}(x) = \int \mathbf{F}_n(x-y)\mathbf{B}(dy)$. Assuming that each element of the matrix $\int y^2 \mathbf{B}(dy)$ is finite, the authors prove that

$$\lim_{n\to\infty} \mathbf{F}_n(x\sqrt{n} + \mu n, n) = \Phi(x\sigma^{-1})\mathbf{e},$$

where $\Phi(x)$ is the standard normal distribution, μ and σ^2 are constants associated with the first two moments of $\mathbf{B}$, and where $\mathbf{e}$ is the vector satisfying $\mathbf{eB}(+\infty) = \mathbf{e}$. The theorem also shows that $X(n)$ and $R(n)$ are asymptotically independent.

Assuming the entries in the matrix of Fourier transforms $\hat{\mathbf{B}}(z) = \int \exp(izy)\mathbf{B}(dy)$ have a common strip of convergence, there is a "conjugate process" $\{\hat{R}(n), \hat{X}(n)\}$ which permits a discussion of the asymptotic behavior of $\mathbf{F}_n(x)$ near the tails.

Extensions are given to the case when $\mathbf{B}(+\infty)$ is irreducible but not necessarily aperiodic, and to the corresponding process in continuous time.

D. A. Darling (Ann Arbor, Mich.)

Mitalauskas, A. 6524

An integral limit theorem for convergence to the stable limiting law. (Russian. Lithuanian and German summaries)

Litovsk. Mat. Sb. **4** (1964), 235–240.

Consider independent random variables ξ_n, $1 \leq n < \infty$, and put $S_n = (\sum_{k=1}^n \xi_k - A_n)/B_n$, $B_n > 0$. To find a necessary and sufficient condition and to determine A_n, B_n such that $\mathbf{P}(S_n < x)$ tends to an assigned stable law is now a completely solved problem, provided that the ξ_n are identically distributed. When they are not identically distributed, the present paper gives a sufficient condition which enables one to determine normalizing constants A_n, B_n.

G. Maruyama (Fukuoka)

Petrov, V. V. 6525

Limit theorems for large deviations violating Cramér's condition. II. (Russian. English summary)

Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. **19** (1964), *no.* 1, 58–75.

This paper continues the author's previous work [same Vestnik **18** (1963), no. 4, 49–68; MR **28** #1646] on limit theorems for large deviations without assumption of Cramér's condition. There are eight theorems which are partly generalizations of results obtained by Linnik. The theorems are too complicated to be stated here in detail.

J. E. Cigler (Groningen)

Rjauba, B. 6526

The central limit theorem for sums of series of weakly dependent random variables. (Russian. Lithuanian and English summaries)

Litovsk. Mat. Sb. **2** (1962), *no.* 2, 193–205.

Let $\{X_1^{(n)}, \cdots, X_n^{(n)}, n = 1, 2, \cdots\}$ be a sequence of sets of random variables with finite second moments. If the dependence between the random variables is sufficiently weak in a certain sense, the author shows that the distribution of the sum $(X_1^{(n)} + X_2^{(n)} + \cdots + X_n^{(n)})$, suitably normalized, converges to the normal distribution. The author's sufficient conditions cannot be stated precisely in a short space; they involve various conditional probabilities of "future" events, given the "past".

S. G. Ghurye (Bloomington, Ind.)

Vilkauskas, L. 6527

Two integral theorems on large deviations in the higher-dimensional case. (Russian. Lithuanian and English summaries)

Litovsk. Mat. Sb. **3** (1963), *no.* 2, 53–67.

The integral limit theorem (i.l.t.) is said to hold for a sequence of regions Q_n of s-dimensional space and for a sequence of identically distributed s-dimensional random variables X_n, if $P(X_n \in Q_n)/\Phi(Q_n) \to 1$, where Φ is the standard Gaussian measure in s-dimensional space. In the spirit of Ju. V. Linnik's work on large deviations in one dimension [Teor. Verojatnost. i Primenen. **6** (1961), 145–163; MR **25** #1572; ibid. **6** (1961), 377–391; MR **27** #804] the author first considers the regions $Q_n = \{x \,|\, |x| \gtrsim R_n\}$, where $0 \leq R_n \leq n^\alpha/\rho(n)$, $\rho(n) \to \infty$, $\alpha < \frac{1}{6}$. When the random variables X_n have mean 0 and identity covariance matrix, the s-dimensional analogue of Linnik's condition $\mathrm{E}\exp|X_n|^{4\alpha/(2\alpha+1)} < \infty$ is shown to be sufficient for the i.l.t. to hold. Subsequently the methods are extended to prove the i.l.t. for a larger class of regions Q_n, defined in terms of the asymptotic behavior of their Gaussian measures.

F. L. Spitzer (Ithaca, N.Y.)

Beilinson, A. A. 6528

On the solution of a non-uniform problem in probability for distributed systems. (Russian. English summary)

Teor. Verojatnost. i Primenen. **9** (1964), 519–523.

Author's summary: "A dynamical system is considered which is described by a parabolic equation in a circle of length $2l$ when acted upon by an undistributed stochastic force $f(t)$ (white noise)

$$\frac{\partial W(x,t)}{\partial t} - \frac{\partial^2 W(x,t)}{\partial x^2} = \delta(x) f(t).$$

"The Green's function for this system (a countable additive measure in the phase space) is constructed. This measure is not quasi-invariant. It is proved that almost all profiles $W(x)$ are infinitely differentiable."

J. Chover (Madison, Wis.)

Dudley, R. M. 6529

Singular translates of measures on linear spaces.

Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **3**, 128–137 (1964).

Let F be the field of real or complex numbers, and let Q be any linear space over F. Let Q^a be the algebraic dual space of Q. For any n-tuple $(q_1, \cdots, q_n)$ of members of Q and Borel set B in n-space F^n, let $S(q_1, \cdots, q_n, B) = \{q' \in Q^a : \langle q'(q_1), \cdots, q'(q_n)\rangle \in B\}$. Let $\mathscr{L}_Q$ be the σ-algebra of subsets of Q^a generated by all such sets. A random linear functional (r.l.f.) on Q is an assignment to each n-tple $(q_1, \cdots, q_n)$ of a Borel probability measure μ on F^n such that for any $q_i, r_j \in Q$ and Borel sets $B \subset F^n$ and $C \subset F^m$, $S(q_1, \cdots, q_n, B) = S(r_1, \cdots, r_m, C)$ implies that $\mu(q_1, \cdots, q_n)(B) = \mu(r_1, \cdots, r_m)(C)$. There is a one-one correspondence between such r.l.f.'s on Q and probability measures P on $\mathscr{L}_Q$ (Theorem 1 reproves this briefly). An r.l.f. μ is called quadratic if $\int_F |x|^2 \mu(q)\,dx < \infty$ for each $q \in Q$. Then its mean

$$m(q) = \int_F x\mu(q)(dx) = \int_{Q^a} q'(q)\,dP(q')$$

is in Q^a. Its covariance

$$B(q, r) = \int_F \int_F x\bar{y}\mu(q, r)(dx, dy) = \int_{Q^a} q'(q)\overline{q'(r)}\,dP(q')$$

defines a topology on Q by the pseudo-norm $\|q\|_P = B(q, q)^{1/2}$. The main result is Theorem 2: If μ is a quadratic r.l.f. corresponding to the measure P on $\mathscr{L}_Q$, and if $z \in Q^a$ is a linear functional on Q which is not continuous in the $\|\ \|_P$ topology, then the measures P and P_z, the translate of P by z, are singular. The proof is brief. The author then applies this result for Q a topological linear space. In particular, when Q is a real Hilbert space he gets a simpler proof of a result of V. N. Sudakov. The author briefly treats the Gaussian case, and also deduces consequences of Theorem 2 for random distributions, and, in particular, stationary ones.

J. Chover (Madison, Wis.)

Has'minskii, R. Z. 6530

Diffusion processes with a small parameter. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **27** (1963), 1281–1300.

In the present paper the author has studied the asymptotic behavior of some functional of the trajectory of the Markov process $X_\epsilon(t)$ connected with the differential operator

$$L^{(\epsilon)}(x) = \sum_{k=1}^{2} A_k(x)\frac{\partial}{\partial x_k} + V(x) + \epsilon L(x).$$

Here $x = (x_1, x_2)$ denotes the point of the two-dimensional plane and

$$L(x) = \sum_{i,j=1}^{2} a_{ij}(x)\frac{\partial^2 x}{\partial x_i \partial x_j} + \sum_{i=1}^{2} b_i(x)\frac{\partial}{\partial x_i}.$$

Many of the results proven in this paper were already stated by the author [Dokl. Akad. Nauk SSSR **142** (1962), 560–563; MR **26** #477]. *R. G. Laha* (Washington, D.C.)

Kahane, Jean-Pierre 6531

Sur les sommes vectorielles $\sum \pm u_n$.

C. R. Acad. Sci. Paris **259** (1964), 2577–2580.

L'auteur considère la série (1) $V = \sum_1^\infty \varepsilon_n u_n$, où les u_n sont des éléments d'un espace de Banach B, et où les ε_n sont des variables aléatoires de Bernoulli symétriques et indépendantes. Il pose $V_n = \sum_1^n \varepsilon_m u_m$, $S = \sup_n \|V_n\|$. Il démontre les théorèmes suivants. (I) Si (1) est p.s. (presque sûrement) convergente et si $p(\|V\| > r) < \alpha/2$, alors $p(\|V\| > 2r) < \alpha^2$. Si (1) est p.s. bornée et si $p(S > r) < \alpha$, alors $p(S > 2r) < 2\alpha^2$. (II) Soit $T = (a_{nm})$ une matrice telle que $\lim_{n\to\infty} a_{nm} = 1$. Soit $W_n = \sum_{m=1}^\infty a_{nm}\varepsilon_m u_m$. Si W_n existe pour tout n et a une limite dans B [est bornée dans B] quand $n \to +\infty$, on dit que (1) est T-sommable [T-bornée]. Si cette propriété a lieu presque sûrement, la série (1) est p.s. convergente [bornée]. (III) Soit λ_n une suite de nombres complexes tels que $0 \le |\lambda_n| \le 1$. Si (1) est p.s. convergente [bornée], la série $\sum_1^\infty \varepsilon_n \lambda_n u_n$ est p.s. convergente [bornée].

Il en résulte que, si les ω_n sont des variables aléatoires indépendantes équiréparties sur $[0, 1]$, la série (1) et la série $\sum_1^\infty \exp(2i\pi\omega_n)u_n$ sont en même temps p.s. convergentes [bornées]. *J. Bass* (Paris)

Krzyżański, M. 6532

Principe d'extremum relatif aux solutions de l'équation intégro-différentielle du processus stochastique markovien purement discontinu.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 531–534.

Let A be a Borel set, T_0 a positive number, $\Sigma_0 = (0, T_0) \times A$. The author considers the equation

$$(1)\qquad u_t'(t, x) = p(t, x)\int_A [u(t, z) - u(t, x)]P(t, x, dz),$$

where $p(t, x)$ is a bounded non-negative function on Σ_0 and $P(t, x, dz)$, for $(t, x) \in \Sigma_0$, a probability distribution on A. He shows that every solution $u(t, x)$ of (1), continuous on $[0, T_0] \times A$, is less than a constant M if $u(0, x)$ has this property, and deduces some consequences from this maximum principle. (1) is the Kolmogorov-Feller equation of a purely discontinuous Markov process.

Petr Mandl (Prague)

Lamperti, John 6533

On a class of stochastic processes.

Ann. Math. Statist. **34** (1963), 206–212.

The author studies a stochastic process $\{Y_t\}$ in R^n which can be obtained as a limit, $\lim_{s\to\infty}\{X_{t+s}/f(s)\}$. Here f is a positive function, $\{X_t\}$ is an arbitrary process in R^n, and the convergence is in the sense of convergence of finite-dimensional distributions. He proves that unless $P(Y_t = 0) \equiv 1$, the scaling function f must be $f(s) = e^{\alpha s}M(s)$, where α is real and $M(s)$ is slowly varying as $s \to \infty$. The limiting process then is such that for each fixed u the process $\{Y_{t+u}\}$ has the same finite-dimensional distributions as the process $\{e^{\alpha u}Y_t\}$. Next he assumes that the limiting process is Markovian with stationary transition function, that the state space is R^1, and that Y_t is never 0, and obtains the following additional information about Y_t. For a constant $A > 0$ define $S_t = -1$ if $Y_t < 0$ and $S_t = A$ if $Y_t > 0$. Then S_t is a stationary Markov chain, and for a suitable choice of A the process $\{Y_t\}$ has the same distributions as the process $\{S_t e^{\alpha t} Y\}$, with Y an appropriate position random variable independent of the process $\{S_t\}$. The author also provides converses to these

assertions and gives some discussion of the Markovian situation when the state space is R^n with $n > 1$.

R. M. Blumenthal (Seattle, Wash.)

Mandl, Petr 6534

Über die asymptotischen Verteilungen der Erst-Passage-Zeiten.

Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (*Liblice*, 1962), *pp.* 475–498. *Publ. House Czech. Acad. Sci., Prague*, 1964.

In this paper first-passage times for one-dimensional diffusion processes are studied. The author considers two cases, diffusion on the real line and on $(0, \infty)$ with a reflecting barrier in 0. Let $P(t, x; E)$ denote the probability that the process assumes some value in E during the time interval $[0, t)$ if it starts at x. It is assumed that for every real-valued twice continuously differentiable function ψ with compact support the function $w(x, t) = \int P(t, x; dy)\psi(y)$ satisfies

$$\frac{\partial}{\partial t} w = \frac{\partial^2}{\partial x^2} w + b(x) \frac{\partial}{\partial x} w.$$

Let $Q(x) = \exp \int_0^x b(s)\, ds$. The author studies the asymptotic behaviour of the probability $F(t, x; Z)$ that the process starting at x assumes values $\geqq Z$ in the time interval $[0, t)$ under various conditions on $Q(x)$.

J. E. Cigler (Groningen)

Parthasarathy, K. R. 6535

A note on mixing processes.

Sankhyā Ser. A **24** (1962), 331–332.

Author's summary: "In this paper it is shown that in the space of discrete stationary stochastic processes under the weak topology, finite Markov chains are dense and the set of weakly mixing processes is a dense G_δ."

J. G. Wendel (Ann Arbor, Mich.)

Shepp, L. A. 6536

The singularity of Gaussian measures in function space.

Proc. Nat. Acad. Sci. U.S.A. **52** (1964), 430–433.

If Ω is the space of all real-valued functions on $[0, T]$, $\mathfrak{B}$ is the σ-field generated by the coordinate functions, and ρ_0, ρ_1 are two real-valued positive definite functions on $[0, T] \times [0, T]$, then two zero mean Gaussian measures μ_0, μ_1 are induced on $\mathfrak{B}$ in the usual way. Jaglom has proved that the Gaussian measures are either mutually singular or equivalent [Proc. Sympos. Time Series Analysis (Brown Univ., 1962), pp. 327–346, Wiley, New York, 1963; MR **26** #5635b]. Using martingale methods, the author proves the following necessary and sufficient condition for singularity. If π is a partition $0 \leqq t_1 \leqq \cdots \leqq t_n \leqq T$ of $[0, T]$, let ρ_i^{π} be the matrix $\{\rho_i(t_j, t_k)\}$. For $0 < \alpha < 1$, let $\rho_\alpha^{\pi} = (1-\alpha)\rho_0^{\pi} + \alpha\rho_1^{\pi}$. Then $\mu_0 \perp \mu_1$ if and only if $\inf_\pi |\rho_0^{\pi}|^{1-\alpha} |\rho_1^{\pi}|^{\alpha} / |\rho_\alpha^{\pi}| = 0$. If ρ_0 and ρ_1 are continuous, then $\inf_\pi$ can be replaced by a sequential limit. A similar theorem holds if mean value functions μ_0 and μ_1 are given.

L. L. Helms (Urbana, Ill.)

Takeuchi, Junji 6537

On the sample paths of the symmetric stable processes in spaces.

J. Math. Soc. Japan **16** (1964), 109–127.

The author solves a number of problems concerning the sample functions of symmetric stable processes $x(t)$ of index α, $0 < \alpha \leqq 2$, in N-dimensional space. The first problem concerns the rate of escape to infinity in the transient case, i.e., when $N > \alpha$. A positive non-increasing function $g(t)$ is in the upper [lower] class if $|x(t)| \leqq g(t)t^{1/\alpha}$ for arbitrarily large t with probability one [zero]. It is shown that g is in the upper [lower] class according as the integral of $t^{-1}[g(t)]^{N-\alpha}$ diverges [converges] at infinity. The proof depends on the Borel-Cantelli lemma together with potential-theoretical estimates of hitting probabilities. The second problem concerns the existence of double points. Using results of Blumenthal and Getoor [Illinois J. Math. **4** (1960), 370–375; MR **22** #12610] it is shown that there are infinitely many double points with probability one if $N \leqq 3$ and $N/2 < \alpha \leqq 2$. In all other cases almost every path is free from double points. The final result deals with the N-dimensional Lebesgue measure of the sample path (the set of $x(t)$, $0 \leqq t < \infty$). It is shown to be zero with probability one when $N \geqq \alpha$, and infinite with probability one when $N < \alpha$. All except the case $N = \alpha = 1$ follow easily from the above-mentioned results of Blumenthal and Getoor concerning the Hausdorff measure of the paths; the result as a whole is stated to be a corollary of unpublished work of S. J. Taylor and D. Ray who determined the exact Hausdorff measure.

F. L. Spitzer (Ithaca, N.Y.)

Takeuchi, Junji 6538

A local asymptotic law for the transient stable process.

Proc. Japan Acad. **40** (1964), 141–144.

The problem solved in the paper reviewed above [#6537] concerning upper and lower classes determining the rate of escape as $t \to \infty$ suggested the present paper which concerns the local behavior as $t \to 0$. When $x(0) = 0$, $\alpha < N$, a positive non-increasing g is in the upper [lower] class according as $|x(t)| \leqq t^{1/\alpha} g(1/t)$ for arbitrarily small positive t with probability one [zero]. The analytic criterion is shown to be exactly the same integral criterion as that established in the previous paper for the case when $t \to \infty$. For Brownian motion this is a well-known consequence of P. Lévy's law of projective invariance ($x(t)/\sqrt{t}$ and $\sqrt{t}\,x(1/t)$ are equivalent processes). No such law is available in general, but the results are those which would follow if $x(t)/t^{1/\alpha}$ and $t^{1/\alpha}x(1/t)$ were equivalent processes. This is not the case. *F. L. Spitzer* (Ithaca, N.Y.)

Tutubalin, V. N. 6539

Compositions of measures on the simplest nilpotent group. (Russian. English summary)

Teor. Verojatnost. i Primenen. **9** (1964), 531–539.

Let G be the group of matrices $g = g(\alpha, \beta, \gamma) = \begin{pmatrix} 1 & \alpha & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix}$, α, β, γ real. If μ is a probability measure on G, then α, β, γ can be regarded as random variables. The distance between μ_1 and μ_2 is defined as $\sup_{a,b,c} |(\mu_1 - \mu_2)(\{g : \alpha < a, \beta < b, \gamma < c\})|$. Let μ_1 and μ_2 be such that α, β, γ have variances under either of them. Necessary and sufficient conditions are obtained in order that the distance between the convolutions μ_1^n and μ_2^n should tend to zero as n tends to infinity. Theorem 1: *If α and β have zero means under both distributions, then necessary and sufficient conditions*

are that each of the four quantities α^2, $\alpha\beta$, β^2, γ should have the same expectation under either μ_1 or μ_2. Theorem 2: If one of the means of α or β is different from zero, then necessary and sufficient conditions are that each of the five quantities α, β, α^2, $\alpha\beta$, β^2 should have the same expectation under either μ_1 or μ_2. (There is no condition on γ in the second case.)

J. G. Wendel (Ann Arbor, Mich.)

Blumenthal, R. M.; Getoor, R. K. 6540

A theorem on stopping times.

Ann. Math. Statist. **35** (1964), 1348–1350.

This note gives, under suitable conditions, a characterization of the Borel field associated with a stopping time T for a strong Markov process. Roughly speaking, the characterization shows that such a field is generated by the random variables of a "stopped process" which agrees with the original one up to time T and remains constant thereafter. *J. W. Lamperti* (Hanover, N.H.)

Molčanov, S. A. 6541

On a problem in the theory of diffusion processes. (Russian. English summary)

Teor. Verojatnost. i Primenen. **9** (1964), 523–528.

Author's summary: "In the paper some Markov processes associated with diffusion processes are discussed. A diffusion process x_t defined on l-dimensional Euclidean space E is considered only at moments when its trajectory belongs to a given set S (a new time is introduced which changes only when the process is in S). If S is a domain with differentiable boundary, the generator $\tilde{\mathfrak{A}}$ of the new process y_t is the same as for x_t at all interior points of S. On the boundary of S non-classical boundary conditions are obtained. These boundary conditions are described in Theorem 1. If S is an $(l-1)$-dimensional surface, we obtain on S a discontinuous process of the Cauchy type. The generator of this process is investigated in Theorem 2."

G. Maruyama (Fukuoka)

Nagasawa, Masao 6542

Time reversions of Markov processes.

Nagoya Math. J. **24** (1964), 177–204.

Consider a temporally homogeneous Markov process x and a positive time $T < \infty$. Then the backwards process $x(T-t-0)$ $(0 < t < T)$ is also a temporally homogeneous Markov process if T is a last exit time, as was proved by G. Hunt [Illinois J. Math. **4** (1960), 313–340; MR **23** #A691]; the same is true if T is a killing time, but for constant T the temporal homogeneity is lost. A wider class of times T for which Hunt's result holds under mild technical conditions on the process x is now described: it is the class of all positive Borel functions T of the path, not exceeding the lifetime, such that $T-s=T(w_s^+)$ if either side is positive and $<\infty$, w_s^+ being the shifted path $t\to x(t+s)$. For this class, the backwards transition probabilities depend upon the law of x but not upon T itself. A similar result is proved for the approximate Markov processes of G. Hunt.

H. P. McKean, Jr. (Cambridge, Mass.)

Nagasawa, Masao 6543

The adjoint process of a diffusion with reflecting barrier.

Kôdai Math. Sem. Rep. **13** (1961), 235–248.

Let $X(t)$ be a time homogeneous Markov process with state space $\bar{D}$, a compact set in R^n. Under some additional conditions the process $\hat{X}(t)$ obtained by reversing the time direction is also a time homogeneous Markov process. In this case call $\hat{X}$ the adjoint of X. In the present paper the author assumes that $\bar{D}$ is the closure of a domain D with a quite smooth boundary, Bdy D, that X is a diffusion process with infinitesimal generator A, a quite smooth elliptic differential operator, and that the domain of A consists essentially of those functions f in $C^2(\bar{D})$ such that the outer normal derivative of f vanishes on Bdy D. Because of the boundary condition X is called a "reflecting barrier process". He proves that the adjoint process exists in this case, constructs its infinitesimal generator, and shows that the adjoint process is also a reflecting barrier process. For diffusions arising in this way there is another process, $X_{\mathrm{Bdy}\,D}$, called the "process on the boundary". The author considers this also, and shows that (with an obvious notation) $(\hat{X})_{\mathrm{Bdy}\,D}=(X_{\mathrm{Bdy}\,D})^\wedge$.

R. M. Blumenthal (Seattle, Wash.)

Nagasawa, Masao; Sato, Keniti 6544

Remarks to "The adjoint process of a diffusion with reflecting barrier".

Kôdai Math. Sem. Rep. **14** (1962), 119–122.

The basic set-up and results of this paper are quite similar to those in the one reviewed above [#6543], except that considerably more general boundary conditions are allowed. *R. M. Blumenthal* (Seattle, Wash.)

Tanaka, Hiroshi 6545

Existence of diffusions with continuous coefficients.

Mem. Fac. Sci. Kyushu Univ. Ser. A **18** (1964), 89–103.

Es sei B der Raum der reellen beschränkten Baireschen Funktionen auf R^d mit $d \geq 2$. Die Arbeit befaßt sich mit einer auf dem Hille-Yosida'schen Halbgruppentheorem fußenden Konstruktion einer Halbgruppe linearer Transformationen in B und eines Diffusionsprozesses in R^d zu gegebenem Operator $A=\sum_{i,j=1}^d a^{ij}\partial^2/\partial x^i\partial x^j+\sum_{i=1}^d b^i\partial/\partial x^i$ mit überall positiv definiter Matrix (a^{ij}), wenn die Funktionen a^{ij} nur als stetig und die Funktionen b^i als meßbar und auf jeder kompakten Menge beschränkt vorausgesetzt werden. Dies geschieht zunächst lokal in einer Kugel, wobei A durch "glattere" Operatoren approximiert wird. Der so gewonnene Diffusionsprozeß wird dann identifiziert mit dem, den man erhält, wenn man den entsprechenden Prozeß in einer größeren Kugel konstruiert und dann auf die erste Kugel einschränkt. Der Greensche Operator des auf diese Weise global in R^d bestimmten Prozesses bildet dann den Raum C der stetigen und beschränkten Funktionen auf R^d in sich ab, und seine Erzeugende ist eine Einschränkung einer gewissen natürlichen Fortsetzung von A. Sind auch die b^i stetig, so hat die zugehörige Halbgruppe T^t, $0 \leq t < +\infty$, die Eigenschaft: Bei beliebigem $f \in C^2$ mit kompaktem Träger konvergiert $t^{-1}(T^tf-f)$ gleichmäßig auf R^d gegen Af. *K. Krickeberg* (New York)

Vorob'ev, N. N. 6546

On a topologization of the set of interior consistent families of measures. (Russian. English summary)

Teor. Verojatnost. i Primenen. **8** (1963), 444–451.

The author extends to consistent families of interior measures the theorem proved by him and Faddeev earlier [Teor. Verojatnost. i Primenen. **6** (1961), 116–118; MR **27** #779]. The definition of a consistent family in this context is the same as that used by the author earlier [ibid. 7 (1962), 153–169].

S. G. Ghurye (Bloomington, Ind.)

Vorob'ev, N. N. 6547

Markov measures and Markov extensions. (Russian. English summary)

Teor. Verojatnost. i Primenen. 8 (1963), 451–462.

The author extends the idea of a Markov process to the case when the index set ("time domain") is a complex, and proves an extension theorem for consistent families of measures as defined by him earlier [Teor. Verojatnost. i Primenen. 7 (1962), 153–169].

S. G. Ghurye (Bloomington, Ind.)

Priestley, M. B. 6548

Estimation of the spectral density function in the presence of harmonic components.

J. Roy. Statist. Soc. Ser. B **26** (1964), 123–132.

Let $X_t = Y_t + Z_t$ be a stationary process, where Y_t is a linear process and Z_t is a process of pure harmonic components of the form $Z_t = \sum_{i=1}^k A_i \cos(\omega_i t + \varphi_i)$. Here k and $A_1, \cdots, A_k$ are unknown constants and $\varphi_1, \cdots, \varphi_k$ are independent and uniformly distributed on $[-\pi, \pi]$. The author provides a technique for detecting the presence of such harmonic components and uses this to construct a consistent estimate of the autocovariance function of the Y_t process. Finally he gives a technique which gives a direct estimate of the spectral density of the Y_t process from the sample autocovariance function.

J. R. Blum (Albuquerque, N.M.)

Šmul'jan, Ju. L. 6549

The optimal factoring of non-negative matrix functions. (Russian. English summary)

Teor. Verojatnost. i Primenen. **9** (1964), 382–386.

In der Theorie der stationären n-dimensionalen stochastischen Prozesse mit diskreter Zeit treten, als Spektraldichten regulärer Prozesse, $n \times n$ Matrizen $f(\lambda)$, $-\pi \leq \lambda \leq \pi$, in L_1 auf, die Faktorisierungen der Form $A(e^{-i\lambda})A^*(e^{-i\lambda}) = 2\pi f(\lambda)$ zulassen, wobei A eine im Einheitskreis holomorphe $n \times q$ Matrix der Klasse H_2 bildet. Existiert eine solche Faktorisierung, so hat f fast überall ein und denselben Rang m, und Kreĭn zeigte, daß es dann unter den $n \times m$ Faktorisierungen eine maximale A_0 im folgenden Sinne gibt: (*) $A(z)A^*(z) \leq A_0(z)A_0^*(z)$, $|z| \leq 1$, bei jeder $n \times m$ Faktorisierung A [Ju. A. Rozanov, Teor. Verojatnost. i Primenen. 5 (1960), 399–414; MR **24** #A2432]. Der Verfasser betrachtet nun allgemeiner $n \times q$ Matrizen A der Klasse H_2, wobei nicht notwendig $q = m$, für die $A(e^{-i\lambda})A^*(e^{-i\lambda}) \leq 2\pi f(\lambda)$ gilt, und beweist folgendes: Es ist $A(z) = A_0(z)E(z)$, wobei $E(z)$ holomorph in $|z| \leq 1$ mit $E(z)E^*(z) \leq I_m$ ist; die Ungleichung (*) gilt wiederum, und trifft darin das Gleichheitszeichen in einem Punkt z_0 zu, so trifft es überall zu und man kann E als konstante Matrix so wählen, daß $EE^* = I_m$ wird.

K. Krickeberg (New York)

Rogozin, B. A. 6550

On the distribution of the first jump. (Russian. English summary)

Teor. Verojatnost. i Primenen. **9** (1964), 498–515.

A Fourier analytical treatment of certain problems in the fluctuation theory of sums of identically distributed independent random variables. The methods are standard (Wiener-Hopf factorization and the renewal theorem); the results, concerning the distribution of functionals such as the supremum and the first positive partial sum, supplement theorems contained in Kemperman's book [*The passage problem for a stationary Markov chain*, Univ. Chicago Press, Chicago, Ill., 1961; MR **22** #9992].

F. L. Spitzer (Ithaca, N.Y.)

Borovkov, A. A.; Rogozin, B. A. 6551

Boundary-value problems for some two-dimensional random walks. (Russian. English summary)

Teor. Verojatnost. i Primenen. **9** (1964), 401–430.

The first author has been engaged in the study of asymptotic expansions concerning boundary problems for the sums of independent random variables [Borovkov, Sibirsk. Mat. Ž. **3** (1962), 645–694; MR **26** #3099]. Consider now two independent systems of identically distributed independent variables, $\xi_1^{(i)}, \xi_2^{(i)}, \cdots, i = 1, 2$, with $\xi_k^{(2)} \geq 0$, their respective sums $S_k^{(i)}$, $k = 1, 2, \cdots, S_0^{(i)} = 0$, and put $\bar{S}_n = \max_{1 \leq k \leq n} S_k^{(1)}$, $\eta_t = \sup\{k : S_k^{(2)} < t\}$. Assuming that the $\xi_k^{(1)}$'s are lattice variables the authors obtain asymptotic expansions for the joint distribution of $\bar{S}_{\eta_t}, S^{(1)}_{\eta_t+1}, S^{(2)}_{\eta_t+1}$, e.g., an asymptotic expansion of $\mathbf{P}(\bar{S}_{\eta_t} < x, S^{(1)}_{\eta_t+1} = x + y, S^{(2)}_{\eta_t+1} = t + s)$ when $t \to \infty$, $x/t \to 0$. Those expansions are effective to wide ranges of deviations including the limit theorems of Cramér's type. The analytical tools are similar to those in the paper cited above. The results obtained are also applied to get asymptotic expansions for the Kolmogorov-Smirnov test.

G. Maruyama (Fukuoka)

Bretagnolle, Jean; Dacunha-Castelle, Didier 6552

Marches aléatoires récurrentes sur R^p.

C. R. Acad. Sci. Paris **259** (1964), 2765–2768.

The authors study recurrent random walk (sums of identically distributed independent random walk) on R^p (Euclidean space of dimension p). The transition probability measure μ is assumed to have an absolutely continuous component, and when $p = 1$ additional conditions are imposed (in order to ensure the success of the Fourier analytical methods employed). Under these conditions the random walk has a potential kernel $U_x = \sum_{n=1}^{\infty} [\mu_x^n - \mu^n]$ (where μ^n is the nth convolution of μ, and μ_x^n the translate of μ^n by x). Just as in the reviewer's work on Z^p, it is shown that $U_{x+y} - U_x$ tends to 0 as $|x| \to \infty$ when $p = 2$, while the limit is $\pm y\sigma^{-2}\lambda$ as $x \to \pm\infty$ when $p = 1$. Here $\sigma^2 \leq \infty$ is the variance of μ, and λ is Lebesgue measure.

F. L. Spitzer (Ithaca, N.Y.)

Klinger, Hanns 6553

Ein Wahrscheinlichkeitsmodell für die symmetrische, lineare Irrfahrt mehrerer Teilchen.

Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (*Liblice*, 1962), pp. 405–416. *Publ. House Czech. Acad. Sci., Prague*, 1964.

The author calculates the invariant distribution of a finite Markov chain (irreducible and aperiodic) with continuous time parameter which describes a certain physical model. In this model, when the physical system is of size n, the number of possible states is 2^n and the infinitesimal generator $q_n(i,j)$ is given explicitly. Thus the invariant distribution $P_n(i)$ is the unique non-negative solution of the system of equations

$$\sum_{i=1}^{2^n} P_n(i)q_n(i,j) = 0, \qquad j = 1, 2, \cdots, 2^n,$$

with the normalized condition $\sum_{i=1}^{2^n} P_n(i) = 1$. The author obtains the recurrence formula for $P_n(i)$ and, using it, evaluates expectations of related physical quantities.

T. Watanabe (Princeton, N.J.)

Neuts, Marcel F. 6554

Coordinate-wise symmetric random walks in n-space.

Simon Stevin **36** (1962/63), 131–139.

Consider a spatially homogeneous random walk on an n-dimensional Euclidean lattice space, such that one-step transitions occur only to neighboring states. The walk is assumed symmetric in the sense that the probability of a step to the right parallel to any given coordinate axis equals the probability of a step to the left. The author derives a representation for $P_{xy}^{(k)}$ (the k-step transition probability from a lattice point x to a lattice point y), which is similar to the standard inversion formula for the characteristic function of $P_{xy}^{(k)}$. This representation is used to give another proof of the fact that the walk is recurrent in dimensions 1 and 2 [Chung and Fuchs, Mem. Amer. Math. Soc. No. 6 (1951); MR **12**, 722].

P. E. Ney (Ithaca, N.Y.)

Reimann, J. 6555

Zweidimensionale Irrfahrt auf einem Gitterpunktsrechteck mit reflektierenden Wänden.

Monatsb. Deutsch. Akad. Wiss. Berlin **5** (1963), 560–565.

Let P be the transition matrix of a (two-dimensional) reflecting-barrier random walk with the state space consisting of the lattice points of a rectangle. Using the fact that P can be decomposed into sums of Kronecker products of simpler matrices, the author gives the explicit formulae for P^N (the N-step transition probabilities) and $\lim_{N\to\infty} P^N$ (the limiting distribution).

T. Watanabe (Princeton, N.J.)

Reimann, J. 6556

Unsymmetrical random walk on the plane and in the space with absorbing barriers.

Acta Math. Acad. Sci. Hungar. **15** (1964), 339–354.

Let R be a rectangle of lattice points on the plane and let each lattice point on the boundary be an absorption point. In this paper the author considers an (absorbing barrier) random walk on R, more general than the usual one, for which the points that can be reached in one step from an interior point (i,j) are the starting point itself and 8 neighboring points, $(i\pm1, j)$, $(i, j\pm1)$, and $(i\pm1, j\pm1)$, with certain restrictions on the one-step transition probabilities in diagonal directions. According to the author's opinion, "this increase of the number of neighboring points is important, e.g., from the point of view of the solution of the Poisson differential equation with the lattice method, where it greatly increases the goodness of the approximation for a given lattice density". Using a matrix method, he computes the n-step transition probabilities of such a random walk with application to absorption problems. The three-dimensional case is also discussed.

T. Watanabe (Princeton, N.J.)

Shepp, L. A. 6557

Recurrent random walks with arbitrarily large steps.

Bull. Amer. Math. Soc. **70** (1964), 540–542.

A random walk $S_n = X_1 + \cdots + X_n$ (the X_i are identically distributed independent random variables with distribution function F) is recurrent if $P[|S_n| < 1$ for infinitely many $n = 1, 2, \cdots] = 1$. It is shown that the distribution F of a recurrent random walk (rec.r.w.) may have arbitrary large tails, viz., given $\varepsilon(x)$, $x \geq 0$, with $\varepsilon(x) \to 0$ as $x \to \infty$, there exists a rec.r.w. with d.f. F, such that for some x_0, $1 - F(x) = F(-x) \geq \varepsilon(x)$, $x \geq x_0$. Under additional assumptions on $F(x)$ (such as convexity for x sufficiently large) such a transient random walk cannot exist [the author, Trans. Amer. Math. Soc. **104** (1962), 144–153; MR **25** #2648].

F. L. Spitzer (Ithaca, N.Y.)

Sherman, S. 6558

Fluctuation and periodicity.

J. Math. Anal. Appl. **9** (1964), 468–476.

Some connections are deduced between the distribution of certain functionals of random walk on the integers and an identity due to Witt concerning the dimension of certain subspaces of a free Lie algebra. Specifically, the Witt identity in the context of free semigroups deals with "words" and their permutations, and this suggested analogous identities concerning permutations of random walk paths. The characteristic function of the least concave polygon majorizing a random walk path of length n (defined somewhat differently than in the work of Sparre Andersen) is shown to satisfy an identity formally identical to that of the maximum M_n of a path of length n. Similar methods, also based on the Witt identity, lead to a new proof of known facts concerning M_n.

F. L. Spitzer (Ithaca, N.Y.)

Tanaka, Hiroshi 6559

Note on continuous additive functionals of the 1-dimensional Brownian path.

Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **1** (1962/63), 251–257.

Let $a(t)$, $t \geq 0$, be a continuous additive functional of Brownian motion in R^1. The author proves that if I is any bounded interval in R^1 and T is the first time the Brownian motion leaves I, then for each $t \geq 0$, $a(\min(t, T))$ has finite moments of all orders. This result gives the author enough finiteness to obtain the representation $a(t) = f(x(t)) - f(x(0)) + \int_0^t g(x(s))\,dx(s)$ ($x(s)$ is the Brownian motion in question) for any such functional. (In this representation f is continuous, g satisfies some finiteness restrictions, and the integral is a stochastic integral.) The author comments that this representation has been obtained by others using different methods. The most noteworthy reference is to A. D. Ventcel' [Dokl. Akad. Nauk SSSR **142** (1962), 1223–1226; MR **27** #4265].

R. M. Blumenthal (Seattle, Wash.)

Fieger, Werner 6560

Zwei Verallgemeinerungen der Palmschen Formeln.

Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (*Liblice*, 1962), *pp.* 107–122. *Publ. House Czech. Acad. Sci., Prague*, 1964.

For a call process $x(t)$ Palm has shown that we have

$$p(x(a+t)-x(a) = 0) = 1-\lambda\int_0^t \varphi_0(\tau)\,d\tau,$$

$$p(x(a+t)-x(a) = k) = \lambda\int_0^t \varphi_{k-1}(\tau)\,d\tau-\lambda\int_0^t \varphi_k(\tau)\,d\tau$$

$(k=1, 2, \cdots)$ for $0\leqq a<+\infty$, $0\leqq t<+\infty$, where

$$\varphi_i(t) = \lim_{\tau\searrow 0} p(x(t+\tau)-x(\tau) = i\,|\,x(\tau)\geqq 1),$$
$$i = 0, 1, 2, \cdots,$$

if the process is (1) stationary and (2) ordinary. Condition (1) implies that $\lim_{t\searrow 0}[p(x(t)\geqq 1)/t] = \lambda<+\infty$. Define

$$v_k(t) = p(x(t) = k), \qquad \psi_k(t) = p(x(t)\geqq k) = \sum_{i=k}^{\infty} v_i(t).$$

Condition (2) implies that if there exists a $\delta>0$ for an arbitrary $t>0$ and $\varepsilon>0$ so that for $0\leqq t_1<t_2\leqq t_1+\delta<t$, then $\psi_2(t_1, t_2)\leqq\varepsilon\psi_1(t_1, t_2)$, where

$$\psi_k(t_1, t_2) = p(x(t_2)-x(t_1)\geqq k) = \sum_{i=k}^{\infty} v_i(t_1, t_2)$$

and

$$v_k(t_1, t_2) = p(x(t_2)-x(t_1) = k).$$

In the paper it is shown that the Palm formula is applicable also in case only one of the two conditions (1) or (2) is fulfilled. *S. Ekberg* (Stockholm)

Mott, J. L. 6561

The distribution of the time-to-emptiness of a discrete dam under steady demand.

J. Roy. Statist. Soc. Ser. B **25** (1963), 137–139.

This note gives a simple combinatorial proof of the discrete analogue of Kendall's result [same J. **19** (1957), 207–212; discussion, ibid. **19** (1957), 212–233; MR **19**, 1092] for the probability of first emptiness of a dam subject to random integral inputs and steady unit demand.

J. Gani (E. Lansing, Mich.)

Yeo, G. F.; Weesakul, B. 6562

Delays to road traffic at an intersection.

J. Appl. Probability **1** (1964), 297–310.

Authors' summary: "A model for road traffic delays at intersections is considered where vehicles arriving, possibly in bunches, in a Poisson process in a one-way minor road yield right of way to traffic, which forms alternate bunches and gaps, in a major road. The gap acceptance times are random variables, and depend on whether or not a minor road vehicle is immediately following another minor road vehicle into the intersection or not. The transforms of the stationary waiting time and queue size distributions and the mean stationary delay for minor road vehicles are obtained by substitution of determined service time distributions into results for a generalisation of the $M/G/1$ queueing system. Some numerical results are given to illustrate the increase in the mean delay for variable gap acceptance times for a Borel-Tanner distribution of major road traffic, and a partial solution is given for a two way major road."

This is the most general model for the major-minor road intersection considered so far although it is closely related to the work of Gaver [Operations Res. **11** (1963), 72–87]. Any further generalizations of this are likely to be quite cumbersome to solve exactly.

G. Newell (Providence, R.I.)

Bartholomew, D. J. 6563

An approximate solution of the integral equation of renewal theory.

J. Roy. Statist. Soc. Ser. B **25** (1963), 432–441.

Author's summary: "An approximation is given to the solution of the integral equation for the renewal density. The approximation is of particular value when the renewal distribution has a high degree of positive skewness. For an important class of such distributions it also provides an upper bound to the renewal density. Some numerical comparisons are given for renewal processes in discrete time. Other methods of obtaining an approximate solution to the equation are briefly reviewed and compared with the new method." *G. Weiss* (Bethesda, Md.)

Davis, A. W. 6564

On the characteristic functional for a replacement model.

J. Austral. Math. Soc. **4** (1964), 233–243.

Consider a system of n "ancestors" existing at time $t=0$, with ranked ages $0\leqq x_1<x_2<\cdots<x_n$. At successive instants (regeneration points) one individual in the system (the ith with probability p_i) is replaced by another of age zero. The author gives a scheme for computing the characteristic functional of $N(u, t)$, the number of individuals of age $\leqq u$ at time t. This is used in the evaluation of the characteristic function of $N(x, t)$ (fixed x), and the age distribution of the ith ranked individual at any time t. Limit distributions (as $t\to\infty$) are given.

P. E. Ney (Ithaca, N.Y.)

Campbell, L. L. 6565

On a class of polynomials useful in probability calculations.

IEEE Trans. Information Theory **IT-10** (1964), 255–256.

The author discusses well-known properties of the Hermite polynomials that may be useful in the analysis of some noise problems. *M. Rosenblatt* (La Jolla, Calif.)

Borel, Émile 6566

★Probabilities and life.

Translated from the French by Maurice Baudin.

Dover Publications, Inc., New York, 1962. vi+87 *pp.* \$1.00.

This is a new translation of the fourth edition of *Les probabilités et la vie* [Presses Universitaires de France, Paris, 1943].

Kristiansson, Lars 6567

On the semi-Markov process and its block diagram representation with applications to aircraft mission analysis.

Saab Tech. Notes TN 55 (1964), 16 *pp.*

Author's summary: "The semi-Markov process is defined as a stochastic process in which the time is the independent variable and the dependent variable can assume only an enumerable number of discrete values ('states'), and where transition from one given state to another depends on the length of time spent in the given state. Provided that the process is described with the aid of frequency functions $d_j(t)$ which specify the probability of a certain state j being reached in the time interval $(t; t+\Delta t)$, simple process equations are obtained. These equations can readily be represented by means of block diagrams. There is perfect correspondence between the process and the electrical network with positive feedback. The utility of the process is illustrated by means of an example concerned with aircraft mission analysis."

STATISTICS

See also 6504, 6548, 6619, 6957, 6958.

Ehrenfeld, Sylvain; Littauer, Sebastian B. 6568

★Introduction to statistical method.

McGraw-Hill Series in Probability and Statistics.

McGraw-Hill Book Co., New York-Toronto-London. 1964. x + 533 *pp.* $9.75.

This introductory text is designed for mature undergraduates in science and engineering programs. It gives a substantial background in statistical inference, with illustrative examples and exercises drawn from a variety of applications. *W. H. Fleming* (Providence, R.I.)

Fabian, Václav 6569

A new one-dimensional stochastic approximation method for finding a local minimum of a function.

Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (*Liblice*, 1962), *pp.* 85–105. *Publ. House Czech. Acad. Sci., Prague,* 1964.

Let $R(x)$ be a real-valued function of a real variable and suppose for each x there is an observable random variable Y_x with $E[Y_x] = R(x)$. There has been a good deal of literature in the last ten years on the problem of finding sampling schemes which asymptotically locate the extrema of the function $R(x)$. In this paper the author discusses a method which in some cases converges to the set of local minima of the function $R(x)$. *J. R. Blum* (Albuquerque, N.M.)

Bhat, B. R. 6570

On the distribution of certain quadratic forms in normal variates.

J. Roy. Statist. Soc. Ser. B **24** (1962), 148–151.

The author considers a general vector normal variate $\mathbf{X}$, mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$, and derives the following principal theorems. Theorem 2: Let $\mathbf{G}_0 = \mathbf{I}, \mathbf{G}_1, \cdots, \mathbf{G}_k$ be $k+1$ real, symmetric, linearly independent $n \times n$ matrices such that $\mathbf{G}_i\mathbf{G}_{i'} = c\mathbf{G}$ $(i, i' = 0, 1, 2, \cdots, k)$, where $\mathbf{G}$ is either the zero matrix or some $\mathbf{G}_i$ and c is some constant. Let $Q_i = \mathbf{X}'\mathbf{A}_i\mathbf{X}$ $(i = 1, 2, \cdots, l)$ be l quadratic forms, where $\mathbf{A}_i$ are linear functions of $\mathbf{G}_j$ $(j = 0, 1, \cdots, k)$, distributed independently as $c_i\chi^2(n_i)$ such that $\sum n_i = n$. Then $\boldsymbol{\Sigma}$ is also a linear function of $\mathbf{G}_j$. If $k+1 = l$ and $\mathbf{A}_i$ are linearly independent, $\boldsymbol{\Sigma}$ is a linear function of $\mathbf{A}_i$. (Here $\chi^2(n)$ denotes a non-central χ^2 variate with n degrees of freedom.) Theorem 3: If Q_1 and Q_2 are two real symmetric non-negative or non-positive quadratic forms, then a quadratic form or a linear form is distributed independently of $Q_1 + Q_2$ if and only if it is distributed independently of Q_1 and of Q_2. *P. Whittle* (Manchester)

Park, John H., Jr. 6571

Variations of the non-central t and beta distributions.

Ann. Math. Statist. **35** (1964), 1583–1593.

Let X be a normal variate with mean ξ and variance 1, and let Y be an independent non-central chi variate. Consider the variates $U_1 = X/Y$ and $U_2 = X/(X^2+Y^2)^{1/2}$. Their distributions are doubly non-central t and beta distributions, respectively, whose densities can be expressed as difficult integrals [or as double series]. The author derives simple asymptotic expressions for the densities, their modes, means, and variances. When Y has 1 degree of freedom, the densities of U_1 and U_2 can be represented fairly simply in terms of the error function. *S. W. Nash* (Vancouver, B.C.)

Siromoney, Gift 6572

Entropy of logarithmic series distributions.

Sankhyā Ser. A **24** (1962), 419–420.

Author's summary: "The logarithmic series distribution was introduced by Fisher and Williams. We show that there are two types of logarithmic series distributions, which we shall call Type I and Type II. We evaluate the entropies of the two distributions and present the corresponding interpretations."

Saunders, Sam C. 6573

On the use of a distribution-free property in determining a transformation of one variate such that it will exceed another with a given probability.

Sankhyā Ser. A **24** (1962), 239–250.

Author's summary: "Let X and Y be independent random variables, with continuous distributions $F \in \mathscr{F}$ and $G \in \mathscr{G}$, on the sample space $\mathscr{X}$. Let ω be a homeomorphism from $\mathscr{X}$ onto itself and define

$$H(\omega) = \int F(\omega)\, dG.$$

From samples of X and Y we form $\hat{F}$ and $\hat{G}$ which are estimates of F and G, respectively, and define an estimate of H, say $\hat{H}$,

$$\hat{H}(\omega) = \int \hat{F}(\omega)\, d\hat{G}$$

for $\omega \in \Omega$, a class of homeomorphisms linearly ordered by H. Interpreting $\omega(Y)$ as the strain under use ω and X as the strength (of some device), then $H(\omega) = P[X < \omega(Y)]$ represents the unreliability and $\hat{H}(\omega)$ is an estimate of it. If we use $\hat{H}$ to determine a use $\hat{\omega}$, then what is the probability that the true unreliability $\hat{H}(\omega)$ is too large? We examine this problem under the assumptions that $\hat{F}(F^{-1})$ and $\hat{G}(G^{-1})$ are distribution-free with respect to $\mathscr{F}$ and $\mathscr{G}$, respectively. This provides an answer in some cases and allows one to obtain stochastic bounds in others."

Doss, S. A. D. C. 6574

On uniqueness and maxima of the roots of likelihood equations under truncated and censored sampling from normal populations.

Sankhyā Ser. A **24** (1962), 355–362.

Author's summary: "It is shown that when estimating from truncated and censored samples both the parameters of normal populations jointly or only one of them, the other being known, the maximum likelihood estimating equations possess a unique solution and that solution provides the absolute maximum of the likelihood function for all samples of any size."

Atiqullah, M. 6575

On the randomization distribution and power of the variance ratio test.

J. Roy. Statist. Soc. Ser. B **25** (1963), 334–347.

Author's summary: "A particular matrix representation is given for the usual sum of squares of treatments in some commonly used designs. This representation simplifies appreciably the calculation of randomization moments under null and non-null hypotheses. Fisher's Z-transformation is applied to study the randomization distribution of the F-criterion. The behavior of power curves under randomization and normal theory conditions is studied." *J. Kiefer* (Ithaca, N.Y.)

Birch, M. W. 6576a

A new proof of the Pearson-Fisher theorem.

Ann. Math. Statist. **35** (1964), 817–824.

Birch, M. W. 6576b

Acknowledgment of priority.

Ann. Math. Statist. **36** (1965), 344.

From the author's summary: "This paper is concerned with the theorem that the χ^2 goodness of fit statistic for a multinomial distribution with r cells and with s parameters fitted by the method of maximum likelihood is distributed as χ^2 with $r-s-1$ degrees of freedom. . . . In this paper the theorem is proved under more general conditions than the combined conditions of Rao and Cramér [Rao, Sankhyā **20** (1958), 211–218; MR **21** #6059; and Cramér, *Mathematical methods of statistics*, Princeton Univ. Press, Princeton, N.J., 1946; MR 8, 39]." {The author considers his approach to contain two points of novelty. First, appropriate differentiability is assumed only at the "true" parameter value. (The second paper acknowledges Rao's prior use [Sankhyā Ser. A **23** (1961), 25–40; MR **25** #718] of a sufficient statistic for this.) The reviewer wonders how this weaker assumption is ever to be used, in view of the fact that this "true" value can be any point of the parameter space. Secondly, the author uses what he calls a "slightly unconventional" definition of maximum likelihood to include the case where the supremum of the likelihood is not attained, a definition which has in fact been used by several previous authors. The paper contains a large number of unclear passages; for example, in the statement of the theorem reference is made to the multinomial probability vector $\pi(\theta)$ at points θ of the compactification of the parameter space where π has not been defined.} *J. Kiefer* (Ithaca, N.Y.)

Fraser, D. A. S. 6577

On local unbiased estimation.

J. Roy. Statist. Soc. Ser. B **26** (1964), 46–51.

The concept "local unbiasedness" is introduced and investigated: A statistic $h(x_1, \cdots, x_n)$ is locally unbiased for θ at θ_0 if

$$Eh((x_1, \cdots, x_n)\,|\,\theta_0\} = \theta_0 \text{ and } dE\{h(x_1, \cdots, x_n)\,|\,\theta\}/d\theta_0 = 1.$$

The statistic $S(\mathbf{x}) = \sum_{i=1}^n \partial \log f(x_i\,|\,\theta)/\partial\theta_0$ is termed locally sufficient, and it is shown that $\theta_0 + S(\mathbf{x})/nI(\theta_0)$ (a misprint omits the n in the denominator), where $I(\theta)$ is Fisher's information, is the minimum-variance locally unbiased estimate at θ_0. The discussion is extended to vector-valued parameters. It is shown that the locally unbiased estimates with minimum variance "piece together" to form a single overall estimate only if the population distributions form an exponential family. An estimation programme is suggested in which the parameter space is split into small neighbourhoods.

H. D. Brunk (Columbia, Mo.)

Fraser, D. A. S. 6578

Local conditional sufficiency.

J. Roy. Statist. Soc. Ser. B **26** (1964), 52–62.

Consider a family of distributions $F(x\,|\,\theta)$ stochastically increasing in the neighbourhood of a parameter value θ_0. A linearizing transformation is introduced, yielding a random variable $\ell(x)$ which is locally translation-invariant. The vector of differences in observed values of ℓ is treated as an ancillary statistic. Local sufficiency, conditioned on this ancillary statistic, is introduced and studied. The large sample distribution of the local conditionally sufficient statistic is found to be normal. The conditional large sample variance is discussed.

H. D. Brunk (Columbia, Mo.)

Giri, N.; Kiefer, J. 6579

Minimax character of the R^2-test in the simplest case.

Ann. Math. Statist. **35** (1964), 1475–1490.

Authors' summary: "In the first nontrivial case, dimension $p=3$ and sample size $N=3$ or 4 (depending on whether or not the mean is known), it is proved that the classical level α normal test of independence of the first component from the others, based on the squared sample multiple correlation coefficient R^2, maximizes, among all level α tests, the minimum power on each of the usual contours where the R^2-test has constant power. A corollary is that the R^2-test is most stringent of level α in this case." *E. L. Lehmann* (Berkeley, Calif.)

Miller, Rupert G., Jr. 6580

A trustworthy jackknife.

Ann. Math. Statist. **35** (1964), 1594–1605.

A method for the reduction of bias in a parametric situation was introduced by Quenouille [J. Roy. Statist. Soc. Ser. B **11** (1949), 68–84; MR **11**, 262] and referred to by Tukey as a "jacknife". The author examines this method in a variety of situations, pointing out where it is effective and where it is of little use.

M. Rosenblatt (La Jolla, Calif.)

Shenton, L. R.; Bowman, K. 6581

Higher moments of a maximum-likelihood estimate.

J. Roy. Statist. Soc. Ser. B **25** (1963), 305–317.

Authors' summary: "For a distribution depending on a single parameter the first four sampling moments of the maximum-likelihood estimate to orders N^{-2}, N^{-3}, N^{-3} and N^{-4}, respectively, are given. Expressions for the measures of skewness γ_1 and γ_2 are also given. Several illustrative examples are included as a check on the heavy algebra. The paper extends earlier work by Haldane and Smith." *H. D. Brunk* (Columbia, Mo.)

Tallis, G. M. 6582

Further models for estimating correlation in discrete data.

J. Roy. Statist. Soc. Ser. B **26** (1964), 82–85.

Author's summary: "This note considers the distribution of the sum of n identically distributed multinomial variates X_i, the correlation coefficient for X_i and X_j being ρ for all i, j, $i \neq j$. The initial model, with n and $\mathbf{p}' = (p_1, p_2, \cdots, p_k)$ fixed, is generalized by allowing first n, then $\mathbf{p}$ and finally both n and $\mathbf{p}$ to be random variables." *H. E. Reinhardt* (Missoula, Mont.)

Halperin, Max 6583

Confidence interval estimation in non-linear regression.

J. Roy. Statist. Soc. Ser. B **25** (1963), 330–333.

The results of this paper are related to those of Williams [same J. **24** (1962), 125–139; MR **25** #1597], which are clearly applicable to a regression function with only one parameter entering non-linearly. In this paper it is shown that the procedure of Williams is applicable to a quite large class of functions in which more than one parameter enters non-linearly. The discussion implies that the regions or intervals obtained are confidence regions and intervals. *P. S. Dwyer* (Ann Arbor, Mich.)

Kabe, D. G. 6584

Multivariate linear hypothesis with linear restrictions.

J. Roy. Statist. Soc. Ser. B **25** (1963), 348–351.

This paper features a normal multivariate linear regression model and gives a criterion for the simultaneous testing of a set of linear estimable parametric functions. The introductory section states the problem and cites recent articles which the paper generalizes. The evaluation of three general integrals, useful in the derivation of the sections following, is presented in the second section. The third and fourth sections establish the criterion, its hth moment, and allied results. *P. S. Dwyer* (Ann Arbor, Mich.)

Wilk, M. B.; Gnanadesikan, R. 6585

Graphical methods for internal comparisons in multiresponse experiments.

Ann. Math. Statist. **35** (1964), 613–631.

The authors, who believe that "the objectives of statistical analysis are neither so narrow nor so formal as described and implied by some statistical theories of estimation and testing hypotheses", propose a procedure for graphical analysis of multivariate data which is an analogue and extension of the method of half-normal plotting due to Daniel [Technometrics **1** (1959), 311–341; MR **22** #A3009]. *S. G. Ghurye* (Bloomington, Ind.)

Mardia, K. V. 6586

Some results on the order statistics of the multivariate normal and Pareto type 1 populations.

Ann. Math. Statist. **35** (1964), 1815–1818.

The author finds the distribution of the vector of smallest order statistics in a random sample of size n from a k-variate Pareto type 1 population [see the author, same Ann. **33** (1962), 1008–1015; MR **27** #871]. He then derives the distributions of the vectors of ranges in random samples of sizes $n = 2, 3$ from k-variate Pareto type 1 and k-variate normal populations. *S. W. Nash* (Vancouver, B.C.)

★Contributions to order statistics. 6587

Edited by Ahmed E. Sarhan and Bernard G. Greenberg. *John Wiley & Sons, Inc., New York-London*, 1962. xxv + 482 *pp*. $11.25.

The editors state in a preface that "the first aim of this monograph has been to assemble scattered materials to help applied research workers learn to use the tools of order statistics. A secondary aim is to provide materials and references that will facilitate further research in methodology". It is indicated that the contributors were asked to prepare articles following an outline of planned contents. The emphasis is less in the direction of (for example) rank-order methods than in that of asymptotically efficient estimation of scale and location parameters using linear combinations of order statistics. The latter developments are introduced in Chapters 3–5 by E. H. Lloyd, J. Jung, G. Blom, and J. Ogawa, after an introduction by H. A. David and a chapter on distributions and moments by Ogawa.

The remainder of Part I (the "theoretical part") of the book consists of a chapter on extreme values by E. J. Gumbel; a section on short-cut tests in normal theory by H. A. David, and a short one on the closest two out of three observations by J. Lieblein; a chapter on nonparametric confidence intervals and tolerance regions by J. E. Walsh; and one on multiple decisions and multiple comparisons by H. A. David.

There is little new theoretical development in these chapters; for the most part, they survey known results in chosen areas. The level of rigor and ratio of statements to detailed proofs varies from chapter to chapter. It would have been more useful if one comprehensive article had replaced the closely related and somewhat overlapping efforts of the four authors of Chapters 3–5, especially since this work is much less well known than the rest of Part I.

Part II ("specific applications") is concerned mostly with the normal distribution (H. Ruben, D. Teichroew, Sarhan and Greenberg, Ogawa, W. J. Dixon), but there are also chapters on the exponential distribution (Sarhan and Greenberg, B. Epstein, Ogawa) and on "other distributions", including the rectangular, gamma, and extreme value (Sarhan and Greenberg, Lieblein, Gumbel, S. Gupta, Lloyd). Typically, moments of order statistics are given and computations are made which implement the theory of Part I, for example, in the determination of optimum coefficients for the estimators discussed in Chapters 3–5. A major value of the book is in the 75 tables and numerous charts, many gathered from previous publications. There is also an extensive bibliography.

The practical worker will certainly find this a useful handbook and reference work, although he will be

disappointed if he seeks either detailed proofs of all developments or, perhaps more important, many detailed practical examples of a less than routine nature.
J. Kiefer (Ithaca, N.Y.)

Rosengard, Alex 6588
Indépendance limite uniforme d'un quantile et des valeurs extrêmes d'un échantillon.
C. R. Acad. Sci. Paris **259** (1964), 2955–2956.
The author continues his study of "asymptotic uniform independence" of certain order statistics from a sample of independent observations [same C. R. **258** (1964), 5786–5788]. He shows that the extreme values and the sample quantile of order k, $0<k<1$, are asymptotically uniformly independent. The result is more general and the proof much simpler than that in an earlier, unpublished paper of the reviewer. *S. M. Berman* (New York)

Sugiura, Nariaki 6589
The bivariate orthogonal inverse expansion and the moments of order statistics.
Osaka J. Math. **1** (1964), 45–59.
Denote by $X_{r/n}$ the rth member in ascending order of magnitude in a random sample of size n from a distribution function with mean μ and finite variance. In a previous paper [same J. **14** (1962), 253–263; MR **26** #859] the author derived bounds for $\mathrm{E}(X_{r/n})$. For $1\leqq r<s\leqq n$, he now uses bivariate extensions of his earlier techniques to derive similar bounds for $\mathrm{E}(X_{r/n}X_{s/n})$; and for symmetric distributions with $\mu=0$, bounds are also derived for $\mathrm{E}(X_{r/n}^i X_{s/n}^j)$ when $i,j=1,2$. Some numerical comparisons are made with exact values available when $i=j=1$.
R. L. Plackett (Newcastle upon Tyne)

Takács, Lajos 6590
The use of a ballot theorem in order statistics.
J. Appl. Probability **1** (1964), 389–392.
The author's generalized ballot theorem [Ann. Math. Statist. **33** (1962), 1340–1348; MR **25** #4580; J. Appl. Probability **1** (1964), 69–76; MR **28** #5495] is applied to obtain the exact distribution of

$$\sup_{\alpha\leq F(u)\leq\beta}[F_n(u)-\gamma F(u)]G(u)\quad\text{for } 0\leqq\alpha<\beta\leqq 1\leqq\gamma,$$

in the two cases $G(u)\equiv 1$ and $G(u)=1/F(u)$, where F_n is the empirical d.f. of n independent random variables with common continuous d.f. F. For special values of α,β,γ the problem had been considered by many authors beginning with Smirnov, but this is the first general treatment.
J. Kiefer (Ithaca, N.Y.)

Takács, Lajos 6591
Fluctuations in the ratio of scores in counting a ballot.
J. Appl. Probability **1** (1964), 393–396.
Suppose that in a ballot candidate A scores a votes and candidate B scores b votes and that all the possible $\binom{a+b}{b}$ voting records are equally probable. Denote by α_r and β_r the number of votes registered for A and B, respectively, among the first r votes recorded. Let μ be a non-negative integer. Denote by $P_j(a,b)$ the probability that the inequality $\alpha_r>\mu\beta_r$ holds for exactly j subscripts $r=1,2,\cdots,a+b$. Special results have been obtained by many authors, going back to the 19th century. The author previously [Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **2** (1963), 118–121; MR **28** #3490] determined $P_j(a,b)$ for $a>\mu b$, and here obtains the complete result. This result is based on a combinatorial calculation of the distribution of the number of r $(1\leq r\leq M)$ for which $\nu_1+\cdots+\nu_r<r$, where the ν_i are interchangeable non-negative lattice random variables.
J. Kiefer (Ithaca, N.Y.)

Dayhoff, Eugene 6592
On the equivalence of polykays of the second degree and Σ's.
Ann. Math. Statist. **35** (1964), 1663–1672.
Hooke introduced bipolykays to apply to sampling from a two-way array [same Ann. **27** (1956), 55–79; MR **17**, 868]. Wilk and Kempthorne introduced a set of functions, denoted by Σ's, in connection with certain structural patterns in the analysis of variance [J. Amer. Statist. Assoc. **52** (1957), 218–236; MR **19**, 474]. These were extended to include all balanced population structures by Zyskind [Sankhyā Ser. A **24** (1962), 115–148; MR **26** #1973], who conjectured the equivalence of extended polykays with the set of Σ's for all balanced structures. In this paper the theory of "n-way polykays" is developed so as to apply to all balanced structures, and the equivalence of these with the Σ's is established. *P. S. Dwyer* (Ann Arbor, Mich.)

Atkinson, F. V.; Church, J. D.; Harris, B. 6593
Decision procedures for finite decision problems under complete ignorance.
Ann. Math. Statist. **35** (1964), 1644–1655.
The authors contribute to the axiomatization of decision-making under complete ignorance as to the state of nature, a continuation of the earlier work of Chernoff [Econometrica **22** (1954), 422–443; MR **16**, 271] and Milnor [*Decision processes*, pp. 49–59, Wiley, New York, 1954; MR **16**, 605]. Let the element a_{ij} of the matrix A be the loss under decision i if state j prevails. Since the individual can mix strategies, he can achieve any loss vector in $C(A)$, the convex hull of the rows of A. Let $Q(A)$ be the set of chosen loss vectors in $C(A)$. The authors present a set of desirable properties for $Q(A)$. $Q(A)$ must be non-null and convex, and each element must be admissible. If $A_1=\lambda A_0+C$, where each row of C is the same vector, c, then $Q(A_1)$ is the image of $Q(A_0)$ under the linear transformation, $\lambda x+c$. The concept of complete ignorance (all a priori distributions over states of nature have equal status) is rendered by the following assumption (combining two of the authors'). If A_1 and A_2 are two matrices, and if the convex hull of the columns of A_1 is the same as that of A_2, then the chosen strategies (rows) of the two matrices are the same. It is also assumed that $Q(A_1)=Q(A_2)$ if the admissible extreme points of $C(A_1)$ and $C(A_2)$ are the same, and, finally, that the point-to-set mapping $Q(A)$ is upper semi-continuous (the last property does not hold, for example, for the minimax of regret criterion). It is not assumed that the choice of decisions is transitive. The main result is the exhibition of a class of decision procedures satisfying these criteria, modifying an earlier result of Milnor's. Let $\{\varepsilon_i\}$ be a positive sequence approaching zero. For given A, define inductively the sequence of sets

$\{Q_k\}$, as follows. Let $Q_1 = O(A)$, $d(x, y) = \max_i |x_i - y_i|$, $v_i^{(k)} = \min_{x \in Q_k} x_i$, $v^{(k)}$ the vector with components $v_i^{(k)}$, $z_k = \min_{x \in Q_k} d(v^{(k)}, x)$, and $Q_{k+1} = \{x \in Q_k : d(v^{(k)}, x) \leq z_k + e_k z_1\}$. The sets Q_k have an intersection $Q(A)$, which is the desired procedure, satisfying all the conditions and, in fact, consisting of a single point. *K. J. Arrow* (Stanford, Calif.)

Hannan, E. J. 6594

Systematic sampling.

Biometrika **49** (1962), 281–283.

The author considers the estimate of the population mean of a stationary process obtained from observations distributed over an interval. He compares the variances of estimates based upon regularly spaced and randomly distributed observations, and derives recommendations for a preferable procedure in terms of the spectral density function of the process. The generalisation of the calculation to several dimensions is indicated.

P. Whittle (Manchester)

Hanumantha Rao, T. V. 6595

An existence theorem in sampling theory.

Sankhyā Ser. A **24** (1962), 327–330.

Author's summary: "A (1, 1) correspondence is established here, between sampling designs and sampling schemes, for sampling from a finite population. This result enables us to search for optimum sampling procedures in any particular case, through a unified general set-up."

Hanurav, T. V. 6596

Some sampling schemes in probability sampling.

Sankhyā Ser. A **24** (1962), 421–428.

Author's summary: "In this paper we give a sampling scheme for finite populations, resulting in a given set of inclusion probabilities. Also, we give another scheme with the property that it can be stopped at any stage without distorting the proportionate values of the inclusion probabilities."

Matérn, Bertil 6597

★Spatial variation: Stochastic models and their application to some problems in forest surveys and other sampling investigations.

Meddelanden Fran Statens Skogsforskningsinstitut, Band 49, *Nr.* 5, *Stockholm*, 1960. 144 *pp*. 8.00 *kr*.

The author is concerned with stationary stochastic processes $\{z(x)\}$ in several dimensions (very often on a plane in physical space) with the ultimate aim of discussing the design and accuracy of forest surveys.

In Chapter 2 special attention is paid to the correlation function $c(v)$ of an isotropic process, for which some interesting representations and inequalities are presented. The variance of the integral of $z(x)$ over a region is discussed.

The author considers a number of special models in Chapter 3; these are largely derived from a Poisson process by the association of a weight function or a geometrical region with each "event", but lead to an interestingly large class of correlation functions. For some processes which divide the space R_n into random sets, $c'(0)$ is found to be relatable to the expected set boundary-content per unit volume of R_n. A corollary of this result is that the expected length of the level curves of a Gaussian process is related to $\sqrt{-c''(0)}$.

The author now considers the estimation of a local mean of $z(x)$ (in R_2) by stratified sampling (i.e., by random sampling within a "stratum" which consists of a spatial region of prescribed geometric shape). He gives an asymptotic argument to show that the circle is the optimal stratum shape for all $c(v)$, and conjectures that the regular hexagon will be optimal if it is required that the strata should fill R_2. Numerical comparisons are given.

The same problem is also examined for the case of systematic sampling and an investigation made of the optimal configuration of sample points. Numerical studies seem to indicate the hexagonal lattice as having a uniform advantage. Edge effects are considered briefly. Consideration is also given to the total distance travelled in taking the sample, which will sometimes be a component of sampling cost. If it is the only component of cost, then the optimal scheme is to sample continuously along parallel straight lines.

The final chapter is devoted to a number of topics important for sample surveys in R_1 and R_2: estimation of sampling error, the use of a two-phase sample (e.g., the combination of visual observations and more exact measurements), and the optimal location of observations confined to curves in R_2 (when the circle appears to be the best closed curve, a straight line the best open one).

P. Whittle (Manchester)

Murthy, M. N. 6598

Almost unbiased estimators based on interpenetrating sub-samples.

Sankhyā Ser. A **24** (1962), 303–314.

Author's summary: "In this paper a technique is given for estimating unbiasedly any non-linear function of estimable parameters. The technique consists in estimating the bias of the usual estimator using estimates based on interpenetrating sub-samples and then correcting the estimator for its bias."

Murthy, M. N. 6599

A note on determination of sample size.

Sankhyā Ser. A **25** (1963), 381–382.

Author's summary: "In this note a procedure of determining the sample size is proposed, where the idea is to fix the sample size in such a way that the probability (P) of the length (L) of the confidence interval (associated with a specified confidence coefficient) for the parameter (μ) being less than a given value ($k\mu$) is a pre-specified quantity."

Särndal, Carl-Erik 6600

★Information from censored samples.

Almqvist & Wiksell, Stockholm-Göteborg-Uppsala, 1962. 120 *pp*. *Sw. Kr.* 18.00.

From a purely theoretical point of view, the author studies the properties of linear unbiased estimates of α_1 and α_2, where $F[(z-\alpha_1)/\alpha_2]$ is a continuous distribution function with a location parameter α_1 and a scale parameter α_2, by censored sample from that population (having

$F[(x-\alpha_1)/\alpha_2]$ as the distribution function). He distinguishes between two different types of censoring. One is natural censoring that arises when, by the nature of the experiment, the individual values of observations with certain known ranks are not available; another is elective censoring where the observations which are to be deleted are by choice, whereas in natural censoring they are involuntarily lost. In the latter case, it is an important problem to determine the optimum spacing of observations. He defines evaluation of the loss (efficiency of the estimators). In Chapter I, he gives a review of well-known results in the theory of linear estimation, with fundamental definitions, conditions and prerequisites. Then in Chapter II he gives the measures of information and loss of information due to censoring and studies them in some cases of censoring in the unit interval when the size of the sample n is large. In Chapter III, he treats the case of small samples as a complement to the large-sample theory of the previous chapter. In Chapter IV, which is the most important part of this book, he studies the spacing of observations (elective censoring) in the case of no natural censoring in the original sample. Here he defines a function $f^*(x)$ by which the spacings are generated; he refers to it as the generating function. Thus, he derives some theorems on the efficiency of estimators by $f^*(x)$ and requires the generating function of optimum spacing which maximizes the efficiency. In Chapter V, he gives optimum and nearly optimum spacing, for small samples, corresponding to Chapter IV. In Chapter VI, he treats the problems on spacing of observations when part of the original sample is naturally censored (the information of the available sample has been already lost by natural censoring). In the last chapter, some applications to the problem of grouping of observations and that of stratification in sample survey are briefly described.

C. Hayashi (Tokyo)

Pathak, P. K. 6601

On simple random sampling with replacement.

Sankhyā Ser. A **24** (1962), 287–302.

Author's summary: "In simple random sampling with replacement, Basu [Sankhyā **20** (1958), 287–294; MR **21** #4517] and Des Raj and Khamis [Ann. Math. Statist. **29** (1958), 550–557; MR **20** #1377] showed that for estimating the population mean, the average of distinct units is more efficient than the overall sample mean. In this paper, a detailed treatment of the above problem is given, and the exact expression for the variance of above estimator is derived. The relative efficiency of the above estimator with other estimators is also considered. An improved estimator of the population variance is obtained. Finally, a comparison between the two simple random sampling schemes (with and without replacement) is made."

Pathak, P. K. 6602

On sampling with unequal probabilities.

Sankhyā Ser. A **24** (1962), 315–326.

Author's summary: "This paper deals with the problem of deriving improved estimators in sampling schemes with unequal probabilities of selection. The improved estimator of the population total Y [D. Basu, Sankhyā **20** (1958), 287–294; MR **21** #4517] is derived. In addition, two sets of estimators of Y and Y^2 are given. The first set of estimators is unwieldy to compute, while the second set is simple. The second set of estimators, though less efficient than the first, is more efficient than the usually employed estimators. It is proved in subfield terminology that if $\mathcal{S}_1$ and $\mathcal{S}_2$ are two sufficient subfields and K is a set common to $\mathcal{S}_1$ and $\mathcal{S}_2$, then $\mathcal{S}_1K+\mathcal{S}_2K'$ is also a sufficient subfield. Hence the subfield $\mathcal{S}_1K+\mathcal{S}_2K'$ can be used to derive improved estimators by Rao-Blackwell theorem. Generalisation of this is also given in case of a countable number of subfields. Application of this result to sampling with unequal probabilities is given."

Ruben, Harold 6603

Studentisation of two-stage sample means from normal populations with unknown variances. II. Confidence estimation for the mean of a stratified population.

Sankhyā Ser. A **24** (1962), 251–254.

Author's summary: The two-stage sampling procedure discussed in the first paper of the present series [Sankhyā Ser. A **24** (1962), 157–180; MR **27** #3045] is used to obtain confidence intervals of predetermined length and confidence coefficient for the mean of a stratified population with normal components. The efficiency of the procedure in relation to the confidence estimation with fixed sample sizes, when the variances of the component strata are known, is discussed briefly."

Ruben, Harold 6604

Studentisation of two-stage sample means from normal populations with unknown variances. III. Joint confidence estimation of a set of means.

Sankhyā Ser. A **24** (1962), 255–258.

Author's summary: "Confidence regions of predetermined dimensions and confidence coefficient are obtained for the joint estimation of the means of a finite set of unconnected and unknown normal populations. The confidence estimation procedure is shown to be conservative in character, and its efficiency in relation to the corresponding estimation procedure with fixed sample sizes, when the variances of the populations are known, is discussed briefly. The paper concludes with a short discussion of the possibility of increasing the efficiency of the two-stage sampling procedure (utilised in the current paper as well as in the preceding two papers of this series [Sankhyā Ser. A **24** (1962), 157–180; MR **27** #3045; see also #6603 above]), and of some other matters."

Tukey, John W.; McLaughlin, Donald H. 6605

Less vulnerable confidence and significance procedures for location based on a single sample: Trimming/Winsorization. I.

Sankhyā Ser. A **25** (1963), 331–352.

Authors' summary: "The vulnerability of Student's t, insofar as efficiency and power are concerned, leads to consideration of substitutes. Among the most promising are ratios of trimmed means to square roots of suitable quadratic forms involving the same order statistics. Matching, across underlying distributions, of ratios of average of denominator to variance of numerator leads to selection of the Winsorized sum of squared deviations as the basis for a denominator. The resulting trimmed t

should prove more useful when the amount of trimming is made to depend on the individual sample in a suitably prescribed manner. Exact critical values for the resulting tailored t seem to require Monte Carlo computation, but use of a simple modified denominator for trimmed t allows us to use the conventional t tables as a reasonable approximation."

Rao, J. N. K.; Tintner, G. 6606

The distribution of the ratio of the variances of variate differences in the circular case.

Sankhyā Ser. A **24** (1962), 385–394.

Authors' summary: "In time series analysis, the variate difference method is used to test the order of the finite difference at which the trend or the systematic part in the time series is approximately eliminated. There is no exact test available in the literature except for the one proposed by Tintner based on a method of selection which uses only a portion of the observations. In this paper, the statistic V_{k+1}/V_k is proposed to test that the trend is approximately eliminated at the kth finite differencing of the series, where V_k is the variance of the series of the kth differences. Its exact distribution assuming that the observations are $NI(0, \sigma^2)$ is derived under a circular definition of the universe. The lower 5% and 1% points of the statistics V_2/V_1 and V_3/V_2 are tabulated for various values of N, the size of the sample. In practice, one uses the non-circular statistic with these percentage points for the circular statistic as an approximation, especially with long time series."

NUMERICAL METHODS

See also 5998a–b, 6068, 6075, 6199, 6228, 6233, 6238, 6272, 6569, 6919, 6972.

Traub, J. F. 6607

★Iterative methods for the solution of equations.

Prentice-Hall Series in Automatic Computation.

Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964. xviii + 310 *pp*. $12.50.

Consider the problem of approximating a real root of a real equation $f(x) = 0$. If $x_i, x_{i-1}, \cdots, x_{i-n}$ are $n+1$ approximants of α, and ϕ is a real function of $n+1$ variables, then $x_{i+1} = \phi(x_i, \cdots, x_{i-n})$ gives rise to an iteration algorithm; and particular interest centers, of course, around iteration functions (I.F.) ϕ for which the process converges to α. This is a very old problem and a great variety of such iteration algorithms have been discovered and rediscovered.

This book aims at a systematization of known I.F. and at classifying them according to the efficiency of the algorithm and the amount of computational labor involved. Although basic convergence theorems are included, the stress is almost exclusively on methods for constructing I.F. and on determining their principal properties. Except for Chapter 11, only the special case is considered where f is a real function of one real variable and α an isolated real root. Moreover, except for Chapter 7, α is even assumed to be a simple root in many discussions.

In Chapter 1 the general classification of I.F. is introduced. A basic distinction is made between old and new information by considering I.F. where at the points $x_i, \cdots, x_{i-k}$ new data are used while at $x_{i-k-1}, \cdots, x_{i-n}$ old information is reused. Special cases are named as follows: $n = k = 0$, one-point I.F.; $n > k$, $k = 0$, one-point I.F. with memory; $n = k > 0$, multipoint I.F., and the general case is called a multipoint I.F. with memory. The classification of I.F. is further based on the "informational efficiency", $\mathrm{Eff} = p/d$, of the algorithm. Here p is the order of convergence of the iteration and d the total number of new derivative evaluations of f per iteration ($f = f^{(0)}$).

Chapter 2 begins with the basic contraction mapping theorem and a discussion of linear and superlinear convergence. Then the so-called iteration calculus is introduced, consisting of a number of fundamental theorems on I.F. without memory, as for example, theorems comparing the order of different I.F., etc.

Chapter 3 treats difference relations, including particularly the asymptotic behavior of the solution of certain difference equations. Later, these results form the basis for the order analysis of one-point I.F.

Chapter 4 provides the fundamental material for much of the remainder of the book by introducing interpolatory I.F. These are I.F. generated by direct or inverse hyperosculatory interpolation. The major results of this chapter concern the convergence and order of these interpolatory I.F.

Chapter 5 discusses one-point I.F. Generally, an infinite sequence of I.F. is called basic if its pth member is of order p. A basic sequence of one-point I.F., E_s, is obtained by inverse interpolation. Other techniques for finding one-point I.F. include the use of direct interpolation and the application of rational approximations to E_r. An important result for all one-point I.F. states that always $\mathrm{Eff} \leq 1$. Accordingly, a one-point I.F. is called optimal if $\mathrm{Eff} = 1$.

Chapter 6 studies two classes of one-point I.F. with memory, namely, interpolatory I.F. and so-called derivative estimated I.F. These are I.F. obtained by estimating the $(s-1)$st derivative of an optimal one-point I.F. of order s with the aid of one new and n old values of the first $s-2$ derivatives. For these two types of I.F. it is shown that the old information adds always less than unity to the order.

Throughout Chapters 4–6 the root α was mostly assumed to be simple. Chapter 7 now generalizes the results to multiple roots. Every E_s turns out to be of linear order for non-simple roots, and a new basic sequence with multiplicity-independent order has to be introduced. Then further techniques for constructing one-point I.F. with memory for non-simple α are discussed.

Chapters 8 and 9 turn to multipoint I.F. These have various computational advantages; e.g., there is no limit for Eff, some of them have free parameters permitting sampling at desirable points, etc. Various techniques for generating multipoint I.F. are introduced. One is based on a special type of interpolation formula, others use recursively defined I.F., the composition of I.F., and the estimation of the highest derivative in a one-point I.F. Then two particular families of multipoint I.F. are investigated in detail.

Chapter 10 contains a brief discussion of various material on I.F. which involve no derivatives, and Chapter 11 presents an introduction to the case where f is replaced by a system of equations. Inverse interpolation

techniques are again employed and error estimates for some vector-valued I.F. are given.

Chapter 12 contains a very valuable compilation of all I.F. discussed in the book, listing in each case the formula, the order and asymptotic error constant, and the relevant section in the previous chapters.

The book concludes with 6 appendices and an extensive bibliography. Among the appendices, the first one brings a summary of results on hyperosculatory interpolation, and the fifth one provides a selection of the author's computer experimentation with various I.F.

Although the mathematical treatment is rigorous throughout, attention is definitely focused on the computational aspects of the topic. There is a vast amount of material in the book and a great deal is either new or presented in a new form. Many examples are provided to show how well-known I.F. are special cases of the general results. The style is terse and tends to make for somewhat uninspired reading, and yet the author has certainly succeeded in presenting a systematic account of a large class of known I.F., and this makes the work an interesting basic text as well as a valuable reference book for this field. *W. C. Rheinboldt* (Silver Spring, Md.)

Chahine, M. T.; Narasimha, R. 6608
The integral $\int_0^\infty v^n \exp[-(v-u)^2 - x/v]\,dv$.
J. Math. and Phys. **43** (1964), 163–168.

From the authors' introduction: "The function $g_n(x, u)$ [of the title] is not in general reducible to any of the standard tabulated integrals. In this report we first discuss the general properties of the functions and their asymptotic behavior; then we present brief tables for representative values of x and u, as found by numerical integration on a computer. More complete tables can be found in JPL Tech. Rep. No. 32-459."

Zhurina, M. I. [Žurina, M. I.]; Karmazina, L. N. 6609
★Tables of the Legendre functions $P_{-\frac12+i\tau}(x)$. Part I.
Translated by D. E. Brown. Mathematical Tables Series, Vol. 22.
Pergamon Press, Oxford-New York-Paris-Frankfurt, 1964. xv+308 *pp.* £5.

For a review of the original [Izdat. Akad. Nauk SSSR, Moscow, 1960], see MR **22** #12736. The reproduction and translation are satisfactory (except the translation back of "Mehler" into "Meler"). *John Todd* (Pasadena, Calif.)

Emersleben, Otto 6610a
Quadratwurzeln von 1–1000 nebst Reziproken auf 20 Dezimalstellen sowie deren summatorische Funktionen.
Wiss. Z. Ernst-Moritz-Arndt-Univ. Greifswald Math.-Natur. Reihe **12** (1963), 223–228.

Emersleben, Otto 6610b
Quadratwurzeln von 1–1000 nebst Reziproken auf 20 Dezimalstellen sowie deren summatorische Funktionen.
Wiss. Z. Ernst-Moritz-Arndt-Univ. Greifswald Math.-Natur. Reihe **12** (1963), *no.* 3/4, *Beilage,* 41 *pp.*

The first article is an introduction to the tables which are in the second. Apart from historical and bibliographical remarks, it includes an asymptotic expression for $\sum^n k^{-1/2}$ and describes situations in lattice energy studies where many-decimal tables of $k^{-1/2}$ are convenient.

The tables give $n^{\pm 1/2}$ for $n = 1(1)1000$ to 20D, with an indication as to whether the rounding was up or down. The sum functions $\sum_1^n k^{\pm 1/2}$ are also given; those final digits which may have suffered from the rounding in the primary tables are given in small type.
John Todd (Pasadena, Calif.)

Klopfenstein, R. W. 6611
Conditional least squares polynomial approximation.
Math. Comp. **18** (1964), 659–662.

The basic problem of polynomial least squares curve fitting on an automatic computer is effectively solved by the generation of orthogonal polynomials by recurrence, as described by Forsythe [J. Soc. Indust. Appl. Math. **5** (1957), 74–88; MR **19**, 1079]. In the present paper, the author shows, briefly and clearly, how constraints may be imposed upon the fitting polynomial. He seems to be unaware, however, of Forsythe's work, and of the more recent papers on the subject; in particular, there appears to be nothing in the paper that was not earlier given, in rather greater generality, by Cadwell and Williams [Comput. J. **4** (1961/62), 260–264; MR **23** #B2571].
C. W. Clenshaw (Toddington)

Percus, J. K. 6612
A note on extension of the Lagrange inversion formula.
Comm. Pure Appl. Math. **17** (1964), 137–146.

The author extends the Lagrange-Burmann formula [for the one-dimensional case, see Whittaker and Watson, *A course of modern analysis*, fourth edition, Cambridge Univ. Press, Cambridge, England, 1927] to the case for a function of n variables. The method he develops for the reversion of series is done so that it can be adapted to large-scale digital computers. Although the analysis is quite nice, the details are such that they cannot be given here. *J. T. Day* (Zürich)

Brown, K. M.; Henrici, P. 6613
Sign wave analysis in matrix eigenvalue problems.
Math. Comp. **16** (1962), 291–300.

The paper deals with the power method in case of a dominant pair of simple, conjugate complex roots, $\lambda_1 = \rho e^{i\varphi}$, $\lambda_2 = \rho e^{-i\varphi}$, with a non-derogatory starting vector. Let $x_\nu^{(n)}$ denote the νth component of the nth iterated vector, and let $r_\nu e^{i\varphi_\nu}$ denote the νth component of some eigenvector to λ_1. If $r_\nu \neq 0$, then a classical result is that ρ^2 is approached by the quotient $\Delta_\nu^{(n+1)}/\Delta_\nu^{(n)}$ of successive 2×2 Hankel determinants $\Delta_\nu^{(n)} = (x_\nu^{(n)})^2 - x_\nu^{(n-1)}x_\nu^{(n+1)}$. This is supplemented by the following interesting result about the phase φ: the sequence of average distances of iterations between sign-changes from minus to plus in the sequence $x_\nu^{(n)}$ approaches $2\pi/\varphi$. Thus, the sign waves in the iteration sequence give additional information about the phase. This method is numerically stable even for very small absolute values of φ (whereas the determination of λ from a quadratic equation is unstable in this case); however, the smaller φ is, the more iterations will be necessary in order to produce a sign change. A similar result holds for the determination of the relative phase angles $\varphi_\nu - \varphi_1$ in an eigenvector. *F. L. Bauer* (Munich)

Brudno, A. L. 6614

ε-solutions of linear algebraic systems. (Russian)

Problemy Kibernet. No. 8 (1962), 187–190.

A rather labored examination of the question, under what conditions will an approximate solution x^* of the linear system $Ax+b=0$ satisfy $|Ax^*+b|\leq\varepsilon|Ax^*|$, where the inequality and the bars are to be interpreted elementwise. *A. S. Householder* (Oak Ridge, Tenn.)

Descloux, Jean 6615

Bounds for the spectral norm of functions of matrices.

Numer. Math. 5 (1963), 185–190.

Let $A=(a_{i,j})$ be any $n\times n$ complex matrix with eigenvalues $\mu_1, \mu_2, \cdots, \mu_n$, and denote the distinct eigenvalues of A by $\lambda_1, \lambda_2, \cdots, \lambda_m$ of respective multiplicities $n_1, \cdots, n_m$. Let $f(z)$ be any function for which the values $f^{(i)}(\lambda_j)$, $0\leq i\leq n_j-1$, $1\leq j\leq m$, exist. If

$$\sigma(A)\equiv\sup_{x\neq 0}\left[\left(\frac{x^*A^*Ax}{(x^*x)}\right)\right]^{1/2},\qquad \|A\|\equiv\left(\sum_{i,j}|a_{i,j}|^2\right)^{1/2},$$

$\Delta(A)\equiv(\|A\|^2-\sum_i|\mu_i|^2)^{1/2}$, and $g(z_1, z_2, \cdots, z_i)$ denotes the $(i-1)$st divided difference of a real- or complex-valued function $g(z)$ at the points $z_1, z_2, \cdots, z_i$, then the author proves that $\sigma(f(A))\leq\sum_{k=0}^{n-1}\delta_k(\Delta(A))^k$, where

$$\delta_{k-1}\equiv\max_{1\leq i_1<i_2<\cdots<i_k\leq n}|f(\mu_{i_1}, \mu_{i_2}, \cdots, \mu_{i_k})|$$

for $1\leq k\leq n$. This generalizes a result of Henrici [Numer. Math. 4 (1962), 24–40; MR 24 #B1751] for the function $f(z)=z^k$, and makes use of Henrici's notion of "departure from normality". *R. S. Varga* (Cleveland, Ohio)

Lavrov, S. S. 6616

Application of barycentric coordinates to the solution of certain numerical problems. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. 4 (1964), 905–911.

The author introduces barycentric coordinates to transform a function of degree m in n variables $F(x_0, x_1, \cdots, x_n)$ by $x_i=\sum\mu_k M_i^{(k)}$, $\sum\mu_k=1$, in a homogeneous form in μ_k. Taking the values of the function in the points $P(k_1/m, k_2/m, \cdots, k_n/m)$, one disposes of exactly as many values as there are coefficients in the form, and by a solution of a system of linear equations the transformation of the form is effected. Applications are given for the solution of a system of functions and for the determination of extreme values of a single function, by replacing given functions by linear or quadratic approximations in these barycentric coordinates. It might be of use to remark here that the author obtained fractions in his solutions because of the omission of a factor $m!$, which is used in the Clebsch-Weitzenböck symbolical method. The form is generally transformed from $(a'x)^m$ into

$$\sum\frac{m!}{i_0!i_1!\cdots i_n!}(a'M^0)^{i_0}\cdots(a'M^n)^{i_n}\mu_0^{i_0}\mu_1^{i_1}\cdots\mu_n^{i_n}$$

and the symbolical method gives for the values and the coefficients with m different indices the relation

$$(a_1'+a_2'+\cdots+a_l')^m=\sum\frac{m!}{s_1!s_2!\cdots s_n!}a'_{1\cdots 1 2\cdots 2 \cdots l\cdots l}.$$

From this fundamental relation the formulas for coefficients with equal indices are directly obtained. This corresponds to the choosing of the different points P by the author for $k_i\neq 1$ in a procedure which remains unchanged for arbitrary changes by linear transformation of the basic point set. *E. M. Bruins* (Amsterdam)

Lynn, M. Stuart 6617

Some infra-max bounds for the spectral radii of splittings of H-matrices.

Numer. Math. 5 (1963), 152–174.

Let $A=(a_{i,j})$ be an $n\times n$ matrix. A splitting $\{B, C\}$ of A is any pair of $n\times n$ matrices B and C such that $A=B-C$ and B is nonsingular. For the matrix equation $Ax=k$, a splitting $\{B, C\}$ of A then defines a natural iterative method for solving this equation: $Bx^{(n+1)}=Cx^{(n)}+k$. Previously, the reviewer [*Boundary problems in differential equations*, pp. 121–142, Univ. Wisconsin Press, Madison, Wis., 1960; MR 22 #12704] introduced the notion of a regular splitting $\{B, C\}$ of A, i.e., a splitting of A for which B^{-1} and C have only non-negative real entries ($B^{-1}\geq 0, C\geq 0$). Moreover, it was shown that if $A^{-1}\geq 0$, and $\{B, C\}$ is a regular splitting of A, then $\rho(B^{-1}C)<1$ and the iterative method defined above is necessarily convergent. Based on Ostrowski's well-known work [Comment. Math. Helv. 10 (1937), 69–96] on H-matrices and M-matrices, the author introduces here the related notion of an M-splitting $\{B, C\}$ of A, i.e., if $B=(b_{i,j})$ and $C=(c_{i,j})$, then $|a_{i,j}|=|b_{i,j}-c_{i,j}|=|b_{i,j}|+(-1)^{\delta_{i,j}}|c_{i,j}|$, $1\leq i, j\leq n$. For an H-matrix A, which in general does not possess regular splittings, the author shows that an M-splitting $\{B, C\}$ of A is convergent, $\rho(B^{-1}C)<1$, and obtains upper bounds less than unity for $\rho(B^{-1}C)$. On the other hand, it is shown that if A is an M-matrix and $\{B, C\}$ is an M-splitting of A, then $\{B, C\}$ is necessarily a regular splitting of A. Conversely, if A is an M-matrix and $\{B, C\}$ is a regular splitting of A with $b_{i,j}\leq 0$, $i\neq j$, then $\{B, C\}$ is also an M-splitting of A. Consequently, for most applications to elliptic difference equations, M-splittings and regular splittings are the same, and the author is able to rederive many known results in this area.

Working with partitioned matrices, the author then uses basic results of Ostrowski [J. Math. Anal. Appl. 2 (1961), 161–209; MR 24 #A421] to extend the previous results of M-splitting theory to partitioned matrices. Finally, defining restrictive splittings of a matrix A, applications are made to successive overrelaxation methods with relaxation factors ω slightly greater than unity. *R. S. Varga* (Cleveland, Ohio)

Niethammer, Wilhelm 6618

Relaxation bei komplexen Matrizen.

Math. Z. 86 (1964), 34–40.

First the author rediscovers independently a result recently published by Lynn [Comput. J. 7 (1964), 72–75], that if in the system $AX=C$ the matrix A has Property A, then symmetric successive overrelaxation with the factor $1+d$ is equivalent to successive overrelaxation with the factor $1-d^2$, and he extends this to show that unsymmetric overrelaxation with factors $1+d_1$ and $1+d_2$ is equivalent to successive overrelaxation with the factor $1-d_1d_2$. Thereafter he applies results found in a previous paper [Math. Z. 85 (1964), 319–327; MR 30 #705] to obtain optimal factors in the complex cases when $B=A-I$ is Hermitian, skew-Hermitian, and complex symmetric. *A. S. Householder* (Oak Ridge, Tenn.)

Price, Charles M. 6619

The matrix pseudo-inverse and minimal variance estimates.

SIAM Rev. **6** (1964), 115–120.

Author's summary: "This paper reviews and applies certain results concerning the matrix pseudo-inverse to the general theory of estimable functions and minimal variance estimates. The paper is divided into two sections. The first section reviews certain known results concerning the matrix pseudo-inverse. This section is essentially non-statistical. The second section uses results in the first section to state and prove a generalized version of the Gauss-Markoff theorem concerning unbiased linear estimates having minimal variance."

The first section is very useful in that it provides a handy reference. It succinctly lists the properties of the pseudo-inverse, the relations it satisfies, and the basic relevant theorems.

A new theorem is given which, for the non-zero eigenvalues, relates the eigensystems of a matrix A and its pseudo-inverse.

{The second footnote on page 116 has a misprint: for A^2, read A'.} *E. R. Hansen* (Palo Alto, Calif.)

Richards, Paul I. 6620

Symmetry in numerical matrix calculations.

SIAM Rev. **5** (1963), 234–248.

Cord and Sylvester [J. Soc. Indust. Appl. Math. **10** (1962), 632–637; MR **27** #3651] showed that if an ordinary differential equation defined on a finite open interval of the real line possesses mirror-type symmetry (i.e., the differential equation is unchanged when the independent variable is reflected about the midpoint of the interval), then a finite-difference approximation to this differential equation yields an $n\times n$ matrix $A=(a_{i,j})$ which has the property of cross-symmetry: $a_{i,j}=a_{n+1-i,n+1-j}$. If $n=2m$, then all the eigenvalues and eigenvectors of A can be obtained by finding the eigenvalues and eigenvectors to two lower-order matrices of order m. This implies a possible factor of four in reducing computing effort. The author, using the theory of group representations, generalizes these results by considering the group of all such symmetry transformations. An example of a 10×10 matrix is included, where saving factors from 4 to 40 are possible. *R. S. Varga* (Cleveland, Ohio)

Rutishauser, Heinz 6621

Une méthode pour le calcul des valeurs propres des matrices non symétriques.

C. R. Acad. Sci. Paris **259** (1964), 2758.

From a given matrix $A=A_0$, a sequence of equivalent matrices A_i is computed. Each step consists of a Jacobi rotation transformation of A_i with the axis (p,q) such that the (p,q)-element of the commutator $C_i=A_iA_i^H-A_i^HA_i$ is made zero, followed by a suitable diagonal transformation. A sweep consists of $\frac{1}{2}n(n-1)$ steps, where (p,q) are taken in the order $(1,2),(1,3),\cdots,(1,n),(2,3),\cdots,(2,n),(3,4),\cdots,(n-1,n)$. The algorithm is said to approximate a normal matrix in the sense that the norm of C_i approaches zero. *F. L. Bauer* (Munich)

Volpe di Prignano, Ernesto 6622

Considerazioni sopra un procedimento per la stima degli autovalori di una matrice simmetrica.

Rend. Mat. e Appl. (5) **22** (1963), 439–446.

Ostrowski's iteration method [Parts I–VI, cf. Arch. Rational Mech. Anal. **1** (1958), 233–241; ibid. **2** (1958/59), 423–428; MR **21** #427; ibid. **3** (1959), 325–340; MR **21** #4541a; ibid. **3** (1959), 341–347; MR **21** #4541b; ibid. **3** (1959), 472–481; MR **21** #6691; ibid. **4** (1959), 153–165; MR **22** #8654] which determines a sequence

$$l_k=y'_{k-1}Ay_{k-1}/y'_{k-1}y_{k-1}$$

tending to the eigenvalue m of a real symmetric matrix A, where the recurrent vector sequence

$$y_k=(A-l_kI)^{-1}y_{k-1}$$

approximates by its direction that of an eigenvector of A at m, is discussed from the point of view of programming for an electronic computer. *H. Schwerdtfeger* (Montreal, Que.)

Wachspress, Eugene L. 6623

Extended application of alternating direction implicit iteration model problem theory.

J. Soc. Indust. Appl. Math. **11** (1963), 994–1016.

The author extends his algorithm [same J. **10** (1962), 339–350; MR **27** #921] for determining optimum acceleration parameters for alternating direction implicit (ADI) iterative methods. In the commutative case ($HV=VH$), where H and V are real, symmetric and positive definite, consider the solution of the matrix problem $(H+V)\underline{z}=\underline{s}$ by means of the iterative method

$$(H+\omega_{k,H}I)\underline{z}^{(k+1/2)}=(\omega_{k,H}I-V)\underline{z}^{(k)}+\underline{s},$$

$$(V+\omega_{k,V}I)\underline{z}^{(k+1)}=(\omega_{k,V}I-H)\underline{z}^{(k+1/2)}+\underline{s}.$$

If the smallest intervals containing the eigenvalues of H and V are, respectively, $[a,b]$ and $[c,d]$, an algorithm is given for the 2^{m+1} optimum parameters $\omega_{k,H}$, $\omega_{k,V}$, $1\leq k\leq 2^m$, which minimize

$$\max_{\substack{\lambda\in[a,b]\\ \gamma\in[c,d]}}\prod_{k=1}^{2^m}\left|\frac{\lambda-\omega_{k,V}}{\lambda+\omega_{k,H}}\right|\cdot\left|\frac{\gamma-\omega_{k,H}}{\gamma+\omega_{k,V}}\right|.$$

In other words, the algorithm previously given for the "square" model problem is generalized to the "rectangular" model problem. Next, the combination of ADI iterations with Lanczos' method is applied to matrix problems for which $HV\neq VH$, but for which there exists a matrix $M=M_1+M_2$, where $M_1M_2=M_2M_1$, and M_1 and M_2 are real, symmetric, and positive definite. Unfortunately, such problems have application only to elliptic difference equations defined on rectangular domains. Numerical results of such combined iterations are included.

In this paper is found a most interesting appendix which describes W. B. Jordan's complete solution of the problem of determining m optimum acceleration parameters for the model problem for general m, using the theory of elliptic functions. In the special case $m=2^k$, the algorithm proposed by the author for determining the optimum parameters is probably the most efficient numerically. *R. S. Varga* (Cleveland, Ohio)

Robinson, Raphael M. 6624

Algebraic equations with span less than 4.

Math. Comp. **18** (1964), 547–559.

By the span of an algebraic equation all of whose roots are real is meant the difference between its largest and smallest roots. The author considers irreducible equations with integer coefficients and leading coefficient equal to 1, such that the roots form a set of conjugate algebraic integers with span less than 4. There are only finitely many such equations of span less than $4-\varepsilon$. Two such equations are considered equivalent if they are of the same degree and if the roots of one are obtainable from the roots of the other by adding a fixed integer or by changing signs. The representative of such an equivalence class is the one whose average root lies between 0 and $\frac{1}{2}$. The author has searched for all representative polynomials of degree n, completely for $2 \leq n \leq 6$ and nearly completely for $n = 7, 8$, and has found just 96 such polynomials, 19 being the familiar Chebyshev polynomials whose roots are the doubled cosines of rational multiples of 2π. The 77 other equations are distributed as to degree as follows.

Degree	2	3	4	5	6	7	8
No. of equations	1	3	10	14	13	15	21

Table 1 gives for each of the 96 equations the maximum and minimum distances between roots and the largest and smallest roots to 4D, their coefficients and their discriminant and its factorization. The 19 Chebyshev cases are conspicuous for the roundness of their discriminants. The other 77 discriminants are often primes. Table 2 gives all 96 sets of roots to 10D.

D. H. Lehmer (Berkeley, Calif.)

von Holdt, R. E.; Howerton, R. J. 6625

The definite integral of the product of linear functions.

Math. Comp. **17** (1963), 419–425.

The author derives a formula for the integral of a product of linear functions $ax+b$ in terms of the values of the functions at the endpoints of the interval of integration.

S. Haber (Washington, D.C.)

Gates, Leslie D., Jr. 6626

Numerical solution of differential equations by repeated quadratures.

SIAM Rev. **6** (1964), 134–147.

The author discusses one-step methods of solving the first-order ordinary differential equation $y' = f(x, y)$ using quadrature formulas. The three types of quadrature formulas he discusses are Gauss, Radau, and Lobatto. The contribution made is a systematic way of generating formulas of a given order of accuracy. Tables of constants are given for methods through order 6.

P. C. Hammer (Madison, Wis.)

Runck, Paul Otto 6627

Konvergenzfragen bei Formeln für die numerische Differentiation und Integration.

Arch. Math. **15** (1964), 115–131.

In previous papers [J. Reine Angew. Math. **208** (1961), 51–69; MR **26** #1994a] the author showed that the sequence of Lagrange polynomials which interpolate to a continuous function $f(x)$ at equally spaced points of $[-1, 1]$ are good uniform approximations to $f(x)$ on a subinterval, depending on the number of nodes, of $[-1, 1]$. In this work he applies the results of the above-mentioned paper in a thorough fashion to obtain bounds for the error on these subintervals for Lagrangian numerical differentiation and integration formulae using equally spaced nodes in $[-1, 1]$.

T. J. Rivlin (Yorktown Heights, N.Y.)

Stroud, A. H. 6628

Approximate integration formulas of degree 3 for simplexes.

Math. Comp. **18** (1964), 590–597.

Formulas for numerical integration with equal weights and order three are developed for the n-simplex. These formulas involve evaluation at $n(n+1)$ points. For $n \leq 8$ all points of evaluation may be taken in the simplex; for $n \geq 9$ it seems that all points must be exterior to the simplex. The formulas are shown to be related to certain orthogonal polynomials. An example is presented for $n = 3$ comparing with previously derived formulas of the author and of the author in collaboration with P. C. Hammer and O. J. Marlowe [the author, Math. Comp. **15** (1961), 143–150; MR **22** #12717; the author, the reviewer, and O. J. Marlowe, Math. Tables Aids Comput. **10** (1956), 130–137; MR **19**, 177].

P. C. Hammer (Madison, Wis.)

Bellman, R.; Brown, T. A. 6629

On the computational solution of two-point boundary-value problems.

Boll. Un. Mat. Ital. (3) **19** (1964), 121–123.

One method of solving systems of linear two-point boundary-value problems of the form

$$(*) \qquad x'' + A(t)x = 0; \qquad x(0) = c,\ x(1) = d,$$

replaces (*) by an initial-value problem, i.e., the boundary condition at $t = 1$ is replaced by the initial condition $X'(0) = g$, where $g = X_2(1)^{-1}[d - X_1(1)c]$ and X_1 and X_2 are matrix solutions of the differential equation satisfying $X_1(0) = I$, $X_1'(0)$, $X_2(0) = 0$, and $X_2'(0) = I$. It is assumed that $X_2(1)$ is nonsingular. Since $X_2(1)^{-1}$ may be difficult to compute accurately, the authors describe an iterative procedure for the construction of a sequence of vectors $\langle g_n \rangle_{n=1}^{\infty}$ which, under certain conditions, converges to g.

M. Lees (Pasadena, Calif.)

Fair, Wyman 6630

Padé approximation to the solution of the Ricatti equation.

Math. Comp. **18** (1964), 627–634.

The τ-method is used to obtain the main diagonal Padé approximation to the solution of a Riccati differential equation with rational coefficients. Recurrence relations are developed, convergence is studied, and several examples involving confluent hypergeometric and Bessel functions are given.

T. E. Hull (Toronto, Ont.)

Grateloup, Georges; Gumowski, Igor 6631

Étude du comportement d'un système à retour non linéaire au voisinage d'un cas critique de Lyapunov.

C. R. Acad. Sci. Paris **258** (1964), 6069–6072.

To find the solutions of a polynomial equation

$$(1)\qquad \sum_{i=2}^{n} a_i y^i = z,$$

the authors have been experimenting with a differential system (2) $y''' + by'' + cy' + dy + \sum_{i=2}^{n} e_i y^i = 0$, where the equilibrium points of (2) are related to the solutions of (1). For $n > 2$, the analogue computations of (2) have exhibited in certain situations an oscillation with a small amplitude. The authors use the method of the small parameter in ordinary differential equations to investigate periodic solutions of (2), and the results coincide with experiment. *J. K. Hale* (Providence, R.I.)

Lancaster, P. 6632

Bounds for latent roots in damped vibration problems.
SIAM Rev. **6** (1964), 121–125.

This paper is concerned with obtaining bounds on the natural frequencies and rates of decay of the free vibrations of lightly damped systems of the type $A\ddot{\mathbf{q}} + \varepsilon B\dot{\mathbf{q}} + C\mathbf{q} = 0$, where ε is a scalar parameter and (1) $\mathbf{q}$ is a real n-column vector; (2) A, B and C are real symmetric $n \times n$ matrices, A is positive definite and B and C are at least non-negative definite. The problem is reduced to that of obtaining bounds on the latent roots of the algebraic equations $(\lambda^2 I + \varepsilon\lambda\beta + w^2)\mathbf{p} = 0$, where $\beta = \varphi' B\varphi$, $w^2 = \varphi' C\varphi$, φ being the real congruent matrix which simultaneously diagonalizes A and C and which is normalized so that $\varphi' A\varphi = I$. Using the well-known Geršgorin theorem, the author shows that (i) $-\varepsilon\beta_n \leq \operatorname{Re} \lambda_r \leq 0$, $r = 1, 2, \cdots, n$. In the case where ε is sufficiently small, and the undamped natural frequencies, w_{r0}, are all distinct, nonzero, and well-separated, the author shows that $\lambda_r = iw_{r0} - \frac{1}{2}\varepsilon\beta_{rr} + O(\varepsilon^2)$, $r = 1, 2, \cdots, n$. If the latent roots are all complex, the author is able to show that the eigen-equation yields sharper bounds than the Geršgorin theorem, namely, (ii) $-\frac{1}{2}\varepsilon\beta_n \leq \operatorname{Re} \lambda_i \leq 0$, $i = 1, 2, \cdots, n$; (iii) $|\lambda_i| \leq w_{n0}$, where β_n is the largest eigenvalue of β and w_{n0} is the largest undamped natural frequency.

It should be pointed out that the technique used by the author is capable of yielding even sharper bounds, in fact, (ii) and (iii) should read (iia) $-\frac{1}{2}\varepsilon\beta_n \leq \operatorname{Re} \lambda_i \leq -\frac{1}{2}\varepsilon\beta_1$; (iiia) $w_{10} \leq |\lambda_i| \leq w_{n0}$, $i = 1, 2, \cdots, n$, where β_n and β_1 are the largest and smallest eigenvalues of β, and w_{n0} and w_{10} are the largest and smallest undamped natural frequencies of the system. *T. K. Caughey* (Pasadena, Calif.)

Young, Jonathan D. 6633

Linear program approach to linear differential problems. (French, German, Italian and Russian summaries)
Internat. J. Engrg. Sci. **2** (1964), 413–416.

Author's summary: "The numerical solution of a linear differential problem is formulated as an overdetermined linear model with the objective of minimizing approximation errors. The dual of this model is a linear program."

Ansorge, R. 6634

Die Adams-Verfahren als Charakteristikenverfahren höherer Ordnung zur Lösung von hyperbolischen Systemen halblinearer Differentialgleichungen.
Numer. Math. **5** (1963), 443–460.

Vorgelegt sei ein hyperbolisches, halblineares System $Au_x + Bu_y = v$ mit den quadratischen Matrizen $A = (a_{jk}(x, y))$ und $B = (b_{jk}(x, y))$ sowie den Spaltenvektoren $u = (u^j)$ und $v = (v_j(x, y, u^1, \cdots, u^n))$. Es werden 10 Voraussetzungen getroffen, über Differenzierbarkeit, Elementarteiler der Matrix $\rho A - B$ u.a. Es werden Charakteristikenverfahren höherer Ordnung angegeben, die auch bei beliebig vielen (nicht nur bei 2) Charakteristikenscharen anwendbar sind und in einer Richtung äquidistant arbeiten, wobei berücksichtigt wird, daß die gewöhnlichen Differentialgleichungen der Charakteristiken des halblinearen Systems selbst näherungsweise gelöst werden. Es wird die Berechnung eines Anfangsfeldes angegeben und sodann Extrapolations- und auch Interpolationsverfahren aufgestellt und für beide Verfahren die stabile Konvergenz im Sinne von Dahlquist bewiesen. Es folgt ein numerisches Beispiel. *L. Collatz* (Hamburg)

Dekker, Leendert 6635

★Numerical aspects of the one-dimensional diffusion equation. (Dutch summary)
Proefschrift ter Verkrijging van de Graad van Doctor in de Technische Wetenschappen aan de Technische Hogeschool te Delft. Delft, 1964. vii + 88 *pp.*

A special analog computer is introduced and applied to solve the quasi-linear parabolic system $\partial z_i/\partial t - a_i\partial^2 z/\partial x^2 = f_i(x, t, z_1, z_2, \cdots, z_n)$, $a_i > 0$, $z_i(0, t) = \phi_i(t)$, $z_i(1, t) = \eta_i(t)$, $z_i(x, 0) = \gamma_i(x)$, $i = 1, 2, \cdots, n$. The quasi-linear equations are transformed to a difference form based on the concepts of "external" and "internal" solutions. The x-external solution z_e (generalization of the "steady state solution") for the homogeneous parabolic problem $z_t - z_{xx} = 0$, $z(0, t) = \phi(t)$, $z(1, t) = \eta(t)$, and $z(x, 0)$ is defined by

$$z_e(x, t) = \sum_{0}^{\infty} \{g_k(x)\phi^{(k)}(t) + g_k(1-x)\eta^{(k)}(t)\},$$

where the $g_k(x)$ are odd polynomials in $(1-x)$ of degree $2k+1$. Since the external solution is not generally the solution of the problem, the difference between the solution and the external solution is defined as the "internal" solution.

The choice of a finite-difference approximation is discussed via a general Crank-Nicolson implicit method [see Forsythe and Wasow, *Finite-difference methods for partial differential equations*, p. 119, Wiley, New York, 1960; MR **23** #B3156]. The error in various approximations is computed. For small values of u in the growth factor $g_n = [1 + (1-a)u](1 - au)^{-1}$, the difference expression of Crank-Nicolson (with $a = \frac{1}{2}$) is the best one. For large values of $|u|$ the Laasonen system ($a = 1$) is best. Other special forms are also considered.
W. F. Ames (Newark, Del.)

Gunn, James E. 6636

The numerical solution of $\nabla \cdot a\nabla u = f$ by a semi-explicit alternating-direction iterative technique.
Numer. Math. **6** (1964), 181–184.

In the numerical solution of Dirichlet's problem for Poisson's equation on a rectangle by the standard five-point difference scheme, one is led to an (unusually large) system of linear equations $A_0 u = f$, where A_0 is a positive-definite symmetric matrix. One of the most effective methods for the solution of this linear system is the alternating direction method of Peaceman-Rachford. For

more general elliptic equations (self-adjoint) one obtains a linear system $Au=f$ for which the full force of the Peaceman-Rachford method has not been analyzed. E. G. D'jakonov [Dokl. Akad. Nauk SSSR **138** (1961), 522–525; MR **25** #767] proposed the iterative scheme $A_0u^{n+1}=A_0u^n-\rho(Au^n-f)$, where ρ is a positive constant and u^{n+1} is approximated by the Peaceman-Rachford process, essentially carried to completion. For this iterative process there is a $\rho>0$ such that a norm of the error is reduced by a factor ε in $O(h^{-2}(\log h)^2 \log \varepsilon)$ arithmetic operations, where h is the mesh spacing. In this paper the author shows that there is a $\rho>0$ such that if only one cycle of the Peaceman-Rachford process is performed in the calculation of u^{n+1}, then a norm of the error is reduced by a factor ε in $O(h^{-2}\log h^{-1}\log \varepsilon)$ arithmetic operations. An estimate is also given for the percentage increase in arithmetic operations over that needed to solve $A_0u=f$. *M. Lees* (Pasadena, Calif.)

Gunn, James E. 6637

On the two-stage iterative method of Douglas for mildly nonlinear elliptic difference equations.

Numer. Math. **6** (1964), 243–249.

The transformation $v=a^{1/2}u$ reduces the elliptic differential equation $-\nabla\cdot a(x)\nabla u+\phi(x,u)=0$ to the elliptic equation

$$-\Delta v+\frac{\Delta a^{1/2}}{a^{1/2}}v+a^{-1/2}\phi(x,va^{-1/2})=0,$$

where Δ is the Laplace operator. In this paper the author describes a discrete version of this reduction and investigates an iterative method for solving the resulting nonlinear difference equations for rectangular regions. The iterative method is an important generalization of a two-level iterative method proposed by Douglas [Numer. Math. **3** (1961), 92–98; MR **23** #B2177], in which the inner iteration is of the alternating direction type. An estimate of the arithmetic operations is included. Some weaker results are given for the case of non-rectangular regions.

M. Lees (Pasadena, Calif.)

Keller, Herbert B. 6638

On the pointwise convergence of the discrete-ordinate method.

J. Soc. Indust. Appl. Math. **8** (1960), 560–567.

van de Lindt, Willem Jacobus 6639

★Nuclear reactor calculations with the group diffusion equations on digital computers. (Dutch summary)

Proefschrift ter Verkrijging van de Graad van Doctor in de Technische Wetenschappen aan de Technische Hogeschool te Delft, Delft, 1964. x+140 *pp.*

This dissertation deals with the numerical solution of multi-group diffusion equations. There is not a great deal new in these 140 pages, and it is apparent that the writer and his advisors were not too well acquainted with the work that has been carried out at the many atomic energy laboratories.

A standard derivation has been given of the Boltzmann equation and the resulting multi-group diffusion equations. A nine-point finite difference operator is derived as an approximation to the two-dimensional Laplace operator. A discussion of the eigenvalue problem associated with the nine-point operator is presented. The most notable feature of this operator is that it does not possess Young's property A. A computer program for the Burroughs 220 based on the nine-point operator was prepared for a medium sized 2-D multi-group problem.

W. Sangren (San Diego, Calif.)

Lynch, R. E.; Rice, J. R.; Thomas, D. H. 6640

Tensor product analysis of partial difference equations.

Bull. Amer. Math. Soc. **70** (1964), 378–384.

Using tensor products, the authors analyze separable partial difference equations for two-dimensional regions. This allows them to give a compact finite Fourier series type formula for the solution of such difference equations, closely related to the results of Egerváry [Acta Math. Acad. Sci. Hungar. **11** (1960), 341–361; MR **24** #B2559] and J. Heller [J. Soc. Indust. Appl. Math. **8** (1960), 150–173; MR **22** #12705]. Comparisons are then made between the direct (i.e., noniterative) evaluation of this formula and standard iterative methods (SOR and ADI) applied to this separable problem. As is well known, the existing analyses of alternating direction methods depend strongly on the commutativity of two matrix operators H and V: $HV=VH$. The authors then state (without proof) a very interesting necessary and sufficient condition that $HV=VH$ in terms of tensor products, which extends the results of Birkhoff, Varga and Young [Advances in Computers, Vol. 3, pp. 189–273, Academic Press, New York, 1962; MR **29** #5395]. *R. S. Varga* (Cleveland, Ohio)

Nicolovius, R. 6641

Ein Verfahren zur numerischen Behandlung fastlinearer partieller Differentialgleichungen in Zylinderbereichen. (English and Russian summaries)

Z. Angew. Math. Mech. **43** (1963), 523–532.

Let u denote the solution of an almost linear second-order partial differential equation in a cylindrical region in R^N. Let x_N denote the coordinate in the direction of the axis of the cylinder. Assume that it is possible to choose n linearly independent solutions v_i of the homogeneous adjoint equation. Then

$$u\approx U=\sum_{k=1}^{m}A_k(x_N)w_k(x_1,\cdots,x_N),$$

where w_k, $k=1,\cdots,m$, are known C^1 functions and A_k, $k=1,\cdots,m$, are unknown functions. Equations for the determination of A_k, $k=1,\cdots,m$, are derived with the aid of v_j, $j=1,\cdots,n$, and Green's Theorem. Convergence is discussed and an example is given to demonstrate the procedure. *J. R. Cannon* (Upton, N.Y.)

Sobolev, S. L. 6642

Some new problems in the theory of partial differential equations.

Differential Equations and Their Applications (Proc. Conf., Prague, 1962), pp. 167–177. *Publ. House Czechoslovak Acad. Sci., Prague; Academic Press, New York,* 1963.

Two problems arising in numerical analysis, of determining the value of the integral of a given function $f(x)$ and of finding an approximate procedure for evaluation

of $f(x)$ at a given point from its values on a set of N points, are shown to be adjoint problems. It is possible to consider the error estimation in these numerical procedures as a minimax problem on some functional space. The part of this problem concerned with finding a maximum of the error over the unit sphere in the particular function space considered leads to an extreme function which is the solution of a partial differential equation of elliptic type. In three particular function spaces it is indicated how to derive the differential equation and its solution. The author's purpose is to exhibit "the deep lying connections existing between the theory of partial differential equations and numerical analysis".

A. O. Garder, Jr. (St. Louis, Mo.)

Stetter, Hans J. 6643

Maximum bounds for the solutions of initial value problems for partial difference equations.

Numer. Math. **5** (1963), 399–424.

Most of the recent literature on the general stability problem for difference approximations to initial-value problems for partial differential equations is concerned with stability relative to the L^2-norm. In this paper the author investigates the general stability problem, relative to the maximum norm, for multi-step partial difference equations. The results are restricted, essentially, to difference equations with constant coefficients. A sufficient condition (too complicated to state here) for stability in the maximum norm is given in terms of properties of the eigenvalues of the amplification matrix. The results are applied to a number of standard difference schemes for parabolic and hyperbolic differential equations. Results of numerical experiments performed in verification of the theory are also included.

M. Lees (Pasadena, Calif.)

Stetter, Hans J.; Törnig, W. 6644

General multistep finite difference methods for the solution of $u_{xy}=f(x, y, u, u_x, u_y)$.

Rend. Circ. Mat. Palermo (2) **12** (1963), 281–298.

The authors present an adaptation of the general theory of multi-step difference methods for ordinary differential equations to hyperbolic differential equations of the form $u_{xy}=f(x, y, u, u_x, u_y)$. Stability and convergence theorems are established in the maximum norm with the aid of a theorem due to the first author [Numer. Math. **3** (1961), 321–344; MR **25** #1653].

M. Lees (Pasadena, Calif.)

Suschowk, Dietrich 6645

Difference analogues of Green's identities for grids in R^n.

Numer. Math. **6** (1964), 200–210.

Im n-dimensionalen $(x_1, \cdots, x_n)$-Raum R^n sei ein Gitter der Maschenweite h gewählt; $f_{x_j}(P)$ bzw. $f_{\bar{x}_j}(P)$ bezeichnen den vorwärtigen bzw. rückwärtigen Differenzenquotienten einer Gitterfunktion f im Punkte P. Es bedeute $F(u, v)=\sum_{j,k=1}^{n} a_{jk}u_{x_j}v_{x_k}$ für zwei Gitterfunktionen u, v bei gegebenen Gitterfunktionen a_{jk}. $E(u, v)$ entsteht aus $F(u, v)$ durch Hinzunahme von bilinearen Gliedern, die in u, v und ihren ersten Differenzenquotienten linear sind, aber keine Produkte von Differenzenquotienten enthalten. Nach sorgfältiger Festlegung des "Randes" eines zusammenhängenden "Gitterbereiches" G werden die finiten Analoga zu den beiden bekannten Greenschen Formeln aufgestellt, die eine Umformung von $\sum_{P\in G} E(u, v)(P)$ und von Summen über Ausdrücke $u(a_{kj}v_{x_j})_{\bar{x}_k}-v(a_{jk}u_{x_j})_{\bar{x}_k}$ gestatten.

An Figuren im R^3 wird erläutert, welche Gitterpunkte jeweils in den Greenschen Identitäten auftreten.

L. Collatz (Hamburg)

Tee, G. J. 6646

A novel finite-difference approximation to the biharmonic operator.

Comput. J. **6** (1963/64), 177–192.

The author considers the solution by finite-difference methods of problems involving the biharmonic equation where the values of the solution and of its normal derivative are specified on the boundary of a region in two dimensions. He considers three finite-difference representations of the biharmonic operator, including the usual 13-point representation, as well as two 17-point representations. Both of the 17-point representations lead to matrices having Property A [the reviewer, Trans. Amer. Math. Soc. **76** (1954), 92–111; MR **15**, 562]. This greatly facilitates the use of the successive overrelaxation iterative method (S.O.R. method) for solving the linear system arising from the difference equation. However, the author does not appear to be aware of Kahan's results which show that even without Property A one can use the S.O.R. method effectively, provided that the diagonal elements of the matrix are positive and all other elements are nonpositive. (Kahan's work is unpublished, but a very brief summary is given on page 407 in a chapter (by the reviewer) of the book, *A survey of numerical analysis* [pp. 380–438, McGraw-Hill, New York, 1962; MR **24** #B2123].) Actually, however, in all three cases considered by the author the matrices do not satisfy Kahan's properties.

One of the 17-point representations was considered by Henrici in 1960 in an unpublished memorandum. The other was introduced by the author in order to achieve stronger diagonal dominance, but at the cost of using mesh points more distantly placed from the center point. For his procedure the author shows that, under certain assumptions, the local truncation error, the convergence rate of the S.O.R. method, and the truncation error of the finite-difference solution are all of order $O(h^2)$ as $h\to 0$. The 13-point formula leads to somewhat smaller truncation errors, but based on numerical experiments, the convergence of the S.O.R. method appears to be several times slower. The convergence rate of the S.O.R. method using the author's 17-point representation appears to be slightly faster than that obtained using the other 17-point representation.

D. M. Young, Jr. (Austin, Tex.)

Veyo, S. E.; Hornbeck, R. W. 6647

The numerical solution of the biharmonic equation using an automatic iterative process.

Trans. ASME Ser. E. J. Appl. Mech. **31** (1964), 704–705.

Zenkin, O. V. 6648

Some remarks on the stability of iteration processes. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 745–748.

The stability of the iterative process

(1) $$X^{(n+1)} = AX^{(n)} + B$$

for the solution of the finite linear system $X = AX + B$ with respect to round-off error is studied. All numbers involved are assumed to be fixed point fractions, s digits in length, with radix b. It is also assumed that the row norm $\|A\|$ of the matrix A is less than or equal to one, and that (1) converges if $\|A\| = 1$. The process (1) is said to be really stable if s_1 digits of each component of the result of the mth iteration can be guaranteed to be accurate for some m, with $0 < s_1 < s$. Simple conditions for real stability involving m and the order n of A are given, and applied to find a lower bound for the mesh length for the solution of the Dirichlet problem for Poisson's equation for which the Liebmann method will be really stable. *L. B. Rall* (Madison, Wis.)

Chen, Kuo-Wang 6649

Generalization of Steffensen's method for operator equations.

Comment. Math. Univ. Carolinae **5** (1964), 47–77.

Das Verfahren von Steffensen

$$x_{n+1} = Fx_n + \delta F(Fx_n, x_n)(x_{n+1} - x_n),$$

eine Zusammenfassung von gewöhnlicher Iteration und der Regula falsi, wird derart verallgemeinert, daß es zur iterativen Lösung von Fixpunktgleichungen $x = Fx$ im Banachraum geeignet ist. Dabei wird die zur Beschreibung des Verfahrens benötigte Steigung δF in der vom Referenten vorgeschlagenen Weise [J. W. Schmidt, Z. Angew. Math. Mech. **43** (1963), 1–8; MR **26** #5442; ibid. **43** (1963), 97–110; MR **27** #616] als linearer beschränkter Operator erklärt, der gewissen Forderungen genügt. Es werden verschiedene Sätze angegeben, welche neben der Existenz einer Lösung die Konvergenz des Verfahrens sichern und eine Fehlerabschätzung erlauben. Man kann ablesen, daß die Konvergenzgeschwindigkeit quadratisch ist. Die stets geforderte Kontraktionsbedingung an F läßt sich vermutlich noch abschwächen. Als Abschluß wird ausführlich auf die Anwendung der allgemeinen Theorie auf Systeme von nichtlinearen Gleichungen und auf nichtlineare Integralgleichungen eingegangen. *J. W. Schmidt* (Dresden)

Miellou, Jean-Claude 6650

Un procédé récurrent de calcul de la solution des problèmes aux limites variationnels elliptiques, linéaires, sous-forme d'un développement en série.

C. R. Acad. Sci. Paris **256** (1963), 1438–1440.

Let H be a Hilbert space with inner product $(\cdot,\cdot)$ and the associated norm $|\cdot|$. Let V be a linear subspace dense in H. On V, let a second inner product $((\cdot,\cdot))$ and its associated norm $\|\cdot\|$ be given, such that the two norms are topologically equivalent on V. Let B be a bilinear form (not necessarily symmetric) defined on $V \times V$ such that $|B(u,v)| \le c_1\|u\|\cdot\|v\|$ and $B(v,v) \ge c_2\|v\|^2$ for all $u, v \in V$. Given $f \in H$, in order to find $u \in V$ satisfying (1) $B(u,v) = (f,v)$ for all $v \in V$, the author observes that from any basis of V, another basis $\{e_i\}$ of V can be constructed such that $B(e_i, e_i) = 1$ and $B(e_i, e_j) = 0$ for $i > j$. Then the expansion of the solution u of (1) in the basis $\{e_i\}$ can be obtained by solving an infinite lower triangular system of linear equations. *Ky Fan* (Evanston, Ill.)

Miellou, Jean-Claude 6651

Remarques sur l'application de la méthode de Galerkin aux problèmes aux limites elliptiques variationnels, linéaires.

C. R. Acad. Sci. Paris **256** (1963), 3946–3948.

Let V be a closed linear subspace of the Sobolev space $H^m(\Omega)$ such that V contains all infinitely differentiable functions on Ω with compact support. Let B be a bilinear form defined on $V \times V$ such that $|B(u,v)| \le c_1\|u\|\cdot\|v\|$ and $B(v,v) \ge c_2\|v\|^2$ for all $u, v \in V$. Consider a basis $\{w_n\}$ of V and denote by V_n the linear subspace spanned by $\{w_1, w_2, \cdots, w_n\}$. For $f \in L^2(\Omega)$, let u_n be the unique vector in V_n satisfying $B(u_n, v) = (f, v)$ for all $v \in V_n$. The main result states that $\{u_n\}$ converges strongly to the unique $u \in V$ satisfying $B(u,v) = (f,v)$ for all $v \in V$. Several estimates of the error $\|u - u_n\|$ are also given. *Ky Fan* (Evanston, Ill.)

Petryshyn, W. V. 6652

On the generalized overrelaxation method for operation equations.

Proc. Amer. Math. Soc. **14** (1963), 917–924.

The well-known necessary and sufficient condition for the convergence of the Gauss-Seidel iterative method ($\omega = 1$) due to Reich [Ann. Math. Statist. **20** (1949), 448–451; MR **11**, 136] was generalized by Ostrowski [Rend. Mat. e Appl. (5) **14** (1954), 140–163; MR **16**, 1155] who analogously gave a necessary and sufficient condition for the successive overrelaxation iterative method ($0 < \omega < 2$). More recently, Kreĭn and Prozorovskaja [Voronež. Gos. Univ. Trudy Sem. Funkcional. Anal. **5** (1957), 35–38; MR **20** #2081] extended Reich's result to operator equations of the form $(D - S - S^*)u = f$ in a Hilbert space. Although apparently unaware of Ostrowski's work, the author similarly extends Ostrowski's result by giving a necessary and sufficient condition that the successive overrelaxation method be convergent in a Hilbert space, using the notions of K-positive definite operators and K-symmetric operators introduced by the author [Trans. Amer. Math. Soc. **105** (1962), 136–175; MR **26** #3180]. *R. S. Varga* (Cleveland, Ohio)

Poljak, B. T. 6653

Some methods of speeding up the convergence of iterative methods. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 791–803.

Consider the functional equation $P(x) = 0$, where P is an operator on a Banach space B with range in B. Most methods for solving this equation consist in forming a sequence $x^n \in B$ which converges to the desired solution x^* as $n \to \infty$. Continuous analogues of these methods are also known, whereby a trajectory $x(t)$ in B is obtained such that $x(t) \to x^*$ as $t \to \infty$. The present paper deals with the convergence of two such methods. The first, discrete, method generates the sequence $\{x^n\}$ by a k-step iteration of the form

(1) $$x^{n+1} = \sum_{i=0}^{k-1} \alpha_i x^{n-i} + \sum_{i=0}^{k-1} T_{i+1} P(x^{n-i}),$$

where α_i are numerical constants satisfying $\sum_{i=0}^{k-1} \alpha_i = 1$, and T_{i+1} are continuous linear operators. The second method defines the trajectory $x(t)$ as a solution of a kth-order differential equation in B,

$$(2)\quad d^k x/dt^k = T_1(d^{k-1}x/dt^{k-1}) + \cdots + T_{k-1}(dx/dt) + T_k P(x).$$

The main result relating to (1) is as follows. Let x^* be a solution of $P(x) = 0$, and let P be Fréchet-differentiable at x^*, with derivative $U = P'(x^*)$. Assume, moreover, that the operators $T_1, \cdots, T_k$ commute with each other, and with U. Denote by $\rho_1, \cdots, \rho_k$ the roots of the algebraic equation $\rho^k = \sum_{i=0}^{k-1} (\alpha_i + \lambda_{i+1}\lambda)\rho^i$, where λ is in the spectrum $\sigma(U)$ of U, and λ_i in $\sigma(T_i)$. Let

$$\sup_{\lambda, \lambda_i} \max_s |\rho_s| \leq q < 1,$$

where the supremum is taken over the respective spectra. Then, for arbitrarily small $\varepsilon > 0$, one has $\|x^n - x^*\| = O((q+\varepsilon)^n)$ as $n \to \infty$, provided the initial elements $x^0, x^1, \cdots, x^{k-1}$ are sufficiently close to x^*. An entirely analogous result is obtained for (2). In the special case of a linear equation $Ax - b = 0$ (A a continuous linear operator), convergence can be established for arbitrary initial elements. More detailed results are presented for the case where P is the gradient of a functional in a Hilbert space, and the methods have the special form

$$x^{n+1} = x^n - \alpha P(x^n) + \beta(x^n - x^{n-1}),$$

$$d^2x/dt^2 = \alpha_1(dx/dt) + \alpha_2 P(x).$$

Convergence of these two-step methods ($k = 2$) is shown in certain situations to be substantially faster than the convergence of corresponding one-step methods.

Walter Gautschi (Lafayette, Ind.)

Tihonov, A. N.; Glasko, V. B. 6654

An approximate solution of Fredholm integral equations of the first kind. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 564–571.

The method of regularization developed by A. N. Tihonov [Dokl. Akad. Nauk SSSR **151** (1963), 501–504; MR **28** #5576; ibid. **153** (1963), 49–52; MR **28** #5577] is applied numerically to the Fredholm equation of the first kind,

$$\int_{-1}^{1} K(x, s)z(s)\,ds = u(x), \qquad -1 \leqq x \leqq 1,$$

with

$$K(x, s) = \frac{1}{\pi}\{(x-s)^2 + 1\}^{-1}.$$

Using the case with known solution $z(s) = (1-s^2)^2$, the optimum value of the regularization parameter for minimum error is determined experimentally. Without regularization, the error in the numerical solution is about 10^6 in magnitude, even though the residue is 10^{-8} in magnitude. Use of regularization with the optimum parameter reduces the error to about 10^{-3}. The effect of perturbation of $u(x)$ on the error is studied empirically for the original equation, and also for the case that the integration extends over some smaller interval $-k \leqq s \leqq k$ with $k < 1$. In the latter case, the conclusion is that the numerical solution is unstable with respect to errors in $u(x)$ even for optimum regularization.

L. B. Rall (Madison, Wis.)

Twomey, S. 6655

On the numerical solution of Fredholm integral equations of the first kind by the inversion of the linear system produced by quadrature.

J. Assoc. Comput. Mach. **10** (1963), 97–101.

This paper has been listed by title in MR **26** #5757. The numerical solution of linear Fredholm integral equations of the first kind presents great difficulties and severely limits the usefulness of such equations in describing physical phenomena. Upon use of a quadrature formula the original equation is approximated by a system of linear algebraic equations, $Ax = g$, but frequently the exact solution of this system, x, has properties which conflict with our a priori knowledge. For example, the components of x may oscillate wildly, whereas the solution of the original integral equation may be quite smooth, even monotone. The author (and in an earlier paper D. L. Phillips [J. Assoc. Comput. Mach. **9** (1962), 84–97; MR **24** #B534]) suggests replacing the linear algebraic system by various minimization problems, e.g.,

$$\min_x [(x - c, x - c) + \lambda(Ax - g, Ax - g)],$$

where c is a preliminary estimate of x, and several values of the parameter λ are to be examined.

The point is that we should incorporate all of our knowledge of the physical situation before attempting to solve a system of possibly ill-conditioned linear algebraic systems. For a further discussion, in which the components of x are determined sequentially using dynamic programming, reference may be made to a report by Bellman, the reviewer and Lockett [RAND Rep. RM-3815-PR (1963)].

R. Kalaba (Santa Monica, Calif.)

Olver, F. W. J. 6656

Error analysis of Miller's recurrence algorithm.

Math. Comp. **18** (1964), 65–74.

This paper is concerned with a device for computing accurately, from a second-order linear homogeneous recurrence relation, a solution which diminishes to zero as the argument becomes infinite, from a given starting value and in the presence of another solution that increases indefinitely relatively to the desired solution.

The method is to use the recurrence relation in the reverse direction, with more or less arbitrary starting values (but chosen in such a way that the desired solution is not swamped by the unwanted one), at an argument large enough so that, on working backwards, the unwanted solution dies away to relative insignificance before any argument is reached for which a final solution is desired.

The author discusses the error analysis of this process, with particular attention to the rounding-errors inevitable at every stage of the process, and the final normalisation whereby the results of backward recurrence are collated with the given "starting" value, which is in general at the final argument reached.

J. C. P. Miller (Cambridge, England)

Bogoljubov, Ju. I. 6657

The representation of a function of five variables by a nomogram on an oriented transparency in the form of a rectangle. (Russian)

Čuvaš. Gos. Ped. Inst. Učen. Zap. No. **15** (1963), 56–59.

The necessary and sufficient conditions for a function F

to be of the form $F = A_{12}(x_1, x_2) + A_{34}(x_3, x_4) + A_5(x_5)$ are $F_1 = f(x_1, x_2)$, $F_2 = \varphi(x_1, x_2)$, $F_5 = \psi(x_5)$, $f_2 = \varphi_1$. From these the author obtains the conditions for a function of the form $\Phi = \sigma(x_1, x_2, x_3, x_4) \cdot F$ in $\Phi_{55} = X(x_5)\Phi_5$,

$$(\Phi_5\Phi_1 - \Phi\Phi_{15})/\Phi_5^2 = f/\psi, \qquad (\Phi_5\Phi_2 - \Phi\Phi_{25})/\Phi_5^2 = \varphi/\psi,$$

and $f_2 = \varphi_1$, $X_5 = \psi_5/\psi$. *E. M. Bruins* (Amsterdam)

Bogoljubov, Ju. I. 6658

Representation of a function of six variables by a nomogram with triangular transparency. (Russian)

Čuvaš. Gos. Ped. Inst. Učen. Zap. No. 15 (1963), 63–70.

The author obtains the necessary and sufficient conditions for a function F to be of the form $F = A_{12}(x_1, x_2) + A_{34}(x_3, x_4) + A_{56}(x_5, x_6)$ in $F_1 = f(x_1, x_2)$, $F_2 = \varphi(x_1, x_2)$, $F_3 = \psi(x_3, x_4)$, $F_4 = \chi(x_3, x_4)$, $f_2 = \varphi_1$, $\psi_4 = \chi_3$. He generalises these conditions to functions of the form

$$\sigma(x_3, x_4, x_5, x_6) \cdot F = \Phi(x_1, x_2, x_3, x_4, x_5, x_6).$$

E. M. Bruins (Amsterdam)

Bogoljubov, Ju. I. 6659

Necessary and sufficient conditions for the representation of functions of six variables by a nomogram with oriented transparency. (Russian)

Čuvaš. Gos. Ped. Inst. Učen. Zap. No. 15 (1963), 71–78.

The author obtains the necessary and sufficient conditions for a function to be of the form $F = A_{12}(x_1, x_2) + A_{34}(x_3, x_4) + A_5(x_5) + A_6(x_6)$ in $F_1 = f(x_1, x_2)$, $F_2 = \varphi(x_1, x_2)$, $F_3 = \psi(x_3, x_4)$, $F_4 = \chi(x_3, x_4)$, $F_5 = \eta(x_5)$, $f_2 = \varphi_1$, $\psi_4 = \chi_3$. He generalises these conditions to functions of the form $\sigma(x_1, x_2, x_3, x_4, x_5, x_6) \cdot F$. *E. M. Bruins* (Amsterdam)

COMPUTING MACHINES

See also 6967, 6968, 6980, 6997, 7000.

Büchi, J. Richard 6660

Regular canonical systems.

Arch. Math. Logik Grundlagenforsch. 6, 91–111 (1964).

The systems of the title are those involving rules of production of the form $aP \to bP$ only. The author studies the sets of words derivable from finitely many axioms, together with finitely many such rules of production. One of his main results is that such sets are the accepted sets of automata. (His other results are mainly concerned with alternate characterizations of such sets.) Chomsky [Information and Control 2 (1959), 137–167; MR 21 #4107] showed that a one-sided linear language is equivalent to an accepted set; a footnote in the paper under review, however, fixes the original publication of this material (as a report of the University of Michigan) at the same time. [The printers seem to have applied a semi-Thue system to the text, and the editors seem to have taken the result as an accepted set; on p. 93, e.g., in Definition 1 the union and intersection symbols should be switched, in line 12 the u_i, should be u_{i+1}, and in line -6 all arrows should be translated one symbol to the right, etc.]

R. M. Baer (Berkeley, Calif.)

Cyprus, Joel Howard 6661

Optimal synthesis of the Boolean functions of four variables with majority logic.

Rice Univ. Studies 50 (1964), no. 2, v+110 pp.

The author considers the synthesis of Boolean functions by majority and minority logic. Restricting attention to four or fewer variables, these functions are decomposed into equivalence classes under the group consisting of complementations and permutations of the variables and functional complementation. Under this group, the 65,536 Boolean functions of four variables fall into 222 equivalence classes. For each class the author finds an optimal realization of a representative function by an exhaustive search using a digital computer.

The more difficult problem of efficiently determining if a given function is equivalent to a representative is sidestepped by listing all the functions of fewer than four variables with their equivalent representatives.

The extensive tables (57 pages) will prove useful to logical designers, while the lucid discussion may be of interest to those interested in programming logical problems.

M. A. Harrison (Berkeley, Calif.)

Lindamood, George E.; Shapiro, George 6662

Magnitude comparison and overflow detection in modular arithmetic computers.

SIAM Rev. 5 (1963), 342–350.

Modular arithmetic, in which an integer is represented by its residues with respect to a set of relatively prime moduli, would permit addition and multiplication to be executed in digital computers with comparative ease and speed, since "carry's" do not have to be performed. Unfortunately, other operations are more cumbersome in modular arithmetic than in the conventional radix notation. Two such operations are magnitude comparison and overflow detection. In this paper, the authors present methods for performing these two operations using only the basic modular arithmetic operations of addition, subtraction, multiplication and comparison of residues.

Magnitude comparison is effected by the rather interesting technique of converting the residue representation of an integer x with respect to moduli $r_0, \cdots, r_{n-1}$ into a "mixed radix" representation of the form

$$x = a_n(r_{n-1} \cdots r_0) + a_{n-1}(r_{n-2} \cdots r_0) + \cdots + a_1 r_0 + a_0,$$

with $0 \le a_i < r_i$. *E. K. Blum* (Middletown, Conn.)

Naur, Peter 6663

Using machine code within an ALGOL system.

Nordisk Tidskr. Informations-Behandling 4 (1964), 115–117.

Author's summary: "The use of machine code within programs run in the GIER ALGOL system is made possible through the provision of standard procedures which transfer the control of the machine to the instruction held as a variable of the ALGOL program. The paper describes these and some associated standard procedures."

F. L. Bauer (Munich)

Naur, Peter 6664

The design of the GIER ALGOL compiler. I, II.

Nordisk Tidskr. Informations-Behandling 3 (1963), 124–140; *ibid.* 3 (1963), 145–166.

A detailed, clear and instructive description of the ALGOL compiler for the Danish computer GIER, a medium size computer with a core store of only 1024 words of 42 bits

and a backing magnetic drum of 12,800 words. The resulting multipass compiler has 9 passes and occupies 5199 words; organization of these passes is nearly optimal under the conditions given by the non-homogeneous store. Parsing techniques and especially questions of the running system: storage allocation, addressing, procedure calling are thoroughly discussed; whenever possible, reasons for the solution adopted are given and connections to methods used elsewhere are mentioned. Thus, this study is exemplary in its presentation, and the GIER ALGOL compiler demonstrates the practicability of ALGOL translation even for a smaller computer. *F. L. Bauer* (Munich)

Stein, Marvin L.; Munro, William D. 6665

★Computer programming: A mixed language approach.
Academic Press, New York-London, 1964. xiv + 459 *pp.* $11.50.

This book is a text on digital computer programming. The authors take the view that it is logically and pedagogically sound to begin such a text with a discussion of machine language programming. Accordingly, they do this with the CDC 1604 as the machine for illustration. They proceed through the usual elementary chapters on (1) Number systems, (2) Machine organization, (3) Elementary coding, (4) Fixed and floating point arithmetic, (5) Nonarithmetic operations, in some 214 pages. This brings them to the first sophisticated idea in programming, the subroutine, which they treat in a quite ad hoc and inadequate fashion in a twenty-page Chapter 6. In Chapter 7, "Input-output" is treated briefly. Chapter 8 is entitled "Assembly of complete programs," Chapter 9, "FORTRAN; mixed language programs"; there is a further discussion of subroutines.

The reviewer questions the authors' suggestion that the book is suitable for the "junior, senior, and early graduate level". Certainly, material as elementary as this can and should be presented at lower levels. There are aspects of programming that are appropriate for the higher levels, but these are not approached in this text.

E. K. Blum (Middletown, Conn.)

Stein, P. R.; Ulam, S. M. 6666

Non-linear transformation studies on electronic computers.
Rozprawy Mat. **39** (1964), 66 *pp.*

This is a paper in the field of experimental mathematics using arithmetic. Experimental mathematics has always been widely practiced, but essentially only the arithmetic branch, centered around number theory, has achieved any status worth mentioning. The modern electronic computer is a handy sorcerer's apprentice for those who would enter the field.

The paper considers mainly the three-dimensional iteration transformations

$$x_i = P_i(x_1, x_2, x_3), \qquad i = 1, 2, 3,$$

where the P_i are cubics in x_1, x_2, x_3. Some related transformations are also discussed. Many photographs of cathode ray tube displays are given, a fondness for citing large numbers of iterations and machine time used is revealed, and a crude classification of the limited results is offered, but there appear to be no firm new results of general mathematical interest.

It is an attractive idea that computers can produce many specific cases of mathematical situations from which we can obtain insight, but usually (though not always) it is the insight, not the specific cases, that has been the criterion for public presentation. One can only wonder what will happen to mathematics if we allow the undigested outputs of computers to fill our literature. The present paper shows only slight traces of any digestion of the computer output.

R. W. Hamming (Murray Hill, N.J.)

GENERAL APPLIED MATHEMATICS

Comstock, Craig 6667

On the autocorrelation of random inhomogeneities.
J. Acoust. Soc. Amer. **36** (1964), 1534–1536.

The author treats singularities at the origin of temperature fluctuation autocorrelation functions. The usual autocorrelation function $N(\rho)$ is decomposed into two functions $N(\rho|a)$ and $g(a)$, the former representing the autocorrelation function for scattering regions of fixed size a while $g(a)$ is a probability density function for the inhomogeneity size a. By a particular choice of form of $N(\rho|a)$ and $g(a)$, both analytic at the origin, it is shown that a family of functions $N(\rho)$, none of which is analytic at the origin, may be constructed. By suitable choice of $g(a)$, a useful $N(\rho)$ may be generated which, the author concludes, will shed light on the physics of the process involved. *G. E. Lord* (Seattle, Wash.)

MECHANICS OF PARTICLES AND SYSTEMS

See also 5980, 6132, 6917, 6926.

Geise, Gerhard 6668

Über ähnlich-veränderliche ebene Systeme. I, II.
Wiss. Z. Techn. Univ. Dresden **12** (1963), 883–888; *ibid.* **12** (1963), 1607–1611.

The motion of variable systems in the plane under preservation of similitude is described, employing complex numbers for the coordinatization. Well-known concepts are developed: poles, higher-order characteristics, Bresse circles and their generalizations. The generation of motions and the approximation of a motion by another motion are discussed. *R. Artzy* (Princeton, N.J.)

Mayne, Georges [Mayné, Georges] 6669a

Le problème de la séparation des variables pour les systèmes tridimensionnels scléronômes dont la fonction hamiltonienne n'est pas homogène en les momentoïdes.
C. R. Acad. Sci. Paris **257** (1963), 4129–4132.

Mayne, Georges [Mayné, Georges] 6669b

Le problème de la séparation des variables pour les systèmes dynamiques scléronômes de fonction hamiltonienne quadratique mais non homogène en les momentoïdes.
C. R. Acad. Sci. Paris **258** (1964), 61–63.

In a previous paper [same C. R. 257 (1963), 1671–1674; MR 28 #754], the author classified dynamical systems

with two or three degrees of freedom for which the variables can be separated. The Hamiltonian was assumed to have two terms: one quadratic in the momenta, one of general form in the coordinates. In the present papers the results are extended first to the case when the Hamiltonian has a third term, linear in the momenta, and then to the case of such a Hamiltonian for a system of n degrees of freedom. *W. Kaplan* (Ann Arbor, Mich.)

Alojan, M. A. 6670

On the dynamic stability of a sinusoidal arch. (Russian. Armenian summary)

Izv. Akad. Nauk Armjan. SSR Ser. Fiz.-Mat. Nauk 17 (1964), no. 4, 23–33.

Die Differentialgleichungen für die Biegeschwingungen eines Trägers mit sinusförmig gekrümmter Mittellinie und sinusförmiger Belastungwerden aufgestellt und durch einen Separationsansatz vereinfacht. Bei periodisch wechselnder Last führt die Berechnung des Zeitverhaltens auf eine Differentialgleichung vom Mathieu'schen Typ, deren Beiwerte aus den Randbedingungen zu bestimmen sind. Für zwei spezielle Fälle werden sie ausgerechnet: für einen beidseitig gelenkig gelagerten und für einen beidseitig fest eingespannten Träger. Dabei wird das Galerkin'sche Näherungsverfahren verwendet, um kritische Belastungen und Eigenfrequenzen zu erhalten. Diese Näherungswerte erlauben die Bestimmung der Grenzen der instabilen Bereiche. Der Vergleich mit bekannten Ergebnissen, die für Träger mit parabolisch gekrümmten Mittellinien erhalten wurden, zeigt nur geringe Unterschiede für beide Trägerformen. *K. Magnus* (Stuttgart)

Kolovskiĭ, M. Z. 6671

The effect of high-frequency perturbations on the resonance vibrations in a non-linear system. (Russian)

Leningrad. Politehn. Inst. Trudy No. 226 (1963), 7–17.

Consider the equation

$$(1)\qquad \ddot{x}+g(\dot{x})+f(x) = F(t),$$

where x denotes the displacement, $g(\dot{x})$ the dissipative force, $f(x)$ the restoring force and $F(t)=A\sin\nu t+B\sin\omega t$, $\omega\gg\nu$, the external force. Following the method of harmonic linearization the solution of (1) is sought in the form (2) $x=x_0+b\sin(\omega t+\varphi)$, where x_0 is a slowly varying function. Upon substitution of (2) into (1) and separation of the slowly varying terms from the first harmonics of vibration with frequency ω, two equations are obtained. The influence of high-frequency perturbations on resonance vibrations is investigated for several particular systems, such as, for example, a system with linear elastic characteristic and dry friction, a system with viscous friction and cubic characteristic, and so on.

E. Leimanis (Vancouver, B.C.)

ELASTICITY, PLASTICITY

See also 6188, 6429.

Eringen, A. Cemal; Suhubi, E. S. 6672

Nonlinear theory of simple micro-elastic solids. I. (French, German, Italian and Russian summaries)

Internat. J. Engrg. Sci. 2 (1964), 189–203.

Authors' summary: "The present work is concerned with the formulation of the basic field equations, boundary conditions and constitutive equations of what we call 'simple micro-elastic' solids. Such solids are affected by the 'micro' deformations and rotations not encountered in the theory of finite elasticity.

"The theory, in a natural fashion, gives rise to the concept of stress moments, inertial spin and other types of second-order effects and their laws of motion. The mechanism of the surface tension is contained in the theory. In a forthcoming paper (Part II) explicit expressions of constitutive equations of several simple micro-elastic solids will be given and applied to some special problems."

R. A. Toupin (Yorktown Heights, N.Y.)

Marris, A. W.; Villanueva, J. 6673

Dependence of stress upon temperature gradient for an ideally elastic solid isotropic and homogeneous in the reference state.

Trans. ASME Ser. E. J. Appl. Mech. 31 (1964), 419–422.

Authors' summary: "This paper postulates that the stress tensor of a Cauchy-elastic material depends upon the first temperature-gradient vector in addition to the deformation gradient and local temperature. Consequences of this postulate are examined for the particular cases of a body constrained to no deformation and to simple isochoric shear." *A. C. Pipkin* (Providence, R.I.)

Toupin, R. A. 6674

Theories of elasticity with couple-stress.

Arch. Rational Mech. Anal. 17 (1964), 85–112.

A valuable comment on the theories of elasticity with couple-stress is made, starting from Cosserat's ideas and ending with the latest studies in this direction. Continuous media with microstructure, stress and couple-stress are considered. A special discussion of nonsimple elastic materials and of Cosserat materials with constrained rotations is given. Initial stresses and hyperstresses are also introduced. *P. P. Teodorescu* (Bucharest)

Barbarito, Bruno 6675

Sulle ipotesi a sostegno dei principii di Kirchhoff. (English summary)

Rend. Accad. Sci. Fis. Mat. Napoli (4) 28 (1961), 403–405.

Author's summary: "Kirchhoff's theorem (existence and uniqueness of the set of loads corresponding to any set of displacements, defined throughout a linear-elastic body) is demonstrated. The starting point is the writing down of the body and surface equilibrium equations taking into account the deformation of the body. It is also pointed out that—as in many other cases in the field of elasticity—the converse theorem holds under far less restrictive assumptions than the direct theorem."

Ruhadze, Ž. A. 6676

Boundary-value problems for the vibration of an infinite, two-dimensional, non-homogeneous, elastic isotropic body. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR 35 (1964), 531–538.

In this paper (following earlier work by V. D. Kupradze,

M. O. Bašeleĭšvili and others), a proof is given of the existence theorem for problems of the vibration of an infinite, plane, piecewise inhomogeneous, isotropic, elastic body. The techniques used involve applications of potential theory, and no restriction is placed upon values of either the frequency or elastic constants.

H. G. Hopkins (Sevenoaks)

Bašeleĭšvili, M. O. 6677

Solution of the third and fourth boundary-value problems in the statics of an anisotropic elastic body. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **35** (1964), 277–284.

This paper concerns the same problems discussed in an earlier paper by the author [same Soobšč. **34** (1964), 283–290; MR **29** #5437], their solutions now being derived by means of different techniques involving Fredholm integral equations. Thus, existence theorems are proved for the solutions of the third and fourth boundary-value problems for simply connected finite and infinite regions. It is stated that the extension to multiply connected regions involves no analytical difficulty. Furthermore, it is stated that similar analysis can be developed for the theory of bending of anisotropic plates. *H. G. Hopkins* (Sevenoaks)

Kvinikadze, G. P. 6678

Existence theorems for the exterior third and fourth dynamics problems of elasticity theory. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **33** (1964), 293–300.

The third and fourth boundary-value problems of the dynamic plane theory of elasticity, as defined by V. D. Kupradze in his book, *Boundary problems of the theory of vibrations and integral equations* (Russian) [GITTL, Moscow, 1950; MR **15**, 318] are discussed for the case when

$$\sqrt{r}\,u_j = O(1), \quad \sqrt{r}\,(\partial u_j/\partial r - ik_j u_j) = o(1), \qquad j = 1, 2,$$

as $r \to \infty$. Uniqueness of the solutions is proved and the problems are reduced to singular integral equations, the solutions of which are studied.

J. R. M. Radok (Adelaide)

Maz'ja, V. G.; Sapožnikova, V. D. 6679

A remark on the regularization of a singular system in the isotropic theory of elasticity. (Russian. English summary)

Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. **19** (1964), *no.* 2, 165–167.

A potential of a double layer is constructed by means of which the second fundamental problem of the theory of elasticity (given displacements) reduces to systems of Fredholm and singular integral equations. The regularization of these systems can be achieved explicitly.

J. R. M. Radok (Adelaide)

Parasjuk, E. N. 6680

On a theorem of Tamarkin type and its application to the second fundamental problem of two-dimensional elasticity theory. (Russian)

Ukrain. Mat. Ž. **16** (1964), 537–538.

The following theorem is proved and discussed with regard to elasticity. Suppose that for the equation

$$(1) \qquad u(x, \lambda) - \int_S K(x, y; \lambda) u(y, \lambda)\, d_y S = f(x, \lambda)$$

(S = a surface) the following conditions are satisfied: (a) The kernel $K(x, y; \lambda)$ and the function $f(x, \lambda)$ are analytic functions of λ in a domain D of the λ-plane; (b) The kernel $\partial K/\partial\lambda$ is of the same type as K, in particular, an integral operator with this kernel is bounded in some Banach space; (c) For each $\lambda \in D$, (1) has a unique solution. Then each solution of (1) for $\lambda \in D$ is an analytic function of λ in D. *J. Nowinski* (Newark, Del.)

Ufljand, Ja. S. [Уфлянд, Я. С.] 6681

★Integral transforms in problems of elasticity theory [Интегральные преобразования в задачах теории упругости].

Izdat. Akad. Nauk SSSR, Moscow, 1963. **367** *pp.* 2.08 *r.*

241 references, including a large number by the author himself and covering the world literature up to 1962, form the basis of what might effectively be called a "dictionary of the applications of integral transforms" in the linear theory of elasticity. Fourier, Mellin, Hankel, Kontorovich-Lebedev, and Mehler-Fok transforms are considered, where the last are based on Legendre polynomials of the first kind with variable complex index. This monograph should be consulted by anyone considering publication of results in this field, in order to avoid duplication.

J. R. M. Radok (Adelaide)

Zorski, Henryk 6682

On the equations describing small deformations superposed on finite deformation.

Second-order Effects in Elasticity, Plasticity and Fluid Dynamics (Internat. Sympos., Haifa, 1962), *pp.* 109–128. *Jerusalem Academic Press, Jerusalem; Pergamon, Oxford,* 1964.

Conditions of ellipticity and strong ellipticity are derived and discussed for the field equations appropriate to the theory of small deformations superposed on large deformations of elastic isotropic solids. The particular finite deformations of simple shear, extension, and torsion of a circular cylinder are used as examples to show what sort of restrictions the assumed existence of superposed displacements will impose on the strain energy function and the initial finite deformation field. The results are presented in the form of inequalities, and for the case where the material is a Mooney material or, more generally, where the strain energy function depends only on the first powers of the principal strain invariants, it is shown that simple, physically meaningful restrictions appear.

General questions of uniqueness, existence and stability are briefly treated, and requirements sufficient to insure each of these properties are compared.

R. L. Fosdick (Chicago, Ill.)

Bragg, Lincoln E. 6683

Monotonicity on curves in lieu of the C-N inequalities for finite elasticity.

Arch. Rational Mech. Anal. **17** (1964), 327–338.

The author shows that the GCN inequality for the strain energy function is equivalent to the monotonicity of some scalar-valued functions defined over curves in the six-dimensional space of symmetric tensors. The theory of those curves is developed. *P. P. Teodorescu* (Bucharest)

Bressan, Aldo 6684

Qualche teorema di cinematica delle deformazioni finite.

Ist. Veneto Sci. Lett. Arti Atti Cl. Sci. Mat. Natur. **121** (1962/63), 235–244.

Stojanovitch, R. [Stojanović, Rastko D.]; Djuritch, S.; Vujoshevitch, L. 6685

On finite thermal deformations. (Polish and Russian summaries)

Arch. Mech. Stos. **16** (1964), 103–108.

The authors present a differential-geometric discussion concerning the deformations in a material which are induced by an arbitrary temperature field without treating the thermodynamic aspects of the problem. They recall that in the usual linear theories treating thermal deformations the total (compatible) strain is represented as the sum of two incompatible strains; the thermal strain and the elastic strain. Since this decomposition is not generally possible in the nonlinear theories, the authors introduce a model for treating the incompatible deformations, which has found application in the continuum theory of dislocations. Finally, the authors remark that the stress state of a thermally deformed body is described, in general, by the stress tensor field σ^{ij} and the couple stress tensor field τ^{ijk}, and briefly mention the equations which are available in order to find the unknown deformations in any given problem. *R. L. Fosdick* (Chicago, Ill.)

Verma, P. D. S. 6686

Electrical conduction in finitely deformed isotropic materials. (Polish and Russian summaries)

Arch. Mech. Stos. **15** (1963), 3–6.

An attempt is made to write down the form taken by the constitutive equation for electrical conduction in deformed isotropic materials, in the special case of a spherically symmetrical deformation. It is unaccountably assumed that only one strain component is different from zero in such a deformation. Consequently, the attempt fails. *A. C. Pipkin* (Providence, R. I.)

Konovalov, A. N. 6687

On an iteration scheme for solving statics problems in elasticity theory. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 942–945.

Difference solutions of problems of the plane theory of elasticity in the presence of inertia forces are derived by iteration, using an alternating direction scheme. Convergence is proved and an asymptotic formula for the rate of convergence obtained. *J. R. M. Radok* (Adelaide)

Neuber, H. 6688

Spannungstheorie der Zahnräder. I. Aussenverzahnung. (English and Russian summaries)

Z. Angew. Math. Mech. **44** (1964), 285–299.

Using the Airy stress function for plane stress problems, the author deals with problems of stress in discs and gear wheels under single concentrated loads on the rims and also periodic concentrated loads. Exact formulae are given for the boundary stress in all cases. *R. M. Morris* (Cardiff)

Sinha, D. K. 6689

Radial deformation of a composite piezoelectric annular disk.

J. Acoust. Soc. Amer. **36** (1964), 1926–1928.

An annular disc is made of two annular discs of different piezoelectric materials of the type of quartz. It is stressed by mechanical pressures on the inner and outer boundaries, and by electrical force on the inner boundary. Considering the problem as one of plane stress, the radial deformations are obtained by the usual procedure. {Even the radial deformation which the author seeks has not been completely calculated by solving the simultaneous equations involving the unknown constants, nor have any numerical results been presented. Consideration of actual numerical values for the physical parameters surely increases the value of such problems. Moreover, it is worthwhile to examine what happens to the remaining three strain components not considered in the paper, and also the axial electrical displacement, the nature of the medium being anisotropic.} *G. Paria* (Indore)

Teodorescu, P. P. 6690

On the plane problem of the theory of elasticity in arbitrary curvilinear co-ordinates. III. Mathematical formulation of the problem. Methods of calculation.

Rev. Méc. Appl. 8 (1963), 953–969.

Part II appeared in same Rev. 8 (1963), 589–609. The author formulates, in the two-dimensional case, Problem I (given boundary stresses) for the stress function in classical terms and then in isogonal, orthogonal, harmonic curvilinear coordinates. The same treatment is then accorded to Problem II (given boundary displacements) for the displacement function. Methods of calculation are considered. *L. M. Milne-Thomson* (Tucson, Ariz.)

Teodorescu, P. P. 6691

On the plane problem of the theory of elasticity in arbitrary curvilinear co-ordinates. III. Mathematical formulation of the problem. Methods of calculation. (Romanian. French summary)

Acad. R. P. Romîne Stud. Cerc. Mec. Apl. **14** (1963), 37–54.

Part II appeared in same Stud. **13** (1962), 1387–1406. {The same as #6690 but in Romanian.} *L. M. Milne-Thomson* (Tucson, Ariz.)

Bertrand, P. L. 6692

Note sur l'équilibre élastique d'un milieu indéfini percé d'une cavité cylindrique sous pression. (English summary)

Ann. Ponts Chaussées **134** (1964), 473–521.

Author's summary: "Faisant suite à une précédente étude qui portait sur une cavité cylindrique percée en milieu semi-indéfini et soumise à une pression constante, l'auteur

étudie ici l'état de contrainte et de déformation d'un milieu indéfini au voisinage d'une cavité cylindrique soumise à une pression non constante, mais présentant une symétrie de révolution par rapport à l'axe de la cavité. La solution de ce problème s'exprime par des séries ou intégrales de Fourier faisant intervenir des fonctions de Bessel modifiées. Elle s'applique plus généralement au cas de plusieurs milieux annulaires concentriques de caractéristiques mécaniques différentes. On peut étendre cette solution aux cas réels de galeries souterraines sous pression moyennant, d'une part une certaine approximation, d'autre part des précautions dans le choix des paramètres des séries de Fourier."

Guz', O. M. 6693
Representation of solutions of three-dimensional axisymmetric problems of elasticity theory for a transversely isotropic body. (Ukrainian. Russian and English summaries)
Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 1592–1595.

In the problem of the title displacements and stresses are expressed by derivations of two functions satisfying the following differential equations of second order:

$$(\partial^2/\partial r^2 + r^{-1}\,\partial/\partial r + \nu_j^2\,\partial^2/\partial z^2)\varphi_j = 0 \qquad (j = 1, 2).$$

In the case of real parameters ν_j, the functions φ_j become harmonic in two different regions obtained by affine transformation of the region considered. No illustrative examples are given. *Z. Kączkowski* (Warsaw)

Herrmann, L. R. 6694
Stress functions for the axisymmetric, orthotropic, elasticity equations.
AIAA J. **2** (1964), 1822–1824.

The author extends Southwell's stress function solution of the torsionless axisymmetric problem of isotropic and homogeneous linear elasticity to the case of cylindrical orthotropy with body force and temperature effects. The resulting solution is in terms of two stress functions which must satisfy a pair of coupled second-order partial differential equations. It follows from the method of introduction of the stress functions that the solution is complete in regions which meet certain convexity requirements.
D. E. Carlson (Urbana, Ill.)

Attia, M. S.; Nassif, M. 6695
Transverse bending of a thin circular plate eccentrically loaded and supported along a concentric circle.
Bull. Calcutta Math. Soc. **54** (1962), 131–150.

Complex potentials are used to solve the biharmonic equation for the deflection of an elastic plate, simply supported and subjected to a point load.

For a circular plate supported round a concentric circle, the result is given as a Fourier series. For the simpler case of an infinite plate the result appears in closed form.
J. W. Craggs (Melbourne)

Tomar, J. S. 6696
On flexural vibrations of isotropic elastic thin square plates.
Bull. Calcutta Math. Soc. **55** (1963), 1–10.

Author's summary: "A solution for flexural vibrations of isotropic elastic plates under conditions which are more general than those of the Timoshenko theory given by the author himself is worked out. Here we include the effects of rotatory inertia and shear but do not assume that the displacements are according to a linear law as was assumed by Timoshenko in his one-dimensional theory of bars. Numerical solution for the case of a simply supported thin square plate for various ratios of thickness to the side are obtained and compared with Mindlin's result for the same plate."

Koltunov, M. A. 6697
On the analysis of flexible shallow orthotropic shells with linear heredity. (Russian. English summary)
Vestnik Moskov. Univ. Ser. I Mat. Meh. **1964**, *no.* 5, 79–88.

Author's summary: "The equations are developed and a method is proposed for solving problems dealing with flexible shallow orthotropic shells whose material possesses linear heredity."

La Tegola, Antonio 6698
Premesse alla teoria non lineare per le volte a doppia curvatura di forma qualsiasi. (English summary)
Rend. Accad. Sci. Fis. Mat. Napoli (4) **30** (1963), 181–215.

Concerning the shells of double curvature and referring to the lines of principal curvature, the author deals with the calculus of the strain components and of the first and second order changes of curvature and twist in terms of the displacements of the shell's middle surface. Results are particularized for cylindrical shells and for plane plates. Relations to other results in this direction are discussed. *P. P. Teodorescu* (Bucharest)

Roth, Robert S.; Klosner, Jerome M. 6699
Nonlinear response of cylindrical shells subjected to dynamic axial loads.
AIAA J. **2** (1964), 1788–1794.

This is primarily an extension, to include dynamic effects when the loading is applied suddenly, of the 1950 Donnell and Wan [J. Appl. Mech. 17 (1950), 73–83] static study of the buckling of thin elastic cylinders with imperfections under uniform axial compression. Only radial inertia effects are considered. The initial imperfection is assumed to be a sinusoidal radial deviation with an amplitude of 1, 0.5, 0.1 or 0 times the thickness, and wave lengths the same as the primary term of the buckling deflection. The dynamic critical loading is taken as that at which there is a sudden large increase in the first maximum value, with time, of the amplitude of this primary term. The buckling deflection has four terms like those originated by von Kármán and Tsien [J. Aeronaut. Sci. 8 (1941), 303–312; MR 4, 63] but with time-dependent coefficients, the wave lengths being constant with time and chosen to minimize the critical loading.

The dynamic critical loading is found to have the classical static value for zero initial imperfection and to decrease with increasing imperfections, much as in the 1950 static study. Initial radial velocities, rather than displacements, are found to produce similar effects, but the magnitude of them needed to produce an appreciable effect

leads the authors to conclude that initial imperfections play the dominant role in practical cases. The dynamic critical loading is found to reach a minimum for a duration of loading of $60R(E/\rho)^{-1/2}$, where R is the radius, E is the Young's modulus and ρ is the mass density of the material. Removal of the load before this time progressively increases the critical loading.

L. H. Donnell (Chesterton, Ind.)

Suvernev, V. G. 6700

Small eigen-vibrations of three-layer shells of revolution. (Russian)

Izv. Sibirsk. Otd. Akad. Nauk SSSR Ser. Tehn. Nauk **1964**, *no.* 2, 93–98.

This paper is a generalisation for thick shells of results obtained previously for elastic thin shells by Grigoljuk and Čulkov [Dokl. Akad. Nauk SSSR **150** (1963), 1012–1014]. In the present work each layer is assumed to be transversely isotropic. The special cases of cylindrical and spherical shells are discussed, and the frequencies of vibration corresponding to various values of the geometric and elastic parameters are displayed graphically.

J. Hubert Wilkinson (Battersea)

Nowacki, Witold 6701

★Dynamics of elastic systems.

Translated from the Polish by Henryk Zorski.

John Wiley & Sons, Inc., New York, 1963. 396 *pp.* $15.75.

The aim of this book, which the reviewer believes to have been achieved, is to present "a wide class of dynamics problems occurring in Civil Engineering" in a form suitable for "designers . . . and students, as an auxiliary book in the Theory of Structures". A unified approach to the subject is made in which the dynamics of bar structures forms a link between the vibrations of strings and beams and the vibrations of plates and shells. Problems of wave propagation are discussed in detail, and thermoelastic effects are considered in some problems. Since anelastic effects are of increasing importance in practical analysis, consideration has also been given to materials whose mechanical behaviour is linearly viscoelastic.

The main mathematical tool, which is used throughout the book, is the method of integral transforms. An introduction to Operational Calculus is given in Chapter 13, and fifteen pages of Tables of relevant integral transforms are given in Chapter 14. The mathematical knowledge required of the reader is not beyond that normally taught to undergraduate engineering students.

Chapter 1 is a very readable introduction to the linear theory of elasticity (and thermoelasticity) in which are presented the main concepts: stress and deformation tensors, principal axes, invariants; equations of motion in rectangular and curvilinear coordinates and in terms of displacements; work, energy and uniqueness theorems. A very adequate introduction to linear viscoelasticity is given in Chapter 2. The reviewer regrets that no mention is made of direct numerical solutions of the viscoelastic equations using experimentally determined creep (or other material) functions. Although simple spring-and-dashpot models are useful in giving a qualitative picture of viscoelastic behaviour, their introduction into a numerical solution may be an unnecessary complication.

The remainder of the book is devoted to solutions of particular dynamical problems. Vibrations of strings are treated in Chapter 3, and Chapters 4–6 deal, respectively, with longitudinal, torsional, and transverse vibrations of rods. Free and forced vibrations of continuous beams and of frame structures are considered in Chapters 7 and 8. Chapters 9 and 10 deal with transverse vibrations of plates and shells.

The basic equations of elastic wave propagation are given in Chapter 11, together with the solutions of Rayleigh, Lamb and Pochhammer, the solutions of other elastic wave problems, and the theory of thermoelastic and viscoelastic waves. Chapter 12 deals with approximate methods of analysis for structures.

In summary, the book achieves its main aims, although a set of unworked problems would have increased its worth as a teaching text. It is a worthwhile addition to the existing literature in its field.

F. J. Lockett (Teddington)

Sensenig, Chester B. 6702

Instability of thick elastic solids.

Comm. Pure Appl. Math. 17 (1964), 451–491.

The author investigates stability of equilibrium of several homogeneous isotropic compressible elastic solids using a special form of strain energy function. A non-linear three-dimensional theory of elasticity, equivalent to the theory of small deformations superposed on a large deformation, is used, with no restrictive assumptions concerning the thickness of the bodies. Values of the parameter (critical strain, critical pressure, etc.) are found at which families of buckled solutions branch off from the family of exact simple solutions of the non-linear theory, together with the corresponding modes of buckling.

A. E. Green (Newcastle upon Tyne)

Wilde, Piotr 6703

The thermal buckling of a thin plate in the form of a minimal surface. (Polish and Russian summaries)

Arch. Mech. Stos. **16** (1964), 81–92.

By considering the case of very thin plates, the bending stresses are neglected and the problem of thermal buckling reduces to the determination of a surface. Since the bending stresses are neglected, the plate buckles for any non-linear temperature distribution in the plate's middle surface.

This paper considers surfaces (minimal surfaces) whose mean curvature is zero everywhere, and the corresponding temperature to cause this surface to buckle. A procedure is given by which it is possible to determine from a specified temperature field if such a minimal surface results.

A solution is given for a plate which is heated along one edge, and an experimental verification of the theory is included. The agreement of the analytical predictions and measured values of displacements is encouraging.

P. R. Paslay (Houston, Tex.)

Kaliski, Sylwester 6704a

The Čerenkov radiation in an elastic dielectric contained in a magnetic field. (Polish and Russian summaries)

Proc. Vibration Problems **4** (1963), 215–233.

Kaliski, Sylwester 6704b

Čerenkov radiation in a perfect elastic conductor in a magnetic field of anisotropic action, excited by a moving impulse. (Polish and Russian summaries)

Proc. Vibration Problems **4** (1963), 301–315.

In these two papers the author is interested in information about discontinuity surfaces for elastic media in the state described by the titles. In the linearized theory as applied to an unstressed solid, the steady motion of a point source of energy will result in the presence of Mach cones for dilatation waves if the source velocity V is in the range $V^2>(\lambda+2r)/\rho$ and for both shear and dilatation waves if the velocity is in the range $(\lambda+2\mu)/\rho>V^2>\mu/\rho$. When electromagnetic effects are included in the linearization, the resulting equations indicate the possibility of three types of discontinuity surface, two of them are essentially modifications of the purely elastic Mach cones, and the third is a modification of the Čerenkov cone, set up when a charge moves with sufficiently high speed through a material. The author is interested in finding the degree of coupling between the various effects; in the first paper there is an external magnetic field directed normally to the plane of motion, and in the second it is directed parallel to the plane of motion.

V. M. Papadopoulos (Madison, Wis.)

Kaliski, Sylwester 6705

Absorption of magneto-viscoelastic surface waves in a real conductor in a magnetic field. (Polish and Russian summaries)

Proc. Vibration Problems **4** (1963), 319–330.

Theory of the damping of surface waves on a viscoelastic medium of finite conductivity in a magnetic field parallel to the surface. The resulting dispersion equation is rather complicated: simpler expressions are given for very large or small ratios of mechanical to magnetic viscosity. *L. J. F. Broer* (Eindhoven)

Valanis, K. C.; Lianis, G. 6706

Thermal stresses in a viscoelastic cylinder with temperature dependent properties.

AIAA J. **2** (1964), 1642–1644.

An approximate method of solution is proposed for the transient thermal stress problem of a hollow circular cylinder composed of thermo-rheologically simple viscoelastic material. The approximation consists of employing a constant Poisson's ratio instead of the time-dependent Poisson's ratio (which is a consequence of viscoelastic response in shear and elastic response in dilatation) based on the fact that the deviation of the time-dependent Poisson's ratio from its mean value is small in comparison with the sum of the mean value and unity.

R. Muki (Tokyo)

Zavoĭšinskiĭ, B. I. 6707

On a deformation problem for an elastic-visco-plastic material. (Russian. English summary)

Vestnik Moskov Univ. Ser. I Mat. Meh. **1964**, no. 5, 29–38.

Author's summary: "The paper considers a hypothesis relating the stress and strain components for elastic-visco-plastic bodies. An integral operator is derived for a two-layered cylinder and sphere subjected to internal pressure and variable heat flow."

★Rheological problems in the mechanics of rock strata [Реологические вопросы механики горных пород]. 6708

Edited by Ž. S. Eržanov.

Izdat. Akad. Nauk Kazah. SSR, Alma-Ata, 1964. 156 *pp.* 1.16 *r.*

A collection of 15 papers, some experimental in character, on the topic of the title.

Eržanov, Ž. S. [Ержанов, Ж. С.] 6709

★Theory of creep of rock strata and its applications [Теория ползучести горных пород и ее приложения].

Izdat. "Nauka". Alma-Ata, 1964. 175 *pp.* 0.84 *r.*

Mandel, J.; Parsy, F. 6710

Surfaces caractéristiques des équations de l'équilibre plastique pour un milieu rigide-parfaitement plastique.

J. Mécanique **2** (1963), 313–340.

For a rigid, perfectly plastic material the authors study the characteristic surfaces for three-dimensional stress distributions. It is shown that when the plastic potential is strongly convex, real characteristic surfaces exist only when the deformation is plane. A proof is given that for a material that satisfies Mohr's criterion, real characteristic surfaces exist even for non-plane deformation. A systematic study of this last case is made and applied to problems of symmetry of revolution.

A. Phillips (New Haven, Conn.)

Smith, E. 6711

The spread of plasticity between two cracks. (French, German, Italian and Russian summaries)

Internat. J. Engrg. Sci. **2** (1964), 379–387.

From the author's summary: "A simple model is used to consider the spread of plasticity between two identical coplanar cracks in the interior of an infinite body subject to an applied shear stress which causes the body to deform in an anti-plane strain mode. The applied stress required for the plastic zones from the two cracks to just meet is determined, together with the displacement at the crack tips, and the results are compared with those for an isolated crack and an infinite number of identical and equally spaced coplanar cracks."

Hetnarski, Ryszard B. 6712

The fundamental solution of the coupled thermoelastic problem for small times. (Polish and Russian summaries)

Arch. Mech. Stos. **16** (1964), 23–31.

Transient temperature and thermal stresses in an infinite elastic solid due to a point heat source are considered, taking into account dynamic and coupling terms in the field equations, and the solution limited for small time values is obtained through the usual Laplace transform technique. (The reviewer thinks that this type of coupled thermoelastic problem is already a settled issue.)

R. Muki (Tokyo)

Puri, Pratap 6713

One-dimensional dynamic problems of thermoelasticity. (Polish and Russian summaries)

Arch. Mech. Stos. **16** (1964), 93-102.

The problems treated are shock heating and ramp-type heating of one of the bounding planes of an infinite slab while the other plane is kept at constant temperature. A mistake made in a preceding paper by A. Singh and the author [same Arch. **15** (1963), 77–88; MR **27** #4431] is corrected. The method used is the same as in the earlier paper.

H. Parkus (Vienna)

FLUID MECHANICS, ACOUSTICS

See also 6133, 6199, 6795, 6882, 6883.

Alacevich, Ferruccio 6714

L'energia "migrante" in seno ad un fluido in moto. Sua rappresentazione analitica e sua natura. I, II. (English summary)

Atti Accad. Ligure **19** (1962), 73–83 (1963); *ibid.* **19** (1962), 315–322 (1963).

In Part I the various terms in the Navier-Stokes equations are re-interpreted in terms of "borrowed" and "transmitted" energy. In Part II it is shown that transmitted energy is of a kinetic nature, and these ideas are discussed in relation to some laminar flows.

D. R. Breach (Toronto, Ont.)

Berker, Ratip 6715

Transformation relative aux solutions des équations du mouvement d'un fluide.

C. R. Acad. Sci. Paris **259** (1964), 1295–1298.

Let $\mathbf{v}(x_i, t)$, $p(x_i, t)$ be the velocity and pressure fields arising from a solution of the Navier-Stokes equations. If a rigid body transformation, relative to fixed axes Ox_i, is applied to the system consisting of the fluid and the axes Ox_i, then the velocity field relative to the fixed axes is not in general a solution of the Navier-Stokes equations. For it to be a solution, $\mathbf{v}$ has to satisfy a further equation involving the instantaneous rotation vector for the axes Ox_i. If this equation is satisfied then, by this transformation process, it is possible to derive from $\mathbf{v}$ further solutions of the Navier-Stokes equations. An application to the motion of a solid in a fluid is given.

D. R. Breach (Toronto, Ont.)

Eringen, A. Cemal 6716

Simple microfluids. (French, German, Italian and Russian summaries)

Internat. J. Engrg. Sci. **2** (1964), 205–217.

Author's summary: "The basic field equations, jump conditions and constitutive equations of what we call 'simple microfluent' media are derived and discussed. These fluids are shown to be a generalization of the Stokesian fluids in which local micro-motions are taken into account. Special cases in which gyrations are small and micro-deformation rates are linear are discussed. The partial differential equations of the constitutively linear theory are obtained."

R. A. Toupin (Yorktown Heights, N.Y.)

Pareschi, Mirella 6717

Su una dimostrazione del teorema di unicità nel moto dei fluidi.

Boll. Un. Mat. Ital. (3) **19** (1964), 206–207.

Si dimostra che il procedimento usato dal recensore [Ann. Mat. Pura Appl. (4) **50** (1960), 379–387; MR **22** #12924] e dal Serrin [*Nonlinear Problems* (Proc. Sympos., Madison, Wis., 1962), pp. 69–98, Univ. Wisconsin Press, Madison, Wis., 1963; MR **27** #442] per dimostrare il teorema di unicità relativo alle equazioni del moto dei fluidi (in un dominio illimitato) è valido solo ammettendo la pressione infinitesima all'infinito di ordine maggiore di $\frac{1}{2}$.

D. Graffi (Bologna)

Smirnov, N. V. 6718

Conical flows of an ideal incompressible fluid. (Russian. English summary)

Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. **19** (1964), *no.* 2, 107–115.

Solutions are found for the incompressible potential flow of a fluid past a rigid half-cone. At large distances from the axis of the cone the flow behaves like a uniform stream, in planes perpendicular to the axis, plus a non-zero axial component. If the cone is slender (i.e., small included angle), the axial component vanishes. In addition, there is a brief discussion of the flow past a cone, the two halves of which include different angles, and of the class of flows which are independent of the radial distance from the vertex.

J. N. Newman (Washington, D.C.)

Uldrick, J. P.; Siekmann, J. 6719

On the swimming of a flexible plate of arbitrary finite thickness.

J. Fluid Mech. **20** (1964), 1–33.

The swimming of sea animals as simulated by an infinitely thin two-dimensional waving plate is treated by developing a complex velocity potential method and taking into account the thickness of the animal. The measure of the thickness enters through the conformal mapping function which transforms the profile of the fish onto the unit circle. For the basic configuration the symmetric Joukowski profile is employed to represent the profile of the fish. The thrust is assumed to be generated by displacements forming a train of travelling waves of small amplitude which pass down the body of the fish, and the envelope of these waves varies arbitrarily along the length of the fish. The amplitude of these displacement waves is taken as a harmonic function of time. The unsteady boundary conditions are satisfied by introducing a source distribution on the circle, and the problem is linearized by assuming a small unsteady perturbation theory. The velocity induced by the source distribution possesses a singularity in the physical plane, and this is removed by introducing a fluctuating vortex distribution along the wake streamline of the steady base flow in such a way that the induced velocities of the source and vortex distributions combined vanish at the wall. From the base, source and vortex potentials the pressure is calculated from Bernoulli's equation. Numerical results are given from which the authors make the following observations: (1) The thickness of two-dimensional fish tends to reduce the available thrust generated by the swimming motion; (2) The

thickness effect is more pronounced at higher reduced frequencies than at smaller ones; (3) The argument of the Theodorsen function is increased by an amount depending upon the thickness of the fish (this results from the slowing-up effect of fluid particles in the wake); (4) The present theory yields identical results for the lift-moment and thrust with existing thin-plate theories when the thickness parameter vanishes.

K. B. Ranger (Toronto, Ont.)

Basu, Jayati 6720

Unsteady waves due to disturbances originating at the surface of liquid of uniform depth.

Indian J. Math. **6** (1964), 45–49.

Assuming linearized boundary conditions and an inviscid heavy fluid, the author derives the velocity potential for motion resulting when a free surface is subject to a time-dependent pressure distribution of either the form $p(x,z,t)=p(x)f(t)$ or $p([x^2+z^2]^{1/2})f(t)$. Although the results are known, she believes her derivation to be more straightforward than the usual ones.

J. V. Wehausen (Berkeley, Calif.)

Cuthbert, Jerry W. 6721

Integral hull form parameters and a high-speed approximation to Michell's integral.

J. Ship Res. **7** (1963/64), *no.* 3, 12–15.

The author attempts to find an asymptotic expression for large values of the Froude number for Michell's integral for the wave resistance of a thin ship. The answer is in error because of an unallowable interchange of order of integration and summation. The correct expression is given in a paper by J. N. Newman [same J. 8 (1964/65), no. 1, 10–14].

J. V. Wehausen (Berkeley, Calif.)

Long, Robert R. 6722

The initial-value problem for long waves of finite amplitude.

J. Fluid Mech. **20** (1964), 161–170.

From the author's summary: "A set of four partial differential equations that govern the propagation of an arbitrary, long wave disturbance of small but finite amplitude are derived. The equations reduce to that of Boussinesq when the assumption is made that the disturbance is propagating in one direction only. The equations are hyperbolic with characteristic curves of constant slope. The initial-value problem can be solved very rapidly by numerical integration along characteristics. Some examples are given."

D. G. Hurley (Nedlands)

Rubatta, Antonello 6723

L'onda cilindrica generata da un battente piano.

Atti Accad. Sci. Ist. Bologna Cl. Sci. Fis. Rend. (11) **10** (1962/63), *no.* 1, 103–118.

The author considers the gravity waves generated by a plane wave-maker at one end of a tank of rectangular cross-section of depth h and whose motion is described by

$$x=f(y,t)=(E_1+F_1y/h)\sin kt+(E_2'+F_2'y/h)\sin 2kt+(E_2''+F_2''y/h)\cos 2kt.$$

The tank extends to infinity and no incoming waves are allowed. The author formulates the problem for an inviscid fluid, proposes solving it approximately by a perturbation series and writes out the conditions to be satisfied by the first two terms. He then assumes irrotationality and solves explicitly through the second order. The asymptotic form of the surface is given by

$$\begin{aligned} y=\eta(x,t) &= h+(E_1+F_1)A_0\cos(kt-m_0x) \\ &+(E_1+F_1)^2A_0^2\frac{m_0^2g}{4k^2}\left(\frac{3m_0^2g^2}{k^4}-1\right)\cos 2(kt-m_0(k)x) \\ &+\{(E_1+F_1)^2B_0'+(E_2'+F_2')A_0'\}\cos(2kt-m_0(2k)x) \\ &+\{(E_1+F_1)^2B_0''-(E_2''+F_2'')A_0''\}\sin(2kt-m_0(2k)x), \end{aligned}$$

where $m_0(k)$ is defined by $k^2=m_0g\tanh m_0h$. Graphs are given which allow computation of A_0, A_0', A_0'', B_0', B_0'' as functions of appropriate parameters. The author calls attention to the fact that the part of the velocity potential associated with the last two terms above contributes a mass transport just cancelling the mass transport known to be associated with the third term. It is this which seems to distinguish the present treatment of the problem from a much more thorough one of Fontanet [Houille Blanche **16** (1961), 3–31; ibid. **16** (1961), 174–197; MR **22** #B107], who imposes a vorticity distribution which just cancels mass transport locally.

J. V. Wehausen (Berkeley, Calif.)

Bhatnagar, P. L. 6724

Superposability and harmonic analysis of a viscous liquid in the presence of magnetic fields.

Calcutta Math. Soc. Golden Jubilee Commemoration Vol. (1958/59), *Part I, pp.* 205–216. *Calcutta Math. Soc., Calcutta,* 1963.

Incompressible magnetohydrodynamic flows with constant viscosity ν and finite, constant electrical conductivity σ are introduced in the first section of this paper. The two remaining sections, which are totally independent of one another, discuss, respectively, the superposability and the harmonic analysis of such flows, generalizing the purely hydrodynamic results of Ballabh [Proc. Banaras Math. Soc. (N.S.) **2** (1940), 69–79; MR **3**, 283] in the first case and of Kampé de Fériet [Quart. Appl. Math. **6** (1948), 1–13; MR **9**, 631] in the second.

Two flows of the same fluid are called superposable if there exists a third flow of this fluid whose velocity vector V, magnetic field vector $H=\mu^{-1}B$, and (scalar) potential Ω of external forces are each equal to the sum of the corresponding quantities for the two original flows. The following pair of vector equations is found to be necessary and sufficient for the flows indicated by the subscripts 1 and 2 to be superposable:

$$\operatorname{curl}[V_1\times B_2+V_2\times B_1]=0,$$

$$\operatorname{curl}[V_1\times\bar\omega_2+V_2\times\bar\omega_1+(4\pi\rho)^{-1}\{(H_1\cdot\nabla)B_2+(H_2\cdot\nabla)B_1\}]=0.$$

Here ρ denotes the fluid density and $\bar\omega$ the vorticity. Three special cases, involving flows which are irrotational and/or free from magnetic field, are discussed in detail.

The section on harmonic analysis is much less fruitful, being confined mostly to straightforward Fourier transformations of the flow equations and derivation of the

spectral function $\gamma(\bar{k}, t)$ for the kinetic energy (K.E.). The inequality

$$\gamma(\bar{k}, t) \leq (\tau/8\pi^3)\,(\mathrm{K.E.})$$

is obtained; τ denotes the total volume of the flow region. The special case of a flow filling all of space is treated separately; the author writes down the transform equations and suggests an iterative procedure by which they may possibly be solved.

H. C. Kranzer (Garden City, N.Y.)

Bretherton, F. P. 6725

Inertial effects on clusters of spheres falling in a viscous fluid.

J. Fluid Mech. **20** (1964), 401–410.

Author's summary: "A compact cluster of 3 to 6 rigid equal spheres is falling under gravity in a viscous liquid. The small effects of inertia on a horizontal regular polygonal configuration are that the polygon expands as it falls and small perturbations from this configuration die out, although when the polygon is large enough it becomes weakly unstable. This is an extension of the analysis of Hocking [same J. **20** (1964), 129–139] which was applied to the experiments of Jayaweera, Mason and Slack [ibid. **20** (1964), 121–128]."

Childress, Stephen 6726

The slow motion of a sphere in a rotating, viscous fluid.

J. Fluid Mech. **20** (1964), 305–314.

From the author's summary: "The uniform, slow motion of a sphere in a viscous fluid is examined in the case where the undisturbed fluid rotates with constant angular velocity Ω and the axis of rotation is taken to coincide with the line of motion. The various modifications of the classical problem for small Reynolds numbers are discussed. The main analytical result is a correction to Stokes's drag formula, valid for small values of the Reynolds number and Taylor number and tending to the classical Oseen correction as the last parameter tends to zero. The rotation of a free sphere relative to the fluid at infinity is also deduced."

Reviewer's additional comments: The alteration of the flow field in the wake due to the Coriolis force is discussed. An interesting feature of the analysis is the use of a three-dimensional Fourier transform to find a solution of Oseen's equation (with a Coriolis term included) when there is a concentrated force located at the origin.

R. C. Ackerberg (Farmingdale, N.Y.)

Kapur, J. N.; Shukla, J. B. 6727

On the unsteady flow of two incompressible immiscible fluids between two plates.

Z. Angew. Math. Mech. **44** (1964), 268–269.

The two fluids (which are also viscous) occupy the regions $0 \leq y \leq h$ and $-h \leq y \leq 0$ and move in the x-direction under the same pressure gradient Pe^{at}. The velocity profile is determined, and the flux, the skin frictions at the plates $y = \pm h$, and the maximum velocity are then deduced.

C. R. Illingworth (Manchester)

Kumar, Ram; Warsi, Z. U. A. 6728

Steady motion of viscous compressible fluid through circular and co-axial pipes.

Bull. Calcutta Math. Soc. **55** (1963), 131–137.

Authors' summary: "In this paper the problems of the steady flow of a perfect gas through circular and co-axial pipes have been studied by the method of finite Hankel-transform, taking into account viscosity and conductivity. The form of ω follows from the necessity of the consistency of the equations. It has been found that the adiabatic flow is possible but either μ has to be assumed variable or Stokes' condition has to be modified." {It seems to the reviewer that the expressions given for the velocity do not satisfy the conditions imposed by the authors across the end of the tube.}

D. R. Breach (Toronto, Ont.)

Peube, Jean-Laurent 6729

Sur l'écoulement radial permanent d'un fluide visqueux incompressible entre deux plans parallèles fixes.

J. Mécanique **2** (1963), 377–395.

In terms of cylindrical polar coordinates r, θ, y the axially-symmetric radial viscous flow takes place between two fixed planes $y = \pm h$, and the r and y components of velocity are expressed in the forms $\sum_{n=1}^{\infty} f_n(y)r^{-n}$ and $\sum_{n=1}^{\infty} g_n(y)r^{-n}$, while the pressure is written as $h_0(y)\ln r + \sum_{n=1}^{\infty} h_n(y)r^{-n}$. The terms up to f_5, g_6, and h_4 are determined, and the resulting velocity and pressure distributions are discussed, particularly in relation to the case of a parabolic radial velocity profile.

C. R. Illingworth (Manchester)

Saffman, Philip G. 6730

The displacement of a viscous fluid from a porous medium.

Proc. 1963 *Heat Transfer and Fluid Mech. Inst.* (*Calif. Inst. Tech., Pasadena, Calif.*, 1963), pp. 176–182. *Stanford Univ. Press, Stanford, Calif.*, 1963.

The paper deals with the flow through a porous medium when a fraction of the fluid being displaced is left behind but is not immobile. If this fraction is constant, the interface moves as if it separated two fluids of modified viscosity, density and interfacial tension, one of which completely expels the other. Assuming that when the macroscopic interface advances the fluid is left behind by trapping and deposition, the author calculates the way the fraction depends on the velocity of the interface according to the model of a porous medium as a random network of capillaries. At the end it is shown that the slower the motion, the smaller is the amount of fluid left behind.

Şt. I. Gheorghiţă (Bucharest)

Sparrow, E. M.; Lin, S. H.; Lundgren, T. S. 6731

Flow development in the hydrodynamic entrance region of tubes and ducts.

Phys. Fluids **7** (1964), 338–347.

A new method has been devised for determining the developing incompressible laminar flow and corresponding pressure drop in the entrance region of tubes and ducts. In a cylindrical duct parallel to the x-axis the x-momentum equation

$$\mathbf{v}\cdot\nabla u = -\partial(p\rho^{-1})/\partial x + \nu\nabla^2 u \tag{1}$$

is simplified by standard assumptions that $p=p(x)$ and $\partial^2u/\partial x^2$ is negligible relative to $\nabla^2 u$ and the novel linearization $\mathbf{v}\cdot\nabla u=s(x)U\partial u/\partial x$, where U is the constant mean velocity and $s(x)$ remains to be determined. With these simplifications and other minor manipulation, (1) yields

$$(2)\qquad U\partial u/\partial x^* + \nu A^{-1}\oint_C (\partial u/\partial n)ds = \nu\nabla_1{}^2 u,$$

where A is the cross-sectional area of the tube, C its boundary, $dx=s(x)dx^*$, and $\nabla_1{}^2=\partial^2/\partial y^2+\partial^2/\partial z^2$. The desired solution of (2) is expressed as

$$u = u_{fd}+u^*,$$

where u_{fd} is the fully-developed axial velocity. The correction u^* is expanded in an eigenfunction series of product solutions $e^{-\alpha_i{}^2x^*}g_i(y,z)$ of (2). The function $s(x)$ is determined by insisting that $\partial p/\partial x$ shall have the same value (despite all of the preceding simplifications) when determined by both momentum and mechanical energy considerations. Extensive computations and comparisons with results of earlier investigators have been made for circular tubes and parallel plate channels.

J. H. Giese (Aberdeen, Md.)

Ahuja, G. C. 6732

Boundary layer on a yawed semi-infinite flat plate in converging and diverging flows with suction and injection. (German and Russian summaries)

Z. Angew. Math. Mech. **44** (1964), 301–313.

A modified Pohlhausen method is used to calculate the development of the laminar boundary layer on a yawed semi-infinite flat plate in either a particular converging or diverging flow with normal suction or injection at the surface. To simplify the analysis only a particular distribution of normal suction or injection is considered. The results include skin friction and separation position and are displayed graphically.

D. G. Hurley (Nedlands)

Ermak, Ju. N.; Neiland, V. Ja. 6733

On the theory of the three-dimensional laminar boundary layer. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 950–954.

First the authors indicate the difficulties of the three-dimensional boundary layer theory. Then by using $ds^2=dh_1{}^2dx^2+dh_2{}^2dy^2+dz^2$ and $H=c_pT+\frac{1}{2}(u^2+v^2)$ and writing the equations for u, v, p and H, they obtain in a certain case a generalization of Crocco's integral in the form $H=Au+Bv+C$ (A, B and C constants). At the end they consider cases when the three-dimensional problem reduces to the two- or one-dimensional problem.

St. I. Gheorghiţă (Bucharest)

Fife, Paul 6734

A note on exterior Dirichlet problems and an application to boundary layer theory.

J. Differential Equations **1** (1965), 95–113.

The motivating interest of this paper lies in the following problem: Given a simple closed curve Γ in the (x,y)-plane, to find a solution of the Prandtl (boundary-layer) equations corresponding to prescribed tangential velocity $w(\tau)$ on Γ and suitably matched with a potential flow defined in the exterior e of Γ and vanishing at infinity. The matching is accomplished by means of the pressure $p(\tau)$ arising from the (multivalued) potential solution $\varphi(x,y)$ of the exterior Neumann problem, with the condition $\nabla\varphi\to 0$ at infinity. This solution contains an arbitrary multiplicative constant μ, which then appears in the boundary-layer equation. The author shows that under certain restrictions on $w(\tau)$ and on Γ, this simultaneous problem admits to first-order approximation a solution pair $[w,\mu]$ such that $w(\tau)$ is periodic with period equal to the circumference of Γ. The author remarks that a general theorem on existence of periodic solutions of parabolic equations could be obtained by the same method.

{The interpretation of the author's result in terms of the strict theory of the Navier-Stokes equations is not clear. The author seems to have the impression that the existence of strict solutions has been proved. However, the known existence theorems—including those to which the author refers—apply only to flows in dimension $n=3$. The case $n=2$ is, in the reviewer's opinion, one of the basic open questions of the theory. Nevertheless, the reviewer had arrived independently at the view that a problem similar to the one studied in this paper should be accessible from the standpoint of finding a strict solution of the hydrodynamical equations. It remains to be seen how these ideas will develop, and in what ways the results will relate to those of the author.} *R. Finn* (Stanford, Calif.)

DiPrima, Richard C.; Pan, Coda H. T. 6735

The stability of flow between concentric cylindrical surfaces with a circular magnetic field. (German summary)

Z. Angew. Math. Phys. **15** (1964), 560–567.

From the authors' summary: "The stability problem for Taylor flows with circular magnetic field has been formulated. The well-known results have been supplemented and described briefly. New results corresponding to the stability of Dean flows with and without conducting walls have been found. The eigenvalue problem has been solved by means of the Galerkin method."

M. N. L. Narasimhan (Bombay)

Riley, N. 6736

The heat transfer from a rotating disk.

Quart. J. Mech. Appl. Math. **17** (1964), 331–349.

Author's summary: "The heat transfer from a rotating disk maintained at a constant temperature is considered for large and small values of the Prandtl number. On the assumption that there is a linear relationship between the viscosity and the temperature the momentum equations are reduced to their incompressible form, and attention is then concentrated upon the energy equation. The effects of wall heating and heating due to viscous dissipation are considered separately, and the heat transfer is expressed as a series in terms of the Prandtl number for both large and small Prandtl numbers."

Upon reading the paper it is found that this summary is precise and that the paper is an interesting and scholarly one citing several related and pertinent references. Comparisons between asymptotic and exact solutions are given as a function of fluid Prandtl number and once again the usefulness and validity of asymptotic solutions to the boundary layer equations are illustrated.

W. H. Dorrance (Ann Arbor, Mich.)

Selig, F. 6737

Variational principle for Rayleigh-Taylor instability.

Phys. Fluids 7 (1964), 1114–1116.

In this paper, the author presents a new variational formulation for the stability problem of superposed fluids.

An eigenvalue problem is formulated by the usual technique of analyzing the disturbance into normal modes. By grouping the terms in a slightly different way from what was done previously by other authors, the author obtains the following formula:

$$-n_i \int_L \rho\left[w_i w_j + \frac{1}{k^2}(Dw_i)(Dw_j)\right] dz + \frac{g}{n_i}\int_L (D\rho) w_i w_j \, dz =$$

$$\int_L \frac{\mu}{k^2}\{[(D^2+k^2)w_i][(D^2+k^2)w_j] + 4k^2(Dw_i)(Dw_j)\}\, dz.$$

This formula enables the author to establish that overstability cannot occur (without the restriction $D^2\mu > 0$).

The variational formulation of the problem is then obtained by putting $i = j$ in the above formula. The author also shows that the variational principle expresses an energy balance. *W. Lai* (Troy, N.Y.)

Sparrow, E. M.; Munro, W. D.; Jonsson, V. K. 6738

Instability of the flow between rotating cylinders: The wide-gap problem.

J. Fluid Mech. 20 (1964), 35–46.

Consideration is given to cylinders rotating in the same direction and in opposite directions, with radius ratios ranging from 0.95 to 0.1. The stability problem is solved by a method based on direct numerical integration of the differential equation for the disturbance amplitude function. Detailed numerical results are presented for the critical Taylor numbers and wave-numbers. These results are used to determine the range of applicability of the closed-form predictions of Taylor [Philos. Trans. Roy. Soc. London Ser. A **223** (1923), 289–343] and Meksyn [*New methods in laminar boundary-layer theory*, Pergamon, New York, 1961], which were derived for narrow-gap conditions. *D. W. Dunn* (Ottawa, Ont.)

Sastry, U. A. 6739

Heat transfer by laminar forced convection in a pipe of curvilinear polygonal section.

J. Sci. Engrg. Res. 7 (1963), 281–292.

The author treats the steady, fully developed laminar flow in a pipe of polygonal cross-section using a complex variable method developed by N. I. Muskhelishvili [*Some basic problems of the mathematical theory of elasticity*, Noordhoff, Groningen, 1963]. The cases worked out in detail include the heat transfer in a pipe of hexagonal cross-section. *L. J. Crane* (Dublin)

Stuart, J. T. 6740

On the cellular patterns in thermal convection.

J. Fluid Mech. 18 (1964), 481–498.

A detailed examination is made of the problem of describing what is actually observed in a thermal convection experiment. The existing rough classification of the cellular patterns on the basis only of their periodicity is sharpened by the introduction of further criteria for deciding which streamlines are singled out by the eye (or camera). Since no one has ever seen a thermal convection cell of infinitesimal amplitude, it seems appropriate that features of known approximations for the finite-amplitude motions are employed. The author points out that his criteria indicate that we may "see" cells of hexagonal outline but not of square or rectangular outline. The distinction between the hexagonal case and the others appears to rest upon the existence of a certain positive definite feature of the linearized solutions for the former, corresponding examples of which are merely non-negative definite in the latter. This appears to be a rather fine point in the scheme of the complicated and approximate theory of finite-amplitude convection. Furthermore, in advocating his argument for the result that hexagonal cells enjoy a preferred status insofar as mere visibility is concerned, the author has forgotten to convince us of the result; we wish he had included some photographs or some further experimental evidence to show that Rayleigh's square cell is not a possible convection cell. *F. E. Bisshopp* (Providence, R.I.)

Butler, Terence 6741

A mathematical example by Hopf with features of turbulence.

J. Math. Anal. Appl. 9 (1964), 215–233.

L'auteur considère l'équation fonctionnelle suivante:

$$(1) \qquad w_t = T(w) - L(w) + \mu w_{xx}, \qquad \mu > 0,$$

où

$$T(w) = -\frac{\pi^2}{4}\int_0^{2\pi}\int_0^{2\pi} w(y)w(y')\bar{w}(y+y'-x)\,dy\,dy',$$

et $L(w) = F * w$ est la convolution de w par une distribution F définie sur $(0, 2\pi)$, et dont les coefficients de Fourier sont $o(n^2)$. $w = u + iv$ est une inconnue complexe, fonction de x, t. u et v jouent le rôle des composantes de la vitesse dans un écoulement visqueux bidimensionnel. L'équation (1) est un modèle des équations de l'hydrodynamique, qui comporte les principales particularités des équations véritables, mais qui est plus accessible. Si $F = \sum (a_n - ib_n)e^{inx}$, la fonction

$$W(x, t) = \sum (a_n - n^2\mu)^{1/2} \exp i(nx + \psi_n), \qquad \psi_n = b_n t + c_n,$$

où la somme est étendue aux N valeurs de n pour lesquelles $a_n > n^2\mu$, est solution de (1) quels que soient les c_n. On peut associer à toute solution $w(x, t)$ (dont les coefficients de Fourier ne sont pas tous nuls pour $t = 0$) une fonction $W(x, t)$ telle que $w - W$ et $w_x - W_x$ tendent vers 0 lorsque $t \to \infty$, uniformément en x.

Soit $\mathscr{F}$ une fonctionnelle continue dans l'espace $-w$, normé par $\|w\|_1 = \sup_x |w(x)| + \sup_x |w_x(x)|$, des fonctions périodiques en x ayant deux dérivées continues. Si les b_n sont linéairement indépendants sur les entiers, on a

$$\bar{\mathscr{F}} = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathscr{F}(W(t))\,dt$$

$$= \int_0^{2\pi} \cdots \int_0^{2\pi} f(\psi_{n_1}, \psi_{n_2}, \cdots, \psi_{n_N}) \prod \frac{d\psi_{n_i}}{2\pi},$$

où f représente $\mathscr{F}(W(t))$, considéré comme fonction de $\psi_{n_1}, \cdots, \psi_{n_N}$.

La moyenne temporelle $\bar{\mathscr{F}}$ de la fonctionnelle $\mathscr{F}(W(t))$

se représente ainsi par une moyenne statistique $\int \mathcal{F}\, dP$ étendue à une variété de dimension finie de l'espace $-w$. On va étudier plus particulièrement les moments de la vitesse en k points et leur comportement lorsque $\mu \to 0$.

On peut poser $w = u + iv$, $u = \sum_{a_n > n^2\mu} u_n$, $v = \sum_{a_n > n^2\mu} v_n$, avec

$$u_n = (a_n - n^2\mu)^{1/2} \cos(\psi_n + nx),$$

$$v_n = (a_n - n^2\mu)^{1/2} \sin(\psi_n + nx).$$

Ensuite, on considère dans R_{2k} le vecteur $\{u(x_1), v(x_1), \cdots, u(x_k), v(x_k)\}$ qui est une somme de vecteurs indépendants dont le nombre augmente indéfiniment quand $\mu \to 0$.

On associe au vecteur $\{u_n(x_1), v_n(x_1), \cdots, u_n(x_k), v_n(x_k)\}$ sa projection normée $X_n(\mu)$ sur un vecteur arbitraire $\{\lambda_1, \cdots, \lambda_{2k}\}$. Les $X_n(\mu)$ sont des variables aléatoires indépendantes, et on exprime la variance de leur somme. On en déduit une condition pour que la somme $\sum X_n(\mu)$ ait, quand $\mu \to 0$, une distribution limite normale, puis une condition pour qu'il en soit de même de la loi de probabilité jointe des $2k$ variables aléatoires $u(x_1), v(x_1), \cdots, u(x_k), v(x_k)$ (préalablement normées). La condition essentielle est que

$$\lim_{\mu \to 0} \left[\max_{a_n > n^2\mu} \frac{a_n - n^2\mu}{\sum \frac{1}{2}(a_n - n^2\mu)} \right] = 0.$$

Elle est liée aux propriétés de la distribution F. On indique un procédé de construction des a_n qui satisfait à la condition ci-dessus.

Si les points x_i sont fixes, les coefficients de corrélation entre les u ou les v en deux points distincts tendent vers 0 avec μ, et l'on peut trouver des suites a_n pour lesquelles ce résultat reste vrai, même si les x_i dépendent de μ, à condition que $x_i - x_j$ appartienne à un intervalle fermé intérieur à $(0, 2\pi)$. On peut au contraire chercher des conditions pour que ces coefficients de corrélation soient indépendants de μ. On définit des suites a_n pour lesquelles il en est ainsi, à condition que les points x_i varient en fonction de μ (ou du nombre N), de telle sorte que les $N(x_i - x_j)$ soient asymptotiquement constants: cela a lieu par exemple si $a_n = n^s$ $(s < 2)$ pour $n > 0$, et $a_n \leq 0$ pour $n \leq 0$. Si $N(x_i - x_j) \to \beta$, on trouve dans ce cas que les coefficients de corrélation de question sont à la limite de la forme $\cos t\beta$, $0 < t < 1$.

J. Bass (Paris)

Deardorff, J. W. 6742

A numerical study of two-dimensional parallel-plate convection.

J. Atmospheric Sci. **21** (1964), 419–438.

The Boussinesq equations for two-dimensional convection between horizontal plates are integrated numerically for Rayleigh number 6.75×10^5 and Prandtl number 0.71. The numerical method uses a grid in physical space and finite-difference equations which preserve exactly a restricted form of energy conservation. The results for the total heat transport agree qualitatively with experiment but are quantitatively too large. Sharp boundary-layering occurs, but there is no breakdown into small turbulent eddies. This is ascribed (correctly, in the reviewer's opinion) to the assumption of two-dimensional motion. Detailed graphical results are presented for the flow and temperature fields resulting from several initial conditions.

R. H. Kraichnan (Peterborough, N.H.)

★**Mécanique de la turbulence.** 6743

Marseille, 28 août–2 septembre, 1961. Colloques Internationaux du Centre National de la Recherche Scientifique, No. 108.

Éditions du Centre National de la Recherche Scientifique, Paris, 1962. 470 *pp.* 60 *F.*

The papers of mathematical interest will be reviewed individually.

Kraichnan, Robert H. 6744

Approximations for steady-state isotropic turbulence.

Phys. Fluids 7 (1964), 1163–1168.

The direct interaction hypothesis for stationary, isotropic turbulence is formulated in the wave-number, frequency domain, rather than in the wave-number, time domain, used previously. The resulting equations are simpler, but less complete in their specification of the statistical properties of the turbulence, than in the original formulation. They involve the wave-number energy spectrum, a correlation frequency for each wave-number and the characteristic response frequency for each wave-number. These equations exhibit some similarities with those derived by Edwards [J. Fluid Mech. 18 (1964), 239–273], who used a Fokker-Planck type of approximation; these (and the important differences) are discussed.

O. M. Phillips (Baltimore, Md.)

Kraichnan, Robert H. 6745

Mixed Lagrangian-Eulerian approach to turbulent dispersion.

Phys. Fluids 7 (1964), 1717–1719.

On considère un champ scalaire turbulent $\psi(x, t)$ qui diffuse suivant l'équation

$$(1) \qquad \frac{\partial \psi}{\partial t} = -u \cdot \nabla \psi,$$

où u est un champ de vitesse incompressible. Si $\xi(x, t|s)$ est le déplacement des particules entre les instants s et t, on a

$$\psi(x, t) = \psi[x - \xi(x, t|0), 0]$$
$$= \exp[-\xi(x, t|0) \cdot \nabla]\psi(x, 0).$$

On prend ensuite les moyennes, en supposant ψ et u aléatoires et indépendants et l'on obtient les diverses corrélations spatio-temporelles de ψ. Puis on suppose que la vitesse u^L des particules (au sens de Lagrange) est normalement distribuée. On trouve par exemple

$$\langle \psi(x, t) \rangle = \exp \tfrac{1}{2}\Xi_{ij}(x, t; x, t|0, 0)\nabla_i \nabla_j,$$

où $\Xi_{ij} = \langle \xi_i(x, t|s)\xi_j(x', t'|s') \rangle$ s'obtient par intégration du tenseur de corrélation $U_{ij}^L(x, t, x', t'|r, r')$ du vecteur u^L. Deux expressions approchées de U_{ij}^L sont données et discutées. L'expression des corrélations de ψ, grâce à l'hypothèse de normalité, est calculable. Les exemples de $\psi(x, 0) = \delta^3(x)$ et $\langle \psi(x, 0) \rangle = 0$,

$$\langle \psi(x, 0)\psi(x', 0) \rangle = \cos[k \cdot (x - x')]$$

sont examinés.

Ces méthodes sont étendues à l'équation

$$(\partial/\partial t - k\nabla^2)\psi = u \cdot \nabla \psi + f,$$

qui contient un terme de diffusion moléculaire, et un

terme extérieur f indépendant de u. La diffusion moléculaire est représentée par un champ de vitesse complémentaire homogène, isotrope, stationnaire, dont la longueur et le temps de corrélation sont très courts, et entre comme un terme correctif dans les formules ci-dessus.

Quelques suggestions sont ensuite données sur l'application de ces méthodes aux équations de Navier-Stokes et sur la valeur des hypothèses de normalité.

J. Bass (Paris)

Kraichnan, Robert H. 6746

Diagonalizing approximation for inhomogeneous turbulence.

Phys. Fluids **7** (1964), 1169–1177.

The author extends his promising theory to turbulent heat convection between horizontal plates and to channel flow. The off-diagonal correlations and the triple correlations are determined by a procedure resembling the direct interaction scheme developed by the author. A complicated but closed system of equations is obtained; the solutions are not yet available. Applicability and limitations are discussed. *R. Betchov* (Los Angeles, Calif.)

Lumley, J. L. 6747

The mathematical nature of the problem of relating Lagrangian and Eulerian statistical functions in turbulence. (French summary)

Mécanique de la Turbulence (*Marseille*, 1961), *pp.* 17–26. *Éditions Centre Nat. Recherche Sci., Paris*, 1962.

Discussion sur les rapports entre les concepts d'homogénéité, isotropie, stationnarité et incompressibilité envisagés du point de vue de Lagrange et du point de vue d'Euler dans un écoulement turbulent, suivie de quelques indications sur l'étude statistique des trajectoires de points. *J. Bass* (Paris)

Novikov, E. A. 6748

The relative motion of liquid particles in turbulent flow.

Izv. Akad. Nauk SSSR Ser. Geofiz. **1964**, 426–429 (*Russian*); *translated as Bull.* (*Izv.*) *Acad. Sci. USSR Geophys. Ser.* **1964**, 258–260.

This paper is a continuation of previous work by the author [Ž. Èksper. Teoret. Fiz. **44** (1963), 2159–2168; MR **28** #1846] in which a method for the statistical description of relative particle motion was proposed. In the present work the third moments for relative motion of two particles are calculated, and a method for calculation of higher-order moments for a system of N particles is suggested. Turbulence is assumed to be locally homogeneous and locally isotropic. Only the inertial time interval, during which viscosity effects are assumed negligible, is considered. It is found that the mean cube of the separation distance of the two particles is proportional to the cube of the time for large times.

R. G. Deissler (Cleveland, Ohio)

Townsend, A. A. 6749

Remarks on the Malkus theory of turbulent flow. (French summary)

Mécanique de la Turbulence (*Marseille*, 1961), *pp.* 167–180. *Éditions Centre Nat. Recherche Sci., Paris*, 1962.

This paper presents an outline of the Malkus theory of turbulent flow and the experimental evidence supporting the theory. Malkus's concepts are applicable to stationary flows, statistically homogeneous on planes parallel to the bounding surfaces, and are based on four principles: (1) The mean velocity and temperature fields can approach, but never exceed, the condition for marginal stability of an inviscid fluid; (2) There is a smallest scale of motion which contributes to the transport of momentum or of heat; (3) This smallest scale is that scale of motion which is marginally stable on the mean field; (4) The flow satisfies an integral condition, like the maximization of the rate of energy release for a fixed volume of channel flow.

One of the main theoretical interests of the author's discussion, in the case of channel flow, is the distinction—clearly made for the first time—between assertions 1 and 2, on one side, which allow the prediction of the form of the velocity distribution, and 3 and 4, on the other side, which are required to determine the scale. (Using assertions 1 and 2 alone, together with a simple assumption very close to 4, the author actually succeeded in determining the velocity profile qualitatively.) The discussion of the experimental evidence is very thorough and convincing. The Malkus predictions for heat convection are qualitatively in agreement with measurements of mean temperature in the laboratory and in the atmosphere, which is not true of the similarity predictions. For Poiseuille flow, the advantage of the Malkus theory is to provide a numerical prediction which is in very fair agreement with experiments.

In two additional sections, the author discusses the relation of Malkus's concepts to the concepts of the statistical theory and the possibility of describing developing flows by a modified form of the theory.

J. C. J. Nihoul (Cambridge, England)

Walz, A. 6750

Compressible turbulent boundary layers (with discussion). (French summary)

Mécanique de la Turbulence (*Marseille*, 1961), *pp.* 299–352. *Éditions Centre Nat. Recherche Sci., Paris*, 1962.

Expository article. *L. S. G. Kovasznay* (Baltimore, Md.)

Yalin, M. S. 6751

On the velocity distribution in turbulent shear flow.

Z. Angew. Math. Mech. **43** (1963), 561–563.

The author derives logarithmic velocity distributions for smooth and for fully rough passages with radially varying shear stress. The assumption is made that velocity is a function of the size of the cross-section, the size of the roughness element, kinematic viscosity, friction velocity, and distance from wall. When the passage is fully rough, the kinematic viscosity is assumed to drop out of the functional relation. It appears that the derivation is slightly more general than those given by previous authors.

R. G. Deissler (Cleveland, Ohio)

Schäfer, Wolfgang 6752

Behandlung von Freistrahlproblemen mittels der Hodographenmethode bei quadratischer Approximation der Adiabate. I. Das Ausströmen aus einem gradlinig begrenzten Gefäss.

Wiss. Z. Techn. Hochsch. Chem. Leuna-Merseburg **5** (1963), 47–52.

In a steady non-viscous subsonic compressible flow let $we^{i\theta}$ be the velocity, M the local Mach number, and Ψ the stream function. Let $\log \omega = \int_{w_*}{}^{w} \tau w^{-1} dw$ and $K^2(\omega) = \tau\rho_\infty/\rho$. Then $\psi = K(\omega)\Psi$ satisfies

$$(1) \qquad \psi_{\omega\omega} + \omega^{-1}\psi_\omega + \omega^{-2}\psi_{\theta\theta} + F(\omega)\psi = 0,$$

where $-KF = d^2K/d\omega^2 + \omega^{-1}dK/d\omega$. The author makes the approximation $2F = k^2\omega^{2m}$, where k and m are constants. Note that $k = M = 0$ corresponds to incompressible flow; $k = 0$, $M \neq 0$ to the von Kármán-Tsien equation of state. The case $m = 0$, which corresponds to a second-order approximation to $p/p_\infty = (\rho/\rho_\infty)^\gamma$, is of interest for comparison with theoretical and numerical results of other authors mentioned in the paper. For general m the author constructs solutions of (1) by Bergman's integral operator method which reduce in the limit $k = M = 0$ to stream functions of incompressible flows. The process is carried out in detail for efflux of a symmetrical jet from an infinite reservoir with straight walls, for which the incompressible stream function is known. *J. H. Giese* (Aberdeen, Md.)

Schäfer, Wolfgang 6753

Behandlung von Freistrahlproblemen mittels der Hodographenmethode bei quadratischer Approximation der Adiabate. II. Der Stoss eines Strahls auf eine Platte.

Wiss. Z. Techn. Hochsch. Chem. Leuna-Merseburg **5** (1963), 147–151.

The author applies the process described in Part I [#6752 above] to another problem with known incompressible stream function, the symmetrical flow produced by impact of a jet on a fixed rigid plate of finite width.

J. H. Giese (Aberdeen, Md.)

Džanybekov, Č. 6754

On the solution of Chaplygin's equation for transonic flows with a local supersonic zone terminated by a shock wave. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1963**, no. 4 (35), 56–60.

F. I. Frankl' [Prikl. Mat. Meh. **19** (1955), 385–392; MR **17**, 550] has constructed an example of a flow with a locally supersonic region terminated by a straight shock. The author has attempted to generalize this result by expanding the stream-function in a series of the form

$$\psi = \sum_{k=0}^{\infty} \rho^{(2k+8)/3} S_k(t),$$

where the term for $k = 0$ corresponds to the example of Frankl', and ρ and t are functions of the magnitude and inclination of the flow velocity. In fact, $\rho = \text{const}$ are characteristics in the supersonic region. The author shows that the required function S_2 does not exist. In an addendum Frankl' has proposed an alternative expansion for ψ. *J. H. Giese* (Aberdeen, Md.)

Fal'kovič, S. V.; Černov, I. A. 6755

On the theory of self-simulating transonic flows. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, no. 1 (38), 125–128.

Certain self-similar solutions of the transonic small disturbance equation, due to Guderley [*The theory of transonic flow*, Pergamon, Oxford, 1962; MR **26** #1750], Germain [C. R. Acad. Sci. Paris **253** (1961), 2511–2513; MR **22** #B2729] and others, are extended and applied to calculate flow fields at large distances from a profile. As an example, the solution of the problem of flow of a sonic stream past a symmetrical airfoil is given, including the asymptotic behavior of streamlines and the trailing edge shock wave.

M. Holt (Paris)

Lun'kin, Ju. P.; Popov, F. D. 6756

Effect of non-equilibrium dissociation on the supersonic flow around blunt bodies. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 896–904.

Les auteurs présentent des applications du premier schéma de la méthode de Dorodnitsyn en seconde approximation (une ligne intermédiaire), dans le cas de l'oxygène avec relaxation chimique. En plus de la présentation de résultats numériques, il est à noter que l'équation de cinétique chimique a été écrite sur quatre lignes supplémentaires sur lesquelles vitesse, pression et température ont été obtenues par interpolation quadratique. *J. P. Guiraud* (Paris)

Nikol'skii, A. A. 6757

Some non-stationary gas motions and their stationary hypersonic analogues. (Russian)

Inž. Ž. **2** (1962), no. 2, 246–253.

A general discussion is given of hypersonic similitude, the analogy between steady flow at very high Mach number and unsteady flow in one fewer dimensions. To illustrate this the author considers the exact solution of a problem of unsteady motion with spherical symmetry, in which the distributions of flow variables are prescribed functions of distance from the center of symmetry. The hypersonic equivalent of this motion is determined.

M. Holt (Paris)

Bagdoev, A. G. 6758

On the validity of a solution of simple wave type. (Russian. Armenian summary)

Izv. Akad. Nauk Armjan. SSR Ser. Fiz.-Mat. Nauk **17** (1964), no. 1, 75–78.

The problem of propagation of a weak plane shock wave in a half-plane is considered, resulting from the application of a prescribed pressure pulse along one of the coordinate axes. The solution to this problem has previously been obtained by simple wave theory [the author and É. M. Nersisjan, Izv. Akad. Nauk SSSR Otd. Tehn. Nauk Meh. Mašinostr. **4** (1960), 3–6], and the validity of the theory is tested by reference to the exact shock equations. It is shown that the shock conditions are satisfied to within terms of the third order in the shock strength.

M. Holt (Paris)

Sagomonjan, A. Ja. 6759

The interaction of a body and a semi-infinite barrier at high impact velocities. (Russian. English summary)

Vestnik Moskov. Univ. Ser. I Mat. Meh. **1964**, no. 5, 39–48.

Author's summary: "A simplified scheme of the phenomenon of interaction of a body of small mass and an immovable semi-infinite obstacle with the formation of shock

waves is considered. The velocity of the contact surface relative to the obstacle and the velocity of the striking body relative to the contact surface are assumed to be supersonic. The material behind the shock waves is simulated by an incompressible fluid. The positions of the front part of the contact surface and of the shock wave are determined approximately at the completion of the interaction."

Warsi, N. A. 6760

On curved shock wave in three-dimensional conducting gas flows.

Proc. Nat. Inst. Sci. India Part A **29** (1964), 7–17.

Author's summary: "The first and the second derivatives of the density, the pressure, the velocity and the magnetic field vector behind the shock, in the case of steady flow of conducting gases in the absence of viscosity and heat conduction, have been determined. Consequently, the curvatures and torsions of the streamlines and the magnetic lines of force, the current density and the vorticity behind the shock have been found."

Ericksen, J. L. 6761

Nilpotent energies in liquid crystal theory.

Arch. Rational Mech. Anal. **10** (1962), 189–196.

Hydrostatic theories are discussed, in which the free energy is a function of a vector **n** and its gradients. The form of the free energy affects the forms of the equations and boundary conditions satisfied by **n**. One of the material constants in the Oseen-Frank expression for the free energy drops out of the differential equations. Consequently, experiments to determine this constant must take the boundary conditions into account. If, as in this case, two different free energy functions yield identical equations for **n**, their difference is called a nilpotent energy. An explicit expression is obtained for the most general form of a nilpotent energy.

A. C. Pipkin (Providence, R.I.)

Herbert, D. M. 6762

On the stability of visco-elastic liquids in heated plane Couette flow.

J. Fluid Mech. **17** (1963), 353–359.

Author's summary: "The stability of thermally stratified plane Couette flow of visco-elastic liquids, with respect to disturbances of small amplitude, is considered. It is found that an initial state of finite elastic stress is necessary for elasticity to effect stability. The critical Rayleigh number of the flow is shown to be decreased for all non-zero rates-of-strain. This greater instability is due solely to the variations of the apparent viscosity with shear rate. In this way, the presence of elasticity can be said to have a destabilizing effect on the flow."

A. C. Pipkin (Providence, R.I.)

Lakshmana Rao, S. K. 6763

Axisymmetric solutions of the equations of motion of non-linear viscous flows.

Proc. Roy. Irish Acad. Sect. A **62**, 55–61 (1962).

The Navier-Stokes equations based on the stress-rate of strain relation $t_{ij} = -p\delta_{ij} + 2\mu e_{ij} + 2\mu_c e_{ik} e_{kj}$ are expressed in terms of the Stokes stream function ψ and the circulation Ω around a circle of radius r centered on the axis of symmetry. One or two formal solutions for ψ and Ω are determined, without introducing boundary conditions or discussing the physical significance of the solutions.

C. R. Illingworth (Manchester)

Leslie, F. M. 6764

Hamel flow of certain anisotropic fluids.

J. Fluid Mech. **18** (1964), 595–601.

Author's summary: "An exact solution is given for the flow in a convergent or divergent channel of a class of anisotropic fluids in which the fluid has a preferred direction."

A. C. Pipkin (Providence, R.I.)

Markovitz, Hershel; Coleman, Bernard D. 6765

Incompressible second-order fluids.

Advances in Applied Mechanics, Vol. 8, *pp.* 69–101. *Academic Press, New York*, 1964.

This is an expository article, with the following section headings. Introduction. Relation to General Simple Fluids. Steady Simple Shearing Flow. Viscometric Flows. Steady Extension of a Cylinder. Relation to Classical Viscoelasticity. Nonsteady Shearing Flows.

A. C. Pipkin (Providence, R.I.)

Markovitz, Hershel; Coleman, Bernard D. 6766

Nonsteady helical flows of second-order fluids.

Phys. Fluids 7 (1964), 833–841.

Authors' summary: "Some exact solutions of the third-order vector partial differential equation describing nonsteady flows of incompressible second-order fluids are presented. Although periodic simple shearing flow is also discussed, the main emphasis here is on helical flows between moving coaxial cylinders, particularly periodic flows of the Couette and Poiseuille type. The calculations suggest that, for such periodic helical flows, measurements of the radial thrusts on the bounding cylindrical walls supply practicable methods for the determination of normal-stress coefficients."

A. C. Pipkin (Providence, R.I.)

Misra, Shankar Prasad 6767

Elastico-viscous fluid flow past a circular cylinder or a flat plate with suction.

Ganita **14** (1963), 98–103.

The author considers first-order effects of a particular elastico-viscous fluid flowing over (i) a circular cylinder with axisymmetry, and (ii) a flat plate as a limiting case of the cylinder. The problem is examined with suction at the boundaries.

W. O. Criminale, Jr. (Princeton, N.J.)

Pipkin, A. C. 6768

Annular effect in viscoelastic fluids.

Phys. Fluids 7 (1964), 1143–1146.

From the author's summary: "In the oscillating motion of an incompressible viscoelastic fluid in a pipe, at high Reynolds number the root-mean-square axial velocity has its maximum value in a boundary layer at the wall of the tube. The elasticity of the fluid can make this annular effect

much more pronounced than it is in the case of a Newtonian viscous fluid. The analysis is carried out for small-amplitude oscillations of a general viscoelastic fluid, and the exact solution for this case is compared with the solution obtained by the boundary-layer approximation. The solution in the case of slow motion of arbitrary amplitude is also considered, and it is shown that no annular effect occurs in this case." *M. N. L. Narasimhan* (Bombay)

Rintel, L. 6769

Flow of non-Newtonian fluids at small Reynolds number between two discs: One rotating and the other at rest.

Second-order Effects in Elasticity, Plasticity and Fluid Dynamics (*Internat. Sympos., Haifa*, 1962), *pp.* 467–472. *Jerusalem Academic Press, Jerusalem; Pergamon, Oxford*, 1964.

The Reiner effect in air thrust bearings is discussed on the basis of the Reiner-Rivlin constitutive equation. It is concluded that the Reiner-Rivlin equation is not adequate to account for the observed phenomena.

A. C. Pipkin (Providence, R.I.)

Segawa, Wataru 6770

Rheological equations of viscoelastic liquid.

J. Phys. Soc. Japan **19** (1964), 206–210.

A constitutive equation for stress as a function of strain and strain-rate is applied to fluids by taking the strain to be an undefined quantity. The equation is simplified by postulating that the principal axes of stress and undefined strain coincide. Problems of steady simple shearing motion and flow through pipes are solved by postulating values for the strain. *A. C. Pipkin* (Providence, R.I.)

Srivastava, P. N. 6771

Propagation of small disturbances in a visco-elastic fluid contained in an infinite cylinder due to the slow angular motion of its base.

Indian J. Math. **6** (1964), 29–38.

A special case of Oldroyd's stress-strain-rate relation for a second-order visco-elastic fluid is considered, with the parameters restricted to a single relaxation-time constant λ and kinematic viscosity ν. The author then proceeds to solve the problem of the motion of such a fluid, initially at rest, in a circular cylinder subject to either impulsive or sinusoidal angular motion of the base for $t>0$. Because of the symmetry of the problem, a linear wave equation is obtained for the azimuthal component of velocity. {Reviewer's note: From the linear form of this differential equation, it is clear that the non-linear terms in the stress-strain-rate relation can have no effect on the fluid motion for this problem.} Solutions are obtained by transform techniques in the form of infinite series. The resulting motion exhibits the behavior of an elastic shear wave with propagation velocity $\sqrt{(\lambda/\nu)}$, as should be expected from the form of the differential equation. The author obtains the ordinary viscous fluid results for the limit of zero relaxation time.

O. R. Burggraf (Columbus, Ohio)

Srivastava, A. C. 6772

Torsional oscillations of an infinite plate in second-order fluids.

J. Fluid Mech. **17** (1963), 171–181.

Author's summary: "The flow of an incompressible second-order fluid due to torsional oscillations of an infinite plate when the fluid is infinite in extent, as well as the case when it is bounded by another stationary parallel plate, has been considered by expanding the velocity components and the pressure in powers of the amplitude of oscillation of the plate. In both cases the first-order solution consists of a transverse velocity and the second-order solution gives a radial-axial flow composed of a steady part and a fluctuating part. In the case of the unbounded plate the steady part of the radial flow does not vanish outside the boundary-layer region. Hence the equations are solved by another approximate method for the steady part of the flow. The effects of the non-Newtonian terms are to increase the non-dimensional boundary thickness and the shearing stress on the plate. In the case of two plates the velocity components and the shearing stresses on the plates have been expressed in powers of the Reynolds number R for its small values. Their asymptotic behaviour for large R has also been studied. The asymptotic expansion of the fluctuating part of the radial-axial flow shows that the boundary layer is developed at both the plates." *A. C. Pipkin* (Providence, R.I.)

Ferrero, Giorgio M. 6773

Teoremi di confronto relativi alla pressione e all'energia in magnetofluidodinamica.

Boll. Un. Mat. Ital. (3) **19** (1964), 185–192.

A classical theorem in fluid dynamics [see, e.g., Serrin, *Handbuch der Physik*, Band 8/1, pp. 125–263, esp. p. 261, Springer, Berlin, 1959; MR **21** #6836b] is extended to magnetohydrodynamics as follows: If in a steady, incompressible, and viscous flow $\nu\nabla\cdot(\mathbf{H}\cdot\nabla\mathbf{H})\gtrsim\nabla\cdot(\mathbf{v}\cdot\nabla\mathbf{v})$, the total pressure (including the magnetic contribution) takes its maximum value on the boundary of the flow region. Notice that both terms in the inequality are positive.

Some properties of flows in which $\mathbf{H}\cdot\nabla\mathbf{H}=0$ are then established. *B. Bertotti* (Frascati)

Fucks, W.; Uhlenbusch, J. 6774

Hydromagnetische Lagertheorie. (English and Russian summaries)

Z. Angew. Math. Mech. **43** (1963), 553–560.

Hydromagnetic flow between two flat plates with a small gap and a small inclination between each other is discussed in the presence of a transversal magnetic field perpendicular to the plate. Induced pressure difference due to the Couette motion of one plate is determined. Similar study is made on the flow between cylindrical journal and bearing in the presence of a magnetic field perpendicular to the surface of the journal. The force and moment on the bearing can be reduced for large Hartmann numbers H to forms similar to those in the non-magnetic case except for factors proportional to H.

H. Hasimoto (Baltimore, Md.)

Gundersen, Roy M. 6775

The effects of heat addition to one-dimensional magneto-hydrodynamic flow. II. (French, German, Italian and Russian summaries)

Internat. J. Engrg. Sci. **2** (1964), 291–304.

Part I appeared in same J. **1** (1963), 359–369 [MR **27**

#5464]. From the author's summary: "The transients due to uniform heat addition to the steady subtransonic, one-dimensional flow of an ideal, inviscid, perfectly conducting compressible fluid, subjected to an oblique magnetic field with two nonzero components, are determined by the use of a linearization based on small heat addition. Both the slow and fast wave cases are studied. Although the solution obtained ceases to be valid in the neighborhood of transonic flow, and this difficulty is due to the inadequacies of the linearization which approximated the characteristics of the base flow by parallel families of lines, a solution valid for all flow speeds may be obtained by making a more detailed examination of the negative characteristics. The solutions obtained include, as special limiting cases, the solutions for the corresponding problems in conventional gas dynamics and when the applied field is transverse." *M. N. L. Narasimhan* (Bombay)

Infeld, E. 6776

Some exact solutions of the equations of magnetohydrodynamics for magnetic plane-symmetrical fields.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 233–238.

Linearized yet exact solutions having plane symmetry are derived for the equations of magnetohydrodynamics. Several geometries are considered, including the flow in a 2-dimensional quadrupole field. Time-dependent solutions are obtained by Fourier superposition, and the application to boundary-value problems is briefly considered.

J. S. Goldstein (Waltham, Mass.)

Jukes, J. D. 6777

On the Rayleigh-Taylor problem in magnetohydrodynamics with finite resistivity.

J. Fluid Mech. **16** (1963), 177–186.

In an attempt to elucidate how the assumption of infinite conductivity may mask important effects, the author considers a modification of the Rayleigh-Taylor stability problem. This consists of two incompressible, inviscid fluids of different densities in a gravitational field and supported one on the other. The upper fluid is electrically conducting and the system is placed in a uniform, horizontal magnetic field. For waves propagating along field lines, solutions appear which do not occur in the idealized infinite conductivity analysis. Moderately long "stabilized" waves are now found to grow aperiodically at a rate $\alpha\rho^{1/3}$ (ρ is the resistivity). Other modes are found to be periodic and damped at a rate $\alpha\rho^{1/2}$.

S. A. Berger (Berkeley, Calif.)

Kellogg, Paul J. 6778

Solitary waves in cold collisionless plasma.

Phys. Fluids **7** (1964), 1555–1571.

Author's summary: "The propagation of finite amplitude waves in a magnetized plasma, in which thermal motion and collisions are neglected, is studied for arbitrary direction of propagation. Solutions for oblique propagation are presented, which complement the analytical solutions for propagation perpendicular to the magnetic field obtained by Adlam and Allen and others, and for parallel propagation obtained by Saffman. Oblique propagation is much more complicated than the two limiting cases in that (1) In a given direction and at a given speed higher than the Alfvén speed there are a large number of different waves, and (2) As shown by Saffman, there are additional waves whose speed is less than the Alfvén speed. These submagnetic waves are investigated in the limit of infinite mass ratio, and the results are compared with computer calculations. The stability of the waves against the two-stream instability is investigated, and it is shown that (1) The supermagnetic waves (speed higher than Alfvén speed v_A) are destroyed by the two-stream instability unless v_A/c is sufficiently large, and (2) The submagnetic waves are destroyed by the two-stream instability unless the pressure is non-negligible."

P. G. Saffman (Pasadena, Calif.)

Moreau, René 6779

Jet laminaire rasant une paroi en présence d'un champ magnétique transversal.

C. R. Acad. Sci. Paris **259** (1964), 2177–2180.

Author's summary: "Il est montré qu'une solution affine exacte ne peut exister, au plus, que pour certaines distributions monomes du champ magnétique. Par contre, une affinité approchée est justifiée pour une distribution arbitraire. Dans le cas particulier du champ magnétique uniforme, les calculs sont achevés avec cette hypothèse."

R. P. Kanwal (University Park, Pa.)

Nardini, Renato 6780a

Su un caso particolare di onde magnetoacustiche.

Magnetofluidodinamica (Centro Internaz. Mat. Estivo, 3° Ciclo, Varenna, 1962), Conferenza 4a, 8 pp. Edizioni Cremonese, Rome, 1964.

Nardini, R. [Nardini, Renato] 6780b

Su un particolare campo magnetofluidodinamico sinusoidale in un mezzo viscoso.

Magnetofluidodinamica (Centro Internaz. Mat. Estivo, 3° Ciclo, Varenna, 1962), Conferenza 4b, 17 pp. Edizioni Cremonese, Rome, 1964.

These are expository lectures which describe in detail specific solutions of the Lundquist equations, with and without viscous terms, in certain highly symmetric geometries. *H. C. Kranzer* (Garden City, N.Y.)

Ostrovskii, L. A. 6781

On a certain type of magnetohydrodynamic waves.

Ž. Èksper. Teoret. Fiz. **44** (1963), 1587–1589 (*Russian. English summary*); *translated as Soviet Physics JETP* **17** (1963), 1068–1069.

The author considers the general problem of the non-linear one-dimensional unsteady magnetohydrodynamic flow in which the magnetic field lies in the transverse plane and is allowed to vary in magnitude and orientation in the transverse plane. Special well-known cases of such flows involve the propagation of magneto-acoustic and Alfvén waves. New simple solutions are obtained; one of these describes the interaction between transverse magneto-acoustic waves and Alfvén waves. *S. A. Berger* (Berkeley, Calif.)

Pacholczyk, A. G. 6782

Sulla instabilità gravitazionale e magnetogravitazionale di sistemi compressibili.

Magnetofluidodinamica (*Centro Internaz. Mat. Estivo*, 3° *Ciclo, Varenna*, 1962), *Conferenza* 5, 66 *pp.* *Edizioni Cremonese, Rome*, 1964.

Revue de résultats récents (dont plusieurs sont dus à l'auteur) sur le problème de l'instabilité magnétogravitationnelle et sur quelques applications en astrophysique (distribution des masses des étoiles, limite supérieure de l'intensité du champ magnétique galactique, instabilité des branches spirales galactiques).

R. Thibault (Sotteville-lès-Rouen)

Peyret, Roger 6783

Solution uniformément valable des équations de l'écoulement dans un accélérateur de plasma à ondes progressives.

C. R. Acad. Sci. Paris **259** (1964), 2592–2595.

In a previous note [the author and Moulin, same C. R. **259** (1964), 1810–1813; MR **30** #1715] a solution was obtained by a formal linearization. Since that solution is not uniformly valid, a uniformization technique is utilized to study the asymptotic behavior of the solution for large values of the space variable.

R. M. Gundersen (Milwaukee, Wis.)

Pratelli, Aldo M. 6784

Analogia tra "viscosità magnetica" e "conducibilità termica" nelle piccole perturbazioni di fluidi comprimibili.

Magnetofluidodinamica (*Centro Internaz. Mat. Estivo*, 3° *Ciclo, Varenna*, 1962), *Conferenza* 6, 18 *pp.* *Edizioni Cremonese, Rome*, 1964.

L'auteur compare les deux systèmes d'équations définissant la propagation des ondes sonores dans un fluide compressible, visqueux, et en tenant compte séparément, d'abord de la conductivité thermique, et ensuite de la conductivité électrique. Les conditions dans lesquelles ces deux systèmes se recoupent sont étudiées, dans le cas d'un mouvement plan avec champ perpendiculaire, en fonction du nombre de Kirchhoff et des nombres de Reynolds cinématique et magnétique.

R. Thibault (Sotteville-lès-Rouen)

Pratelli, Aldo M. 6785

Sulla influenza delle varie viscosità nella propagazione di piccole perturbazioni in magnetofluidodinamica.

Ist. Lombardo Accad. Sci. Lett. Rend. A **97** (1963), 699–716.

Author's summary: "It is shown that in an ionized, viscous, and heat-conducting gas, perturbations propagating along the magnetic field obey one differential equation of the 4th and another of the 5th order.

"When the magnetic field is perpendicular to the direction of the propagation, it is shown that the velocity, the concentration, the temperature, the magnetic field, and the pressure fulfil a differential equation of the 7th order; particular cases in which the order is lower are mentioned.

"A theory concerning weak shocks in fluid dynamics is extended to magnetohydrodynamics, with no assumption on the adimensional numbers."

B. Bertotti (Frascati)

Roberts, Charles S.; Buchsbaum, S. J. 6786

Motion of a charged particle in a constant magnetic field and a transverse electromagnetic wave propagating along the field.

Phys. Rev. (2) **135** (1964), A381–A389.

Authors' summary: "The relativistic equation of motion is examined for a charged particle in a constant magnetic field and a transverse electromagnetic wave propagating along the field. A general discussion is given of the effects at cyclotron resonance of the magnetic field of the wave and the relativistic mass increase with energy. An exact solution to the equation of motion is found for the case of a circularly polarized wave. The solution shows that when the index of refraction of the medium in which the wave propagates is not unity, the energy of the particle is a periodic function of time, the exact relationship being expressible in terms of elliptic integrals. When the index of refraction is unity, the effect of the magnetic field of the wave just compensates for the change in mass with energy, and the energy of the particle increases indefinitely at resonance. Several possible applications of this solution to classical cyclotron resonance phenomena are pointed out. As a numerical example, the case of an electron in a constant magnetic field of 1000 G initially at resonance with microwaves whose **E** field is 0.1 esu is considered."

S. A. Berger (Berkeley, Calif.)

Sandler, Stanley I.; Dahler, John S. 6787

Nonstationary diffusion.

Phys. Fluids 7 (1964), 1743–1746.

Authors' summary: "A simple proof is given that the telegrapher's equation provides a more rigorous description of diffusion than does the familiar diffusion equation. Solutions of these two equations are compared and it is shown that they are numerically indistinguishable for most cases of interest. The signal velocity of a concentration pulse is found to be $(p/\rho)^{1/2}$ with p the pressure and ρ the fluid density. It is found that in special circumstances the reflection and/or dispersion of concentration waves can occur."

OPTICS, ELECTROMAGNETIC THEORY, CIRCUITS

See also 6187, 6686, 6704a–b, 6819, 6882, 6883, 6884, 6908, 6990, 6991.

Barakat, Richard 6788

Theory of the coherency matrix for light of arbitrary spectral bandwidth.

J. Opt. Soc. Amer. **53** (1963), 317–323.

A coherency matrix of a light beam is introduced on the basis of the general theory of stationary stochastic processes with orthogonal increments. The Stokes parameters are re-interpreted as power spectral densities and their generalizations for $N \times N$ systems are studied. The transfer function matrix of a non-image-forming optical instrument is also discussed.

In the first part of this paper, the author distributes compliments and also provides a good deal of criticism of earlier work (including some by the reviewer) relating to the representation of a light beam by a coherency matrix.

In the reviewer's opinion the criticism reflects the following viewpoints: (1) That it is always better to use a frequency domain rather than a time domain representation (the two representations being related via Fourier transforms), (2) That it is preferable to use a real rather than a complex representation of the electic field, and (3) That it is always better to employ ensemble averaging rather than time averaging. *E. Wolf* (Rochester, N.Y.)

Beran, M.; Parrent, G., Jr. 6789
[Parrent, George B., Jr.]
The mutual coherence of incoherent radiation. (Italian summary)
Nuovo Cimento (10) **27** (1963), 1049–1065.

Authors' summary: "In this paper it is shown that the Dirac δ-function form, often assumed for the mutual power spectrum, $\hat{\Gamma}(S_1, S_2, \nu)$, to represent incoherent radiation, may be replaced by the finite form:

$$\hat{\Gamma}(S_1, S_2, \nu) = I(S_1, \nu)\,\frac{2J_1(k|S_1-S_2|)}{k|S_1-S_2|},$$

where $I(S_1, \nu)$ is the power spectrum of the radiation. It is then pointed out that there are many other similar finite forms that would serve to represent sources termed incoherent, the proper form depending upon the particular source properties." *E. Wolf* (Rochester, N.Y.)

Fragstein, C. 6790
Zur Frage der Reversibilitätsprinzipien in der Optik. Eine Bemerkung zu einer Arbeit von Antonin Vašíček.
Z. Physik **166** (1961/62), 257–260.

From the author's summary: "In contrast to the view expressed by Vašíček, this paper attempts to show that there is only one principle (not two) of reversibility in optics, namely, the one which Vašíček attributes to Vlasow." *E. Wolf* (Rochester, N.Y.)

Hauser, H. 6791
Optische Übertragung bei partiell-kohärenter Beleuchtung. I. Die Grenzen der linearen Übertragung.
Optica Acta **9** (1962), 121–140.

This paper is concerned with optical imaging in partially coherent illumination. As is well known in such cases, the relation between the intensity (and also the phase) of the object and the image distribution is not linear. Some estimates of the non-linear contributions of the object structure are obtained. The results are illustrated by several examples, including observation of phase objects by the use of Zernike's phase contrast method. *E. Wolf* (Rochester, N.Y.)

Hauser, H. 6792
Optische Übertragung bei partiell-kohärenter Beleuchtung. II. Die Übertragungstheorie mehrstufiger Abbildungssysteme.
Optica Acta **9** (1962), 141–148.

This investigation is closely related to that of the preceding paper [#6791]. It is mainly concerned with transmission properties of optical systems composed of two or more lenses. Imaging of transilluminated, as well as self-luminous, objects is considered. *E. Wolf* (Rochester, N.Y.)

Parrent, George B., Jr.; Shore, Robert A.; Skinner, Thomas J. 6793
On the mutual coherence function in an inhomogeneous medium.
J. Mathematical Phys. **3** (1962), 678–684.

This paper is concerned with the behaviour of the second-order coherence function $\Gamma_{12}^{(\tau)}$ (known as the mutual coherence function) of a stationary scalar wavefield in an inhomogeneous medium. It is shown that Γ obeys two wave equations which are natural generalizations of the wave equations well known (but not mentioned) in the corresponding theory for propagation in homogeneous media. Approximate expressions for Γ in a field propagated from a plane source are derived in terms of Green's functions. Propagation in a statistically homogeneous medium is studied in some detail. *E. Wolf* (Rochester, N.Y.)

Barakat, Richard 6794
Stochastic generalization of the Green-Wolf complex scalar potential of electromagnetic theory.
J. Opt. Soc. Amer. **53** (1963), 252–255.

In a paper by Green and the reviewer [Proc. Phys. Soc. Sect. A **66** (1953), 1129–1137; MR **22** #6385a] a new complex scalar potential for representing an electromagnetic field in free space was introduced. In a subsequent paper [the reviewer, Proc. Phys. Soc. 74 (1959), 269–280; MR **22** #6385b] some properties of this new representation were discussed. The present paper considers a statistical ensemble of such potentials for representing a stochastic electromagnetic field. Use is made of the spectral representation theorem of Cramér and Kolmogorov. Some results relating to the average energy density and the average energy flow vector are obtained. *E. Wolf* (Rochester, N.Y.)

Denisse, J. F.; Delcroix, J. L. 6795
★Plasma waves.
Translated from the French by Marcel Weinrich and David J. BenDaniel. Interscience Tracts on Physics and Astronomy, No. 17.
Interscience Publishers [John Wiley & Sons], New York-London-Sydney, 1963. xii + 143 *pp.* $7.75.

The subject of waves in plasmas is one that is complicated in itself, and these complications have been frequently aggravated by problems of notation. Several books on the subject have appeared recently and have been invaluable in introducing some order into a rather tangled subject. The present book is a translation from a recent French text which sets out to describe plasma wave motion "within as general and systematic a framework as possible". Lest this phrase be misinterpreted, it should be made clear that the authors have adopted a somewhat restricted approach to their subject. There are many important topics connected with plasma wave motion that are not to be found; to give but one example, there is no discussion of the Landau damping of longitudinal plasma waves. None the less, the authors have organized their material well and this little book is a valuable introduction—but no more than that—to the subject.

In Chapter I the plasma is characterised by the two-fluid equations; from these and the Maxwell equations, the dispersion relation for plane waves is obtained. This relation is then examined in special cases for the study of

transverse and longitudinal waves in a plasma; in the latter case both electron plasma oscillations and the low-frequency ion oscillations are discussed briefly. Later chapters deal with the presence of a transverse magnetic field in the plasma and the more complicated situation arising from the now coupled transverse and longitudinal motions. The last two chapters discuss briefly wave propagation in cold plasmas (Appleton-Hartree theory) and low-frequency magnetohydrodynamic waves.

T. J. M. Boyd (Culham)

Karczewski, B. 6796

Coherence theory of the electromagnetic field. (Italian summary)

Nuovo Cimento (10) **30** (1963), 906–915.

The complex degree of coherence of an optical wavefield has previously only been defined within the context of a scalar representation of the field. In the present paper a generalization is suggested, which is applicable to any stationary quasi-monochromatic electromagnetic field. This generalization is suggested by the results of a theoretical analysis, based on vectorial diffraction theory, of a two-beam interference experiment. The generalized degree of coherence is expressible in terms of the trace of the second-order electric correlation matrix of the field. Relation of the present investigation to that of K. Germey [Ann. Physik **10** (1962), 141–152] is briefly discussed.

E. Wolf (Rochester, N.Y.)

Tai, C. T. 6797

A study of electrodynamics of moving media.

Proc. IEEE **52** (1964), 685–689.

There appears to be increasing interest among electrical engineers in the phenomenological electrodynamics of moving media. This paper was apparently written to provide some preliminary orientation in this field: some of the principal features of the theory are briefly sketched. Pauli's book [*Theory of relativity*, Pergamon, New York, 1958; MR **22** #3534] is unfortunately not cited among the references.

T. Erber (Chicago, Ill.)

De Socio, Marialuisa 6798

Su alcune questioni di unicità in elettromagnetismo.

Boll. Un. Mat. Ital. (3) **19** (1964), 146–152.

L'autore dimostra il teorema di unicità per le soluzioni delle equazioni di Maxwell (con le usuali condizioni iniziali e al contorno) in un dominio limitato o illimitato, qualora la densità di corrente sia funzione, anche non lineare, del campo elettrico e del campo magnetico. Dal teorema, valido per tutti i valori (positivi e negativi) del tempo, si deduce che, in un dominio limitato in cui non entra nè si produce energia, un campo elettromagnetico può annullarsi solo dopo un tempo infinito.

D. Graffi (Bologna)

Galli, Mario G. 6799

Sopra alcune proprietà del campo elettromagnetico generato dal moto iperbolico. (English summary)

Riv. Mat. Univ. Parma (2) **4** (1963), 127–136.

Author's summary: "The electromagnetic field generated by the hyperbolic motion of a charge presents, at first sight, some strange properties, from which a few authors inferred the absence of radiation. One of these properties is as follows: The magnetic field at the instant $t=0$ is zero everywhere in the whole space, although the particle is moving in the whole interval $(-\infty, 0)$ with accelerated motion. In this article the author gives a new proof of this property using a method by which it is also possible to demonstrate that this property is characteristic of the hyperbolic motion. This new method puts in a clear light the fact that it is completely unjustified to infer the absence of radiation."

N. D. SenGupta (Bombay)

Krauss, Morris 6800

Gaussian wave functions for polyatomic molecules: Integral formulas.

J. Res. Nat. Bur. Standards Sect. B **68B** (1964), 35–41.

Chen, Yung Ming 6801

On scattering of waves by objects imbedded in random media: Stochastic linear partial differential equations and scattering of waves by conducting sphere imbedded in random media.

J. Mathematical Phys. **5** (1964), 1541–1546.

The author considers a medium for which both the operator L fixing the left-hand side $L(u)$ of a very general wave equation, and the right-hand side g representing the source distribution, depend on a random parameter q. Both L and g are given in terms of expansions with respect to a small quantity ε. The resulting formal equation, up to second-order terms, for the expectation value $\langle u \rangle$ constitutes an extension of the corresponding equation of Keller's theory for a random medium with a non-random source distribution. The general equation is further worked out for a medium with a refractive index which can be represented by the product of a non-random (inhomogeneous and non-stationary) factor and another random factor.

The theory is applied next to the scattering of a plane wave by a perfectly conducting sphere imbedded in a random medium. This special case amounts to a wave equation for $\langle u \rangle$ which contains an integral term representing the interaction between the scattering sphere and the surrounding random medium. An examination of the integral term with the aid of a Watson transformation (applied to the Green's function associated with the scattering problem) shows that the integral term can be neglected for large values of ka (ratio of the circumference of the sphere and the wavelength); its neglect for small ka proves to be trivial. The remaining terms of the wave equation for $\langle u \rangle$ can be interpreted with the aid of an effective refractive index $\tilde{n}$ of the random medium.

H. Bremmer (Eindhoven)

Mathur, N. C.; Yeh, K. C. 6802

Multiple scattering of electromagnetic waves by random scatterers of finite size.

J. Mathematical Phys. **5** (1964), 1619–1628.

This paper treats the scattering of a plane electromagnetic wave incident on a half-space of identical, similarly oriented, randomly positioned scatterers. The general theory follows very closely the work of Waterman and Truell on scalar scattering [same J. **2** (1961), 512–537; MR

22 #B768], with obvious changes from scalars to vectors and dyadics. The case of spherical scatterers (having ε and μ) is treated in detail. Using the Mie theory of vector scattering, the authors consider: (a) sparsely packed spheres, where the exciting field on a particle is taken to be the incident field; and (b) dense packing in which the exciting field is found by the truncation of a certain hierarchy of integral equations. Among the results obtained are expressions for the refractive index, transmission and reflection coefficients for the loosely packed case, and a verification of the extinction theory for closely packed scatterers. Various special cases are considered, for which agreement is obtained with the results of other authors.

R. A. Hurd (Ottawa, Ont.)

CLASSICAL THERMODYNAMICS, HEAT TRANSFER

Morozova, T. V. 6803

Asymptotic inequalities applicable to certain thermodynamic functions. (Russian)

Ukrain. Mat. Ž. **15** (1963), 71–76.

Ergodic theorems of statistical mechanics are discussed in general terms as asymptotic relations between volumes and integrals over specified parts of related spaces. The author then gives a detailed proof of a relation whose formal expression is

$$\frac{1}{N}\log\left\{\iint\cdots\int dx_1 dx_2\cdots dx_N\right\}-\int\rho(x)\log\{1/\rho(x)\}\,dx\to 0$$

as N tends to infinity. The integral on the left represents the volume where certain "distribution functions" lie within prescribed limits. The function $\rho(x)$ is determined by the same set of distribution functions.

A. J. Macintyre (Cincinnati, Ohio)

Giulianini, Arturo 6804

Un contributo allo studio della propagazione del calore in una particolare regione cilindrica.

Atti Accad. Sci. Ist. Bologna Cl. Sci. Fis. Rend. (11) **9** (1961/62), *no.* 2, 79–100.

The author considers two problems: (a) the temperature history in a cylindrical region moving with constant axial velocity under a given initial temperature distribution and with all its surfaces at zero temperature; (b) the same region, axial velocity zero, under a prescribed distribution of constant heat sources. *B. A. Boley* (New York)

Glansdorff, P.; Dekeuster, E.; Haesendonck, Y. 6805

Application du potentiel local de production d'entropie à la recherche de distributions stationnaires de températures et de concentrations. (English summary)

Acad. Roy. Belg. Bull. Cl. Sci. (5) **50** (1964), 60–73.

Authors' summary: "The authors use the local potential of entropy production introduced by one of them and by I. Prigogine to derive a variational method allowing them to obtain stationary distribution laws. A comparison is established between several processes of approximation, and a generalization of the Rayleigh-Ritz method is introduced (self-consistent method). The one-dimensional and bidimensional problems in the field of heat conductivity are approached for a thermal conductivity depending on temperature. Similarly the same processes are then applied to the case of the one-dimensional problem of diffusion associated to a bimolecular chemical reaction."

D. Mattis (Yorktown Heights, N.Y.)

QUANTUM MECHANICS

See also 6092, 6180, 6272, 6328.

Blokhintsev, D. I. [Blohincev, D. I.] 6806

★Quantum mechanics.

Translated from the third and fourth Russian editions by J. B. Sykes and M. J. Kearsley.

D. Reidel Publishing Co., Dordrecht, 1964. xvi+535 *pp.* *Dfl.* 53.00; $14.85.

This outstanding textbook has so far had several editions, both in the original Russian [e.g., *Foundations of quantum mechanics* (Russian), 3rd ed., Gosudarstv. Izdat. "Vysšaja Škola", Moscow, 1961; MR **24** #B1896] and in other foreign languages. A German translation has been reviewed [*Grundlagen der Quantenmechanik*, Deutscher Verlag der Wissenschaften, Berlin, 1953; MR **16**, 430].

The level of this book goes somewhat beyond that of a first introduction to quantum theory, and covers several topics which are usually treated in an intermediate graduate course. The primary concern of the author is to provide the student with a clear and correct understanding of both the physical basis and the mathematical formalism of quantum theory. The goal of clarity and unambiguity is achieved in an exemplary manner indeed.

After a historical review concerning the origins of quantum theory, the fundamental principles are outlined in great generality and much of the mathematical framework is illuminated in connection with representation theory. Quantum dynamics, however, is given a somewhat too short treatment. The next two chapters discuss the motion of particles in potential fields and in the electromagnetic field. This is followed by the brief theory of angular momentum. Stationary perturbation theory for bound states is the next topic, followed by an outline of collision theory and transition probabilities. By way of application, emission, absorption, and the scattering of light, passage through potential barriers, the highlights of the many-body problem (including the discussion of systems of identical particles) are given. After an introduction to second quantization, further applications follow, such as multi-electron atoms, formation of molecules, magnetic phenomena, and some comments on nuclei and elementary particles. In the last chapter a review of the fundamental ideas is given, together with an excursion into ideology.

This book will prove to be a useful addition to the existing spectrum of texts on quantum theory.

P. Roman (Boston, Mass.)

Clutton-Brock, M. 6807

Feynman's kernel and the classical path.

Proc. Cambridge Philos. Soc. **61** (1965), 201–205.

Author's summary: "The quantum mechanical kernel $K(q'', t''; q', t')$ in Feynman's form of a sum over paths is evaluated by means of the canonical transformation to the

system Q, P for which the Hamiltonian vanishes identically. The kernel is found to be completely determined by the classical paths, and is given by

$$K(q'', t''; q', t') = A \sum_c \frac{\partial(Q, P)}{\partial(q', q'')} \exp\left[\frac{i}{\hbar} S(q'', P, t''; q', P, t')\right].$$

The sum is over all classical paths leading from q' at t' to q'' at t''. $S(q'', P, t''; q', P, t')$ is the classical action along the classical path characterized by the constant momentum P." *A. Peres* (Haifa)

Dubrovskiĭ, V. A.; Skuridin, G. A. 6808

Asymptotic expansions in wave mechanics. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 848–870.

The authors investigate asymptotic expansions for the partial differential equations of wave mechanics with a small coefficient (Planck's constant, $\hbar$) at the highest derivative by means of an unified asymptotic method, which can be applied also to analogous equations in other fields of mathematical physics. The essence of the method consists of the following: the reciprocal of the small coefficient ($1/\hbar$) is considered a constant of separation of variables and a new variable (the action) is introduced on the basis of physical considerations.

Schrödinger's, Klein-Gordon's, and Dirac's equations are transformed to the five-dimensional form with the action as the fifth coordinate. The asymptotic series for the five-dimensional equations as $\hbar \to 0$ are obtained and Cauchy's problem (scattering problem) is solved. The relation between the asymptotic expansions and the theory of perturbation is established. In the case of Schrödinger's equation, the results obtained are used to derive the formulae for the motion of a wave packet and to study the transition from wave mechanics to classical mechanics.

Y. Shimizu (Tokyo)

Goldberger, Marvin L.; Watson, Kenneth M. 6809

Measurement of time correlations for quantum-mechanical systems.

Phys. Rev. (2) **134** (1964), B919–B928.

Correlations in a time series of measurements in quantum-mechanical systems exist by virtue of the incomplete precision of measurements, or, more generally, their incompleteness relative to the set of measurements required to define the dynamical state of the system. Suppose the method of measurement to be so well-designed that phase relations between different state vectors compatible with the same result are to some extent unaffected by the process. Then the values, before and after measurement, of unmeasured variables will be correlated. With some qualifications, the authors assume the ideal situation to be that of maximum overlap between the wave functions before and after measurement, and they call this assumption "the principle of least interference". They also use the term "morally best", quoted from a private communication with E. P. Wigner. (The reviewer considers it semantically careless thus to adopt into science a term belonging to the area of ethics, an area which needs protection from the encroachment of science.) In simple cases at least, this principle is equivalent to exact preservation of the phase relations referred to above.

Subject to this principle, the authors develop an expression for the correlation function of two observables measured at different times. They also extend their results to the case in which the measurement is not instantaneous, assuming that operators for the observables measured then have the form

$$\mathcal{J}(t) = \int_{-\infty}^{t} L(t-\tau) J(\tau)\, d\tau,$$

where J is an instantaneous operator (i.e., of the usual type), and L is of the nature of a response function. Various cases of measurements of this kind are discussed. The most important is that in which it is shown that through the use of a composite detector, it is possible to study fluctuations having a period much shorter than the response time of an individual detector.

The final application is to a measurement of the Hanbury-Brown and Twiss type, proposed by Fano [Amer. J. Phys. **29** (1961), 539–545]. A result essentially equivalent to Fano's is derived.

A. Siegel (Boston, Mass.)

Moldauer, P. A. 6810

Problem of measurement.

Amer. J. Phys. **32** (1964), 172.

Attention is directed to earlier work by Wigner on the theory of measurement in quantum mechanics [e.g., same J. **31** (1963), 6–15; MR **27** #1112], in which the conceptual difficulties associated with the link between measurement and observation are emphasized. Rather than submit to the solipsism which is claimed to be implicit in these considerations, a structure is devised in which the wave-function can be subjective, while the physical universe remains objective. The statistical interpretation is central to the discussion. *H. W. Lewis* (Madison, Wis.)

Moses, H. E. 6811

A note on the equivalence of certain realizations of boson annihilation and creation operators due to K. O. Friedrichs.

Comm. Pure Appl. Math. **17** (1964), 355–356.

In his book [*Mathematical aspects of the quantum theory of fields*, Interscience, New York, 1953; MR **15**, 80] K. O. Friedrichs indicated how to construct certain families of boson annihilation and creation operators which differed from one another by the operation of multiplication by a scalar function. In the present note it is shown that two such sets of operators are unitary equivalent if and only if the scalar function is square integrable with respect to a certain measure occurring in Friedrichs' original definition. (In the opinion of the reviewer, the argument is incomplete because it does not specify the domains of the unbounded operators involved.)

A. S. Wightman (Princeton, N.J.)

Roos, Matts 6812

On the construction of a unitary matrix with elements of given moduli.

J. Mathematical Phys. **5** (1964), 1609–1611.

The author investigates the problem of finding a unitary $n \times n$ matrix (U_{jk}) when the moduli $u_{jk} = |U_{jk}|$ are given.

If D_1 and D_2 are diagonal unitary matrices, D_1UD_2 has the same u_{jk} as U. Thus the author considers the problem

of determining all U for given u_{jk} up to a phase transformation $U \to D_1 U D_2$. The author exhibits a concrete method of determining U_{jk} from given u_{jk} for the case of $n=4$. It is indicated that the method works for general n in principle. The author divides unitary matrices into two classes, regular and irregular. It is asserted (Theorem) that a regular unitary matrix is determined by the u_{jk} up to a phase transformation.

The reviewer feels that the author's discussion is incomplete and, in particular, the theorem is not valid in the form stated, because of an explicit counterexample. In general, $\cos\gamma_{13}$ is determined (as a function of θ_{23}) from (10) of the paper. However, a given value of $\cos\gamma_{13}$ has two solutions γ_{13} in the regular case, which differ in sign, and they give rise to different U's which are not related by a phase transformation. In the irregular case ($|\cos\gamma_{13}|=1$), this ambiguity does not occur. Other points which are not substantiated are the uniqueness of solution of equation (11) for a regular case and the assertion that irregular matrices are not completely determined by the u_{jk} up to a phase transformation. In fact, if we restrict U_{jk} to be real (completely irregular case with $|\cos\gamma_{jk}|=1$ for all j, k), then the u_{jk} determine U_{jk} up to a phase transformation, in general. *H. Araki* (Kyoto)

Zucker, I. J. 6813

Quantum mechanics of the isotropic three-dimensional anharmonic oscillator.

Proc. Cambridge Philos. Soc. **60** (1964), 273–278.

The author treats the calculation of the eigenvalues of a three-dimensional, anharmonic oscillator with a spherically symmetric potential which has a power series expansion. He shows that when the power series expansion of the potential has terms of even order only; then a method of McWeeney and Coulson [same Proc. **44** (1948), 413–422; MR **10**, 225] yields the eigenvalues to any desired degree of accuracy. He treats in some detail the case where $V=ar^2+br^4+cr^6$. In essence, his technique is to expand the exact wave function in terms of the wave functions of the harmonic oscillator. This yields a secular determinant of infinite order, which can then be solved by successive approximations. *J. McKenna* (Murray Hill, N.J.)

Aramaki, Seiya 6814

On the forward peak of scattering in the high energy limit.

Progr. Theoret. Phys. **30** (1963), 265–266.

This short note presents a simple proof that the forward scattering amplitude $A(s, \cos\theta=1)$ must dominate in the high-energy limit provided (i) there is a finite domain of analyticity in the $\cos\theta$ plane at $s=\infty$ enclosing the real region $-1+\varepsilon<\cos\theta<1-\varepsilon$ $(\varepsilon>0)$, and

(ii) $$\int^{\infty} \frac{\operatorname{Im} A(s,1)}{|A(s,1)|^2}\frac{ds}{s}<\infty.$$

The author justifies the latter condition by an appeal to low-energy pion-nucleon data. This implies a once subtracted dispersion for $A(s,1)$ (at least for the non-spin, non-isotopic-spin flip amplitude), and hence (ii). This condition may also be true in a more general context, but is not satisfied by an arbitrary Herglotz function.

A. C. Hearn (Oxford)

Arthurs, A. M.; Holt, A. R. 6815

Rotating coordinates in scattering theory.

Proc. Phys. Soc. **82** (1963), 1073–1074.

In a paper by Arthurs and Lewis [Proc. Roy. Soc. Ser. A **269** (1962), 585–588; MR **26** #1053] it was shown that if H is the Hamiltonian of a scattering system, and if $H'=e^{iu}He^{-iu}$ is a unitary transformation of H, then under appropriate circumstances, the corresponding exact cross-sections σ and σ' are equal. In this letter it is shown that, even though the exact cross-sections σ and σ' are equal, in first Born approximation the corresponding cross-sections σ_B and σ_B' can differ enormously. The authors discuss a case in which σ_B' even diverges although σ_B remains finite.

J. McKenna (Murray Hill, N.J.)

Domokos, G. 6816

High energy scattering in systems with anomalous thresholds.

Acta Phys. Polon. **24** (1963), 345–350.

In this paper the author considers the high-energy behaviour of the scattering amplitude in the presence of anomalous thresholds. He considers mainly $\pi\Sigma$ and πd scattering, in which cases the amplitudes are determined by the Pomeranchuk pole, just as in cases without anomalous thresholds. In the case of scattering on a nucleus, the amplitude is proportional to the atomic number. The role of many-nucleon intermediate states is briefly discussed.

J. N. Islam (College Park, Md.)

Finn, Albert C. 6817

Theorem on the shrinking of diffraction peaks.

Phys. Rev. (2) **132** (1963), 836–837.

Author's summary: "It is shown that the width of a diffraction peak divided by $\frac{1}{2}\sigma(s,t=0)$ cannot decrease faster at high energies than a constant times $(\ln s)^{-6}$. This follows from unitarity and analyticity in the largest Lehmann ellipse consistent with perturbation theory."

O. W. Greenberg (Princeton, N.J.)

Fradkin, D. M.; Hammer, C. L.; Weber, T. A. 6818

Scattering wavefunctions for a Dirac particle in a central potential.

J. Mathematical Phys. **5** (1964), 1645–1651.

Authors' summary: "The scattering wavefunction for a Dirac particle in a central potential is written in terms of a matrix acting on a plane-wave spinor whose momentum direction $\hat{p}$ and polarization direction are the assigned directions for the asymptotic incident plane wave. The matrix involves four functions, independent of polarization direction, which multiply the matrices 1, $\alpha\cdot(\hat{p}-\hat{r})$, $\alpha\cdot(\hat{p}+\hat{r})$, $\hat{p}\wedge\hat{r}\cdot\sigma$. Differential relations for these four functions are directly obtained and asymptotic relations are given. In particular, the function multiplying $\alpha\cdot(\hat{p}+\hat{r})$ is asymptotically zero, so the potential scattering formulation is identical with that given previously for the Coulomb potential. The scattering wavefunction is a solution of the general differential relations subject to appropriate boundary conditions. For the Coulomb potential, these differential relations simplify, and an iterative solution is developed based on a Green's function technique with the Sommerfeld-Maue approximation as the zero-order solution." *T. Erber* (Chicago, Ill.)

Kalikstein, Kalman; Spruch, Larry 6819
Scattering of electromagnetic waves by a ferrite in a waveguide.
J. Mathematical Phys. 5 (1964), 1261–1272.
Authors' summary: "As for any multichannel scattering problem, variational techniques can be utilized in the determination of the elements of the scattering matrix or of the equivalent network elements for a gyromagnetic obstacle in a waveguide. As always, however, it can be quite difficult to interpret numerical results which in general are neither upper nor lower bounds. A variational bound originally developed for the determination of the phase shift for a given angular momentum in a quantum mechanical central potential scattering problem is here adapted to the solution of a transversally magnetized, lossless ferrite slab in a rectangular waveguide propagating only one mode, the TE_{10} mode. With a simple trial function and with the aid of a comparison scattering problem which need not be tensor in character (so that the determination of upper and lower bounds is not really difficult), close bounds are obtained on $\cot \eta_e$ and $\cot \eta_o$, the cotangents of the real uncoupled phase shifts associated with the even and odd standing waves, respectively. The bounds obtained on $\cot \eta_e$ and $\cot \eta_o$ determine bounds on the equivalent π network. A second variational bound, which can be simpler to apply and which can be applied to a wider class of problems, is also developed. This, too, is an adaptation of a formalism originally introduced in quantum mechanical scattering problems, and depends upon a consideration of the spectrum of the fundamental operator of the theory, the Hamiltonian in the quantum mechanical case and an analogue thereof in the electromagnetic case."
J. W. Moffat (Toronto, Ont.)

López, C. A.; Saavedra, I. 6820
Analyticity of the Jost functions for the Coulomb potential in the complex angular momentum plane.
Nuclear Phys. 53 (1964), 519–528.
Authors' summary: "The Jost functions for the Coulomb potential and complex angular momentum are constructed and their analytic properties discussed in detail."
V. de Alfaro (Princeton, N.J.)

Redmond, Peter J. 6821
Some remarks concerning a pathological matrix of interest in the inverse-scattering problem.
J. Mathematical Phys. 5 (1964), 1547–1554.
This paper is a discussion of some properties of an infinite matrix which arose in R. G. Newton's investigation of an inverse scattering problem. The motivation for the author's analysis apparently rests on the fact that he was previously unaware that it is possible for an infinite matrix to annihilate a vector which does not belong to its range. The author points out that since the matrix in question has this property, the inverse scattering problem of Newton does not have a unique solution.
I. Kay (Ann Arbor, Mich.)

Regge, Tullio 6822
Recent progress in potential scattering theories.
Lectures in Theoretical Physics, Vol. VI (Summer Inst. Theoret. Phys., Univ. Colorado, Boulder, Colo., 1963), pp. 373–378. Univ. Colorado Press, Boulder, Colo., 1964.
This short survey can be considered as a continuation of the detailed lectures given earlier by the author [*Theoretical physics* (Seminar Theoret. Phys., Trieste, 1962), Internat. Atomic Energy Agency, Vienna, 1963; MR 27 #3339]. The following topics are discussed: (1) generalizations toward relativistic extensions of the theory; (2) extensions to the many-channel problem; (3) more satisfying treatments of already known results, in particular, the discussion of the asymptotic behavior for large complex angular momenta; (4) the problem of dealing with three- and more-particle systems.
P. Roman (Boston, Mass.)

Bialynicki-Birula, I. 6823
On the electron propagator in the two-component formulation of quantum electrodynamics.
Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 12 (1964), 231–232.
While studying the electron propagator two-component formulation of quantum electrodynamics, the author has found that there exists a gauge in which the electron self-energy function in the lowest order reduces simply to a first-order polynomial in p^2, and a simple gauge transformation totally eliminates the spectral function of the self-energy. A few related remarks also appear in the note.
G. Bandyopadhyay (Kharagpur)

Bilhorn, D. E.; Foldy, L. L.; Thaler, R. M.; Tobocman, W.; Madsen, V. A. 6824
Remarks concerning reciprocity in quantum mechanics.
J. Mathematical Phys. 5 (1964), 435–441.
In classical physics "reciprocity" generally refers to a symmetry condition on a Green's function, while in quantum mechanics reciprocity usually refers to a symmetry condition on scattering amplitudes. In this paper the authors generalize and combine these two ways of looking at reciprocity. Let H be the Hamiltonian matrix (which need not be Hermitian), and let Ψ be the Green's function (matrix) satisfying $(H-E)\Psi = I$. If U is a unitary matrix, we define $\tilde{H} = UH^TU^\dagger$ and $\tilde{\Psi} = U\Psi^TU^\dagger$, where H^T denotes the transpose and $U^\dagger$ denotes the Hermitian conjugate. Then it is shown that if H has the invariance property $\tilde{H} = H$, the Green's function also satisfies $\tilde{\Psi} = \Psi$. If ψ^a is a state vector, let $\psi^{\tilde{a}} = U\psi^{a*}$, where * denotes complex conjugation. Then the invariance of H further implies the relations between the matrix elements $H_{a\beta} = H_{\tilde{\beta}\tilde{a}}$, $\Psi_{a\beta} = \Psi_{\tilde{\beta}\tilde{a}}$, where $H_{a\beta} = \langle\psi^a|H|\psi^\beta\rangle$, etc. Finally, it is shown that the reciprocity condition satisfied by the Green's function implies that a symmetry relation is satisfied by the elements of the reaction matrix.
J. McKenna (Murray Hill, N.J.)

Chakrabarti, A. 6825
Applications of the Lorentz transformation properties of canonical spin tensors.
J. Mathematical Phys. 5 (1964), 1747–1755.
Author's summary: "Some applications of the Lorentz transformations of relativistic spin tensors in the canonical representation are discussed. The problem of precession of polarization is discussed in Sec. 2. It is shown that the kinematical equation, obtained quite simply, already contains the 'Thomas factor'. In Sec. 3, applications to polarization

analysis of decay products are considered. The canonical form of S-matrix elements are used and multipole parameters for successive decays of the type $a \to b+c$ are obtained in an arbitrary frame in a relatively simple way. The exact relativistic way in which the multipole parameters depend, in an arbitrary frame, on the particle momenta are discussed for decays of the type $a \to b+c+d$. While the canonical representation is used mainly, the corresponding technique in the spinor representation is discussed."

Curtius, P. 6826

On perturbation expansions in axiomatic quantum field theory.

Helv. Phys. Acta **36** (1963), 389–404.

Author's summary: "It is shown that the perturbation expansions of Lehmann-Symanzik-Zimmermann axiomatic field theory can be described by Lagrangian formalism; and conversely that the perturbation series derived from Lagrangian formalism satisfy the conditions following from a perturbation treatment of the Lehmann-Symanzik-Zimmermann axioms."

O. W. Greenberg (Princeton, N.J.)

Domokos, G.; Surányi, P. 6827

On the behaviour of Green functions at small distances. (Russian summary)

Acta Phys. Acad. Sci. Hungar. **17** (1964), 107–114.

The asymptotic behaviour of Green's functions at small distances is examined in terms of a scalar field model of the four-point function, with the help of the Bethe-Salpeter equation. The Bethe-Salpeter kernel for the lowest non-vanishing approximation is studied, and a close connection is found between the behaviour of the Green's function at small distances and the type of interaction adopted. It is found that according to whether the dimension of the coupling constant involved is a negative power of length, a dimensionless power of length or a positive power of length, the theory corresponds to a "super-renormalizable" theory, a renormalizable theory or an unrenormalizable theory, respectively. For the latter type of theory the Bethe-Salpeter equation gives a well-defined solution, but with an essential singularity at the origin.

J. W. Moffat (Toronto, Ont.)

Dowker, J. S. 6828

A note on Salam's compositeness conditions.

Nuovo Cimento (10) **29** (1963), 551–554.

It has recently been noted that particles are composite when their wave-function renormalization constants are zero. Salam [Nuovo Cimento (10) **25** (1962), 224–227] has further pointed out that the renormalization constants for the dissociation vertices of the composite particles are also zero. In the article under review these conditions for composite particles are examined within the context of the Zachariasen model. The model consists of two particles A and B and one distinct antiparticle $\bar{B}$. The Lagrangian is constructed in such a way that the only basic interactions are $A \to B+\bar{B}$ and $B+\bar{B} \to B+\bar{B}$, the bare coupling constants of which are denoted by g_0 and λ, respectively. The bare mass of the particle A is denoted by M_0. The model leads us to the following conclusions. The propagator of the particle has two singularities, which correspond to two physical masses (M_1, M_2). When the limit $g_0 \to 0$ is performed, one of two masses (say M_1) tends to the bare mass M_0 and the other (M_2) to a mass value M', which is the mass of a composite state. In this limit we find also that both the wave function renormalization constant Z_3 and the vertex renormalization constant Z, for the composite state, tend to zero, in agreement with the above-mentioned conditions for the composite particles. The reviewer would like to note that a similar consideration has been presented also by T. Muta [Progr. Theoret. Phys. **30** (1963), 223–235; MR **27** #6540] by making use of the double Lee model.

H. Umezawa (Tokyo)

Durand, Loyal, III 6829

Remarks on the electromagnetic interactions of massless particles.

Phys. Rev. (2) **128** (1962), 434–440.

This article is aimed at examining a limit of applicability of the Case-Gasiorowicz Theorem, which states that a massless particle of spin larger than or equal to one cannot be charged.

To derive this theorem, the author considers the electromagnetic vertex of the massless charged fields and concludes that the vertex should be zero as long as the spin is larger than or equal to one. The argument is developed in the Breit system, where the momenta of the initial and the final particles are equal in magnitude but oppositely directed. Let us call the direction of the initial momentum the third direction. Then, the helicity conservation law leads to the conclusion that the electromagnetic vertex of the current orthogonal to the third direction is zero when the spin of the charged particle is not equal to $\frac{1}{2}$. The third component of the vertex also appears to be zero because of the invariance of the theory under the reflection and time inversion. Expanding the fourth component of the vertex in terms of the electric multipole moments, one finds that the electromagnetic interaction should contain derivatives of order $2\mathscr{S}+1$ (for the integer spin $\mathscr{S}$) or $2\mathscr{S}-1$ (for the half-integer spin). The coupling constant of such an electromagnetic interaction cannot be dimensionless when $\mathscr{S} \geq 1$.

Thus the conclusion is that the massless particle of spin ≥ 1 cannot be charged as long as we do not introduce any constant of dimension of mass. This means that massless particles can be charged when they interact with massive particles. An example is presented in the case of a massless vector field.

H. Umezawa (Tokyo)

Ginzburg, I. F. 6830

Inelastic interactions of high-energy particles in renormalized strong-interaction theories.

Ž. Èksper. Teoret. Fiz. **44** (1963), 500–513 (*Russian. English summary*); *translated as Soviet Physics JETP* **17** (1963), 342–350.

Author's summary: "The expansion of the Green's functions and differential cross sections for inelastic processes in powers of the reciprocal of the energy $1/s$ is deduced in renormalized theories. In some cases the first terms of the series are the usual peripheral diagrams, whereas in other cases they are somewhat more complicated. The region

of applicability of the results obtained is much larger than that for the usual pole theory of peripheral collisions."
H. Umezawa (Tokyo)

Kamefuchi, S.; Umezawa, H. 6831
The mass of gauge particles and the self-consistent method of quantum field theory. (Italian summary)
Nuovo Cimento (10) **32** (1964), 448–468.

Authors' summary: "The aim of the present paper is to apply the self-consistent method based on inequivalent representations, which was developed in our previous papers, to vector fields in order to see whether the so-called gauge theories can allow nonzero masses for gauge vector fields. The Stueckelberg formalism is exclusively used, which is most convenient for our present purpose. It is shown that the representations for vector fields with nonzero mass are inequivalent to those for vector fields with zero mass even when the cut-off momentum and the normalization volume are finite. In quantum electrodynamics it can be shown that the photon cannot have any stable sister particle with nonzero mass. However, it is possible that there exists such a sister particle which is unstable. Our consideration is further extended to the Yang-Mills field as a simple example of multiplet gauge fields. It can be shown that the three components of this field cannot have an equal, nonzero mass simultaneously, but that they can have nonzero masses in an asymmetric way such that at least one of the three masses is different from the others. Most conclusions reached here seem to be valid in more general cases of gauge theories."

{Reviewer's opinion: The controversy as to whether gauge invariance implies zero-mass particles (Goldstone's theorem) is due in part to the different concepts meant by gauge invariance. For example, the theorem can only be proved if there is a positive definite metric; the present paper is one in which there is an indefinite metric, just as there is in the theory of quantum electrodynamics in two dimensions (Schwinger's model). The present paper is not at all rigorous, but its conclusions are not in conflict with Goldstone's theorem.} *R. F. Streater* (London)

Labko, S. I. 6832
A spin $\frac{1}{2}$-$\frac{3}{2}$ particle in a constant homogeneous electric field. (Belorussian)
Vesci Akad. Navuk BSSR Ser. Fiz.-Tèhn. Navuk **1964**, no. 3, 53–62.

McEwan, J. 6833
Representation of particle states in quantum field theory.
Phys. Rev. (2) **132** (1963), 2353–2362.

This article is aimed at proposing a definition of a state representing an unstable particle. Although a state of a stable particle can be defined in terms of its asymptotic fields, such a definition cannot be applied to the case of unstable particles, because there is no asymptotic field corresponding to unstable particles. To consider an unstable particle produced in a finite region of space-time $V(x)$, the author introduces a test function $X_V(x)$ with the region V as its support. The state of an unstable particle is defined by an integral equation which is derived from a relation for a stable particle with a wave packet $f(x)$ by the replacement $f \to X_V$. The state thus defined depends on the choice of the function X_V. This reflects the fact that the state of an unstable particle should depend on the details of the preparation of the particle. For this reason the function X_V is called the preparation function. Naively speaking, the preparation function plays a role of separating a narrow domain around the resonance mass from the whole region of the continuous mass spectrum. The decay law of the state of the unstable particle deviates from the exponential decay law at very long times and depends on the details of the preparation, in agreement with Schwinger's conclusion [Ann. Physics **9** (1960), 169–193; MR **22** #2362]. *H. Umezawa* (Tokyo)

Ogievetskij, V. I. [Ogieveckiĭ, V. I.]; Polubarinov, I. V. 6834
Interacting fields of spin 1 and symmetry properties.
Ann. Physics **25** (1963), 358–386.

As is well known, any massive free field of the vector type satisfies an equation of the form $\partial_\mu \phi_\mu = 0$ when it belongs to the unit spin representation of the homogeneous Lorentz group. This equation is satisfied by an interacting vector field when and only when the source of the vector field conserves ($\partial_\mu j_\mu = 0$). When this equation is not satisfied, it may be expected that the wave function of the interacting vector field is a superposition of wave functions of spinless and unit spin particles, although, even in such a case, there exists an equation of the form $\partial_\mu \phi_\mu = F(\phi, \psi)$ which reduces the number of degrees of freedom of the field to three. Here ψ stands for the field operators of the source of the vector field.

In the article under review the author proposes a hypothesis which states that any interacting field should possess a definite spin. This hypothesis leads to the conclusion that the source current j_μ of any vector field should conserve and therefore that the theory should possess some symmetry properties of the type of the Lie group. This conclusion holds even in the case of the massless vector field. In this way the author emphasizes that the symmetry of the nature might be intimately related with the space-time property of the field (i.e., the spin property). Taking this viewpoint, the author presents a detailed consideration of the symmetry properties in cases of interacting fields of spin 0, $\frac{1}{2}$ and 1. *H. Umezawa* (Tokyo)

Bonometto, S. 6835
On gauge invariance for a neutral massive vector field. (Italian summary)
Nuovo Cimento (10) **28** (1963), 309–319.

In the Stueckelberg formalism the massive vector field theory is formulated in a way similar to the quantum electrodynamics. In this formalism are introduced a vector field a_μ and a scalar field b. When the source current of the vector field conserves, the theory appears to be invariant under the gauge transformation $a_\mu \to a_\mu + \partial_\mu \Lambda$, $b \to b - m\Lambda$, where m is the mass of the vector field. In the article under review the author presents a simple and elegant derivation of the Stueckelberg formalism and clarifies an intimate relation between the gauge transformation given above and that in quantum electrodynamics. The argument is based on the fact that the equation $(\Box - m^2)a_\mu = 0$ takes the form of the massless field equation, $\Box^{(5)} a_\mu = 0$, in a five-dimensional space, when we consider only solutions

satisfying $\partial_5{}^2 a_\mu = -m^2 a_\mu$. Here, $\Box^{(5)}$ is the five-dimensional d'Alembertian.

This fact led the author to consider the gauge-invariant theory of "massless" vector field ($\Box^{(5)} a_\mu = 0$, $\mu = 1, \cdots, 5$) in five-dimensional space. It turns out that such a theory is just the Stueckelberg formalism for the massive vector field when we denote a_5 by b. The author also points out that his formulation gives us a simple way of deriving the Ogievetski and Polubarinov formalism for the massive vector field. *H. Umezawa* (Tokyo)

Pócsik, G. 6836

Fermion self-masses and Lehmann's spectral representation. (Russian summary)

Acta Phys. Acad. Sci. Hungar. **17** (1964), 103–106.

Lehmann's theorem provides a connection between the bare mass and the physical mass in terms of the spectral representation. This theorem is extended to the case of the spinor self-coupling on the basis of the spectral representation of the propagator, the field equations and the equal-time anticommutators. In a simplified version of the theorem a typical self-consistent mass equation holds.
J. W. Moffat (Toronto, Ont.)

Power, E. A.; Saavedra, I. 6837

Non-linear interactions, causality and the R-matrix.

Proc. Cambridge Philos. Soc. **60** (1964), 935–938.

Authors' summary: "The problem of causality in non-relativistic scattering where a bounded region of non-linear interaction occurs is re-examined in the context of a paper by Ebel. In the model considered it is shown that the acausal behaviour cannot be accounted for in the straightforward manner that is possible for an energy-dependent but linear potential which gives rise to a similar R-matrix." *J. W. Moffat* (Toronto, Ont.)

Pyt'ev, Ju. P. 6838

Asymptotic properties of dynamical variables in the theory of spinor and scalar meson fields. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 871–879.

In broad terms this is a study of the propagation of singular solutions of certain fields which are of current interest in quantum-mechanical field theory. It is an extension of an earlier paper by the author [Dokl. Akad. Nauk SSSR **149** (1963), 298–301] on the relation between classical and wave mechanics, here in the sense of study of the characteristic hypersurfaces of differential equations, and of the associated conservation equations. It is stated that the theory is to be interpreted in the sense of the theory of distributions. (Since the analysis is given very compactly in formal tensor notation, it is difficult to extract from it a statement of specific results.)
E. L. Hill (Minneapolis, Minn.)

Segal, I. E. 6839

Remark on operational formulations of relativistic causality.

Nuovo Cimento (10) **31** (1964), 427–428.

The author asserts that the so-called "usual causality condition" described in an article by Taylor and Toll [Nuovo Cimento (10) **15** (1960), 389–394; MR **22** #B896] criticizing his consideration of another condition [the author, Phys. Rev. (2) **109** (1958), 2191–2198; MR **22** #B897] is vacuous. *E. H. Barsis* (Argonne, Ill.)

Sokolik, H. A. [Sokolik, G. A.] 6840

Unified description of spinors and model of elementary particles.

Nuclear Phys. **50** (1964), 171–176.

Author's summary: "It is shown that a model of elementary particles can be based on the possibility of reducing the basis elements of the representations of the extended Lorentz group to anomalous and normal spinors."
H. Umezawa (Tokyo)

Stipčević, Zdravko 6841

A generalization of the Hartree method. (Serbo-Croatian summary)

Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hrvatske Ser. II **18** (1963), 279–284.

The author uses the variational method to find the energy of a many-body system, using as the trial wave function of a system of $2n$ particles a product of the form

$$\varphi_1(1, 2)\varphi_2(3, 4) \cdots \varphi_i(2i-1, 2i) \cdots \varphi_n(2n-1, 2n).$$

This seems to the present reviewer a rather restricted kind of trial function, and until quantitative results are given of applications of this method, it seems difficult to judge the effectiveness of this approach.
D. ter Haar (Oxford)

Tillmann, Heinz Günther 6842

Zur Eindeutigkeit der Lösungen der quantenmechanischen Vertauschungsrelationen. II.

Arch. Math. **15** (1964), 332–334.

In a previous paper [Acta Sci. Math. (Szeged) **24** (1963), 258–270; MR **28** #1878] the author has given a set of assumptions on which the uniqueness of the representation of operators in a quantum-mechanical system can be proved. In this paper he has improved the consistency of assumptions by redefining the permutability of two operators by $\langle Rf, Sg \rangle - \langle Sf, Rg \rangle = 0$, $f, g \in D_R \cap D_S$.
H. Wakita (Hiroshima)

Visconti, A.; Carmona, J. 6843

Structure of field quantities and S-operator. (French and Italian summaries)

Nuovo Cimento (10) **29** (1963), 742–759.

In this article the authors present an elegant formulation of the quantum field theory based on functional techniques. A key role in the formulation is played by a mapping operator which transforms c-number functionals into operator functionals. By making use of the mapping operator, together with the generator for the propagators, the authors succeed in deriving the well-known formulae for the S-matrix and the R-products in an extremely simple manner.

It is suggested that one might find a possible extension of the quantum field theory by introducing the interaction differential operator which relates the generator of the propagator of the free fields to that of the interacting fields. *H. Umezawa* (Tokyo)

Wergeland, Harald 6844

Photon emission from a given current.

Norske Vid. Selsk. Forh. (Trondheim) **37** (1964), 69–75.

The problem of photon emission from a current which is a given function of space and time has been treated by many authors since the famous study of the "infrared catastrophe" by F. Bloch and A. Nordsieck [Phys. Rev. (2) **52** (1937), 54–59] and is known to be exactly soluble. The present author expounds an elementary method of solving this problem. *H. Araki* (Kyoto)

Freund, Peter G. O. 6845

The method of complex angular moments in relativistic quantum theory.

Lectures in Theoretical Physics, Vol. VI (Summer Inst. Theoret. Phys., Univ. Colorado, Boulder, Colo., 1963), pp. 459–512. Univ. Colorado Press, Boulder, Colo., 1964.

This interesting article is aimed at the theoretical physicist who has already a knowledge of standard dispersion relation techniques and who wishes to obtain an exposition of the method of complex angular momenta and its main applications to high-energy physics. Potential scattering is omitted from consideration.

After a review of the Mandelstam representation, the interpolation problem for partial wave amplitudes is considered, followed by a detailed discussion of the properties of the interpolating functions. The next chapter presents the theory of Regge poles. Chapter 4 investigates the concept of an "elementary" particle. Finally, high-energy scattering is given a fairly detailed discussion.

P. Roman (Boston, Mass.)

Morey Terry, F. 6846

On the nature of nuclear potentials. (Spanish. English summary)

Rev. Un. Mat. Argentina **22**, 22–37 (1964).

Author's summary: "In the present paper, the explicit form of the potentials resulting from the theories of dispersion relations from the point of view of field theory and potential theory is obtained. The calculations are performed for one of the ten necessary potentials as exposition of the results obtained and of the methods necessary to handle these potentials. The results obtained agree with what one may expect from phenomenological analysis if the spreading of the particles, when proceeding to the non-relativistic limit of a relativistic theory, is taken into account."

This paper is not concerned with the explicit derivation of a nucleon-nucleon potential from field theory by the use of dispersion relations. Rather, it is concerned with the reduction of the wave equation for two Dirac particles interacting through a potential to an equivalent non-relativistic Schrödinger equation.

J. M. Charap (London)

Rothleitner, J. 6847

Exakte Lösungen von Low-Gleichungen. (English summary)

Z. Physik **177** (1964), 287–299.

Exact solutions of the Low equation are derived for various one-, two-, three- and four-dimensional crossing matrices. The three-dimensional case corresponds to the physical problem of symmetric pseudoscalar pion-nucleon scattering in the static model. The solution for this case is a special one for which the phase shifts satisfy the relation $\delta_2 = (\delta_1 + \delta_3)/2$, which is quite different from what is usually assumed in approximations to the Low equation. The four-dimensional case corresponds to the same physical problem, but exact solutions are found for which $\delta_{13} \neq \delta_{31}$, as well as the case usually assumed where these phase shifts are equal. *J. Franklin* (Livermore, Calif.)

Schwinger, Julian 6848

Coulomb Green's function.

J. Mathematical Phys. **5** (1964), 1606–1608.

The momentum representation equation for the Green's function of the non-relativistic Coulomb problem is transformed into a non-dimensional Euclidean surface integral equation. This may be solved, and it leads in an elegant fashion to a one-parameter integral representation for the original Green's function. *J. M. Charap* (London)

Falkoff, D. L. 6849

Approach to equilibrium in weakly interacting systems.

Lectures on the many-body problem, pp. 191–198. *Academic Press, New York*, 1962.

This paper consists of extracts from a paper by Mathews, Shapiro and the author [Phys. Rev. (2) **120** (1960), 1–16, MR **22** #7341]. *R. J. Rubin* (Washington, D.C.)

Andrews, M.; Gunson, J. 6850

Complex angular moments and many-particle states. I. Properties of local representations of the rotation group.

J. Mathematical Phys. **5** (1964), 1391–1400.

The rotation matrices, $D_j^{m,m'}(\alpha, \beta, \gamma)$, are investigated as functions of complex values of their parameters. The rotation matrices of the second kind, $E_j^{m,m'}(\alpha, \beta, \gamma)$, which are related to the $D_j^{m,m'}(\alpha, \beta, \gamma)$ in much the same way as the Legendre functions of the first kind, $P_j(\cos\beta)$, are related to those of the second kind, $Q_j(\cos\beta)$, are also studied. The continuation is made from a definition of these matrix elements in terms of the hypergeometric function. The asymptotic behaviours as the variables $|z| = |\cos\beta|$, $|j|$, or $|m'|$ become large are derived. A generalized Clebsch-Gordan reduction of products of these functions is stated, as well as a completeness relation. Finally, a generalization of the Sommerfeld-Watson transform is given and a theorem allowing the expansion of a square-integrable function, $f(\alpha, z, \beta)$, in terms of the $D_j^{m,m'}$ is proved.

J. T. Cushing (London)

Barut, A. O. 6851

Complex Lorentz group with a real metric: Group structure.

J. Mathematical Phys. **5** (1964), 1652–1656.

The group in question is the group of 4×4 matrices Λ satisfying $\Lambda^* G \Lambda = G$, i.e., the transformations act on the space of complex four-vectors and leave invariant the form $\sum_{j,k} Z_j Z_k G_{jk}$, where G is a diagonal matrix, $G_{11} = -G_{22} = -G_{33} = -G_{44} = 1$. The author studies its subgroup structure, Lie algebra and complex extension (which is A_3 in Cartan's notation), the little groups, the inhomogeneous groups associated, and the group invariants. The

group contains the Lorentz group as a subgroup, and the little group for positive mass is the group U_3.

Apart from the purely mathematical analysis of the group, done here apparently for the first time, the paper is of interest in the current problem of connecting the Lorentz group with an internal symmetry group.

R. F. Streater (London)

Beltrametti, Enrico 6852

Group theoretical properties of complex angular momenta.

Lectures in Theoretical Physics, Vol. VI (Summer Inst. Theoret. Phys., Univ. Colorado, Boulder, Colo., 1963), *pp.* 155–168. *Univ. Colorado Press, Boulder, Colo.,* 1964.

It is shown that the extension to complex angular momenta of Weyl's method, to obtain expressed representations of the 3-dimensional rotation group, naturally leads to defining two sets of infinite matrices. These sets can be defined in correspondence with the elements of the rotation group, except for certain singularities. Moreover, it is shown that the product of two matrices does not always exist, but if one considers the two sets together, one obtains a typical structure which can describe the product rules of the rotations. *P. Roman* (Boston, Mass.)

Cook, L. F., Jr.; Lee, B. W. 6853

Unitarity and production amplitudes.

Phys. Rev. (2) **127** (1962), 283–296.

Production amplitudes are studied using the Blankenbecler extension of the N/D method. In contrast with the paper of Meetz [same Rev. (2) **125** (1962), 714–728], Jacob-Wick helicity amplitudes are used [Ann. Physics 7 (1959), 404–428; MR **22** #2396] rather than Ciulli-Fischer partial wave amplitudes [Ž. Èksper. Teoret. Fiz. **38** (1960), 1740–1750; MR **22** #7701; ibid. **39** (1960), 1349–1356, MR **22** #13176; Nuovo Cimento (10) **12** (1959), 264–285; MR **21** #6968], which are nonrelativistic. Special care is devoted to the anomalous singularities which require a suitable modification of the N/D method.

The unitarity is stated directly in terms of absorptive parts actually needed in dispersion relations rather than in terms of Cutkosky's generalized unitarity which connects the imaginary part of a graph to its possible partitions [see also Ball, Frazer and Nauenberg, Phys. Rev. (2) **128** (1962), 478–494; MR **29** #5533].

The N/D solution for the T matrix is obtained in the presence of anomalous singularities by analytic continuation in the external masses. The correctness of the procedure is guaranteed, noticing that the T matrix obtained has correct analytic properties, satisfies unitarity, and has the correct discontinuities across anomalous singularities. *S. Ciulli* (Bucharest)

Gribov, V.; Okun', L. [Okun', L.]; Pomeranchuk, I. [Pomeranĉuk, I.] 6854

On processes determined by fermion Regge poles.

Ž. Èksper. Teoret. Fiz. **45** (1963), 1114–1122 (*Russian. English summary*); *translated as Soviet Physics JETP* **18** (1964), 769–774.

The authors show in a simple and elegant manner (not involving partial wave expansions) that if a process is described by fermion Regge poles, then these poles must occur as conjugate pairs in the physical l-plane corresponding to states of opposite parity. They also discuss polarization phenomena for one- and two-pole pairs and deduce relations between cross-sections of processes governed by the same Regge poles.

A. C. Hearn (Oxford)

Halfin, L. A. 6855

Quantum theory of unstable elementary particles.

Dokl. Akad. Nauk SSSR **141** (1961), 599–602 (*Russian*); *translated as Soviet Physics Dokl.* **6** (1962), 1010–1012.

This article considers the decay of unstable elementary particles from the point of view of dispersion theory. The usual description of an unstable particle in terms of a mass and a width is generalized to include a "preparation function" in the case where the width is not negligible compared to the mass. The main result relates the poles of the mass distribution to a dispersion integral over the parameters of the decay law.

J. Franklin (Livermore, Calif.)

Jacob, Maurice; Chew, Geoffrey F. 6856

★Strong-interaction physics.

A lecture note volume. Frontiers in Physics.

W. A. Benjamin, Inc., New York-Amsterdam, 1964. xi + 154 *pp.* $9.00.

This book consists of two separate lecture course notes which were given at the Middle East Technical University in Ankara during 1962–1963.

The first course by M. Jacob, entitled "An Introduction to the Analysis of Strong-Interaction Processes" covers mainly the phenomenological aspects of pion-nucleon physics and is designed to introduce current notions and mathematical methods for analyzing strong-interaction processes. After a general kinematical introduction and an introduction of partial wave amplitudes, charge independence is discussed and a phenomenological analysis of pion-nucleon scattering is given. This is followed by a survey of other mesons. The next chapter then gives a more systematic introduction to the analysis of pion-nucleon scattering by means of dispersion relation techniques. Finally, the peripheral model and the role of nearby singularities is discussed.

The second contribution, by G. F. Chew, is entitled "Nuclear Democracy and Bootstrap Dynamics" and can be considered as a continuation of the second author's book [G. F. Chew, *S-matrix theory of strong interactions,* Benjamin, New York, 1961; MR **24** #B748], which was published in the same series in 1961. The most significant change in viewpoint is the unequivocal adoption of nuclear democracy as a guiding principle, and the assertion that subtractions can always be avoided and that no arbitrary parameters should appear in the strong interaction S-matrix.

Maximal analyticity of the first and of the second degree is introduced and a non-relativistic model to illustrate these concepts is given. Finally, relativistic bootstrap dynamics is elucidated.

P. Roman (Boston, Mass.)

Källén, Gunnar 6857

★**Elementary particle physics.**

Addison-Wesley Series in Advanced Physics.

Addison-Wesley Publishing Co., Inc., Reading, Mass.-London, 1964. xiv + 546 pp. $15.00.

Most extensive textbooks in elementary particle physics were written between 1956 and 1960 and were devoted to the study of field theory, with the exception of Roman's, which dealt with the symmetry aspect. With the failure of orthodox field theory to account properly for the mechanics of the strong interactions, the series has been discontinued; a large number of short treatises have appeared in 1963–1964, dealing with new developments and successful new ideas.

The present volume is the first full-length textbook to emerge after this "revolution". Indeed, from many points of view the book is really refreshing: it is extremely close to experiment and is probably the best available description of the phenomenology of high-energy physics. Strong and weak interactions are studied in detail, with full explanations of the ways in which our knowledge and characterization are gained: identifying resonances, their spins and other quantum numbers, etc., e.g., a special chapter is devoted to mass measurements.

Interactions studied in detail include π-N scattering, photoproduction, annihilation, electron scattering, and some strong interactions of strange particles. Weak interactions are studied in great detail, leptonic and non-leptonic.

The strength of this textbook is its interest in the phenomenology; most experimental results stand as of mid-1963, with even some more recent ones. What is entirely missing is the theory. Theoretical covering is as of 1958; topics like the Mandelstam representation and all the related development in dispersion theory, Regge poles, etc., are not dealt with. On the symmetry side, things stop at strangeness, with the whole of SU(3) missing.

To sum up, the book is useful as an introduction to basic phenomena. It can best be used in conjunction with some of the existing paperbacks on the theory.

Y. Ne'eman (Pasadena, Calif.)

Lubkin, Elihu 6858

Possible relationship between electric charge and dual charge.

J. Mathematical Phys. **5** (1964), 1603–1606.

The author speculates on a possible geometro-dynamical approach [D. Finkelstein and C. W. Misner, Ann. Physics **6** (1959), 230–243; MR **22** #2369] for eliminating the particles with fractional charge corresponding to the unphysical triplet representations of SU(3) [M. Gell-Mann, Phys. Lett. **8** (1964), 214–215]. This paper is now superseded by those of A. Komar [Phys. Rev. Lett. **13** (1964), 220–221] and C. R. Hagen and A. J. Macfarlane [Phys. Rev. (2) **135** (1964), B432–B433]. *A. Peres* (Haifa)

Mayer, Meinhard E.; Schnitzer, Howard J.; Sudarshan, E. C. G.; Acharya, R.; Han, M. Y. 6859

Concerning space-time and symmetry groups.

Phys. Rev. (2) **136** (1964), B888–B892.

The authors consider several variations and generalizations of the theorem [W. D. McGlinn, Phys. Rev. Lett. **12** (1964), 467–469; MR **29** #976; F. Coester, M. Hamermesh and W. D. McGlinn, Phys. Rev. (2) **135** (1964), B451–B452; MR **29** #1930] which states that in the Lie algebra consisting of the generators of the Poincaré group and some internal symmetry group S (semi-simple), if one or more generators of S commute with the generators of the homogeneous Lorentz group, then one has essentially a direct product of the two Lie algebras. The Galilean group is also considered in place of the Poincaré group, with the result that a "linear" symmetry breaking is now possible.

A. O. Barut (Trieste)

Greenberg, O. W. 6860

Coupling of internal and space-time symmetries.

Phys. Rev. (2) **135** (1964), B1447–B1450.

Another and more global proof of the theorem about the structure of the group consisting of the generators of the Poincaré group and a semi-simple internal symmetry group. (See the above paper by M. E. Mayer et al. [#6859], and the references therein.) *A. O. Barut* (Trieste)

Nuyts, Jean 6861

Remarks on the $|\Delta \vec{I}| = \frac{1}{2}$ rule in weak interactions.

Lectures in Theoretical Physics, Vol. VI (Summer Inst. Theoret. Phys., Univ. Colorado, Boulder, Colo., 1963) *pp.* 169–189. *Univ. Colorado Press, Boulder, Colo.* 1964.

To begin with, a brief survey of the general features of weak interactions is given, stressing some of the yet unanswered questions. Then attention is focussed on special topics related to the $|\Delta \vec{I}| = \frac{1}{2}$ rule, viz., leptonic K-meson decays, strangeness-changing semi-leptonic decays, and non-leptonic decays of strange particles.

P. Roman (Boston, Mass.)

Ryan, C.; Sudarshan, E. C. G. 6862

Representations of parafermi rings.

Nuclear Phys. **47** (1963), 207–211.

Authors' summary. "We consider the operator algebra corresponding to a system consisting of an arbitrary finite number ν of parafermi oscillators with a view to obtaining all irreducible representations of the operator algebra defining it. It is shown that this algebra is isomorphic to the Lie algebra of the orthogonal group B_ν in $2\nu+1$ dimensions. The known results of the representation theory of the orthogonal group are then used to find the representations of the ring of parafermi operators. In particular it is seen that the so-called Green Ansatz for a parafermi ring arises in a natural way and that, in a certain sense, it furnishes the most general 'solution' of the parafermi algebra. The case of a parafermi ring of order 2, which is the simplest non-trivial case, is considered in some detail."

{Reviewer's remark: This article stimulates the question: Is there a type of (second quantized) particle statistics for each Lie algebra?}

O. W. Greenberg (Princeton, N.J.)

Raczka, Jan 6863

On functional formulation of the S-matrix theory.

Acta Phys. Polon. **24** (1963), 763–788.

Author's summary: "A functional formulation of quantum field theory is presented consisting in a unification and generalization of the three- and four-dimensional functional methods. The basis for the formulation are the generating functionals for the on the mass shell S-matrix elements and their out of the mass shell generalizations restricted by the conditions of unitarity and causality. The notions of Hilbert space (of state vectors) and field operators in this space do not occur (at least not as primary notions) and, therefore, also the corresponding assumptions (as, e.g., the asymptotic condition or the assumption about completeness) are avoided."

P. W. Higgs (Edinburgh)

Tietz, T. 6864

A new method for finding the phase shifts for the Schrödinger equation.

Acta Phys. Acad. Sci. Hungar. **16** (1963), 289–292.

The author derives several approximate relations between the phase shifts of the radial Schrödinger equation describing scattering by a central potential. The potential $V(r)$ vanishes at infinity more rapidly than the Coulomb potential, and at the origin can be no more singular than the Coulomb potential. Then if the energy and angular momentum are large enough, $\eta_l = \frac{1}{2}(\eta_{l+1} + \eta_{l-1})$. The author shows that this relation is satisfied quite well for a Fermi-Thomas potential. Several other appropriate formulae for $\eta_l - \eta_{l+1}$ and $\eta_{l-1} - \eta_{l+1}$ are also given.

J. McKenna (Murray Hill, N.J.)

Weidlich, W. 6865

Die feldtheoretische Formulierung der nichtrelativistischen Mehrkanalstreutheorie. (English summary)

Z. Naturforsch. **18a** (1963), 1266–1275.

Author's summary: "Methods for the explicit construction of asymptotic fields in terms of the interpolating field in a nonrelativistic theory are derived in both cases of one sort and many sorts of asymptotic particles. A simple nontrivial multichannel case is discussed explicitly."

T. Erber (Chicago, Ill.)

Levine, A. D. 6866

A note concerning the spin of the phonon.

Nuovo Cimento (10) **26** (1962), 190–193.

Vonsovskii and Svirskii [Fiz. Tverd. Tela **3** (1961), no. 7, 2160–2165] have introduced the phonon spin in isotropic media by using the Lagrangian formalism. The present author simply extends the method to a cubic lattice and finds that the phonon spin is well-defined only along certain restricted directions of propagation of the waves.

T. Arai (Argonne, Ill.)

Mumm, Thiounn 6867

Solutions singulières et théorie de la fusion.

C. R. Acad. Sci. Paris **259** (1964), 2366–2368.

The fusion of two particles (in the sense of de Broglie) is briefly examined. Fusion of two singular solutions is shown to be generally impossible, whereas a singular solution can be combined with a plane wave. The constituent and total masses are related.

R. H. Boyer (Austin, Tex.)

Degens, Luc 6868

Sur une hypothèse de Mott et Massey concernant la forme asymptotique d'une solution d'une équation intégrale d'un problème de diffusion à trois corps.

C. R. Acad. Sci. Paris **259** (1964), 2797–2798.

Author's summary: "Il est montré que l'équation intégrale de Mott et Massey pour la diffusion d'une particule par un système lié de deux particules (dont l'une est identique à la particule incidente) admet une infinité de solutions qui ne vérifient pas l'hypothèse de Mott et Massey concernant la forme asymptotique du terme d'échange."

F. R. Halpern (La Jolla, Calif.)

Fradkin, D. M.; Weber, T. A.; Hammer, C. L. 6869

Coulomb scattering of Dirac particles.

Ann. Physics **27** (1964), 338–361.

Authors' summary: "The continuum Coulomb wave function in the Johnson and Deck form—a matrix operator acting on a plane wave spinor—is discussed. General relations for the elastic differential cross-section, the asymmetry parameters, and the precession and nutation of polarization are developed. An expansion of the asymptotic wave function correct to third order in $\lambda = \alpha Z$ for all Born parameters $\nu = \alpha Z/\beta$ is obtained. This expansion involves a two-parameter function $T(\theta, \nu)$ which in turn may be expanded for small ν to give results previously obtained by Johnson, Weber, and Mullin. On the other hand, a large ν asymptotic expansion of $T(\theta, \nu)$ valid for $|2\pi\nu| > 1$ permits a discussion of the nonrelativistic limit. Both the small and large approximations substantially agree for $|\nu| \simeq \frac{1}{2}$. Consequently, these two approximations may be used to span the whole range of Born parameters. The asymptotic Born parameter approximation predicts markedly different scattering behavior for electrons and positrons. A striking feature is that functions pertaining to electron scattering are oscillatory in the scattering angle θ while the positron functions are not. In particular, the oscillations in the asymmetry parameter for electron scattering obtained numerically by Sherman are contained in the analytical approximate form. Also, the approach to the Rutherford cross-section is different from that quoted by Mott and Massey."

F. R. Halpern (La Jolla, Calif.)

Weber, T. A.; Fradkin, D. M.; Hammer, C. L. 6870

A method of asymptotic expansion of Fourier-type integrals.

Ann. Physics **27** (1964), 362–376.

Authors' summary: "The expansion for large ν of integrals of the type $G(\nu) = \int_{t_1}^{t_2} e^{i\nu f(t)} g(t)\, dt$ is discussed. The method consists of using the mapping $z = f(t)$ so that the integral in question becomes $G(\nu) = \int_a^b e^{i\nu z} F(z)\, dz$. This integral is evaluated by distorting the path of integration into the complex plane in such a way that finite segments, emanating from endpoints or paralleling branch cuts, lie along paths characterized by $\mathrm{Im}(\nu z) = \text{constant}$. Integration over segments involving encircled singularities z_1 (or endpoints) with the minimum $\mathrm{Im}(\nu z_1)$ give the major contributions to the integral. The integral over such a segment is approximated in terms of an expansion with an accuracy depending on the magnitude of ν and the distance between the encircled singularity and the nearest neighboring singularity. A modification of this

procedure gives an accuracy dependent on the distance to the next nearest neighboring singularity. The first approach is applicable to widely separated singularities while the second approach is useful for two closely spaced singularities. As an illustrative example, the method is applied to an integral which arises in connection with Coulomb scattering in the preceding paper [#6869]."

F. R. Halpern (La Jolla, Calif.)

Moskalenko, V. A. 6871

Criterion of superconductivity.

Dokl. Akad. Nauk SSSR **147** (1962), 1340–1343 (*Russian*); *translated as Soviet Physics Dokl.* **7** (1963), 1142–1145.

The normal state of a metal is investigated by using the Frohlich Hamiltonian for the electron-phonon interaction, and also by taking into account the shielded Coulomb interaction between the particles. The stability of the system against formation of bound pairs of electrons of opposite spins is investigated in the ladder approximation. A criterion, including the effect of the Coulomb interaction, is given for the occurrence of superconductivity.

M. J. Stephen (New Haven, Conn.)

Weller, Wolfgang 6872

Zur Superfluidität eines Bose-Systems.

Z. Naturforsch. **18a** (1963), 79–85.

A weakly interacting, dilute Bose gas is studied using the method of thermodynamic Green's functions. The usual approximation of Bogoliubov is made in treating the condensed particles. The elementary excitations of the system are discussed when the condensate is in uniform motion and when it is in vortex motion.

M. J. Stephen (New Haven, Conn.)

STATISTICAL PHYSICS, STRUCTURE OF MATTER

See also 6173, 6174, 6180, 6186, 6639, 6761, 6943.

Griffiths, Robert B. 6873

A proof that the free energy of a spin system is extensive.

J. Mathematical Phys. **5** (1964), 1215–1222.

A spin system means here a system of particles each of which has a finite number of states. The author shows first that the free energy of such a system is a convex function of the Hamiltonian. This is used to study the thermodynamic properties of finite but large subsystems of infinite systems for which the Hamiltonian has the translational symmetry of a lattice on one, two, or three dimensions. The object is to show that the free energy and other derived properties (entropy, specific heat, etc.) are extensive, i.e., the free energy per particle has a limit as the size of the system becomes infinite. In three dimensions these limits are shown to exist if the interactions between pairs of points a distance r apart decrease faster than r^{-3}.

G. Newell (Providence, R.I.)

Snider, R. F. 6874

Perturbation variation methods for a quantum Boltzmann equation.

J. Mathematical Phys. **5** (1964), 1580–1587.

From the author's summary: "For molecules with degenerate internal states, the single-particle distribution function must be replaced, if the translational motion is treated classically, by a Wigner distribution-function density matrix. The modified Boltzmann integro-differential equation for this quantity has been previously derived but so far only limited solutions of the resulting equation have been obtained. Methods are herein discussed which enable the standard methods for the solution of the classical Boltzmann equation to be applied to the solution of this equation. Complications involving commutation properties are resolved."

L. J. F. Broer (Eindhoven)

Zil'berman, P. E. [Zilberman, P. E.] 6875

The variational principle in the theory of Green's functions for inhomogeneous systems.

Fiz. Tverd. Tela **5** (1963), 386–396 (*Russian*); *translated as Soviet Physics Solid State* **5** (1963), 281–288.

The variational principle of Luttinger and Ward in the theory of the electron gas is generalised to the case of an inhomogeneous system in an external field. The thermodynamic potential is expressed as a functional of the single-particle Green's function and of the electrostatic potential. The Hartree approximation is made. As an illustration the motion of degenerate electrons in the field of a neutralising background with finite discontinuities is considered.

M. J. Stephen (New Haven, Conn.)

Huang, Kerson 6876

Imperfect Bose gas.

Studies in Statistical Mechanics, Vol. II, pp. 1–106. *North-Holland, Amsterdam; Interscience, New York*, 1964.

The author gives a rather extensive and very systematic review of the present state of the problem. After an introduction dealing with the ideal Bose gas and some of the properties of liquid helium to be explained by the theory, and after a brief explanation of the cluster expansion and the pseudopotential method, the author goes over to the real problem. Two models of interactions are studied in great detail: the hard-sphere Bose gas, and a hard-sphere gas with a weak, simple, attractive part. In both cases the microscopic properties are derived. Among these the equation of state is discussed with particular emphasis on the condensation properties. The mathematical details are usually left out. Many of the topics appeared in the author's book [*Statistical mechanics*, Wiley, New York, 1963; MR **27** #4605], but are treated here more systematically.

R. Balescu (Brussels)

Kelbg, G. 6877

Quantenstatistik der Gase mit Coulomb-Wechselwirkung.

Ann. Physik (7) **12** (1963/64), 354–360.

Author's summary: "The Slater sum of an N-particle system with Coulomb interaction is found. The Coulomb potentials are replaced by effective or pseudopotentials which depend on the coordinates of two particles each and are functions of $\hbar$ and the absolute temperature T. These pseudopotentials are regular at the origin and are composed of a long-range and a short-range part. The

mathematical formalism of classical statistical mechanics can be employed to calculate the free energy and the equation of state of a quantum plasma. The resulting quantum corrections to the Debye-Hückel limit law are found and discussed. Applications follow also to the thermodynamics of charged impurity centres in semiconductors."
P. W. Higgs (Edinburgh)

Temperley, H. N. V. 6878

On Mayer's hypothesis about the critical region of a classical gas.

Proc. Phys. Soc. **83** (1964), 1013–1019.

Author's summary: "Mayer and Mayer suggested in 1940 that there might be two distinct temperatures that could be called critical, one defined as in the van der Waals theory, the other associated with a singularity in the virial series. The hypothesis is tested by applying the Percus-Yevick approximation proposed in 1958, which is now known to be fairly reliable at high densities, to a model of Gaussian type which allows crudely for the effects of both attractions and repulsions between molecules. The treatment shows that the hypothesis may be correct; the two temperatures are expected to be close together but need not be identical. In certain limiting cases one of them disappears."
G. Newell (Providence, R.I.)

Vallander, S. V.; Egorova, I. A.; Rydalevskaja, M. A. 6879

The statistical Boltzmann distribution as a solution of the kinetic equations for gas mixtures. (Russian. English summary)

Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. **19** (1964), no. 2, 57–70.

Authors' summary: "It is proved that if the principle of detailed balance is true, the equilibrium kinetic equations for a reactive mixture of gases with internal degrees of freedom have only a single solution, i.e., the Boltzmann statistics."
R. Balescu (Brussels)

Vallander, S. V.; Egorova, I. A.; Rydalevskaja, M. A. 6880

An extension of the Chapman-Enskog method to reactive gas mixtures with internal degrees of freedom. (Russian. English summary)

Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. **19** (1964), no. 2, 155–161.

Authors' summary: "The Chapman-Enskog method is extended to the mixture of gases with internal degrees of freedom and chemical exchange reactions. A new system of macroscopic parameters is chosen for the distribution function."
R. Balescu (Brussels)

Wang Chang, C. S.; Uhlenbeck, G. E.; de Boer, J. 6881

The heat conductivity and viscosity of polyatomic gases.

Studies in Statistical Mechanics, Vol. II, pp. 241–268. *North-Holland, Amsterdam; Interscience, New York*, 1964.

This paper, written in the form of a report in 1954, is already considered as a classic among specialists, although it was never published formally. Its publication here therefore fills a gap and is well within the line of the "Studies in Statistical Mechanics" series.

The paper is essentially a rather straightforward extension of the well-known Chapman-Enskog method of kinetic theory of gases to the case of polyatomic molecules. The treatment is semi-classical, in the sense that translation is treated classically, whereas internal degrees of freedom are described by quantum numbers. (It does seem strange, however, that the effect of the internal degrees of freedom is not retained in the flow term of the generalized Boltzmann equation.) Heat conductivity, viscosity, and volume viscosity are calculated in detail.
R. Balescu (Brussels)

Barrett, E. B.; Whitmer, R. F.; Tetenbaum, S. J. 6882

Nonlinear interaction of an electromagnetic wave with a plasma layer in the presence of a static magnetic field. III. Theory of mixing.

Phys. Rev. (2) **135** (1964), A369–A373.

Two monochromatic plane waves of angular frequencies ω_1, ω_2 are incident normally on a plasma layer of uniform density in a uniform magnetic field perpendicular to the direction of propagation. The electric field of each incident wave is linearly polarized in a direction perpendicular to the magnetic field. The motion of the positive ions and thermal effects are neglected and a constant electron-neutral-particle collision frequency is assumed. The novel feature of the present investigation is that the amplitudes of the two incident waves are of comparable magnitude.

Following the methods of Parts I and II by R. F. Whitmer and E. B. Barrett [same Rev. (2) **121** (1961), 661–668; MR **22** #9014; ibid. (2) **125** (1962), 1478–1484; MR **25** #1784], double Fourier series expansions in terms of multiples of ω_1 and ω_2 are substituted into the nonlinear set consisting of the first two velocity moments of Boltzmann's equation, and Maxwell's equations. The first-order approximation is carried out to give the amplitudes of the various components having frequencies $\pm\omega_1$, $\pm\omega_2$. These results are compared with the second-harmonic waves of Part I. First, it is found that the forward and backward waves are coupled, whereas they cancelled to the first order in Part I; secondly, the power at the sum and difference frequencies is proportional to the product of the input powers of the incident waves.

The results of computer calculations of transmitted and reflected powers for the sum and difference frequencies are presented graphically. These show quite sharp resonances at expected values of the magnetic field in terms of the electron density. This provides a means of determining the electron density in the plasma layer from measurements of resonant magnetic-field strength.
K. C. Westfold (Clayton)

Tetenbaum, S. J.; Whitmer, R. F.; Barrett, E. B. 6883

Nonlinear interaction of an electromagnetic wave with a plasma layer in the presence of a static magnetic field. IV. Experimental results.

Phys. Rev. (2) **135** (1964), A374–A381.

Authors' summary: "The nonlinear interaction of an electromagnetic wave with a uniform, weakly ionized

anisotropic plasma layer has been investigated experimentally. The experiment was designed to comply as closely as possible with the assumptions used in the theory of the nonlinear interaction, which is given in Parts I, II, and III [see #6882 above] of this series of papers. Experimental results on sum and difference frequency mixing and harmonic generation are described as a function of the ambient electron density, the electron-neutral-particle collision frequency, the external dc magnetic-field strength, thickness of the plasma layer, and the field strengths and frequencies of the incident waves. The case of linearly polarized small-amplitude microwave signals incident on a layer of helium plasma is examined. Within the plasma, the signals propagate as plane waves with their directions of propagation and polarization normal to the dc magnetic field (i.e., the extraordinary mode of propagation). A detailed comparison is made between the experimental results and the theoretical predictions. Quantitative agreement was obtained over the electron density range permitted by the experiment. The results show that the plasma model assumed in the theory is an accurate representation of the actual plasma, and that the small-signal analysis accurately predicts the effects of the nonlinear terms in the Boltzmann equation on propagation phenomena. However, special care is required to insure that the plane-wave, infinite-medium assumption employed in the theory is satisfied experimentally. The experimental results also establish the validity of using certain characteristics of the nonlinear phenomena as a tool for measuring the electron density in a plasma as proposed in Parts II and III. A resonance was detected in the sum frequency wave which is not predicted by the theory. This resonance occurs for values of the dc magnetic-field strength corresponding to cyclotron resonance at the arithmetic mean of the two incident frequencies. The origin of this resonance is not understood at this time."

K. C. Westfold (Clayton)

Brossier, C. 6884

Modèle non-linéaire d'ondes électromagnétiques progressives en présence d'un champ magnétique.

Nuclear Fusion 4 (1964), 137–144.

Author's summary: "Pour un plasma infini et homogène, on construit un modèle d'ondes progressives électromagnétiques se propageant parallèlement à un champ magnétique extérieur uniforme et constant. Dans la limite où l'amplitude de l'onde tend vers zéro, on retrouve la relation de dispersion classique avec une prescription pour éviter la singularité habituelle."

Ferrari, Carlo; Clarke, Joseph H. 6885

Flusso unidimensionale, non stazionario, e non in equilibrio di un gas in presenza di un campo di radiazioni ionizzanti. Equazioni generali.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) 35 (1963), 3–9.

The authors discuss the one-dimensional time-dependent flow of a gas which is being ionised by radiation. They derive equations describing the transport of radiant energy and the production of ions. These equations are then put in non-dimensional form.

D. R. Breach (Toronto, Ont.)

Ferrari, Carlo; Clarke, Joseph H. 6886

Nuova determinazione della fotoionizzazione a monte di un'onda d'urto intensa.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) 35 (1963), 417–424.

The paper is concerned with the ionisation produced upstream from a plane shock wave of great intensity. The gas upstream from the wave-front is sufficiently rarefied that recombination effects, due to collisions, can be neglected in comparision with those due to radiation. The radiation temperature in this region is assumed to be much greater than the gas temperature so that the ionisation due to collisions can also be neglected. It is also assumed for the upstream region that the density, the pressure, the gas temperature, and the fluid velocity relative to the wave front are all constant. Under these conditions a first integral of the radiant energy equation is derived. The variation of the radiation temperature with the degree of ionisation is considered and illustrated graphically in the case of argon. The directional dependence of the radiation intensity is also investigated.

D. R. Breach (Toronto, Ont.)

Kahn, F. D. 6887

Transverse plasma waves and their instability. II.

J. Fluid Mech. 19 (1964), 210–220.

Applying results of his earlier work [same J. 14 (1962), 321–335; MR 26 #1154], the author studies the instabilities of small phase velocity transverse plasma oscillations with unperturbed distribution functions of the form

$$f_0(u, v, w) = \phi(au^2 + bv^2 + cw^2 + 2(fuv + gvw + hwu)).$$

The fastest growing modes are described. Such instabilities in plasmas with steady flows are considered in an approximate manner.

H. Weitzner (New York)

Mosburg, E. R., Jr.; Persson, K.-B. 6888

Nonlinear ambipolar diffusion of an isothermal plasma across a magnetic field.

Phys. Fluids 7 (1964), 1829–1833.

Authors' summary: "Solutions are presented of the first two moment equations, including nonlinear terms, for the ambipolar diffusion of an isothermal plasma across a magnetic field. The two geometries considered are the plane parallel case and the infinite cylinder with axial symmetry. The Bohm criterion is automatically satisfied by the solutions. It is shown that the singularity in the space derivative of the ambipolar drift velocity at the plasma boundary cannot be removed by an axial magnetic field of any strength. Thus the plasma drift velocity and the plasma density remain monotonic functions of the position coordinate. It is also shown, under the assumptions of this theory, that the ambipolar space charge field is always directed outward and does not reverse direction in this isothermal approximation even for extremely high magnetic fields. One is forced to conclude that a realistic theory of ambipolar diffusion requires the consideration of thermal gradients within the plasma."

Peleya, Roger F. 6889

Bivariate distribution function for a plasma in a magnetic field.

Phys. Fluids 7 (1964), 1719–1720.

The position-velocity distribution function for the Brownian motion of a plasma in a magnetic field is given. Diffusion coefficients across and along the magnetic field are then given. *E. Pinney* (Berkeley, Calif.)

Trilling, Leon 6890

Asymptotic solution of the Boltzmann-Krook equation for the Rayleigh shear flow problem.

Phys. Fluids 7 (1964), 1681–1691.

This paper shows that the similar solution of the compressible Rayleigh flow problem, namely, an impulsive start of an infinite plate, can be obtained as a zero approximation from the BGK model equation. The problem was analysed previously from the continuum point of view in the references given in this paper. The present author uses Hilbert's expansion scheme to construct an asymptotic solution in the limit of small Knudsen's number. In order to produce a non-trivial zero-order solution, as well as to take into account the viscous effect properly, the coordinate and velocity perpendicular to the plate are stretched as in the ordinary boundary-layer theory. The expansion parameter, which is dictated by the boundary-layer theory, is found to be the square root of the Knudsen number (or the inverse of the square root of the Reynolds number by assuming that the Mach number is of the order unity). The zero-order equation is set to satisfy the non-slip fluid mechanical boundary conditions. As a consequence of Hilbert's procedure (in contrast to the Enskog-Chapman procedure) the differential order of the higher-order approximate equations does not increase. The author suggests slip boundary conditions to be used for the higher-order approximate equations. The boundary conditions on the detailed distribution function itself instead of the fluid mechanical quantities such as density, drift, temperature are briefly discussed at the end of the paper.

C. H. Su (Cambridge, Mass.)

Viskanta, Raymond 6891

Some considerations of radiation in magnetohydrodynamic Couette flow. (German summary)

Z. Angew. Math. Phys. 15 (1964), 227–236.

The flow between two relatively moving plane walls under the influence of a constant pressure gradient, of an electrically conducting, radiating, viscous, incompressible fluid is considered, in the presence of an external magnetic field of constant strength acting in a direction perpendicular to the walls. The effect of magnetic field on energy transfer when radiation is also present is studied (1) when the electric field E is negligible, and (2) when E is not negligible. The energy equation is solved numerically for different types of wall conditions.

S. D. Nigam (Madras)

Weitzner, Harold 6892

Plasma oscillations and Landau damping.

Phys. Fluids 6 (1963), 1123–1127.

This paper considers the problem of plane wave oscillations of a plasma in an attempt to clarify some points concerned with the existence of such oscillations and the associated mechanism of Landau damping. Following Landau's treatment [Acad. Sci. USSR J. Phys. 10 (1946), 25–34; MR 8, 401] the author uses the linearized Vlasov and Poisson equations to obtain an integral representation for the electric field for the case of a plane wave oscillation. Two classes of distribution functions are considered. The first consists of those functions which do not vanish for any particle speed and whose first derivatives vanish at only a limited number of points. The second consists of functions which vanish identically for particle speed greater than some constant. It is shown that for stable distributions of the first class, which includes Maxwellians, the electric field damps not exponentially but as some inverse power of time. For stable distributions of the second class, and for wavenumbers less than a certain constant, the plasma oscillations do have an undamped component corresponding to a conventional normal mode with a phase speed greater than the fastest particle speed. For wavenumbers larger than that constant the oscillations again tend to zero as some inverse power of time.

S. A. Berger (Berkeley, Calif.)

Levelt, J. M. H.; Cohen, E. G. D. 6893

A critical study of some theories of the liquid state including a comparison with experiment.

Studies in Statistical Mechanics, Vol. II, pp. 107–239. *North-Holland, Amsterdam; Interscience, New York,* 1964.

This is, in the reviewer's opinion, the best available review paper on liquid state theory. With perfect clarity and simplicity, the authors expose the main theories proposed up to now (they do not claim completeness), as well as all improvements on these theories; at the end of each chapter the crucial experimental evidence is discussed.

After an introduction devoted to the virial expansion and its application to dense gases, the cell model is very extensively discussed. Although quite simple in principle, this model is still probably the one which gives the best results when compared to experiment. All attempts at complicating it have given either meager partial improvements, or have even worsened the agreement with experiment. The hole theories have not been more successful. The cell-cluster theory, although correcting the main drawback of the cell model, i.e., the lack of correlations between molecules, has also led to disappointing results, because of bad convergence: small cell-clusters are insufficient to improve significantly the theory. The two main difficulties of the cell model (and indeed of any existing model) of liquids therefore remain unsolved: the explanation of a large lack in entropy and the understanding of the nature of molecular correlations.

The reading of this masterful paper is strongly recommended, not only to specialists, but to anyone interested in the present state of this great unsolved problem in physics.

R. Balescu (Brussels)

Morita, Tohru 6894

Statistical mechanics of quenched solid solutions with application to magnetically dilute alloys.

J. Mathematical Phys. 5 (1964), 1401–1405.

The atoms in quenched solid solutions and alloys are usually frozen in a metastable configuration. In theories of the properties of internal degrees of freedom of such systems, assumptions are made, either explicitly or implicitly, concerning the spatial arrangement of the component atoms. In this paper the properties of a metastable system are treated in terms of a suitably defined

fictitious equilibrium system. The formalism is applied to the case of a magnetically dilute alloy. The result for the magnetization is the same as that obtained by Brout [Phys. Rev. (2) **115** (1959), 824–835]. Application of the formalism to less dilute systems is promised.
R. J. Rubin (Washington, D.C.)

Birkhoff, G. 6895
Reactor criticality in neutron transport theory.
Rend. Mat. e Appl. (5) **22** (1963), 102–126.
The author continues his investigations [Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 567–569; MR **21** #3131] into the nature of reactor criticality in neutron transport theory. In the related topic of reactor criticality with multigroup diffusion models, the theory of positive operators (e.g., Jentsch's Theorem and the Krein-Rutman Theorem) has been a very effective tool in obtaining theoretical results [G. Habetler and M. Martino, Proc. Sympos. Appl. Math., Vol. XI, pp. 127–139, Amer. Math. Soc., Providence, R.I., 1961; MR **26** #2280]. To obtain the proper setting, the author generalizes these results to the case of a bounded operator on a Banach lattice, one of whose powers is uniformly positive (Theorem 2, p. 118), and then applies this generalization to the transport theory case of monoenergetic neutrons and isotropic scattering in a homogeneous bare reactor. Extensions to inhomogeneous reactors and nonisotropic scattering are also considered.
R. S. Varga (Cleveland, Ohio)

Boffi, V. C.; Knoke, F.; Molinari, V. G.; Scozzafava, R. 6896
Exact and asymptotic solution of the energy-dependent Boltzmann equation in the study of the neutron slowing down.
Ann. Physics **27** (1964), 283–296.
Authors' summary: "In this paper we obtain the exact and asymptotic solution of the Boltzmann equation which governs the stationary energy distribution of neutrons slowing down in an infinite homogeneous medium in the case of a general capture law. Specific results are given for particular cases of variation of the cross sections versus the neutron velocity."
G. M. Wing (Boulder, Colo.)

Boffi, V. C. 6897
Exact and asymptotic time-energy distribution of neutrons slowing down in an infinite homogeneous medium.
Ann. Physics **27** (1964), 297–330.
Author's summary: "In this paper a new method for integrating the time-energy dependent Boltzmann equation for neutrons slowing down in an infinite homogeneous medium is presented. The exact step-by-step energy distribution at the time t is constructed. The behavior of the exact distribution for small values of energy and then for large values of time is derived as the limit of the sequence of the continuous functions, which step-by-step represent the physical solution of the problem. Information on the behavior for large and small values of time of the exact step-by-step time-energy distribution is also given. How the absorption can be accounted for is finally mentioned."
G. M. Wing (Boulder, Colo.)

Depken, S. 6898
Remark to the solution of the neutron transport equation. (Italian summary)
Nuovo Cimento (10) **31** (1964), 7–9.
The author derives a closed-form infinite-medium solution to the multi-energy transport equation, but a solution valid under rather restrictive conditions. The medium contains a point mono-energetic and isotropic source. Scattering is assumed isotropic in the center of mass and the ratio of the scattering cross-section of each element to the total cross-section in the medium is taken to be energy-dependent.

The author introduces Fourier, Legendre, and Mellin transforms (apparently the Mellin transform is mistakenly referred to as a Laplace transform) which eliminate, successively, the spatial, angular and energy variables. The transformed Boltzmann equation is solved, and inversion of the transformed solution yields all Legendre components of the flux. However, these components are given by integrals which are extremely complicated, and it is not clear that the author's method is practical.

The paper is very compact and somewhat hard to follow. It seems to be a summary of work described more fully in the reference he cites.
E. M. Gelbard (Pittsburgh, Pa.)

Hajdu, Janos 6899
Sur la théorie quantique des processus de transport.
C. R. Acad. Sci. Paris **259** (1964), 1019–1021.
In this note a system of free electrons and of scattering fixed centers statistically distributed is considered in electric and magnetic fields with a slight temperature gradient. In a simple manner, starting with the linearized equation of motion of the density operator, the author derives the modified Kubo formula for the transport coefficient as well as a transport equation of Markovian type which reduces to the Boltzmann-Bloch equation in the absence of a magnetic field.
S. Ueno (Kyoto)

Mendelson, M. R.; Summerfield, G. C. 6900
One-speed neutron transport in two adjacent half-spaces.
J. Mathematical Phys. **5** (1964), 668–674.
Authors' summary: "Using Case's method for solving the one-speed transport equation with isotropic scattering, the Milne problem solution, the solution for a constant source in one half-space, and the Green's function solution are obtained for two adjacent half-spaces. These problems have been solved previously by other methods. Here the derivations are greatly simplified by using Case's method."
E. M. Gelbard (Pittsburgh, Pa.)

RELATIVITY

See also 6924, 6926.

Bialas, Andrzej 6901
Equations of motion of a rotating particle with magnetic moment.
Acta Phys. Polon. **22** (1962), 499–510.
Continuing a previous paper [same Acta **20** (1961), 831–844; MR **27** #3250], the author's thesis is as follows.

Charged point particles with spin satisfy the force equation $\dot{P}^\alpha = X^\alpha$ and the torque equation $\dot{\Omega}^{\alpha\beta} + P^\alpha u^\beta - P^\beta u^\alpha = -2X^{[\alpha\beta]}$, with $\Omega^{\alpha\beta} u_\beta = 0$. Starting from the work of Mathisson [ibid. 6 (1937), 163–200; Proc. Cambridge Philos. Soc. 36 (1940), 331–350; MR 2, 207; ibid. 38 (1942), 40–60; MR 3, 158], the author shows that his equations for particles with spin $\Omega^{\alpha\beta}$, charge, and magnetic moment can indeed be put into this form. To this end one must define P^α such that it is no longer parallel to the velocity u^α, but it is argued that this is physically reasonable. *F. Rohrlich* (Syracuse, N.Y.)

Bondi, H. 6902

The contraction of gravitating spheres.

Proc. Roy. Soc. Ser. A **281** (1964), 39–48.

In this important paper, written in the author's usual lucid style, it is first argued that the association between the nucleon number within a gravitating sphere S and the integral of the trace of the energy-momentum tensor, extended over S, is fallacious. The field equations for the contraction of S are written down and physically interpreted, the coordinates used being of the Schwarzschild type. The slow contraction of a completely opaque sphere is considered. In particular, a family of spheres generically described by the Schwarzschild interior solution is investigated, and after obtaining the particle paths it is shown that the simple $\frac{5}{3}$ power law which obtains in Newtonian theory for the pressure-density relation has to be replaced by a steeper dependence of pressure on density for high gravitational potentials. Finally, using radiation coordinates previously introduced by Bondi, van der Burg and Metzner [same Proc. **269** (1962), 21–52; MR **26** #4793], radiating contracting systems are examined, a full set of equations relating to these having been written down. A particular example derived from the Schwarzschild model is discussed.

H. A. Buchdahl (Canberra)

Bondi, H. 6903

Massive spheres in general relativity.

Proc. Roy. Soc. Ser. A **282** (1964), 303–317.

Author's summary: "The exact relativistic form of the equation of hydrostatic support by an isotropic pressure is found in an especially convenient form. A quantity u, the natural generalization of Schwarzschild's m/r ratio at the surface, is used, and it is proved that the critical value $u = \frac{1}{2}$ cannot be attained except perhaps under conditions of severe tension. It is shown that if u is everywhere less than $\frac{1}{2}$, different physical restrictions are significant in well-defined inner and outer zones as far as the attainment of high u values is concerned. A number of limiting configurations are derived for physically significant restrictions. In particular it is shown that if the density is nowhere negative, then $u < 0.485\ldots$, while if in addition the density is nowhere less than three times the pressure and nowhere increasing outwards, then $u \lesssim 0.319$."

Bonazzola, Silvano 6904

Sur le tenseur énergie-impulsion approché de Taub.

C. R. Acad. Sci. Paris **259** (1964), 2605–2608.

The author uses the approximate stress-energy tensor proposed by the reviewer [J. Mathematical Phys. **2** (1961), 787–793] to calculate the exact material plus gravitational energy for a spherically symmetric time-dependent space-time whose metric tensor satisfies certain boundary conditions at spatial infinity ($r \to \infty$). These conditions insure that the metric approaches that of Minkowski space as $r \to \infty$ and hence insure that the exact expression for the total energy is approximated by the approximate one as $r \to \infty$.

A. H. Taub (Berkeley, Calif.)

Braunss, G. 6905

Zur Schwarzschildschen Lösung in einer einheitlichen Feldtheorie. (English summary)

Z. Naturforsch. **19a** (1964), 1032–1033.

In an earlier paper [same Z. **19a** (1964), 401–405; MR **29** #1990] the author introduced the hypothesis that the components of the metric tensor of space-time are functionals of a "world-field" which was, for the sake of simplicity, taken to be a scalar Φ. The Einstein tensor is thus equated to a functional of Φ and its derivatives, chosen in such a way that the Bianchi identities give a non-linear Klein-Gordon equation for Φ. This Ansatz in the first place involves an arbitrarily disposable function $F(\Phi)$. In this paper it is shown that even if one imposes the condition that space-time be flat when $\Phi = 0$, a certain choice of F will, in the static spherically symmetric case, lead to a Schwarzschild field to within terms $O(r^{-2})$. Finally the question of determining particle trajectories is briefly discussed. *H. A. Buchdahl* (Canberra)

Bruhat, Y. [Fourès-Bruhat, Yvonne] 6906

Sur la théorie des propagateurs.

Ann. Mat. Pura Appl. (4) **64** (1964), 191–228.

Starting with the concept of tensor propagators, introduced by Lichnerowicz, the author has shown that the propagator is the sum of a measure carried by the characteristic cone and of a usual function having support inside the cone. This function is given in the form of a convergent series of functions, in which each term is given by an integral relation of its predecessor. Next, the asymptotic properties of this function have been studied for uniformly hyperbolic metric, and, with a further hypothesis tantamount to the asymptotic Minkowskian behaviour of the metric, the function tends to zero on all lines which are uniformly temporal and is summable on such lines. The result is applied to retarded electromagnetic potential of a charged particle. A further example is furnished by determining the scalar and vector propagators for the de Sitter space-time. *N. D. SenGupta* (Bombay)

Dehnen, H. 6907a

Über den Energietensor des Gravitationsfeldes.

Z. Physik **179** (1964), 76–95.

Dehnen, H. 6907b

Über den Energieinhalt statischer Gravitationsfelder nach der allgemeinen Relativitätstheorie in Newtonscher Näherung.

Z. Physik **179** (1964), 96–101.

In the first paper, the author defines the energy-momentum density of the gravitational field relative to an

arbitrarily chosen congruence of observers [cf. the reviewer, Les théories relativistes de la gravitation (Royaumont, 1959), pp. 85–91, Éditions Centre Nat. Recherche Sci., Paris, 1962; MR 29 #2036]. Exploiting analogies between this formalism and electromagnetic theory, he constructs also an energy-momentum "tensor" for the gravitational field. He treats the Schwarzschild field as an example, choosing a congruence of observers adapted to the time-like isometry.

In the second paper, he shows, with the same choice of observers, that in a weak field approximation his expression for the total field energy of a static system agrees with the Newtonian result.

F. A. E. Pirani (Waltham, Mass.)

Dewitt, Cécile Morette 6908

Fonctions de Green dans un espace de Riemann: Méthode de calcul pour les champs faibles; expression explicite, à l'ordre G, pour un champ statique à symétrie sphérique.

C. R. Acad. Sci. Paris **256** (1963), 3827–3829.

This paper calculates the first-order "tail" term or single scattering introduced into the Green's function for the vector wave equation, $g^{ab}A_{c;ab}+R_c{}^dA_d=0$, by a weak gravitational field. It is shown that in a linearised Schwarzschild metric this term is zero for the Green's function between two points of space-time, x, x', if there is insufficient time for a signal leaving one to reach the origin of the Schwarzschild field and return to the other.

S. W. Hawking (Cambridge, England)

Edelen, Dominic G. B. 6909

Circulation in relativistic continuum mechanics.

J. Math. Anal. Appl. **9** (1964), 331–335.

In this paper the author discusses a class of integrals formed from quantities that describe the behavior of material regions of space-time. The author generalizes the discussion given by the reviewer [Arch. Rational Mech. Anal. **3** (1959), 312–324; MR **21** #4732] to stress-energy tensors which may differ from those describing a perfect fluid. The author falsely claims that previous discussions have been based on a rotation tensor that is physically questionable. In the reviewer's paper the notion of rotation is described in terms of a vector in space-time whose vanishing is almost but not quite equivalent to the vanishing of the author's "Fermi rotation tensor".

A. H. Taub (Berkeley, Calif.)

Ferrarese, Giorgio 6910

Sul campo gravitazionale einsteiniano generato da una massa sferica rotante.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **32** (1962), 658–665.

Joshi, R. L.; Husain, S. I. 6911

Total radiation in Einstein's unified field theory.

Tensor (N.S.) **15** (1964), 66–73.

The state of total radiation is said to exist at a given point of a differentiable manifold V_4 of class $(C^2, C^4 \text{ pm})$ if there exists a null vector l such that

$$l_\nu R^\alpha_{\beta,\lambda\mu}+l_\lambda R^\alpha_{\beta,\mu\nu}+l_\mu R^\alpha_{\beta,\nu\lambda}=0,$$

$$l_\alpha R^\alpha_{\beta,\lambda\mu}=l_\beta R_{\lambda\mu}, \qquad l_\alpha R^\alpha_{\beta,\lambda\mu}l^\lambda=0.$$

The authors verify that these conditions can be satisfied everywhere in the case of the wave solutions of Einstein's field equations obtained by Takeno [Tensor (N.S.) **6** (1956), 69–83; MR **19**, 226; ibid. (N.S.) **7** (1957), 143–172; MR **20** #3888; ibid. (N.S.) **8** (1958), 21–37; MR **20** #3888], Hlavatý [*Geometry of Einstein's unified field theory*, Noordhoff, Groningen, 1957; MR **20** #5067] and Vaidya [Progr. Theoret. Phys. **25** (1961), 305–314; MR **22** #10735]. It is found that for a cylindrically symmetric case, the radiation condition can only be satisfied asymptotically as $1/r^2 \to 0$.

A. H. Klotz (Liverpool)

Katz, Joseph 6912

Éléments d'une théorie locale du champ gravitationnel.

C. R. Acad. Sci. Paris **259** (1964), 2609–2611.

A well-known method of introducing the gravitational field in physical theory is to imagine that each space-time point has an orthogonal frame attached to it. The lack of parallelism of neighboring frames is interpreted as gravitation [T. W. B. Kibble, J. Mathematical Phys. **2** (1961), 212–221; MR **23** #B997].

In this note, the author briefly discusses the possibility of using local frames which are not orthogonal, but only affine.

A. Peres (Haifa)

Lichnerowicz, André 6913

Champs spinoriels et propagateurs en relativité générale.

Bull. Soc. Math. France **92** (1964), 11–100.

The author extends to spinor fields the theory of propagators for covariant wave operators developed by him in previous publications [e.g., Inst. Hautes Études Sci. Publ. Math. No. 10 (1961); MR **28** #967]. As he states in the introduction, the principal results of the present work have appeared previously [C. R. Acad. Sci. Paris **252** (1961), 3742–3744; MR **23** #B987; ibid. **253** (1961), 940–942; MR **24** #B1605; ibid. **253** (1961), 983–985; MR **24** #B744].

F. A. E. Pirani (Waltham, Mass.)

Schmutzer, Ernst 6914

Spinorielle Feldtheorien und Noether-Theorem in der gekrümmten Raum-Zeit. (English summary)

Z. Naturforsch. **19a** (1964), 1027–1031.

This paper represents a continuation of the author's recent investigations concerning spinorial field theories [same Z. **15a** (1960), 355–362; MR **22** #603; ibid. **17a** (1962), 707–711; MR **27** #4182]. It is emphasized that of the two points of view that can, as was pointed out by Sommerfeld, be adopted towards spinors in general, the one according to which spinors are treated as invariants with respect to coordinate transformations possesses certain advantages, particularly when a transition to curvilinear coordinates is sought. The alternative method, which is the one which had in fact been generally followed, supposedly fails in this regard. Against the background of 4-dimensional Riemannian space-time a Noether-type theorem is deduced, the fundamental metrical quantities in the latter being the metrical spin tensor and not the usual metric tensor. This procedure rules out the use of the Belinfante tensor, but the method yields the well-known energy-momentum tensor and the correct spin integral. These results are applied to a system consisting

of combined gravitational, electromagnetic and spinor fields with special—but very brief—reference to the theories of Dirac and Heisenberg. *H. Bondi* (Pretoria)

Taub, A. H. 6915

Hydrodynamics and general relativity.

Fundamental Topics in Relativistic Fluid Mechanics and Magnetohydrodynamics (*Proc. Sympos., Michigan State Univ.*, 1962), *pp.* 21–28. *Academic Press, New York,* 1963.

The author discusses the following four topics in relativistic hydrodynamics: (1) a variational principle, and the formulation of the Einstein field equations and conservation of the stress-energy tensor from this principle; (2) the various conservation laws of relativistic fluids, their independence of the variational principle, and their interdependence for compressible non-charged fluids; (3) the "velocity potential" for isentropic irrotational relativistic fluids and the velocity four-vector U^a for such flows; (4) remarks on the intrinsic nature of discontinuities of the metric tensor. In the first topic, the author notes that the theory of relativistic fluids is concerned with a degenerate variational principle. Further, the type of variational principle used leads to no additional equations. Thus, the expressions for the characteristic values and directions of the symmetric stress-energy tensor; the relation between specific internal energy, rest density, and pressure; and the condition for isentropic flow are all independent of the variational principle and are new assumptions. However, the conservation of mass implies conservation of entropy. By studying the vorticity tensor, it is noted that for irrotational isentropic flows, U^a can be expressed in terms of the metric tensor of a special coordinate system. Hence, discontinuities in U^a (such as those due to shocks) are reflected in discontinuities of the metric tensor.

N. Coburn (Ann Arbor, Mich.)

Toró, Tiberiu I. 6916

Sur une généralisation de la connexion de spin pour le cas non local.

C. R. Acad. Sci. Paris **259** (1964), 307–309.

The notion of the spin-affine connection is generalized. The term to be added consists of a four-dimensional integral over all space-time, linear in the field to be differentiated covariantly and containing a kernel which is not further specified in the paper. This work is claimed to be in analogy to a non-local theory proposed by M. Zăgănescu and Gh. Drecin [Lucrăr. Şti. Inst. Ped. Timişoara Mat.-Fiz. 1960, 211–214].

P. G. Bergmann (New York)

ASTRONOMY

See also 6109, 6265, 6782.

Jeffreys, Harold 6917

A canonical transformation for treating small coordinates.

Monthly Notices Roy. Astronom. Soc. **127** (1963), 1–2.

Author's summary: "A simple method is given for allowing for small coordinates in a dynamical system when an approximation to them is obtainable, by direct solution of the Hamiltonian equations or otherwise."

D. Brouwer (New Haven, Conn.)

Kalicin, N. 691[illegible]

On a method of solving the equations of Newton and Einstein in celestial mechanics. (Bulgarian. Russian and English summaries)

Bălgar. Akad. Nauk. Otd. Mat. Fiz. Nauk. Izv. Fiz. Inst. s Aneb **12** (1964), 191–204.

Author's summary: "The variables of Newton's and Einstein's celestial mechanics are developed in power series with respect to the constant of the gravitational interaction (Newton's gravitational constant). Comparatively simple analytic developments are obtained which in certain cases, as for example in the problem of [illegible] bodies of Newton's universal mechanics, can lead to simple quadratures. The convergence of the new method is verified by the problem of two bodies in Newton's celestial mechanics. The calculations show that in this case even the first members of the power series given by us converge rather quickly to Kepler's ellipses, parabolas and hyperbolas.

"The novel feature in our approximate solution of the problem of n bodies within the framework of Newton's celestial mechanics is that it gives, through elementary time functions and the initial conditions of movement, the explicit movement of n bodies. Furthermore, no suppositions are made with respect to the mass of the bodies. As is well known, almost all approximate solutions of the problem of n bodies make use of the supposition that in the system there exists a central body with a mass considerably exceeding the masses of the rest of the bodies. We do not use this hypothesis, and on that account our method can also be applied to such problems as a movement of a body (with comparatively small mass) in the gravitational field of a double, triple or quadruple star. Such problems are difficult to treat by means of classical celestial mechanics.

"It is easy to show that our integrals satisfy the conservation law of the system's momentum and the movement of the system's center of gravity."

Mjačin, V. F. 69[illegible]

A strict error estimate for Störmer's method. I. (Russian. French summary)

Bjull. Inst. Teoret. Astronom. **9** (1963/64), 668–706.

Author's summary: "Il s'agit dans cet ouvrage [illegible] l'intégration numérique des équations différentielles [illegible] type $d^2x/dt^2=f(t,x)$ selon la méthode de Störmer et [illegible] l'appréciation rigoureuse de cette méthode. On représen[te] l'erreur de l'intégration sous la forme $\sum_i \Xi_{i,k}v_i$, où l'indi[ce] k correspond à l'intervalle d'intégration, les $\Xi_{i,k}$ désigne[nt] certaines solutions fondamentales des équations a[ux] variations, et les v_i contiennent les différences négligé[es], les erreurs d'arrondissement et quelques membres com[plémentaires] qui rendent la formule rigoureuse. Le § [illegible] contient la comparaison qualitative de l'appréciati[on] obtenue ici avec une autre, déjà connue [l'auteur, mê[me] Bjull. 8 (1962), 537–540]."

Musen, P.; Bailie, A. 6920

On the motion of a 24-hour satellite. (Russian summary)

The Use of Artificial Satellites for Geodesy (Proc. First Internat. Sympos., Washington, D.C., 1962), *pp.* 62–67. *North-Holland, Amsterdam*, 1963.

This paper is drawn from material already published [the authors, J. Geophys. Res. **67** (1962), 1123–1132; MR **25** #1016; Musen, ibid. **67** (1962), 313–319].

J. M. A. Danby (New Haven, Conn.)

Wilkens, Alexander 6921a

Zur Theorie der nahezu kommensurablen Bewegung der Planetoiden des Hecuba-Typus im Raum.

Bayer. Akad. Wiss. Math.-Natur. Kl. S.-B. **1963**, *Abt. II*, 101–127 (1964).

Wilkens, Alexander 6921b

Zur Theorie der nahezu kommensurablen Bewegung der Planetoiden des Hestia-Typus im Raum.

Bayer. Akad. Wiss. Math.-Natur. Kl. S.-B. **1963**, *Abt. II*, 129–153 (1964).

The problem treated is that of the motion perpendicular to Jupiter's orbital plane of an asteroid in the vicinity of a commensurability with Jupiter. The eccentricities of the orbits of both Jupiter and the asteroid are ignored. Next, the short-period terms are ignored and the equations of the variations of the elements are reduced to two equations of the first order with i, the inclination, and ζ, a long-period or critical argument, as dependent variables. The author treats these equations by using i as independent variable, eliminating the time.

{The solution of the problem, which may be reduced to that of a Hamiltonian system with one degree of freedom, is well known. In general, i will be a periodic function of the time with finite period. (The exceptional case is that of the limitational type of motion in which the period becomes infinite.) The variable i is therefore unsuitable as independent variable, and nearly all of the results of the papers are invalid.}

D. Brouwer (New Haven, Conn.)

Agostinelli, Cataldo 6922

Alcune considerazioni sul problema dei tre corpi.

Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **98** (1963/64), 355–365.

J. Meffroy [C. R. Acad. Sci. Paris **254** (1962), 818–820; MR **24** #B1200] pointed out the possibility of a new particular solution of the three-body problem in which the plane of the three bodies rolls along a cone of revolution with fixed axis and vertex at one of the bodies considered as the central body. The author shows that in general such a solution cannot exist provided that the case in which the plane of the three bodies is of fixed direction be excepted. *E. Leimanis* (Vancouver, B.C.)

Lebrun, C.; Robe, H. 6923

Note sur une extension du problème restreint des 3 corps.

Acad. Roy. Belg. Bull. Cl. Sci. (5) **50** (1964), 315–327.

The authors locate the singular points (Lagrangian or libration points) of the potential function representing a modified form of the restricted problem of three bodies in rotating coordinates. The problem involves, besides the conventional infinitesimal particle, two ellipsoids as the primaries which revolve in circular orbits forming a system which might represent a close binary configuration. Formulation includes the second Legendre polynomial but no higher. Results show that the triangular libration points might be stable and the collinear points are always unstable—in agreement with the situation occurring in the conventional restricted problem, i.e., when the primaries are point masses.

V. Szebehely (New Haven, Conn.)

Chiu, H. Y. 6924

Selected topics in modern theoretical physics.

Lectures in Theoretical Physics, Vol. VI (Summer Inst. Theoret. Phys., Univ. Colorado, Boulder, Colo., 1963), *pp.* 225–291. *Univ. Colorado Press, Boulder, Colo.*, 1964.

These lecture notes deal with the following four topics: (1) general theory of stellar structure, (2) neutrino processes in stars, (3) stellar collapse and supernovae, and (4) neutron stars and gravitational singularities.

P. Roman (Boston, Mass.)

Bonnor, W. B. 6925

On Olbers' paradox.

Monthly Notices Roy. Astronom. Soc. **128** (1964), 33–47.

Author's summary: "A formula is derived for the flux of light at a point in a general cosmological model, taking account of the obscuration of distant galaxies by nearby ones. The results are applied to several particular models, including a perpetually oscillating one. Of those models which are plausible on other grounds, none shows a conflict with the observed light-flux."

D. Brouwer (New Haven, Conn.)

Nesterov, S. V. 6926

The motion of a particle in the gravitational field of a body with periodically oscillating surface. (Russian. English summary)

Vestnik Moskov. Univ. Ser. I Mat. Meh. **1964**, *no.* 5, 89–99.

Author's summary: "In the present paper Bogoljubov's method is used to investigate the motion of a particle in the gravitational field of a liquid body whose surface oscillates under the action of the Newtonian attraction between the particles of the liquid. The amplitude of oscillations of the surface of the gravitating body serves as a small parameter of the problem. The coordinates of the particle are found with accuracy permitting the determination of the motion of apsidal points."

GEOPHYSICS

See also 6043.

Kazakova, L. É. 6927

An approximate solution of the inverse problem of the potential of a simple layer. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 5 (42), 23–29.

Given a known closed smooth surface S, the unknown of the problem considered in this paper is the continuous, non-negative density of a mass distribution (simple layer) over S, the potential created by this distribution being known over another given surface exterior to S.

In general this problem is unstable, but it becomes stable, has a unique solution and is solved in this paper under the additional condition that the total mass over S does not exceed a given constant.

E. Kogbetliantz (New York)

Nedjalkov, I. [Nedjalkov, I. P.]; Bŭrnev, P.; Germanov, M. 6928

On the inverse problem of the potential. (Bulgarian. Russian and English summaries)

Bŭlgar. Akad. Nauk. Otd. Mat. Fiz. Nauk. Izv. Fiz. Inst. s Aneb **12** (1964), 153–164.

Authors' summary: "The inverse problem of the potential consists in determining the form and location of an ore deposit of which the gravitation anomaly on the earth's surface is known. This work relates to a numerical method for solving the problem by successive approximations. Every approximation is carried out by solving one algebraic system. A case in which the anomaly does not change in some fixed direction is examined in greater detail, giving a numerical example."

Nedyalkov, I. [Nedjalkov, I. P.] 6929

Non-uniqueness of certain inverse problems in potential theory.

C. R. Acad. Bulgare Sci. **17** (1964), 781–784.

The non-uniqueness of solution of the inverse problems of potential theory is well known. The author points out that in the case of prospecting for an ore body it is possible to find the true solution S by combining gravimetric measures with an electrical prospecting method: two families of possible solutions of inverse gravimetric and inverse electrical problems coincide on S only. An example follows.

E. Kogbetliantz (New York)

ECONOMICS, OPERATIONS RESEARCH, GAMES

See also 5832, 6467, 6560, 6561, 6562, 6564, 6573.

Derman, Cyrus 6930

On sequential decisions and Markov chains.

Management Sci. **9** (1962/63), 16–24.

A system with several states passes from state to state according to probability laws determined by making one of K decisions; a cost is incurred each time a decision is made. Thus if at time t the system is in state i and the decision k is made, then the probability of passing to state j is $q_{ij}(k)$, and the cost $w_{ik} \geq 0$ is incurred. The author considers the problem of minimizing the limit superior as $t \to \infty$ of the average cost up to time t, and shows that this problem has a solution in stationary pure strategies, i.e., strategies in which the choice of the decision k depends only on the current state i and the dependence is non-probabilistic. Another problem considered is that of minimizing total (unaveraged) cost, and a similar conclusion is reached. It is shown that the problems of finding optimal strategies in the two cases are equivalent to certain linear programming problems.

The existence of an optimal stationary pure strategy for the first problem is a special case of Theorem 2 of Gillette [*Contributions to the theory of games*, Vol. 3, pp. 179–187, Princeton Univ. Press, Princeton, N.J., 1957; MR **19**, 1147] on stochastic 2-person games. The second problem is related to work of Shapley [Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 1095–1100; MR **15**, 887], also on stochastic 2-person games.

R. J. Aumann (New Haven, Conn.)

Leonov, V. V. 6931

A dynamic programming problem for multi-stage processes. (Russian)

Diskret. Analiz. No. 2 (1964), 48–55.

Consider the N-stage process defined by the recurrence relation $m_i = f(k_i, l_i, m_{i-1})$ $(i = 1, 2, \cdots, N)$, where the integer m_i denotes the state, $k_i \in \{1, 2, \cdots, K\} = Q_1$ and $l_i \in \{1, 2, \cdots, L\} = Q_2$ are the controls at stage i. Let $S_N^{k_1, \cdots, k_N}(m_0)$ be the set of all possible terminal states for the initial state m_0 and controls $k_1, \cdots, k_N$. The author is interested in the following problem: Find $(k_1, l_1), \cdots, (k_N, l_N)$ such that

$$R(m_N) = \max_{(k_1, \cdots, k_N) \in Q_1^N} \min_{m \in S_N^{k_1, \cdots, k_N}(m_0)} R(m),$$

where $R(m)$ is monotone increasing in m. From the author's introduction: "The purpose of this paper is to clarify such properties of the multistage process which will permit one, in many instances, to considerably reduce the number of checks when searching for an optimal control process."

B. Bereanu (Bucharest)

Varsavsky, Oscar 6932

Relations triples dans les programmes linéaires généralisés.

C. R. Acad. Sci. Paris **259** (1964), 2585–2588.

Given the linear programming problem: max FE $= xc$, subject to (1) $Ax \leq b$, (2) $x \geq 0$, and given the vectors b and c, which matrix $\bar{A}$ and corresponding vector $\bar{x}$ will maximize the function FE so that $\bar{x}c \geq xc$ for all solutions x of (A, b, c)! This is no longer a linear problem.

The author restricts himself to the case where $\bar{A} \in \mathfrak{A}$, the set $\mathfrak{A}$ itself being defined by linear constraints. The coefficients a_{ij} have then to satisfy the conditions:

$$\text{(3)} \qquad \sum_{i,j} \alpha_{kij} a_{ij} \geq \beta_k \qquad (k = 1, 2, \cdots, r),$$

where the α_{kij} and β_k are known.

Let y, t, z be the Lagrange multipliers associated with (1), (2), (3), respectively. It is then shown that at the maximum value of FE the following three equalities hold:

$$\sum_j c_j x_j = \sum_i y_i b_i = \sum_k z_k \beta_k.$$

From this it is concluded that the constraints (3) on the coefficients a_{ij} are affected by the "prices" z just like the "resources" b in the usual economic interpretation. That is to say that, apart from possible discontinuities, the z_k measure $\partial \text{FE}/\partial \beta_k$. Or, to put it still in another way, the z measure the "scarcity" of the β. In addition, it can be shown that $\sum_k z_k \alpha_{kij} = y_i x_j \geq 0$. Variations in the "resources"

b instead of in the elements of A lead to a similar problem: max FE $= xc$ subject to $Ax \leq b$, $x \geq 0$, and the linear constraint

$$(4) \qquad \sum \alpha_{ki} b_i \leq \gamma_k \qquad (k = 1, 2, \cdots, r).$$

If t_k are the Lagrange multipliers associated with the constraints (4), the analogous result is that at the maximum value of FE one has the "triality":

$$\sum_j c_j x_j = \sum_i b_i y_i = \sum_k \gamma_k t_k,$$

as well as the equality

$$\sum_k t_k \alpha_{ki} = y_i \qquad (i = 1, 2, \cdots, m).$$

Similarly, if the "costs" c are permitted to vary subject to the conditions:

$$(5) \qquad \sum_j \alpha_{kj}' c_j \leq \gamma_k'$$

and the t_k' are the Lagrange multipliers associated with the constraints (5), the result is that at the maximum value of FE the "triality" holds: $\sum_j c_j x_j = \sum_i b_i y_i = \sum_k \gamma_k' t_k'$, as well as the equality $\sum_k t_k' \alpha_{kj}' = x_j$ ($j = 1, 2, \cdots, n$). Finally, if the elements a_{ij}, the resources b_i and the costs c_j are allowed to vary simultaneously and independently of each other, the result is that the following five equalities hold:

$$\mathrm{FE} = \sum_j c_j x_j = \sum_i b_i y_i = \sum_k z_k \beta_k = \sum_k t_k' \gamma_k' = \sum_k t_k \gamma_k.$$

H. F. Karreman (Madison, Wis.)

Thedéen, Torbjörn 6933

A note on the Poisson tendency in traffic distribution.

Ann. Math. Statist. **35** (1964), 1823–1824.

The author extends results of Breiman [same Ann. **34** (1963), 308–311; MR **26** #2326] to show that the number of cars in n disjoint, non-overlapping intervals tends to a multivariate Poisson distribution in the limit of large time, under Breiman's hypotheses.

G. Weiss (Bethesda, Md.)

Verbovskiĭ, B. S. 6934

The existence of a solution to a many-index problem in linear programming. (Russian)

Dokl. Akad. Nauk SSSR **158** (1964), 763–766.

Solvability conditions are established for the following "many-index" linear program: Minimize

$$L = \sum_{k \in M} p_{i_1 \cdots i_s} x_{i_1 \cdots i_s},$$

the variables being subject to the restrictions

$$\sum_{k \in M_j} a^{(j)}_{i_1 \cdots i_s} x_{i_1 \cdots i_s} \leq b_{M_j(i_1 \cdots i_s)},$$

$$= b_{M_j(i_1 \cdots i_s)};$$

$x_{i_1 \cdots i_s} \geq 0$; $j = 1, \cdots, l$; $i_k = 1, \cdots, n_k$; $k = 1, \cdots, s$. Here $a^{(j)}_{i_1 \cdots i_s}$, $p_{i_1 \cdots i_s}$ are given real numbers, the $b_{M_j(i_1 \cdots i_s)}$'s are given non-negative numbers.

P. L. Ivănescu (Bucharest)

de Cani, John S. 6935

A dynamic programming algorithm for embedded Markov chains when the planning horizon is at infinity.

Management Sci. **10** (1963/64), 716–733.

R. A. Howard's policy-space approximation technique of dynamic programming over a Markov chain [*Dynamic programming and Markov processes*, Technology Press of M.I.T., Cambridge, Mass., 1960; MR **22** #9287] is extended to Markov renewal processes with a finite number of states (in addition to the Markovian transitions, the times between successive transitions from state i to state j are non-negative, independent and identically distributed random variables whose distribution depends upon i and j). A return function linear in time, with a fixed return at the end of the transition interval, is assumed. For each state i a choice of a policy from among k_i possible policies determines the return functions, the transition probabilities, and the expected intertransitions times corresponding to that particular state only.

Two models with infinite planning horizon are studied and algorithms for determining the optimal policies, which reduce to Howard's for common cases, are given and their convergence to optimal policies is proved. In the first model the time rate of return is taken as objective function and both ergodic and non-ergodic chains are dealt with. Two numerical examples illustrate the algorithm. In the second model with continuous discounting, the present value of the expected return from the process is maximized. The illustrative numerical example is as elaborate and suggestive as the other two.

The author points out that W. S. Jewel [Operations Res. **11** (1963), 938–948; MR **29** #677; ibid. **11** (1963), 949–971; MR **29** #678] has independently developed dynamic programming algorithms similar to those contained in the reviewed paper.

B. Bereanu (Bucharest)

Balaş, E.; Ivănescu, P. [Ivănescu, Petru L.] 6936

On the transportation problem. II.

Cahiers Centre Études Recherche Opér. **4** (1962), 131–160.

The first four chapters were reviewed earlier [E. Balaş and P. Ivănescu, same Cahiers **4** (1962), 98–116; MR **26** #7447; E. Balaş and L. P. Hammer, Acad. R. P. Romîne Stud. Cerc. Mat. **11** (1960), 439–450; MR **27** #1305; E. Balaş and P. Ivănescu, ibid. **12** (1961), 413–427; MR **27** #1306; ibid. **12** (1961), 429–435; MR **27** #1307; E. Balaş and L. P. Hammer, Com. Acad. R. P. Romîne **11** (1961), 1047–1049; MR **27** #1308]. The authors show that solving the problem of minimizing $\sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij}$ subject to $\sum_{j=1}^n x_{ij} = a_i$ ($i = 1, \cdots, m$), $\sum_{i=1}^m x_{ij} = b_j$ ($j = 1, \cdots, n$), $\sum_{j=1}^r x_{ij} \leq e_i$ ($i = 1, \cdots, m$; $0 < r < n$), $x_{ij} \geq 0$, is equivalent to solving the following ordinary transportation problem: Minimize $\sum_{h=1}^{2m} \sum_{j=1}^n c_{hj}' x_{hj}'$ subject to $\sum_{j=1}^n x_{hj}' = a_h'$ ($h = 1, \cdots, 2m$), $\sum_{h=1}^{2m} x_{hj}' = b_j'$ ($j = 1, \cdots, n$), $x_{hj}' \geq 0$, where

$$\begin{aligned} a_h' &= e_h & (h &= 1, \cdots, m), \\ &= a_{h-m} - e_{h-m} & (h &= m+1, \cdots, 2m), \\ b_j' &= b_j & (j &= 1, \cdots, n), \end{aligned}$$

$$\begin{aligned} c_{hj}' &= c_{hj} & (h &= 1, \cdots, m; j = 1, \cdots, n), \\ &= c_{h-m,j} & (h &= m+1, \cdots, 2m; j = r+1, \cdots, n), \\ &= C & (h &= m+1, \cdots, 2m; j = 1, \cdots, r), \end{aligned}$$

and C is sufficiently large.

The generalization to the case where one has the additional restriction $\sum_{i=1}^s x_{ij} \leq f_j$ ($j = 1, \cdots, n$; $0 < s < m$) is stated, and examples are provided.

L. Lorentzen (Berkeley, Calif.)

Ivanescu, Petru L. [Ivănescu, Petru L.]; Rosenberg, I. 6937

Application of pseudo-Boolean programming to the theory of graphs.

Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **3**, 163–176 (1964).

A pseudo-Boolean function is a function F with domain the n-tuples $(x_1, \cdots, x_n)$, each $x_i = 0$ or 1, the x_i obeying the rules of operation of a Boolean algebra, and with range the real numbers. "Pseudo-Boolean Programming" is a term coined by the authors for a recursive scheme for minimizing such functions. In short the scheme is this. Given $F(x_1, \cdots, x_n)$, write it as $F = F_1(x_1, \cdots, x_n) = x_1\{F_1(1, x_2, \cdots, x_n) - F_1(0, x_2, \cdots, x_n)\} + F_1(0, x_2, \cdots, x_n) = x_1 g_1(x_2, \cdots, x_n) + h_1(x_2, \cdots, x_n)$. Clearly, if F_1 is to be minimized, then $g_1 > 0$ implies $x_1 = 0$; or, letting $S_1 = \{(x_2^*, \cdots, x_n^*) | g_1(x_2^*, \cdots, x_n^*) \leq 0\}$, $x_1 = 1$ only if $x_1^* = \sum_{S_1} \tilde{x}_2 \cdots \tilde{x}_n = 1$, where $\tilde{x}_i = x_i$ if $x_i^* = 1$ and $\tilde{x}_i = 1 - x_i$ if $x_i^* = 0$. Then, substitute $x_1 = x_1^*$ into F_1 to obtain $F_2(x_2, \cdots, x_n)$, a function in one fewer variables. Repeat to obtain finally $F_n(x_n) = x_n g_n + h_n$, g_n and h_n constants. If $g_n > 0$, let $x_n = 0$; if $g_n < 0$, let $x_n = 1$; if $g_n = 0$, take $x_n = 0$ or 1; then, iteratively, find $x_{n-1}, \cdots, x_1$ (there may be many solutions).

In this paper the authors apply this scheme to compute the chromatic number, the number of internal stability, the number of external stability and the kernels of a finite graph (for definitions of these, see Berge [*Théorie des graphes et ses applications*, Dunod, Paris, 1958; MR **21** #1608; English transl., Methuen, London, 1962; MR **24** #A2381]) by formulating appropriate pseudo-Boolean functions whose minima yield the required answers.

M. L. Balinski (Princeton, N.J.)

Wong, Eugene 6938

A linear search problem.

SIAM Rev. **6** (1964), 168–174.

A one-dimensional search problem is considered. A list $(x_i, f(x_i))$, $i = 1, \cdots, n$, is stored, the $\{x_i\}$ forming an ordered set with $x_j > x_k$ if $j > k$. It is desired to find $f(x)$ for some x in the list, given that the search proceeds by comparing x with some sequence of chosen arguments $\{x_{i_k}\}$. The problem is to choose the sequence in order to minimize the number of comparisons needed to find x. The author solves the problem through the recursive, functional equation approach of dynamic programming.

M. L. Balinski (Princeton, N.J.)

Bernard, Georges 6939

Réduction du paradoxe de Saint-Pétersbourg par la théorie de l'utilité.

C. R. Acad. Sci. Paris **258** (1964), 3168–3170.

In the Saint Petersburg Paradox the probability is $p = 2^{-n}$ that the payoff is $v = 2^n$ (for $n = 1, 2, \cdots$). By introducing a utility function $U(v, p) = v^a p^c$, the author computes the fair entrance fee to the game as $1/(1 - 2^{a-c})$, for $a < c$. Some discussion of the reasonableness of this result is given.

H. D. Block (Ithaca, N.Y.)

Bereanu, Bernard 6940

Régions de décision et répartition de l'optimum dans la programmation linéaire.

C. R. Acad. Sci. Paris **258** (1964), 1383–1386.

Let $t = (t_1, \cdots, t_r) \in T = \{t | \delta_j \leq t_j \leq \delta_j', 1 \leq j \leq r\}$ be a stochastic r-vector and $c_0, c_1, \cdots, c_r$ constant n-vectors. Define $c = c_0 + \sum_{i=1}^r c_i t_i$, and consider the linear program (I): Find $\gamma(t) = \min_{x \in X} c'(t)x$; $X = \{x | Ax = b, x \geq 0\} \neq \varnothing$, with the usual meaning for A, b, and x. Then, if X is bounded and all bases are non-degenerate, there exists a set of convex polyhedral sets $S_l \subset T$ $(1 \leq l \leq q)$, called "regions of decision" of (I), associated with a set of basic solutions $\hat{x}_l$ corresponding to basis B_l and a set of $(r+1)$-vectors $\alpha_l = (\alpha_{0l}, \alpha_{1l}, \cdots, \alpha_{rl})$ $(l = 1, 2, \cdots, q)$, such that (i) $\bigcup_l S_l = T$, (ii) $t \in S_l$ implies $\gamma(t) = c(t)\hat{x}_l = \alpha_{0l} + \alpha_{1l}t_1 + \alpha_{2l}t_2 + \cdots + \alpha_{rl}t_r$, (iii) $i \neq j$ implies $P\{t | t \in S_i \cap S_j\} = 0$. The regions S_l are described by systems of linear inequalities and the distribution function, the mean of $\gamma(t)$ and the probability that a given basis be optimal are obtained in terms of the probability distribution of t, the vectors α_l and the regions S_l. Finally, similar developments are obtained via duality, i.e., the vector b is assumed to be a stochastic variable given in the form $b = b_0 + \sum_{i=1}^r b_i t_i$ and the rest of the program coefficients to be constant. The proofs of all these results are to be published in a forthcoming paper.

A. G. Azpeitia (Amherst, Mass.)

Descloux, Jean 6941

Note on convex programming.

J. Soc. Indust. Appl. Math. **11** (1963), 737–747.

The author considers the problem of finding a point $\bar{x} \in E_n$ which minimizes the convex function $F(x) \equiv \max_{A \in \Omega}\{[A, x] - b(A)\}$ subject to $G(x) \equiv \max_{A \in \Lambda}\{[A, x] - c(A)\} \leq 0$. Here b and c are continuous real-valued functions on E_n, $[A, x]$ denotes the scalar product and the sets Ω and Λ of E_n are supposed to satisfy the Haar condition: each set of n vectors of $\Omega \cup \Lambda$ are linearly independent. The author describes an iterative method which yields a converging sequence x_k with $\lim x_k = \bar{x}$, if Ω and Λ are (possibly infinite) compact sets. His algorithm is a generalization of a method proposed by Cheney and Goldstein [Numer. Math. **1** (1959), 253–268; MR **22** #316] in the case of finite sets Ω and Λ. It consists in choosing a sequence of "bases" $I_k = \{A_k^1, \cdots, A_k^{n+1}\} \subseteq \Omega \cup \Lambda$, $I_k \cap \Omega \neq \varnothing$ so that (1) $I_k \cap I_{k+1}$ contains n elements, (2) 0 lies in the convex hull of I_k, and (3) the values $\mu_k = F_{I_k}(x_k)$ at the optimal points x_k of the restricted problem of minimizing $F_{I_k}(x) \equiv \max_{A \in I_k \cap \Omega}\{[A, x] - b(A)\}$, subject to $G_{I_k}(x) \equiv \max_{A \in I_k \cap \Lambda}\{[A, x] - c(A)\} \leq 0$, increase strictly if $x_k \neq \bar{x}$.

J. Stoer (Munich)

Hanson, M. A. 6942

Stochastic non-linear programming.

J. Austral. Math. Soc. **4** (1964), 347–353.

Let Y be a subset of E_m and let $y \in Y$ have the scalar distribution $\psi(y)$, where ψ is C'' on Y. Let X be a subset of E_n and let $S = X \times Y$. Let $\phi(x, y)$ be a scalar-valued function on S with range in E_r and let $h(x, y)$ be a vector-valued function on S with range in E_s. The problem considered is the following one. Determine an x_0 in X that minimizes $\mathscr{E}\phi(x, y)$ subject to the constraints $\mathscr{E}g(x, y) \geq 0$ and $h(x, y) \geq 0$ for all y in Y, where

$$\mathscr{E}f(x, y) = \int_Y f(x, y)\psi(y)\, dy.$$

The analysis of the problem is carried out under certain assumptions on the constraints g and h.

The first theorem of the paper gives necessary conditions for a minimizing x_0, analogous to the multiplier rule and the Kuhn-Tucker conditions. The second theorem asserts that under appropriate concavity and convexity hypotheses on ϕ, g, and h, the necessary conditions are also sufficient. The third theorem is a duality theorem that asserts that under the convexity and concavity hypotheses of Theorem 2, the minimizing x_0 is also the maximizing x_0 for an appropriately defined maximizing problem.

{Reviewer's note: The statements of the three theorems are incorrect. Statement (4) of Theorem 1 should read $\mathscr{E}(\phi_x+\lambda g_x)+\mu(y)h_x(x,y)=0$, and appropriate changes should be made in the corresponding statements of Theorems 2 and 3. The various proofs can be readily modified to accommodate the correct statements. The reviewer finds the proof that $\lambda\leq 0$ and $\mu\leq 0$ somewhat obscure. In particular, the ability to choose x^* as described on page 350 is not clear, since the condition $\|x-x^*\|^2=o(|g^j(x,y)-g^j(x^*,y)|)$ implies that if $x\neq x^*$, then $g^j(x,y)\neq g^j(x^*,y)$ for all components g^j of g such that $g^j(x,y)=0$. Finally, there seems to be some confusion between $\mathscr{E}g^j(x,y)$ and $g^j(x,y)$ throughout the proof of Theorem 1 on page 350.}

L. D. Berkovitz (Lafayette, Ind.)

Messikommer, B. 6943

Die Optimierung eines halbkontinuierlichen chemischen Reaktors mittels dynamischer Programmierung.

Analogue Computation Applied to the Study of Chemical Processes (*Proc. Internat. Seminar, Brussels*, 1960), *pp.* 16–21. *Presses Acad. Européennes, Brussels*, 1961; *Gordon and Breach, New York*, 1962.

In chemical engineering it frequently happens that the output of one stage of processing is used as the input to the next. This suggests that it may be fruitful to view them as dynamic programming processes for the purpose of optimization. Beginnings have been made in recent books [cf., e.g., R. Aris, *The optimal design of chemical reactors*, Academic Press, New York, 1961; MR **23** #B473]. This interesting and well-written paper shows how to formulate the reaction rate and mass balance equations, apply Bellman's principle of optimality, and employ a Lagrange multiplier to simplify the calculations. It is suggested that an analogue computer (with some auxiliary logical circuitry) be used, and some numerical results are quoted. The paper is accessible to both mathematicians and chemical engineers.

R. Kalaba (Santa Monica, Calif.)

Mangasarian, O. L.; Stone, H. 6944

Two-person nonzero-sum games and quadratic programming.

J. Math. Anal. Appl. **9** (1964), 348–355.

Authors' summary: "It is shown that a necessary and sufficient condition that a point be a Nash equilibrium point of a two-person nonzero-sum game with a finite number of pure strategies is that the point be a solution of a single programming problem with linear constraints and a quadratic objective function that has a global maximum of zero. Every equilibrium point is a solution of this programming problem. For the case of a zero-sum game, the quadratic programming problem degenerates to the well-known dual linear program associated with the game."

The quadratic objective function is not necessarily concave, but the authors suggest that, on the basis of computational experience, the gradient projection method works satisfactorily nevertheless; since the value of the global maximum is known, local maxima may easily be rejected.

K. J. Arrow (Stanford, Calif.)

Tsetlin, M. L. [Cetlin, M. L.]; Krylov, V. Yu. [Krylov, V. Ju.] 6945

Examples of games with automata.

Dokl. Akad. Nauk SSSR **149** (1963), 284–287 (*Russian*); *translated as Soviet Physics Dokl.* **8** (1963), 232–234.

This short paper gives a general definition and two examples of games in which the players are finite automata playing a game repeatedly. For each play of the game a strategy is chosen which determines the probability of a win or loss of one unit for each player; information on the wins or losses in any one play is used to determine the strategy for the following play.

In one of the examples, a game is defined between two automata in which one of the players wins and the other loses in every play. The players are assumed to be employing a type of strategy called a "linear tactic", which was defined in a previous paper [Cetlin, same Dokl. **149** (1963), 52–53; MR **26** #3531]. For such a game a limiting expectation W is shown to exist and to satisfy certain properties depending on the payoff matrix. If the payoff matrix has a row (or column) with all entries non-negative, then W is said to be equal to the harmonic mean of the elements of this row (or column), otherwise $W=0$.

J. H. Griesmer (Yorktown Heights, N.Y.)

Bartoszyński, Robert 6946

On a certain concept of value of information in games.

Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (*Liblice*, 1962), *pp.* 29–33. *Publ. House Czech. Acad. Sci., Prague*, 1964.

The author restates the definitions of game value and optimal strategies in terms of a quantity that measures what happens to the sup-inf payoff of a player when the other player unilaterally restricts the set of mixed strategies at his disposal.

R. J. Aumann (New Haven, Conn.)

Fleming, Wendell H. 6947

The convergence problem for differential games. II.

Advances in game theory, pp. 195–210. *Princeton Univ. Press, Princeton, N.J.*, 1964.

In a differential game, the motion of a point ξ in euclidean space E^r during a time interval T is influenced by strategic choices made by two players at each instant of T. The payoff is $U[\xi(T)]+\int_0^T f(\xi(t), y(t), z(t))\,dt$, where y and z are the strategic variables under the control of the two players; the way in which y and z affect the motion of ξ may be described by a differential equation $\dot{\xi}=g(\xi,y,z)$. Rigorizing the above description is a non-trivial task. The approach used by the author is to divide T into N equal time units and to allow the players to make their choices at the beginning of each unit only; denote the value of such a "discretized" game by V_N. In a previous

paper [J. Math. Anal. Appl. **3** (1961), 102–116; MR **25** #5806] the author showed that V_n converges as $n\to\infty$, if certain rather restrictive assumptions are made about f and g. In this paper convergence is proved without any of those restrictive assumptions. The proof makes use of a number of recent results on non-linear parabolic differential equations.

R. J. Aumann (New Haven, Conn.)

Thrall, R. M.; Lucas, W. F. 6948

n-person games in partition function form.

Naval Res. Logist. Quart. **10** (1963), 281–298.

Let N be a set with n members, called players. A partition of N is a family of disjoint subsets of N whose union is N. A partition function F assigns to each partition P and each member P_i of P a real number $F_P(P_i)$; a pair (N, F) is called a game in partition function form. The partition function generalizes the von Neumann-Morgenstern (N-M) characteristic function [von Neumann and Morgenstern, *Theory of games and economic behavior*, Princeton Univ. Press, Princeton, N.J., 1944; MR **6**, 235]; unlike the characteristic function, the partition function allows the payoff to a coalition M to depend on how the players outside M have combined into coalitions.

The notions of imputation, domination, effectiveness, core and solution are defined in a manner analogous to the N-M definitions. The results, however, are quite different. For example, the standard simplex of imputations in the N-M theory is replaced by a finite number of parallel simplices, and a single solution may have pieces in several of these simplices.

All solutions are found for all 2- and 3-person games, and for games in which $F_{(N)}(N)$ is sufficiently large compared to the other values of F (in which case the other values do not matter). The most interesting games discussed are the general 3-person games; it turns out that in comparison with the N-M 3-person results, the individual solutions are often larger, but there are usually fewer solutions and often the solution is unique.

R. J. Aumann (New Haven, Conn.)

Stearns, R. E. 6949

Three-person cooperative games without side payments.

Advances in game theory, pp. 377–406. *Princeton Univ. Press, Princeton, N.J.*, 1964.

The author proves that every 3-person cooperative game without side payments in the sense of the reviewer and Peleg [Bull. Amer. Math. Soc. **66** (1960), 173–179; MR **22** #10802] has a von Neumann-Morgenstern solution, and characterizes all the solutions to each such game. This is a considerable generalization of results of Peleg [Trans. Amer. Math. Soc. **106** (1963), 173–179; MR **26** #1211], who treated the 3-person constant-sum case.

R. J. Aumann (New Haven, Conn.)

Burger, E. 6950

Bemerkungen zum Aumannschen Core-Theorem.

Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **3**, 148–153 (1964).

A simplification of the theory of cooperative games without side payments due to Peleg and the reviewer [Bull. Amer. Math. Soc. **66** (1960), 173–179; MR **22** #10802] (the AP theory) is presented. In the author's approach, a game is characterized entirely by its characteristic function, and the compact polyhedron H of "actually feasible" payoff vectors in the AP theory is replaced by $v(N)$, where N is the set of all players. The author shows that many of the theorems of AP theory carry over to his version. In particular, in a given game let $\bar{E}$ be the set of feasible payoff vectors (H in the AP theory, $v(N)$ in the author's); E and $\bar{A}$ the sets of group rational and individually rational members of $\bar{E}$, respectively; and $A = E\cap\bar{A}$. The reviewer has shown [Trans. Amer. Math. Soc. **98** (1961), 539–552; MR **23** #B483] that the cores associated with all four of these sets coincide in the AP theory. The author shows that a much simpler proof of this theorem can be given in his theory.

R. J. Aumann (New Haven, Conn.)

Radström, Hans 6951

A property of stability possessed by certain imputations.

Advances in game theory, pp. 513–529. *Princeton Univ. Press, Princeton, N.J.*, 1964.

Let v be a constant-sum superadditive characteristic function on a player set I. A coalition structure (c.s.) is a family F of subsets of I with the properties: $I\in F$, $\varnothing\notin F$; every S in F with more than one player splits uniquely into a pair of elements of F (called the parties of S, and co-parties of each other); every S in F other than I is a party of precisely one member of F. Intuitively, the members of F are bargaining units; once it is determined that an S in F receives the total payoff $w(S)$, the parties of S bargain for their share of $w(S)$, and this process continues until the payoff of each player is determined. The author proves that with each c.s. F it is possible to associate an imputation w so that: (1) If I_1 and I_2 are the parties of I, then $w(I_1)=v(I_1)$ and $w(I_2)=v(I_2)$, where $w(S)=\sum_{i\in S} w_i$; (2) Let S_1 and S_2 be parties of an S in F, S_3 the co-party of S, F_1 $[F_2]$ the c.s. obtained from F by deleting S and adding $S_1\cup S_3$ $[S_2\cup S_3]$, and w_1 $[w_2]$ the imputation associated with F_1 $[F_2]$; then $w(S_1)=\frac{1}{2}\{w(S)+w_1(S_1\cup S_3)-w_2(S_2\cup S_3)\}$, and similarly for S_2. Intuitively, (2) says that S_1 and S_2 share $w(S)$ in such a way that any threat by S_1 $[S_2]$ to get a larger payoff by teaming up with S_3 can be matched with similar arguments by S_2 $[S_1]$. The w satisfying (1) and (2) is unique, and the author derives a formula for it in terms of v.

R. J. Aumann (New Haven, Conn.)

Pears, A. R. 6952

On topological games.

Proc. Cambridge Philos. Soc. **61** (1965), 165–171.

The games considered are those introduced by Berge [*Contributions to the theory of games*, Vol. 3, pp. 165–178, Princeton Univ. Press, Princeton, N.J., 1957; MR **19**, 1147], converted into two-person constant-sum games by confining the investigation to maximin questions. The author finds two systems of sufficient conditions for the existence of maximin strategies.

J. R. Isbell (New Orleans, La.)

Kuznecov, I. N. 6953

Optimal distribution of bounded means for convex-concave payoff functions. (Russian)

Izv. Akad. Nauk SSSR Tehn. Kibernet. **1964**, no. 5, 100–106.

Author's summary: "Conditions are found, the fulfillment of which guarantees an optimal distribution $\varphi(x)$ of bounded mean Φ over the space X. The distribution is called optimal if it gives the largest value to the mean payoff. The investigation is carried out for a known a priori probability $p(x)\,dx$ obtained from the payoff $P(\varphi, x)\,dx$, where the payoff function $P(\varphi, x)$ has an interval $(0, \varphi_1)$ of convexity downward and the interval (φ_1, ∞) of convexity upward."

W. Fleming (Providence, R.I.)

Hershkowitz, Martin 6954

A computational note on von Neumann's algorithm for determining optimal strategy.

Naval Res. Logist. Quart. **11** (1964), 75–78.

The author compares the speeds of convergence of the following forms of iterative methods for solving matrix games by applying them to some numerical examples: (1) von Neumann's Predictor-Corrector Algorithm, (2) The Simplex Method, (3) Brown's Fictitious Play, and (4) A Relaxation Method. He draws the conclusion that the von Neumann algorithm is inferior to the others.

L. Lorentzen (Berkeley, Calif.)

BIOLOGY AND BEHAVIORAL SCIENCES

Ewens, W. J. 6955

The diffusion equation and a pseudo-distribution in genetics.

J. Roy. Statist. Soc. Ser. B **25** (1963), 405–412.

Consider the distribution function $F(x;t)$ $(0 \leq x \leq 1)$ of a variate x at time t whose density function $f(x,t)$ satisfies the forward Kolmogorov equation

$$\frac{\partial f(x;t)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}\{v(x)f(x;t)\} - \frac{\partial}{\partial x}\{m(x)f(x,t)\}.$$

Here $m(x)$ and $v(x)$ are the instantaneous drift and diffusion coefficients, respectively. The particular case $m(x) = x(1-x)\phi(x)$, $v(x) = ax(1-x)$, where $\phi(x)$ is an arbitrary polynomial (possibly constant or zero) and a is a constant, is found in genetical applications. This implies that $x=0$, $x=1$ are "exit" boundaries. This corresponds in genetics to the fixation of a certain gene in a population.

If we solve the "stationary" equation

$$\frac{1}{2}\frac{d^2}{dx^2}\{v(x)f(x)\} - \frac{d}{dx}\{m(x)f(x)\} = 0,$$

we obtain successively

$$\frac{1}{2}\frac{d}{dx}\{v(x)f(x)\} - m(x)f(x) = c_1,$$

and

$$f(x) = \frac{c_2\psi(x)}{v(x)} + \frac{2c_1}{v(x)}\psi(x)\int^x \{\psi(y)\}^{-1}\,dy,$$

which we may alternatively write (if $c_1 \neq 0$)

$$f(x) = \frac{2c_1}{v(x)}\psi(x)\int_c^x \{\psi(y)\}^{-1}\,dy,$$

where the constant c is related to c_2.

An interpretation of $f(x)$ as a description, in a certain sense, of the transient behaviour of the diffusion process is given. Interpretations of other solutions, obtained by a different allocation of constants, are given. An analogous function for a certain discrete process is exhibited.

R. G. Stanton (Waterloo, Ont.)

Osadnik, Lucie 6956

Kybernetische Modelle in der Demographie.

Mathematische und physikalisch-technische Probleme der Kybernetik, pp. 97–103. *Akademie-Verlag, Berlin*, 1963.

Cybernetic (self-regulating) models are employed for the elucidation of several well-known relations in biometrics, e.g., sex-ratio at birth, influence in changes in pre- and post-natal mortality and the like.

C. J. Maloney (Bethesda, Md.)

Kruskal, J. B. 6957

Multidimensional scaling by optimizing goodness of fit to a nonmetric hypothesis.

Psychometrika **29** (1964), 1–27.

Let E be a metric space with metric d. Let there be given a rank order $\succ$ on the $\frac{1}{2}n(n-1)$ pairs (i,j) of integers from 1 to n with $i<j$. A configuration is an n-tuple $X = (x_1, \cdots, x_n)$ of points in E, not all equal; it is a perfect scale for the given order if $(i,j) \succ (k,m)$ is equivalent to $d(x_i, x_j) > d(x_k, x_m)$. There may not be a perfect scale. For each X, the author defines the stress $S(X)$ as the infimum of $[\sum_{i<j}(d(x_i,x_j) - \hat{d}_{ij})^2/\sum_{i<j} d^2(x_i,x_j)]^{1/2}$, taken over all choices of $\frac{1}{2}n(n-1)$ numbers $\hat{d}_{ij}$ such that $\hat{d}_{ij} > \hat{d}_{km}$ is equivalent to $(i,j) \succ (k,m)$; $S(X)$ measures the failure of X to be a perfect scale. The stress of the order is defined as the minimum of $S(X)$ over all X; it measures the failure of the order to have a perfect scale in E. E is usually taken to be a euclidean space of fixed dimension with one of the l^p metrics, $1 \leq p \leq \infty$. This definition provides a basis for techniques in psychology where dissimilarities between n objects are matched by distances between n points. A novelty of the author's approach is that only the rank order of the dissimilarities is used, not any cardial measure of their size. Examples are given.

R. J. Aumann (New Haven, Conn.)

Kruskal, J. B. 6958

Nonmetric multidimensional scaling: A numerical method.

Psychometrika **29** (1964), 115–129.

Numerical methods are described for calculating the stress $S(X)$ defined by the author in a previous paper [see #6957 above] and minimizing it over all X.

R. J. Aumann (New Haven, Conn.)

INFORMATION, COMMUNICATION, CONTROL

See also 5717, 5725, 5726, 6126, 6355, 6565, 6567, 6660, 6661, 6943, 6945, 6947, 6953.

Bar-Hillel, Yehoshua 6959

★Language and information. Selected essays on their theory and application.

Addison-Wesley Series in Logic.

Addison-Wesley Publishing Co., Inc., Reading, Mass.-London; The Jerusalem Academic Press, Ltd., Jerusalem, 1964. x + 388 pp. $12.50.

This volume contains a reprinting of twenty of the author's papers, and is arranged as follows. Part I: four papers on the theoretical aspects of language; Part II: five papers on algebraic linguistics; Part III: five papers on machine translation; Part IV: three papers on semantic information; Part V: three papers on the mechanization of information retrieval.

In a 16-page introduction the author reminisces a bit about the writing of various of the papers. More important, the author recounts his experiences with machine translation and the widely discussed problem of information retrieval (which has become stigmatized by the clichés "publication explosion" and "information and documentation explosion"), and the last paper in Part V (entitled "Is Information Retrieval Approaching a Crisis?" and reprinted from American Documentation 14 (1963), 95–98) provides a sensible contrast to many of the amateur *coups d'œil* (e.g., Derek J. de Solla Price, *Little science, big science* [Columbia Univ. Press, New York, 1963]), which now seem to constitute a second-level publication explosion. *A. J. Lohwater* (Providence, R.I.)

Oettinger, A. G. 6960

An essay in information retrieval or the birth of a myth.
Information and Control 8 (1965), 64–79.

Author's summary: "A frequently quoted and still current myth on the ill effects of the so-called information explosion is analyzed. Five years and 250,000 dollars were allegedly spent in the United States to duplicate the result published in the Soviet Union in 1950. Not only does this myth rest on a comedy of errors, but the trivial results in question were surprisingly well known to all specialists most concerned, in at least one instance as early as 1937."

The comedy of errors refers to the use made of a paper of A. G. Lunc [Dokl. Akad. Nauk SSSR 70 (1950), 421–423; MR 11, 574] by over-zealous proponents in the area of information retrieval [for some interesting comments on such zealotry see some of the reminiscences in #6959 reviewed above]. As Oettinger points out, the myth was even carried to the floor of the U.S. Congress by the Hon. Roman C. Pucinski, who asserted that a "team of top-notch mathematicians" worked for five years to obtain an independent solution of a relatively trivial and unimportant result. The paper continues with a selection from the luxuriant growth of the second-level publication concerning scientific publication, so thickly has this rainforest proliferated that highly relevant references have been overlooked [e.g., R. P. Boas, Jr., Science 125 (1957), 1260], fortunately, however, without the loss of enough money to cause comment on the floor of the U.S. Congress. The question may be asked: Would it have been cheaper to train this "team of top-notch mathematicians" in the simplest research techniques, the use of journals and reference tools, etc., than to deluge them with a third-level publication of print-out matter containing (hopefully) all possible applications and corollaries of a given result. Boas's earlier strictures [loc. cit.] still indicate that no easy substitute for competent scholarship is yet in sight.

The second part of the paper contains an illustration of how the Lunc paper in question might be translated.
A. J. Lohwater (Providence, R.I.)

Aczél, J.; Daróczy, Z. 6961

Über verallgemeinerte quasilineare Mittelwerte, die mit Gewichtsfunktionen gebildet sind.
Publ. Math. Debrecen 10 (1963), 171–190.

Let I be an interval on the real line, let φ be a strictly monotone continuous function on I and let f be a non-negative continuous function on I not vanishing on any subinterval of positive length. The generalized quasi-linear mean value of points $x_1, x_2, \cdots, x_n \in I$ is defined by the formula

$$M_\varphi(x_i, f) = \varphi^{-1}\left[\sum_{i=1}^{n} f(x_i)\varphi(x_i) \Big/ \sum_{i=1}^{n} f(x_i)\right].$$

In the first portion of this paper the authors obtain a necessary and sufficient condition that the equality of mean values $M_\varphi(x_i, f) = M_\psi(x_i, g)$ holds. Next they investigate the equation of homogeneity $M_\varphi(tx_i, f) = tM_\varphi(x_i, f)$. They prove that the problem of determining all homogeneous generalized quasi-linear mean values is equivalent to the problem of determining all solutions of a vector-matrix functional equation. After solving this functional equation they are able to give explicit expressions for all those generalized quasi-linear mean values which satisfy the homogeneity equation. In the last paragraph of the paper the authors remark that since, in some cases, $M_\varphi(x_i y_i, f) = M_\varphi(x_i, f) M_\varphi(y_i, f)$, there would appear to be some applications of their mean value theory to the entropy concept of information theory. They show that the two (classical) definitions of entropy

$$I_\alpha(x_i) = \frac{1}{1-\alpha} \log\left(\sum_{i=1}^{n} x_i^\alpha \Big/ \sum_{i=1}^{n} x_i\right) \qquad (\alpha \neq 1),$$

$$I_1(x_i) = -\sum_{i=1}^{n} x_i \log x_i \Big/ \sum_{i=1}^{n} x_i$$

[A. Rényi, *Wahrscheinlichkeitsrechnung. Mit einem Anhang über Informationstheorie*, VEB Deutscher Verlag, Berlin, 1962; MR 26 #5597] are special cases of their results. Finally some open problems are discussed.
G. Maltese (College Park, Md.)

Umegaki, Hisaharu 6962

General treatment of alphabet-message space and integral representation of entropy.
Kôdai Math. Sem. Rep. 16 (1964), 18–26.

Instead of the "message space" A^I consisting of all doubly infinite sequences $a = (\cdots, a_{-1}, a_0, a_1, \cdots)$ (where the a_i's belong to some finite set A) the author considers more generally a totally disconnected compact Hausdorff space X with a fixed homeomorphism S onto itself, playing the role of the shift. Theorems on entropy (more exactly: entropy rate) of information sources are generalized for this abstract set-up, the proofs being similar to the original ones. In particular, for any "clopen" partition $\mathfrak{L}$ (i.e., partition consisting of closed-open sets) the existence of a universal bounded, non-negative, S-invariant Borel-measurable function $h(x) = h(x, \mathfrak{L}, S)$ is proved such that for every S-invariant regular probability measure p on X, $H(p) = \int_X h(x)\,dp(x)$ and

$$h(x) = h_p(x) = -\sum_{U \in \mathfrak{L}} P_p(U | \mathfrak{L}_\infty)(x) \log P_p(U | \mathfrak{L}_\infty)(x)$$

for p—a.e. $x \in X$. Here the entropy $H(p) = H(p, \mathfrak{L}, S)$ is defined—as usual in ergodic theory—as the limit $\lim_{n\to\infty} (1/n) H(\bigvee_{k=0}^{n-1} S^{-k}\mathfrak{L})$, and $P_p(U | \mathfrak{L}_\infty)(x)$ denotes the

conditional probability with respect to the σ-field generated by $\{S^{-k}\mathfrak{Q}\}_{k=1}^{\infty}$ with underlying probability measure p. For the case $X = A^I$ this theorem was proved by K. R. Parathasarathy [Illinois J. Math. **5** (1961), 299–305] and independently by K. Jacobs [Math. Z. **78** (1962) 33–43; MR **26** #286]. The functional $H(\xi) = \int_X h(x)d\xi(x)$ defined on the Banach space of all S-invariant bounded regular signed measures ξ on X coincides with the entropy functional introduced by the author [Kôdai Math. Sem. Rep. **15** (1963), 162–175; MR **28** #2189].

I. Csiszár (Budapest)

Umegaki, Hisaharu 6963a

A functional method for stationary channels.

Kôdai Math. Sem. Rep. **16** (1964), 27–39.

Umegaki, Hisaharu 6963b

Supplement and correction to the preceding paper "A functional method for stationary channels".

Kôdai Math. Sem. Rep. **16** (1964), 189–190.

An abstract generalization of the concept of a stationary channel is given, by replacing the input and output message spaces A^I and B^I by arbitrary compact Hausdorff spaces X and Y with fixed homomorphisms S and T, respectively, playing the role of the shift. Assuming further that X and Y are totally disconnected spaces, the results of the author's paper above [#6962] can be applied, and theorems on the integral representation of the transmission functional are proved. Here the transmission functional $\Re(\xi; \mathfrak{Q}, \mathfrak{G})$ corresponding to given "clopen" partitions $\mathfrak{Q}$ of X and $\mathfrak{G}$ of Y is defined with the aid of the entropy functional [cf. #6962 above] analogously to the usual definition of transmission rate. The exact analogue of K. R. Parathasarathy's theorem [Illinois J. Math. **5** (1961), 299–305] is proved under an additional measurability hypothesis called (Cl'); hence, by introducing

$$C_s = \sup\{\Re(p; \mathfrak{Q}, \mathfrak{G}), p \in \mathbf{P}(X, S)\},$$

the equality

$$C_s = \sup\{\Re(p; \mathfrak{Q}, \mathfrak{G}), p \in \mathbf{P}_e(X)\}$$

follows, where $\mathbf{P}(X, S)$ and $\mathbf{P}_e(X)$ denote the set of all S-invariant probability measures and all ergodic probability measures on X, respectively. Also, the abstract version of R. L. Adler's theorem [Proc. Amer. Math. Soc. **12** (1961), 924–930; MR **24** #B2504] on equality of stationary (C_s) and ergodic (C_e) capacities is stated. In the original paper the capacity C_s is stated to be attained (for some $p \in \mathbf{P}_e(X)$) under a continuity hypothesis; in the supplement, however, the proof of this statement is pointed out to be incorrect. Further, in the supplement, the author points out that the integral representation theorem of Parathasarathy type and its consequence

$$C_s = \sup\{\Re(p; \mathfrak{Q}, \mathfrak{G}), p \in \mathbf{P}_e(X)\}$$

are valid for arbitrary compact metric spaces X and Y and arbitrary finite measurable partitions $\mathfrak{Q}$ and $\mathfrak{G}$.

I. Csiszár (Budapest)

Širjaev, A. N. 6964

Detection of randomly appearing targets in a multichannel system. (Russian)

Trudy Mat. Inst. Steklov. **71** (1964), 113–117.

Results of the author [Teor. Verojatnost. i Primenen. **8** (1963), 431–443; MR **28** #649] are extended to observation of the outputs, not of one, but of n independent channels with equal signal-to-noise ratios (SNR). A target is declared present in a given direction if the output of at least one channel exceeds a threshold after that direction is observed for a fixed time T_0. The mean time $\tau(T, N, n)$ between appearance of a target and its detection is for $T \to \infty$ asymptotically $\tau(T, N, n) \sim \frac{1}{2}(N+2) \ln T$, where N is the number of directions observed and T is the mean time between false alarms, provided $\ln n = o(T_0)$. The unit of time is $\Delta/(\mathrm{SNR})$, where Δ is the sampling interval.

C. W. Helstrom (Pittsburgh, Pa.)

Cooper, David B.; Cooper, Paul W. 6965

Nonsupervised adaptive signal detection and pattern recognition.

Information and Control **7** (1964), 416–444.

Authors' summary: "Adaptive signal detection and pattern recognition can be viewed as a problem in statistical classification wherein the partitioning of an n-dimensional sample space into category (signal) regions is determined through estimation from a set of samples from the categories. When the correct associations of the samples are known, the problem is the commonly treated supervised one. This paper, examining the nonsupervised case wherein the correct associations of the samples are unknown, demonstrates that it is possible under extremely general conditions to achieve effective adaptation without supervision. With particular emphasis on a two-category (binary detection) model, general conditions are described under which nonsupervised adaptation is possible, and specific simple yet rapidly convergent techniques are presented under varying degrees of prior knowledge of the statistical properties of the data.

"Most of the paper is concerned with a two-category case where the corresponding (equiprobable) distributions differ only in location. The paper proceeds by examining the over-all probability distribution comprised of the two-component category distributions, and the adaptation treated is directed toward determining the decision boundary, or the distribution parameters necessary for defining it. For univariate normal distributions, various estimators (and their convergence properties) of the over-all mean are examined. For multivariate monotone (including normal) distributions the over-all sample covariance matrix is used to obtain the component covariance matrices when these are general (including the colored noise case), or simply to obtain the principal eigenvector (of the overall matrix) when the component distributions are spherically symmetric (white noise). A hill-climbing algorithm is included. These results for the important model of binary signal detection in gaussian noise demonstrate that no prior knowledge of the signal or noise parameters is required for nonsupervised adaptation to the optimum detector. It is shown in one dimension that for equiprobable component distributions of almost any functional form and differing only by translation we can obtain category distribution estimators which converge uniformly over the real line with probability 1. Considered also are the case of different a priori probabilities, the problem of tracking, and some aspects of the multiple-category problem."

W. L. Root (Ann Arbor, Mich.)

Marcus, Solomon 6966

★**Lingvistică matematică. Modele matematice în lingvistică [Mathematical linguistics. Mathematical models in linguistics].**

Editura Didactică şi Pedagogică, Bucharest, 1963. 220 *pp*. *Lei* 9.40.

This book is based in part on a course in mathematical linguistics given by the author at the University of Bucharest in the years 1961 through 1963. The orientation of the book is based on the notion of opposition due to F. de Saussure, which led N. Trubetzkoy to construct a classification of different types of phonological opposition, and which was later improved by J. Cantineau. The further investigations of A. Martinet, L. Hjelmslev, Z. Harris, N. Chomsky, A. Reformatski, I. Revzin, R. Dobrušin, and S. Šaumyan form the basis for the author's treatment.

The style is distinctly mathematical (i.e., definition, theorem, proof) and is sprinkled with appropriate introductions, observations, and examples (Roumanian and Russian). The only mathematical preparation needed is a knowledge of elementary set theory, relation theory, combinatorial algebra, logic, and mappings.

The scope of the book is indicated by the following chapter headings: Logical theory of linguistic opposition, The phoneme, morphemes and quasimorphemes, Functional methods in morphemic analysis, Grammatical categories, Models based on partitions and on the relation of government, and Mathematical modeling of the parts of speech. The bibliography contains an extensive list of 301 references covering the work of American and European linguists. Unfortunately, the book will have little influence on English-speaking linguists since it is in Roumanian.

H. P. Edmundson (Pacific Palisades, Calif.)

Gladkiĭ, A. V. 6967

Grammars with linear memory. (Russian)

Algebra i Logika Sem. **2** (1963), no. 5, 43–55.

Let G denote a grammar, let $L(G)$ denote the language generated by G, and let $|W|$ denote the length of the word $W \in L(G)$. Call G a linear bounded grammar ("grammar with linear memory" in the author's terminology) if there exists a natural number k such that for any word $W \in L(G)$, obtained, say, by a derivation $S = W_0 \to W_1 \to \cdots \to W_n = W$ one has that $|W_i| \leq k|W|$, $i = 0, 1, \cdots, n$. (Through a misprint, the "k" is omitted in the definition, but later usage in the paper suggests that this is what the author intended.) It is shown that the class of linear bounded languages coincides with the class of context-sensitive languages, that any language accepted by a bounded linear automaton ("any language recognizable by linear memory" in the author's terminology) is a bounded linear language, hence, context-sensitive, and, finally, that the intersection of any finite set of linear bounded languages is again a linear bounded language. (Thus the results overlap those of Landweber [Information and Control **6** (1963), 131–136; MR **29** #3291]; see also the following review [#6968].)

R. M. Baer (Berkeley, Calif.)

Kuroda, S.-Y. 6968

Classes of languages and linear-bounded automata.

Information and Control **7** (1964), 207–223.

Here a linear bounded grammar is defined as a context-sensitive grammar which is of order 2, is length-preserving, and satisfies: $S \to EF$ implies $E = S$, the initial symbol. (The order of a grammar is the maximum of the lengths of strings occurring in the rules.) For any context-sensitive grammar there is an equivalent linear bounded grammar. The converse of a result of Landweber [see #6967 above for reference], when shifted to the context of nondeterministic linear bounded automata, yields a characterization of the context-sensitive languages as those accepted by (nondeterministic) linear bounded automata. And, among other results, it is shown that the context-free languages are accepted by (deterministic) linear bounded automata.

At one point the author introduces a linear bounded automaton coupled to a certain type of counter (which presumably is just a second tape) and claims that the resulting device is again a linear bounded automaton. This point deserves a bit more discussion than is accorded it in the text.

R. M. Baer (Berkeley, Calif.)

Zítek, František 6969

Quelques remarques au sujet de l'entropie du tchèque.

Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (*Liblice*, 1962), *pp*. 841–846. *Publ. House Czech. Acad. Sci., Prague*, 1964.

The author makes two remarks about certain information-theoretic properties of the Czech language. The first remark concerns the entropy of written Czech using a modification of the method used by M. H. Hansson on Swedish, in which native speakers guess a phrase letter-by-letter. As an estimate of the entropy, a sample mean of $H = 2.40$ is obtained. The second remark concerns the morphological homonymity present in Czech grammar in comparison with that of Latin, German (with the definite article), and German (without the definite article). The analysis uses simple concepts from the theory of partitions, and the entropies for nouns are summarized in a table.

H. P. Edmundson (Pacific Palisades, Calif.)

Rozenvasser, E. N. 6970

Theory of a linear system with stationary time lag and periodically varying parameter. (Russian. English summary)

Avtomat. i Telemeh. **25** (1964), 1067–1074.

Continuing his earlier work [Avtomat. i Telemeh. **21** (1960), 15–19; MR **22** #9385; ibid. **21** (1960), 1279–1292; MR **22** #9422; Prikl. Mat. Meh. **25** (1961), 284–293; MR **24** #A1474; *Analytic methods in the theory of non-linear vibrations*, Proc. Internat. Sympos. Non-linear Vibrations, Vol. I, 1961, pp. 417–424, Izdat. Akad. Nauk Ukrain. SSR, Kiev, 1963; MR **27** #5998], the author studies a linear system written in operational form as follows: (1) $y = W(p)x$, $x = \mu\varphi(t)y$, where (2) $W(p) = \exp(-p\tau)\cdot M(p)/D(p)$, where M and D are polynomials in p, μ is a scalar, and φ is continuous in t, of period T. (1) is derived apparently by Laplace transform techniques. By using results of the paper cited above, the author derives a characteristic equation, whose roots serve to determine the stability of (1). Then a Floquet representation of a certain solution of (1) is obtained, where the roots of the characteristic equation appear as characteristic exponents.

A. Stokes (Washington, D.C.)

Egorov, A. I. 6971

A variational problem in the theory of equations of elliptic type. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 500–508.

Let D be an open bounded subset of 3-dimensional space, with smooth boundary S, and let U be some bounded subset of r-dimensional space. The problem is to choose $u(x)$ with values in U to minimize

$$I = \iint_S \alpha(x)z(x)\,dS + \iiint_D \beta(x)z(x)\,dx,$$

where the function z is determined in D by the partial differential equation $\Delta z = F(x, z, u(x))$, where Δ is the Laplace operator, together with the data $dz/dn + hz = f(x)$ on S, $h > 0$. The function F is assumed to satisfy a Lipschitz condition in (z, u). The main result is the maximum principle in the following form. Let $w(x)$ satisfy the linear partial differential equation

$$\Delta w = \frac{\partial F(x, z(x), u(x))}{\partial z} w - \beta(x)$$

with the data $dw/dn + hw = \alpha(x)$ on S. Let

$$H = w(x)F(x, z(x), u).$$

If $u(x)$ is a piecewise smooth minimizing control, then H is maximum when $u = u(x)$.

W. H. Fleming (Providence, R.I.)

Faulkner, Frank D. 6972

A comparison between some methods for computing optimum paths in the problem of Bolza.

Computing Methods in Optimization Problems (Proc. Conf., Univ. California, Los Angeles, Calif., 1964), *pp.* 147–157. *Academic Press, New York,* 1964.

The author discusses modifications of the method of steepest ascent [Bryson and Denham, J. Appl. Mech. **29** (1962), 247–257; MR **26** #4822] and of the differential methods of Bliss [*Mathematics for exterior ballistics*, Chapter IV, Wiley, New York, 1944; MR **6**, 24] for computing optimal trajectories. Advantages and disadvantages of each are listed, and suggestions are made concerning ways of combining the methods.

L. M. Graves (Chicago, Ill.)

Ewing, G. M.; Haseltine, W. R. 6973

Optimal programs for an ascending missile.

J. Soc. Indust. Appl. Math. Ser. A Control **2** (1964), 66–88.

An admissible program is defined to be an ordered triple of real-valued functions (v, y, m) defined on an interval $[0, T]$ and satisfying the following: (i) $v(0) = y(0) = 0$; (ii) $v(T) = V$, $y(T) = Y$, $m(T) = M$, where $M > 0$, $Y > 0$, $V \geq 0$ are fixed; (iii) v is in $L[0, T]$, m is non-increasing, and $y(t) = \int_0^t v(s)\,ds$; (iv) if $E(t) = \exp(c^{-1}(v(t) + gt))$, where g and c are positive constants, then, for every t_1, t_2 in $[0, T]$,

$$m(t_1)E(t_1) = m(t_2)E(t_2) + c^{-1}\int_{t_1}^{t_2} D[v(s), y(s)]E(s)\,ds.$$

The following problem is then posed. Does there exist an admissible program (v_0, y_0, m_0) such that if (v, y, m) is any other admissible program, then $m_0(0) \leq m(0)$. If so, what is the program (v_0, y_0, m_0), and is it unique.

It is shown, under reasonable assumptions about the drag function D, that a unique optimal program exists and has the following characteristics. If there is no drag or if Y is small and the drag is negligible, then the optimal program consists of an initial velocity boost followed by a coast. If for given Y, V is small, then the optimal program consists of an initial boost followed by a propulsive phase and then a coast. For Y and V both large the optimal program consists of an initial boost, a propulsive phase, and a terminal boost.

The proof of existence is accomplished by first showing that only programs whose velocities v are always nonnegative need be considered and that the velocity functions v_n in a minimizing sequence are of uniformly bounded variation. The existence then follows from Helly's theorem.

To characterize the solution and show uniqueness, the authors determine the trajectory Γ in (t, y)-space of a coasting program, and show that all other trajectories lie on one side of Γ. The analogue of a field in the calculus of variations is then constructed. By comparing admissible trajectories with trajectories of the field, it is shown that an optimal program must correspond to a unique trajectory of the field.

L. D. Berkovitz (Lafayette, Ind.)

Vidal, Pierre 6974

Sur la réponse transitoire d'un système non linéaire échantillonné régi par une équation aux différences finies d'ordre *m*.

C. R. Acad. Sci. Paris **256** (1963), 5058–5061.

This brief note concerns a "sampled system" governed by the mth-order difference equation $x_{n+m} + f_1 x_{n+m-1} + \cdots + f_m x_n = 0$, where the f_i are functions of $x_n, \cdots, x_{n+m}$ and n. The equilibrium point of the system is taken to be the origin. For given initial conditions of the form $|x_1| = |x_2| = \cdots = |x_m| = c$, the "$b$ percent response time" is defined to be the shortest time, aT, such that $|x_{m+a}| \leq (b/100)c$. Here T is the "sampling period", i.e., the time step in the difference equation. It is shown that if the coefficients satisfy the condition $1 - \sum_{i=1}^m k^i|f_i| > d > 0$ with $k \geq 1$, then $|x_{m+j}| \leq k^{-j}|x_m|$, so that the b percent time is given by $bk^a > 100$. Somewhat surprising is the absence of references to the abundant literature on the subject.

E. K. Blum (Middletown, Conn.)

Paraev, Ju. I. 6975

Solution of a problem on analytic controller design. (Russian. English summary)

Avtomat. i Telemeh. **25** (1964), 445–451.

Author's summary: "The application of the method of dynamic programming to the solution of the controller analytical design problem is discussed."

W. T. Reid (Norman, Okla.)

Cesari, Lamberto 6976

Problemi di Lagrange con vincoli unilaterali.

Atti Convegno Lagrangiano (Torino, 1963) [*Atti Accad. Sci. Torino* **98** (1963/64), *suppl.*], *pp.* 88–119. *Accad. Sci., Torino,* 1964.

This is a survey consisting of two parts. The first deals with the theory of linear programming and its duality principle; the second, and much longer, part concerns

Pontryagin's approach to mathematical control theory, his maximum principle and its relation to the classical calculus of variations. It mentions many recent contributions to control theory and also contains a new existence theorem of the author proved in detail elsewhere [Ann. Scuola Norm. Sup. Pisa (to appear)].

H. A. Antosiewicz (Vienna)

Troickil, V. A. 6977

Necessary conditions in variational problems of optimizing control processes. (Russian)

Leningrad. Politehn. Inst. Trudy No. 226 (1963), 44–61.

This paper discusses the relation between the problems of Bolza in the calculus of variations and the problem of optimal control. This is now well known, and better expositions can be found in the papers by L. D. Berkovitz [J. Math. Anal. Appl. **3** (1961), 145–169; MR **25** #2470; ibid. **5** (1962), 488–498; MR **25** #4961] and by M. R. Hestenes [see #6355 above]. *J. P. LaSalle* (Providence, R.I.)

Basile, G.; Marro, G. 6978

Calcolatore analogico per l'ottimizzazione predittiva dei controlli lineari.

Atti Accad. Sci. Ist. Bologna Cl. Sci. Fis. Rend. (11) **10** (1962/63), *no.* 1, 119–138.

The problem considered is to maximize an integral with quadratic integrand constrained by linear differential equations with constant coefficients. The author then discusses the structure of resistive electric networks with which are associated voltages of the same functional form required by first-order necessary conditions for the maximum problem. *G. M. Ewing* (Norman, Okla.)

Gnoenskii, L. S.; Movšovič, S. M. 6979

The application of the methods of mathematical programming to an optimal control problem. (Russian)

Izv. Akad. Nauk SSSR Tehn. Kibernet. **1964**, *no.* 5, 16–29.

The paper considers the problem of time-optimality in control systems. Mathematical programming methods are used to find the control. The number of variables, the form of the constraints, and the method of solving the obtained problem essentially depend on the way in which the corresponding Cauchy problem is approximated. The cases of piecewise-constant and piecewise-linear approximations are considered in detail. Much attention is paid to the development of algorithms for solving the mathematical programming problems that arise. The usual time-optimal problem for the matrix system $\dot{X} = A(t)X + b(t)u(t)$, $X(0) = X_0$, $|u(t)| \leq 1$, is called Problem $\tilde{\mathrm{R}}_1$. If, in addition, the state variables are bounded, $|x_i(t)| \leq m_i$, $i = 1, \cdots, n$, we have Problem $\tilde{\mathrm{R}}_2$. The approach is to minimize the norm of X at fixed instants T (Problems $\tilde{\mathrm{R}}_1(T)$, $\tilde{\mathrm{R}}_2(T)$) and then to seek the minimal T for which the norm of X equals zero. The computational difficulties in these problems stem not only from the order of the state equations but also from the form of the integrand function in the Cauchy formula for the formal solution of the state equations. To obviate these difficulties, the integral in the Cauchy formula is approximated by a sum, and the problem reduces to seeking the minimum of a multivariable convex function in a closed region. The corresponding approximate problems are referred to as Problems $\mathrm{R}_1(T)$, $\mathrm{R}_2(T)$. A modification of the simplex method is used to solve these problems. The Kuhn-Tucker optimality criterion is also utilized for solving Problem $\mathrm{R}_1(T)$. These methods can be extended also to cases of piecewise-parabolic and piecewise-polynomial approximations. A third-order example is considered and solved for both the piecewise-constant and the piecewise-linear approximations. *N. H. Choksy* (Silver Spring, Md.)

Paiewonsky, Bernard; Woodrow, Peter; Brunner, Walter; Halbert, Peter 6980

Synthesis of optimal controllers using hybrid analog-digital computers.

Computing Methods in Optimization Problems (Proc. Conf., Univ. California, Los Angeles, Calif., 1964), pp. 285–303. *Academic Press, New York,* **1964**.

This paper describes how results due to Neustadt [J. Math. Anal. Appl. **1** (1960), 484–493; MR **22** #B1612; J. Soc. Indust. Appl. Math. Control Ser. A **1** (1962), 16–31; MR **26** #2707] can be used to compute for a class of linear systems controls which reduce the error to zero in minimum time. A description of the computer and a report on computer studies are given.

J. P. LaSalle (Providence, R.I.)

Stratonovich, R. L. [Stratonovič, R. L.] 6981

On the theory of optimal control. Sufficient coordinates.

Avtomat. i Telemeh. **23** (1962), 910–917 (*Russian. English summary*); *translated as Automat. Remote Control* **23** (1963), 847–854.

A basic problem in automatic control is the selection of an "optimal" control decision on the basis of what is currently known. Frequently this means making decisions on the basis of the current values of the physical state variables in an effort to minimize the expected value of the future cost. Recently, though, Bellman and Kalaba, Zadeh, and others have emphasized the importance of adaptive control processes, in which the controller's knowledge of the dynamics, objectives, etc., of the process changes in the course of time. This brings with it the attendant problems of describing the controller's state of knowledge and how this changes in the light of experience gained during the process.

An axiomatic treatment is adumbrated, and several interesting examples are provided.

R. Kalaba (Santa Monica, Calif.)

Fattorini, H. O. 6982

Time-optimal control of solutions of operational differential equations.

J. Soc. Indust. Appl. Math. Ser. A Control **2** (1964), 54–59.

Let H be a Hilbert space with norm $|u|$, A a linear operator in H, and $f(t)$ a function on $t \geq 0$ into H. The problem considered is this: Given u, v in H, find an admissible control f such that the weak solution of the operational differential equation

$$(1)\qquad \begin{aligned} u'(t) &= Au(t) + f(t), \qquad t \geq 0, \\ u(0) &= u, \end{aligned}$$

reaches v in the smallest possible time. The definitions are

as follows. Let L_t be the space of functions on $(0, t)$ into H, strongly measurable and bounded, with

$$\text{norm } \|f\|_t = \text{ess sup}(|f(r)|,\ 0 \le r \le t).$$

Let $T(t)$ be the semigroup of bounded operators defined by $T(t)u(0) = u(t)$, where $u(t)$ is the strong solution of the homogeneous equation $u'(t) = Au(t)$. The weak solution of (1) is then defined for f in L_t by

$$u(t) = T(t)u + \int_0^t T(t-r)f(r)\,dr,$$

integration being in the sense of Bochner. Given u, v in H, an f in L_t is admissible if the weak solution of (1) reaches v at some t_0 and $\|f\|_t \le 1$, $0 \le t \le t_0$; it is optimal if t_0 is minimal.

It is proved that if for u and v in H there exists an admissible control, then there exists an optimal control, and that the optimal control uses the maximum available energy ($|f(r)| = 1$ a.e.) and is unique up to sets of measure zero. It is stated that the methods are applicable in wider situations, for example, when H is a reflexive Banach space or $A = A(t)$. No applications are indicated.

K. L. Cooke (Claremont, Calif.)

Senin, A. G. 6983

Passage of random signals through a linear dynamical system with distributed parameters. (Russian)

Izv. Akad. Nauk SSSR Tehn. Kibernet. **1964**, *no.* 2, 81–86.

Author's summary: "We consider the question of the transformation of the correlation functions and spectral densities of stationary random processes by dynamical systems with distributed parameters. The solution of the problem is given under various boundary conditions."

The author illustrates his technique by considering a random process which satisfies the heat equation in a one-dimensional rod of finite length, and whose value or spatial derivative (or some combination thereof) is known at one end of the rod. *A. Feinstein* (Sunnyvale, Calif.)

Duncan, R. L. 6984

Hit probabilities for multiple weapon systems.

SIAM Rev. **6** (1964), 111–114.

The author considers the problem of determining the probability of hitting a target with at least one missile from a random circular salvo.

D. S. Adorno (Wilton, Conn.)

Beneš, V. E. 6985

Permutation groups, complexes, and rearrangeable connecting networks.

Bell System Tech. J. **43** (1964), 1619–1640.

This paper presents an algebraic and combinatorial approach to the study of rearrangeability in connecting networks. Networks of this type are used in telephone systems, and they consist of arrangements of switches and transmission links through which inlets can be connected to outlets in many combinations. A network is rearrangeable if every assignment of inlets to outlets is realizable. The specific network form considered here is a sequence of switching stages connected by link patterns. A stage realizes a subgroup of the symmetric group of permutations, and a link pattern realizes a permutation. To obtain rearrangeability, the product of subgroups and permutations defined by the network should yield the symmetric group. Sufficient conditions (in terms of properties of stages and links) under which this is possible are formulated. The author indicates that these conditions do not provide new design principles, but they are intended to give new insights to the problem. A specific class of networks, that are generalizations of those given by Paull (reference provided in the paper), is also described.

The paper is clearly written, and it presents an interesting application of group theory to the design of routing networks. (A minor comment on presentation: unconventional notations for the empty set and the cardinality of a set are introduced on p. 1627 prior to their definition on the next page.) *S. Amarel* (Princeton, N.J.)

Halilov, T. A. 6986

Design of automatic control systems by a method of successive approximations. (Russian. Azerbaijani summary)

Izv. Akad. Nauk Azerbaĭdžan. SSR Ser. Fiz.-Tehn. Mat. Nauk **1964**, *no.* 2, 115–119.

Brown, Robert Grover; Nilsson, James William 6987

★Introduction to linear systems analysis.

John Wiley and Sons, Inc., New York-London, 1962. xi + 403 *pp.* $10.50.

Possibly the only thing that distinguishes this book from many other similar ones on linear system analysis is the inclusion of material on random processes. In Part I, Systems with Explicit Driving Functions, there are ten chapters: (1) Introduction to the Laplace Transform; (2) Translating the Physical Problem into Mathematical Language; (3) More on Laplace Transform Theory; (4) System Concepts; (5) Systems with Feedback; (6) Fourier Transform Analysis; (7) Some General Properties of Physically Realizable Transfer Functions; (8) Distributed-Parameter Systems; (9) Systems with Time-Varying Parameters; (10) The z-Transform. In Part II, Systems with Random Driving Functions, there are five chapters: (11) Essential Notions of Probability; (12) Mathematical Description of a Random Process; (13) The Analysis Problem; (14) Optimization; (15) Further Topics on Optimization. There are two Appendices: (A) Tables of Laplace Transforms; (B) Calculus of Variations and Lagrange's Equations.

The material presented is straightforward and uninspiring, as are the problems at the end of each chapter, and a bit behind the times for the advanced undergraduate or beginning graduate student for whom it is intended.

N. H. Choksy (Silver Spring, Md.)

Doležal, Václav 6988

★Dynamics of linear systems.

Publishing House of the Czechoslovak Academy of Sciences, Prague, 1964. 224 *pp.* *Kčs.* 26.00.

In der Netzwerktheorie treten gewisse Schwierigkeiten auf, die von den Elektrotechnikern meist entweder nicht aufgeklärt oder mit illegitimen Mitteln behandelt werden: Die äußeren Erregungen können Sprünge haben, sind also nicht durchweg differenzierbar, oder sie stellen Impulse

dar, die nicht mit dem Funktionsbegriff erfaßbar sind; die nach den üblichen Methoden hergestellten Lösungen haben andere als die vorgeschriebenen und bei der Herleitung benutzten Anfangswerte. Der Autor stellt sich die Aufgabe, unter Benutzung der Distributionen eine so allgemeine Theorie zu entwickeln, daß solche Widersprüche nicht auftreten und auch die allgemeinsten Erregungen zulässig werden. Das Gleichungssystem der Netzwerktheorie wird in Matrixform geschrieben: (1) $Ax'+Bx+C\int_0^t x(\tau)d\tau=f(t)$ mit den $r\times r$ Matrizen A, B, C und den r-dimensionalen Vektorfunktionen $x(t)$, $f(t)$. Als Anfangsbedingung wird der r-dimensionale konstante Vektor c vorgeschrieben. Das 1. Kapitel verbleibt noch im Rahmen der klassischen Theorie (Funktionen) und sucht einige der genannten Schwierigkeiten dadurch zu beheben, daß statt des Systems (1) das von 0 bis t integrierte System betrachtet wird, in das der Anfangsvektor c eingetreten ist. Bei der Lösung vermittels Laplace-Transformation spielt bekanntlich die Matrix $Z^{-1}(p)$, wo $Z(p)$ die Impedanz $Ap+B+Cp^{-1}$ ist, eine wichtige Rolle. Deshalb werden im 2. Kapitel derartige Matrizen auf ihr funktionentheoretisches Verhalten untersucht. Im 3. Kapitel geht der Autor zu seinem eigentlichen Thema, der Behandlung des Systems (1) im Rahmen der Distributionstheorie, über. Dazu werden zunächst die grundlegenden Eigenschaften der Distributionen aus dem Raum D im Sinne von L. Schwartz entwickelt und aus D die Unterklasse D_L von Distributionen abgegrenzt, die Laplace-Transformierte besitzen. Wenn f ein Vektor über D_L ist, so wird der Vektor x über D_L die Distributionslösung des Systems (1) mit der Anfangsbedingung c genannt, wenn x das System $Ax'+Bx+Cx^{(-1)}=f+Ac\delta_0$ befriedigt. (Das Integral $x^{(-1)}$ wird vorher in geeigneter Weise definiert.) Damit wird den in $t=0$ eventuell auftretenden Sprüngen zwischen gegebenen und angenommenen Anfangswerten Rechnung getragen. Das so formulierte Problem hat immer eine Lösung, von der gezeigt wird, daß sie ein zutreffendes Bild der physikalischen Wirklichkeit liefert. Im 4. Kapitel wird an Stelle von (1) das Differentialgleichungssystem $Ax''+Bx'+Cx=f$ auf der vollen Achse $-\infty<t<\infty$ ohne Transformationstheorie mit Distributionen behandelt, die zu diesem Zweck zu komplexwertigen verallgemeinert werden. Im 5. Kapitel wird das System mit zeitvariablen Koeffizienten

$$(2)\quad \sum_{k=1}^{r}\sum_{m=0}^{n} a_{ikm}(t)x_k^{(m)}(t) + \sum_{k=1}^{r}\int_0^t W_{ik}(t,\tau)x_k(\tau)\,d\tau = f_i(t) \quad (i=1,2,\cdots,r)$$

nach einer Operatorenmethode gelöst, die vom Heaviside-Kalkül inspiriert ist. Nach eingehendem Studium des Produktes $[Wx]$ einer Funktion $W(t,\tau)$ von zwei Variablen mit einer Distribution x von endlicher Ordnung wird die Menge $\mathfrak{A}_n$ der Operatoren $Ax=[Wx]^{(k)}$ betrachtet. Der in (2) vorkommende Operator $Ax=a_mx^{(m)}+\cdots+a_1x'+a_0x+[Wx]$ gehört zu $\mathfrak{A}_n$. Die Lösung von (2) besteht in dem Nachweis der Existenz des inversen Operators A^{-1}. Im 6. Kapitel werden die Eigenschaften von Multipolen, die sonst nur für Systeme mit konstanten elektrischen Elementen (Widerstand, usw.) formuliert werden, auf allgemeine lineare Systeme übertragen, deren Elemente zeitvariabel sein können.

Das Referat kann nur einen kleinen Teil von dem reichen Inhalt des Buches wiedergeben, das einige bisher unbefriedigend behandelte Teile der theoretischen Elektrotechnik auf eine der modernen Analysis entsprechende Basis stellt. *G. Doetsch* (Freiburg)

Naslin, P. 6989

★**Les régimes variables dans les systèmes linéaires et non linéaires.**

Dunod, Paris, 1962. xvi+524 *pp*. 88 *NF*.

Essentially this is a standard book on linear and nonlinear systems of which many have appeared both in English and in French. It differs from most English books in the prolific use of illustrative diagrams, a typical French textbook characteristic. In an introduction of six pages the author lists the general properties of linear and nonlinear systems and the methods of studying them.

Part I, Linear Systems, contains seven chapters. Chapter 1 introduces the p $(=d/dt)$ operator and block diagrams. The author carefully justifies his use of this p operator instead of the more usual complex variable s of the Laplace transform. Electric and mechanical systems are discussed, as are electromechanical analogues. Linearized nonlinear systems are presented. Signal flow graphs are also dealt with in this chapter. Chapter 2 is a standard exposition of the symbolic calculus. Chapter 3 introduces the E operator defined as the delay operator, $Ef(t)=f(t-T)$, which forms the basis of a numerical calculus. Chapter 4 takes up the question of harmonic modes. Chapter 5 discusses the relationship between the transient response and the frequency response. Chapter 6 presents the standard Nyquist and Hurwitz stability criteria, and a new damping criterion based on a new form of the characteristic polynomial. Chapter 7 deals with sampled-data systems with finite pulse-width sampling. Use is made here of the E-transform developed in Chapter 3. There are several places here that a factor of τ, the pulse-width, is omitted from various formulas—notably in equations (7.8), (7.9) and (7.10), and in the seventh, eighth, ninth and tenth rows of the last column of Table 7.2 on page 281.

Part II: Nonlinear Systems, contains three chapters. Chapter 8 extensively discusses various aspects of nonlinear filters. Topological (phase-space) methods are taken up in Chapter 9. The last chapter presents numerical and graphical methods of computing the transient modes of nonlinear systems.

Appendix A presents the Graeffe method for finding the roots of a polynomial equation. Appendix B treats the Fourier series and integral and their use in harmonic analysis. Appendix C is a good discussion of the Laplace transform. *N. H. Choksy* (Silver Spring, Md.)

Brayton, R. K.; Moser, J. K. 6990

A theory of nonlinear networks. I.

Quart. Appl. Math. **22** (1964), 1–33.

This paper is a new and systematic approach to the study of nonlinear RLC-networks. A network here is an idealised concept defined as a set of nodes and a set of connecting branches; it is also called a graph. The entire report is based on the fact that the system of differential equations for such networks has the form: (1) $L_\rho(di_\rho/dt)=\partial P/\partial i_\rho$, $C_\sigma(dv_\sigma/dt)=-\partial P/\partial v_\sigma$ $(\rho=1,\cdots,r,\quad \sigma=r+1,\cdots,r+s)$, where $P=P(i,v)$ is a function describing the physical properties of the resistive part of the network. Since P has the dimension of voltage times current, it is called a mixed potential function. System (1) seems to form a Hamiltonian system but is not so, since, in distinction to the

latter, it contains dissipative terms. The main difference between linear networks and the networks considered here is that in the latter the voltage-current relations for the resistors may be nonmonotonic, i.e., "negative resistances" may occur. Of course, any or all of the elements may be nonlinear.

The mixed potential function $P(i, v)$ is used to construct Ljapunov-type functions for stability considerations. A procedure for obtaining $P(i, v)$ directly from the network is given. The concepts of a complete set of mixed variables and of a complete circuit are introduced. One of the main purposes of the paper is to draw conclusions on the solutions of system (1) from a consideration of its special forms. System (1) preserves its form under coordinate transformations which leave invariant the indefinite metric: (2) $-\sum_{\rho=1}^{r} L_\rho(di_\rho)^2+\sum_{\sigma=r+1}^{r+s} C_\sigma(dv_\sigma)^2$. Special forms of the network, and thus of system (1), occur, for example, if only resistors are present (R-network) or if inductors are absent (RC-network). For such special forms the metric also takes special forms which are studied.

In general, RLC-networks will admit oscillations even in the linear case. The paper derives criteria which rule out self-sustained oscillations in general nonlinear RLC-networks with time-invariant elements.

The set of variables $i_1, \cdots, i_r, v_{r+1}, \cdots, v_{r+s}$ is said to be complete if they can be chosen independently without a violation of Kirchhoff's laws and if they determine in each branch at least one of the two variables, the current or the voltage. The problem is then to determine a complete set of variables for a given graph. The authors show how to solve this problem in several ways, from the voltages and link currents of a maximal tree; a maximal tree is a subgraph not containing any loops and not capable of being enlarged as a tree, and links are branches not contained in the maximal tree. A circuit with a complete set of variables is said to be complete.

Part I ends with an example of an arbitrarily large ladder network containing nonlinear elements. Part I contains the main results without details of proofs and refinements. Part II [#6991 below] complements Part I with detailed proofs and with several additional results.

N. H. Choksy (Silver Spring, Md.)

Brayton, R. K.; Moser, J. K. 6991

A theory of nonlinear networks. II.

Quart. Appl. Math. **22** (1964), 81–104.

Part II recapitulates the results of Part I [#6990 above] and gives details of the proofs of the results cited there. In addition, certain well-known results of electrical network analysis, such as the Foster reactance theorem, are discussed and rederived from the point of view of the existence of a mixed potential function. In the last section a theorem is proved on the existence of periodic solutions for periodically excited nonlinear circuits, which can be considered as an extension of a theorem of R. J. Duffin [Bull. Amer. Math. Soc. **54** (1948), 119–127; MR **9**, 503].

N. H. Choksy (Silver Spring, Md.)

Ádám, András 6992

On the repetition-free realization of truth functions by two-terminal graphs. I. (Russian summary)

Magyar Tud. Akad. Mat. Kutató Int. Közl. **9** (1964), 11–20.

Beschrieben wird ein Verfahren, mit dessen Hilfe zu einer gegebenen Schaltfunktion f eine Kontaktschaltung ermittelt werden kann, welche von jedem vorhandenen Schaltelement genau einen Kontakt erhält. Vorausgesetzt werden muß dabei allerdings, daß f überhaupt in der angegebenen Form realisierbar ist. Außerdem beschränkt sich der Autor auf Funktionen, die keine "Zerlegung durch (wiederholungsfreie) Überlagerung" gestatten. Das angegebene Verfahren führte nach Angaben des Autors schneller als eine auf Trachtenbrot zurückgehende Methode — aus der es hervorgegangen ist — zum Ziel und ist für die maschinelle Durchführung geeignet.

H. Rohleder (Leipzig)

Steinbuch, Karl 6993

★Automat und Mensch. Kybernetische Tatsachen und Hypothesen.

Zweite erweiterte Auflage.

Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963. xii + 392 *pp.* *DM* 36.00.

The first edition (1961) was reviewed earlier [MR **25** #2937]. This edition consists of a certain "feedback" in response to various criticisms made of the first edition.

Salomaa, Arto 6994

Theorems on the representation of events in Moore-automata.

Ann. Univ. Turku. Ser. A I No. 69 (1964), 14 *pp.*

Bekanntlich (vgl. z.B. V. M. Gluškov [Uspehi Mat. Nauk **16** (1961), no. 5 (101), 3–62; MR **25** #1976]) läßt sich jedes reguläre Ereignis E über einem endlichen Alphabet X in einem passenden endlichen initialen Moore-Automaten mit dem Eingabealphabet X durch eine geeignete Teilmenge S dessen Zustandsmenge A in dem Sinne repräsentieren, daß E aus genau den Wörtern über X besteht, die den Automaten aus seinem Anfangszustand in einen Zustand der Menge S überführen. In der vorliegenden Arbeit werden zunächst für den Fall eines einelementigen Alphabets $X=\{x\}$ Formeln für die minimalen Anzahlen $N_S(E)$ und $N_A(E)$ der Mengen S bzw. A abgeleitet, die für die Repräsentierung eines gegebenen regulären Ereignisses E über $\{x\}$ erforderlich sind. Es wird dazu für die regulären Ereignisse über $\{x\}$ eine Darstellung durch eine eindeutig bestimmte Normalform angegeben, aus der sich die Anzahlen $N_S(E)$ und $N_A(E)$ ablesen lassen. Es wird ferner auch für die regulären Ereignisse über einem beliebigen Alphabet X eine gewisse Art von "Normaldarstellung" hergeleitet (allgemeiner auch für die Ereignisse E, für die $N_S(E)=1$ ist). Sodann werden hinreichende Bedingungen dafür angegeben, daß für ein reguläres Ereignis E über einem beliebigen Alphabet X, bei gegebenem k, $N_S(E)>k$ gilt, und die eine Verallgemeinerung eines von Bodnarčuk [Ukrain. Mat. Ž. **14** (1962), 190–191] für den Fall $k=1$ bewiesenen Resultats sind. Schließlich wird bewiesen, daß ein nichtleeres endliches Ereignis E über einem Alphabet X dann und nur dann irreduzibel ist, d.h. $N_S(E)=|E|$ gilt, wenn E von der Form $E=P_1 \vee P_1P_2 \vee \cdots \vee P_1P_2\cdots P_k$ ist, wobei $P_2, \cdots, P_k$ nichtleere Wörter über X sind.

G. Asser (Greifswald)

Thiele, Helmut 6995

"Klassische" und "moderne" Algorithmenbegriffe.

Mathematische und physikalisch-technische Probleme der Kybernetik, *pp.* 111–146. *Akademie-Verlag, Berlin,* 1963.

Es wird ein allgemeiner Überblick über die bisher entwickelten und verwendeten "klassischen" Algorithmenbegriffe gegeben sowie über die modernen Untersuchungen zur Theorie der algorithmischen Sprachen berichtet. Der Arbeit ist ein umfangreiches Literaturverzeichnis beigefügt. *G. Asser* (Greifswald)

Bloh, A. Š.; Neverov, G. S. 6996

On a method of synthesis of graph-diagrams for algorithms. (Russian)

Dokl. Akad. Nauk BSSR 8 (1964), 568–571.

The authors describe a method by which one can obtain a graph-diagram for certain algorithms which have a specified number of steps and which are originally described by means of characteristic tables. They suggest that the method may be useful in simplifying the programming of certain problems. The method used is similar to a method of Bloh for the synthesis of contact networks [Avtomat. i Telemeh. **22** (1961), 755–764; MR **26** #7473]. *G. N. Raney* (Storrs, Conn.)

Bekišev, G. A. 6997

On the disparallelization of computing algorithms. (Russian)

Vyčisl. Sistemy No. 5 (1962), 22–30.

The author defines for the graph $G=(X,\Gamma)$ an A-partition as an ordered partition (S_α) of X into nonempty subsets such that (for all α) $\Gamma S_\alpha \subset \bigcup_{\beta<\alpha} S_\beta$. The interpretation is that the set X is a collection of operations and that $y\Gamma x$ means that the result of the operation x serves as an argument for the operation y. The results deal with the existence of A-partitions of graphs and estimates of minimal length algorithms corresponding to these. *R. M. Baer* (Berkeley, Calif.)

Moisil, Gr. C. 6998

Entwicklungsstand und Perspektiven der Theorie der konkreten und der abstrakten endlichen Automaten.

Mathematische und physikalisch-technische Probleme der Kybernetik, pp. 454–472. *Akademie-Verlag, Berlin*, 1963.

Es werden Möglichkeiten dargelegt, die abstrakte Automatentheorie auf konkrete Automaten (Schaltungen mit realen Schaltelementen) anzuwenden. Anhand einer Vielzahl von Beispielen wird gezeigt, wie sich die reale Arbeitsweise der verschiedensten Arten von Schaltelementen und damit auch von komplizierten Schaltungen aus solchen Elementen durch kanonische Gleichungen beschreiben lassen, wenn man passende Hilfsvariablen einführt. Damit wird es möglich, die für die Analyse von endlichen abstrakten Automaten entwickelten Methoden unmittelbar zur Analyse von konkreten Automaten zu verwenden. (Bei dem wesentlich wichtigeren Problem der Synthese ist entsprechendes nicht möglich.) Ein umfangreiches Literaturverzeichnis gibt insbesondere einen Überblick über die in Rumänien durchgeführten Arbeiten zu den genannten Fragen. *G. Asser* (Greifswald)

Kálmár, László 6999

Algorithmische Sprachen und Programmierung von Rechenautomaten.

Mathematische und physikalisch-technische Probleme der Kybernetik, pp. 147–176. *Akademie-Verlag, Berlin*, 1963.

Es wird über diejenigen algorithmischen Sprachen (Autokodsprachen) berichtet, zu denen die praktischen Aufgaben der Programmierung von digitalen Rechenautomaten geführt haben. Dabei werden besonders die allgemeinen logischen Prinzipien der Programmierung herausgearbeitet. Grundlage für diese Betrachtungen bildet eine abstrakte Definition der Rechenautomaten die im gewissen Sinne eine "Verfeinerung" des abstrakten Automatenbegriffs von Huffman, Moore, Mealy, Gluškov darstellt, indem nämlich die üblichen Bestimmungsstücke eines abstrakten Automaten zusätzlich so strukturiert werden, wie dies bei realen digitalen Rechenautomaten der Fall ist. Zum Beispiel wird die Menge A der inneren Zustände als Produktmenge der Form $B \times Z^I \times I$ angenommen, wobei B, Z, I gegebene abstrakte Mengen sind. Jeder Zustand ist dann ein Tripel (b, z, i), wobei $b \in B$ den jeweiligen "Betriebszustand" des Rechenautomaten bezeichnet (z.B. Halte- oder Rechenbetrieb), $z \in Z^I$ den Zustand der verschiedenen "Zellen" des Automaten beschreibt (Z ist dabei die Menge der Zustände, die die einzelnen Zellen des Automaten annehmen können während die Menge I als Indexmenge für die verschiedenen Zellen dient, d.h. die Adressen symbolisiert) und $i \in I$ die im "Befehlszähler" eingestellte Adresse angibt. Weitere wichtige Bestimmungsstücke sind die "Eingabefunktion" des Automaten, seine "Adressenzahl", sein "Befehlssystem" und zwei Funktionen zur Dekodierung der Zellenzustände, die sämtlich abstrakt festgelegt werden Dadurch wird es möglich, dem Programmierungsproblem eine ganz abstrakte Fassung zu geben. Der Übergang von einer algorithmischen Sprache zur Programmierungssprache erscheint als ein Übersetzungsproblem, das sich bei bekanntem Übersetzungsalgorithmus unter gewissen Voraussetzungen mittels eines Übersetzungsprogramms (programmierenden Programms) realisieren läßt. Es wird in diesem Zusammenhang die Überzeugung dargelegt, daß die Frage der Existenz einer optimalen algorithmischen Sprache (nach Präzisierung der Gesichtspunkte der Optimalisierung) ein Problem ist, das nur durch eine wissenschaftliche Untersuchung in befriedigender Weise gelöst werden kann. Abschließend werden die Möglichkeiten diskutiert, mit Rechenautomaten auch nichtnumerische Probleme zu lösen. *G. Asser* (Greifswald)

Petrone, Luigi 7000

Linguaggi algoritmici.

Univ. e Politec. Torino Rend. Sem. Mat. **22** (1962/63), 55–77 (1964).

This is a survey article on some concepts and results that arise from the study of the interconnections between grammars of formal languages and types of automata that serve as generating or as recognition devices for various such languages. The paper reports, in particular, on the results concerning regular and context-free languages. *E. Engeler* (Minneapolis, Minn.)

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Journal references in Mathematical Reviews are given in the following form: *Rückschritte der Math.* (6) **5** (**56**) (1960/61), no. 2, 18–53 (1962), where, after the abbreviated title, one has: (series number) volume number (volume number in first series, if given) (nominal date), issue number if necessary, first page—last page (imprint date). If only one date is given, this will usually be interpreted as the nominal date and will be printed immediately after the volume number. If no volume number is given, the year will be used in its place. The symbol ★ precedes the title of a book or other non-periodical which is being reviewed as a whole.

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42 Univalent and p-valent functions
43 Coefficient problems
45 Riemann surfaces, uniformization
46 Algebraic functions, Abelian integrals
47 Quasiconformal mapping
49 Automorphic functions and modular functions
50 Maximum principle. Schwarz's lemma, Phragmén-Lindelöf theorems
52 Extremal problems in complex domain
55 Entire functions
60 Meromorphic functions, general theory

61 Distribution of values, Picard-type theorems, Nevanlinna theory, defect (deficiency) theory
62 Theory of cluster sets, prime ends, boundary behavior
65 Bounded functions, positive real part, Blaschke products
66 Bounded mean modulus, bounded characteristic
67 H_p classes and other restrictions
70 Approximation by polynomials and rational functions, Faber polynomials
75 Boundary-value problems, Dirichlet problem
77 Special boundary-value problems; Riemann-Hilbert problem. Inverse boundary problems
79 Algebroidal functions
80 Topological function theory, functions topologically equivalent to analytic mappings
81 Generalized analytic mappings. Pseudo-analytic functions of Bers, Vekua, etc.
82 Functions of hypercomplex variables and generalized variables
85 Spaces of analytic functions
86 Kernel function and applications
87 Algebras of analytic functions

31. POTENTIAL THEORY
10 Harmonic functions in the plane
11 Harmonic functions, general
15 Subharmonic, superharmonic functions
20 Boundary-value problems, Dirichlet problem
21 Capacity and harmonic measure
22 Generalized capacity and potential
30 Biharmonic and polyharmonic functions and equations
35 Poisson's equation

32. SEVERAL COMPLEX VARIABLES
02 Foundations, general questions
10 Multiple power series, formal theory
15 Analytic and meromorphic curves
20 Holomorphic mappings, domains of holomorphy, analytic spaces (see also 57.00)
25 Automorphic functions
30 Approximation theorems

33. SPECIAL FUNCTIONS
04 Number-theoretic functions, special sequences
05 Tables of special functions
10 Exponential and trigonometric functions
15 Gamma and Beta functions
17 Error function, probability integral
19 Elliptic functions and integrals
20 Hypergeometric functions
21 E-functions
25 Cylindrical functions; Bessel functions
27 Spherical functions; Legendre polynomials and functions, spherical harmonics, ultraspherical polynomials
28 Lamé, Mathieu, spheroidal wave functions
29 Wave functions
32 Gegenbauer functions
40 Orthogonal polynomials, general (Chebyshev, Hermite, Jacobi, Laguerre polynomials and functions)

34. ORDINARY DIFFERENTIAL EQUATIONS
02 Solutions in closed form; integration by quadratures; reduction of differential equations
03 Operational calculus
04 General existence and uniqueness theorems, Picard approximations and generalizations
07 Continuous dependence of solution on parameter
10 Analytic theory
20 Linear equations and systems, general coefficients
21 Linear equations with constant coefficients
22 Linear equations with periodic and almost periodic coefficients
30 Boundary-value problems, linear equations
31 Sturm-Liouville theory
32 Eigenvalues, eigenfunctions
33 Eigenfunction expansions
34 Asymptotic properties of eigenfunctions
36 Boundary-value problems, non-linear
40 Qualitative theory of first- and second-order equations and systems; singular points and limit cycles
41 Periodic and almost periodic solutions
42 Non-linear oscillations
43 Stability of first- and second-order equations and systems
50 Qualitative theory of equations and systems of order greater than 2; asymptotic behavior of solutions
51 Stability; periodic and almost periodic solutions
53 Perturbations
54 Singular perturbations; small parameter with highest derivative
60 Stochastic differential equations
65 Dynamical systems, systems on manifolds
70 Equations and systems of infinite order
75 Difference-differential equations
76 Solution by finite differences
80 Differential equations in Banach and other spaces
85 Optimal control problems

35. PARTIAL DIFFERENTIAL EQUATIONS
05 General properties of solutions, Cauchy-Kowalewski theorems, etc.
10 First-order equations and systems
12 Classification of equations and systems
13 Exact equations and integrable systems. Pfaffian equations
15 Characteristics
20 Separation of variables, elementary properties of the equations of mathematical physics
30 Elliptic equations and systems of first and second order
31 Strongly elliptic systems
32 Quasi-linear elliptic equations and systems
33 Schrödinger equation, wave equation
34 Maxwell's equations, Helmholtz equation
35 Spectral analysis, eigenfunctions, operators
36 Elliptic systems and equations of higher order
37 Biharmonic equation
40 Hyperbolic equations and systems of second order
41 Quasi-linear hyperbolic systems and equations of second order
45 Hyperbolic equations and systems of higher order
Parabolic equations of second order
55 Equations of mixed type and second order
Mixed type, order higher than two
Non-linear equations and systems
62 Approximation of solutions
Solution by finite differences
65 Special equations: Navier-Stokes, etc.
70 Asymptotic properties
75 Infinite systems, equations of infinite order
Equations in Banach spaces and other spaces

39. FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS
10 Finite differences, general
12 Solution of ordinary differential equations by finite differences
13 Solution of partial differential equations by finite differences
20 Difference equations
30 Functional equations

40. SEQUENCES, SERIES, SUMMABILITY
10 Series and sequences of numbers, general
11 Infinite products
12 Continued fractions
13 Convergence
15 Multiple series and sequences
20 Sequences and series of functions, convergence
30 Summability, general methods
31 Summability, matrix methods
32 Cesàro summability

30 Abel, Poisson summability
35 Summability of double series
40 Convergence and summability of integrals
42 Tauberian theorems

41. APPROXIMATIONS AND EXPANSIONS
10 Interpolation
15 Polynomial approximation
17 Approximation by rational functions
30 Approximate quadratures
40 Rate of convergence, degree of approximation, best approximation
41 Constructive theory of functions
45 Approximation by smoothing integrals
50 Asymptotic approximations
55 Other methods of approximation

42. FOURIER ANALYSIS
05 Trigonometric polynomials
06 Best approximation by trigonometric polynomials
08 Trigonometric interpolation
10 Fourier coefficients
11 Convergence of Fourier series
15 Orthogonal functions
16 Expansions in orthogonal functions
17 Completeness of sets of orthogonal functions
18 Completeness, closure, spectral synthesis
20 Summability of Fourier and generalized Fourier expansions
25 Fourier transform
26 Other Fourier-type transforms
27 Trigonometric moment problems
30 Almost periodic functions
35 Positive definite functions
40 Multiple Fourier series and integrals
50 Abstract harmonic analysis

44. INTEGRAL TRANSFORMS, OPERATIONAL CALCULUS
10 Laplace transform
25 Convolution
28 General transforms
30 Special transforms: Legendre, Hilbert, etc.
32 Multiple transforms
40 Operational calculus
42 Operational calculus in several variables
50 Fractional derivatives and integrals
60 Moment problems

45. INTEGRAL EQUATIONS
10 Linear integral equations, general
11 Fredholm equations
12 Eigenvalue problems
13 Volterra equations
15 Singular equations
20 Systems of linear equations
30 Non-linear integral equations
31 Singular non-linear equations
40 Integro-differential equations

46. FUNCTIONAL ANALYSIS
10 Topological linear spaces (including normed spaces and Hilbert spaces)
20 Partially ordered linear spaces
30 Special function spaces
40 Distributions
50 Single linear operators in linear or partially ordered spaces
60 Groups or semigroups of linear operators, representations of topological groups
65 Rings of operators, group algebras, abstract topological algebras, and their representations
70 Banach algebras of analytic functions
80 Differential and integral operators
90 Applications of functional analysis

49. CALCULUS OF VARIATIONS

50. GEOMETRY
05 Foundations
10 Euclidean
11 Triangles, tetrahedra
12 Circles, spheres
14 Constructions
15 Descriptive geometry
17 Analytic geometry
20 Affine geometry, general
25 Transformations
30 Projective geometry, general
35 Transformations
38 Second-order loci
40 Non-Euclidean geometry (i.e., hyperbolic or elliptic)
45 Minkowski geometry
48 Special geometries
50 Classical algebraic geometry
51 Foundations
52 Transformations
55 Curves
56 Surfaces
57 Abelian varieties, genera, integrals
60 Finite geometries
70 Projective planes
80 Configurations
90 Regular figures, divisions of space
95 Paradoxical decompositions

52. CONVEX SETS AND GEOMETRIC INEQUALITIES
10 Convex polyhedra
25 Convex curves
30 Convex regions
34 Helly-type theorems
40 Extremum problems and geometric inequalities
45 Packing and covering
50 Distance geometries

53. DIFFERENTIAL GEOMETRY
01 Curves and surfaces in Euclidean spaces
04 Minimal surfaces
10 Affine differential geometry
20 Projective differential geometry
25 Conformal differential geometry
30 Non-Euclidean differential geometry
32 Other special differential geometries
35 Vector and tensor analysis
38 Spinor analysis
40 Differentiable manifolds (general theory), e.g., definitions jets, tangent bundles
42 Differential invariants (local theory), geometric objects
45 Differential calculus (global theory); forms, currents, integration, tensor fields, etc.
50 Connections (general theory), including G-structures
55 Affine connections
60 Projective connections
65 Conformal connections
70 Local Riemannian geometry
72 Riemannian manifolds
75 Global surface theory (convex surfaces)
78 Lorentz metrics and generalizations
80 Kählerian connections and generalizations
85 Finsler spaces and generalizations (areal metrics)
90 Integral geometry
99 Other generalizations (G-spaces of Busemann, etc.)

54. GENERAL TOPOLOGY
10 Axiomatics; generalizations of topological spaces
20 General topological spaces
21 Convergence notions

22 Compactness and generalizations
23 Compactifications, other extensions
24 Proximity spaces
25 Uniform spaces
26 Metric spaces, metrizability
27 Dimension theory
28 Mappings
29 Many-valued mappings
30 Continua
35 Topological semigroups, lattices, etc.
40 Topology of E_2, 2-manifolds
50 Topology of E_n, n-manifolds ($n > 2$)
60 Transformation groups
70 Topological dynamics
80 Fixed-point theorems, coincidence theorems, etc.

55. ALGEBRAIC TOPOLOGY
10 Graphs, map-coloring
20 Knots, links
30 Homology and cohomology theory
32 Sheaves
34 Cohomology (and homology) operations, Steenrod algebra
36 Periodic transformations, fixed points, coincidences (Smith theory)
38 Dimension theory
40 Homotopy theory, general
42 Algebra of mappings (including cohomotopy groups)
45 Computation of homotopy groups
50 Fiber spaces
52 Spectral sequences
60 Manifolds
62 Classification
64 Mappings
66 Generalized manifolds, local homology
68 Homotopy manifolds
70 Imbedding, immersion

57. TOPOLOGY AND GEOMETRY OF DIFFERENTIABLE MANIFOLDS
10 Differentiable structures, cobordism
20 Differentiable mappings, imbeddings, immersions, diffeomorphisms
30 Fiber bundles, tangent bundles
32 Characteristic classes
34 Topology of vector and tensor fields
36 Foliations
40 Topology of Lie groups
45 Topology of homogeneous spaces
50 Analysis on differentiable manifolds
60 Complex manifolds (also in several complex variables)
70 Infinite Lie groups, deformations

60. PROBABILITY
05 Foundations
10 Combinatorial probability theory
15 Geometric probability theory
20 Distributions
30 Limit theorems
40 Stochastic processes, general theory
45 Markov processes
48 Branching processes
50 Stationary processes
55 Random walks, Brownian motion
60 Queuing theory, telephone traffic, storage
62 Renewal theory
65 Other special processes
70 Random noise
90 Other applications

62. STATISTICS
05 Elementary descriptive statistics
10 Distributions of statistical functions
20 Estimation theory: parametric case
25 Testing of hypotheses: parametric case
30 Multivariate analysis, regression analysis
35 Order statistics
40 Non-parametric methods
50 Analysis of experiments
55 Design of experiments
60 Decision theory
65 Multistage decision procedures, sequential analysis
70 Statistical engineering, quality control
75 Life testing
80 Sampling surveys
85 Statistical analysis of stochastic processes, time series analysis
90 Applications

65. NUMERICAL METHODS
05 General mathematical methods, iteration, tables
15 Monte Carlo methods
20 Interpolation, smoothing, least squares, curve fitting, approximation of functions
25 Computation of special functions, series, integrals
30 Mathematical programming
35 Linear equations, determinants, matrices
40 Eigenvalues, eigenvectors, Rayleigh-Ritz method
50 Solution of algebraic and transcendental equations and systems
55 Numerical differentiation and integration, mechanical quadrature
60 Ordinary differential equations
65 Partial differential equations
70 Difference and functional equations
75 Integral and integro-differential equations
80 Error analysis
85 Graphical methods, nomography
90 Harmonic analysis and synthesis

68. COMPUTING MACHINES

69. GENERAL APPLIED MATHEMATICS

70. MECHANICS OF PARTICLES AND SYSTEMS
10 Foundations
20 Statics
30 Kinematics, mechanisms, linkages
40 Dynamics
50 Oscillations, stability
55 Non-linear oscillation
60 Exterior ballistics
70 Variable mass, rockets

73. ELASTICITY, PLASTICITY
05 Foundations of mechanics of deformable solids
06 Elasticity: general theorems
07 Finite deformation
10 Plane stress and strain
15 Three-dimensional problems
25 Beams and rods
30 Plates
32 Shells and membranes
35 Anisotropic bodies
37 Dislocation theory
45 Vibrations, structural dynamics
47 Aeroelasticity
50 Stability, buckling, failure
55 Wave propagation, impact
65 Visco-elasticity
70 Plasticity, creep
80 Soil mechanics
90 Thermo-mechanics

76. FLUID MECHANICS, ACOUSTICS
- 05 Foundations
- 10 Incompressible fluids: general theory
- 15 Incompressible fluids with special boundaries
- 17 Airfoil theory
- 20 Free surface flows, water waves, jets, wakes
- 25 Viscous fluids
- 30 Boundary layer theory
- 35 Stability of flow
- 37 Convection
- 40 Turbulence
- 45 Rarefied gas flow
- 50 Compressible fluids: general theory
- 55 Compressible fluids: subsonic flow
- 60 Compressible fluids: transonic flow
- 65 Compressible fluids: supersonic and hypersonic flow
- 70 Shock waves
- 75 Aero- and hydrodynamic sound
- 80 Acoustics
- 85 Non-Newtonian fluids
- 90 Magnetohydro- and aerodynamics, ionized gas flow (see also 78.30 ; 82.25 ; 85.50)
- 92 Quantum hydrodynamics
- 93 Relativistic hydrodynamics
- 95 Diffusion, filtration

78. OPTICS, ELECTROMAGNETIC THEORY, CIRCUITS
- 05 Geometric optics
- 10 Physical optics
- 20 Electron optics
- 30 Space charge waves
- 40 Electromagnetic theory
- 45 Electro- and magnetostatics
- 50 Waves and radiation
- 60 Diffraction, scattering
- 70 Antennas, wave-guides
- 80 Circuits, networks (for switching theory, see 94.70)
- 90 Technical applications

80. CLASSICAL THERMODYNAMICS, HEAT TRANSFER
- 10 Classical thermodynamics
- 20 Heat and mass transfer
- 30 Combustion, interior ballistics
- 40 Chemical kinetics

81. QUANTUM MECHANICS
- 10 General theory
- 15 Scattering theory
- 20 Mathematical field theory
- 22 Dispersion relations
- 24 Weak interactions
- 26 Elementary particles
- 28 Nuclear physics
- 30 Atomic physics
- 38 Molecular physics
- 50 Quantum mechanics of many-body systems
- 60 Superconductivity and superfluidity
- 70 Quantum statistical mechanics

82. STATISTICAL PHYSICS, STRUCTURE OF MATTER
- 05 General statistical mechanics
- 10 Quantum statistical mechanics
- 15 Statistical thermodynamics
- 17 Irreversible thermodynamics
- 20 Kinetic theory of gases
- 25 Plasma (see also 76.90)
- 30 Liquids
- 40 Solids
- 45 Crystals
- 50 Metals
- 60 Transport processes
- 65 Nuclear reactor theory

83. RELATIVITY
- 10 Special relativity
- 20 General relativity
- 30 Unified field theories
- 40 Other relativistic theories

85. ASTRONOMY
- 10 Celestial mechanics
- 15 Galactic and stellar dynamics
- 20 3- and n-body problems
- 27 Astronautics
- 30 Stellar structure
- 40 Stellar atmospheres, radiative transfer
- 50 Hydrodynamic and hydromagnetic problems
- 60 Statistical astronomy
- 70 Cosmology
- 80 Special problems
- 90 Radio astronomy

86. GEOPHYSICS
- 10 Hydrology, hydrography, oceanography
- 20 Meteorology
- 30 Seismology
- 40 Potentials, prospecting
- 50 Geo-electricity and magnetism
- 60 Geodesy, mapping problems
- 70 Atmospheric physics

90. ECONOMICS, OPERATIONS RESEARCH, GAMES
- 05 Decisions, utility
- 10 Economic models
- 15 Economic time-series analysis
- 17 Miscellaneous applications to economics
- 30 Management science, operations research
- 35 Highway traffic
- 40 Actuarial theory
- 50 Linear programming
- 52 Transportation problems
- 54 Flows in networks
- 56 Integer programming
- 58 Non-linear programming
- 60 Logistics, inventory, storage
- 70 Games

92. BIOLOGY AND BEHAVIORAL SCIENCES
- 10 Biology
- 20 Genetics
- 30 Population dynamics, epidemiology
- 40 Sociology
- 50 Psychology
- 55 Nervous networks

94. INFORMATION, COMMUNICATION, CONTROL
- 05 Foundations, coding theorem
- 10 Statistical theory of communication channels, filters
- 13 Detection
- 15 Codes, decoding
- 20 Data processing
- 30 Linguistics, machine translation
- 55 Control systems
- 57 Stability of control systems
- 60 Optimal control
- 65 Random control
- 70 Switching theory, relays
- 80 Servomechanisms
- 90 Automata (see also 02.95)

ABBREVIATIONS OF NAMES OF JOURNALS

This list gives the form of references used in MATHEMATICAL REVIEWS and the complete title; the place of publication and other pertinent information are given in parentheses when desirable for clarity.

Abh. Akad. Wiss. Göttingen Math.-Phys. Kl. Abhandlungen der Akademie der Wissenschaften in Göttingen. Mathematisch-Physikalische Klasse. (Göttingen)

Abh. Deutsch. Akad. Wiss. Berlin Kl. Math. Phys. Tech. Abhandlungen der Deutschen Akademie der Wissenschaften zu Berlin. Klasse für Mathematik, Physik und Technik. (Berlin)

Abh. Dokumentationszentrums Technik Wirtschaft. Abhandlungen des Dokumentationszentrums der Technik und Wirtschaft. (Vienna)

Abh. Math. Sem. Univ. Hamburg. Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg. (Hamburg)

Abh. Sächs. Akad. Wiss. Leipzig Math.-Natur. Kl. Abhandlungen der Sächsischen Akademie der Wissenschaften zu Leipzig. Mathematisch-Naturwissenschaftliche Klasse. (Leipzig)

Abstracts Bulgarian Sci. Lit. Math. Phys. Astronom. Geophys. Geodes. Bulgarian Academy of Sciences. Centre of Scientific Information and Documentation. Abstracts of Bulgarian Scientific Literature. Mathematics, Physics, Astronomy, Geophysics, Geodesy. (Sofia)

Acad. R. P. Romîne An. Romîno-Soviet. Ser. Mat.-Fiz. Academia Republicii Populare Romîne. Institutul de Studii Romîno-Sovietic. Analele Romîno-Sovietice. Seria Matematică-Fizică. Rumyno-Sovetskie Zapiski. Fiziko-Matematičeskaja Serija. (Bucharest) (Continued as: *An. Romîno-Soviet. Mat.-Fiz.*)

Acad. R. P. Romîne Baza Cerc. Şti. Timişoara Stud. Cerc. Şti. Tehn. Academia Republicii Populare Romîne. Baza de Cercetări Ştiinţifice Timişoara. Studii şi Cercetări Ştiinţe Tehnice. (Timişoara)

Acad. R. P. Romîne Fil. Iaşi Stud. Cerc. Şti. Fiz. Şti. Tehn. Academia Republicii Populare Romîne. Filiala Iaşi. Studii şi Cercetări Ştiinţifice. Fizică şi Ştiinţe Tehnice. (Iaşi)

Acad. R. P. Romîne Fil. Iaşi Stud. Cerc. Şti. Mat. Academia Republicii Populare Romîne. Filiala Iaşi. Studii şi Cercetări Ştiinţifice. Matematică. (Iaşi)

Acad. R. P. Romîne Stud. Cerc. Mat. Academia Republicii Populare Romîne. Institutul de Matematică. Studii şi Cercetări Matematice. Matematičeskie Trudy i Issledovanija. Études et Recherches Mathématiques. (Bucharest) (Continued as: *Stud. Cerc. Mat.*)

Acad. R. P. Romîne Stud. Cerc. Mec. Apl. Academia Republicii Populare Romîne. Institutul de Mecanică Aplicată "Traian Vuia". Studii şi Cercetări de Mecanică Aplicată. (Bucharest)

Acad. Roy. Belg. Bull. Cl. Sci. Académie Royale de Belgique. Bulletin de la Classe des Sciences. Koninklijke Belgische Academie. Mededelingen van de Klasse der Wetenschappen. (Brussels)

Acad. Roy. Belg. Cl. Sci. Mém. Coll. in-8°. Académie Royale de Belgique. Classe des Sciences. Mémoires. Collection in-8°. Koninklijke Belgische Academie. Klasse der Wetenschappen. Verhandelingen. Verzameling in-8°. (Brussels)

Acad. Serbe Sci. Arts Glas Cl. Sci. Math. Natur. Académie Serbe des Sciences et des Arts. Glas. Classe des Sciences Mathématiques et Naturelles. Nouvelle Série. (Belgrade)

Accad. Naz. Sci. Lett. Arti Modena Atti Mem. Accademia Nazionale di Scienze Lettere e Arti. Modena. Atti e Memorie. (Modena)

Acta Acad. Abo. Math. Phys. Acta Academiae Aboensis. Mathematica et Physica. (Åbo = Turku) (Continued as: *Acta Acad. Abo. Ser. B*)

Acta Acad. Abo. Ser. B. Acta Academiae Aboensis, Ser. B. Mathematica et Physica. Matematik-Naturvetenskaper-Teknik. (Åbo = Turku) (Formerly: *Acta Acad. Abo. Math. Phys.*)

Acta Arith. Polska Akademia Nauk. Instytut Matematyczny. Acta Arithmetica. (Warsaw)

Acta Astronom. Sinica. Acta Astronomica Sinica. (Peking)

Acta Cryst. Acta Crystallographica. (Copenhagen)

Acta Fac. Natur. Univ. Comenian. Acta Facultatis Rerum Naturalium Universitatis Comenianae. (Bratislava)

Acta Math. Acta Mathematica. (Uppsala)

Acta Math. Acad. Sci. Hungar. Acta Mathematica Academiae Scientiarum Hungaricae. (Budapest)

Acta Math. Sinica. Acta Mathematica Sinica. (Peking) (Translated as: *Chinese Math.*)

Acta Mech. Sinica. Acta Mechanica Sinica. (Peking)

Acta Philos. Fenn. Acta Philosophica Fennica. (Helsinki)

Acta Phys. Acad. Sci. Hungar. Acta Physica Academiae Scientiarum Hungaricae. (Budapest)

Acta Phys. Austriaca. Acta Physica Austriaca. (Vienna)

Acta Phys. Polon. Polska Akademia Nauk. Instytut Fizyki. Acta Physica Polonica. (Warsaw)

Acta Polytech. Scand. Math. and Comput. Mach. Ser. Acta Polytechnica Scandinavica. Mathematics and Computing Machinery Series. (Stockholm)

Acta Sci. Math. (Szeged). Acta Universitatis Szegediensis. Acta Scientiarum Mathematicarum. (Szeged)

Acta Sci. Natur. Univ. Pekinensis. Acta Scientiarum Naturalium Universitatis Pekinensis. (Peking)

Acta Tech. Acad. Sci. Hungar. Acta Technica Academiae Scientiarum Hungaricae. (Budapest)

Acta Univ. Lund. Sect. II. Acta Universitatis Lundensis. Sectio II. Medica, Mathematica, Scientiae rerum naturalium. (Lund) (Formerly: *Lunds Univ. Årsskr. Avd. 2*)

Actes Soc. Helv. Sci. Natur. Actes de la Société Helvétique des Sciences Naturelles. Verhandlungen der Schweizerischen Naturforschenden Gesellschaft. Atti della Società Elvetica di Scienze Naturali. (Basel)

Actualités Sci. Indust. Actualités Scientifiques et Industrielles. (Paris)

Acustica. Acustica. (Zürich)

Advances in Math. Advances in Mathematics. (New York)

Advances in Phys. Advances in Physics. A Quarterly Supplement of the Philosophical Magazine. (London)

Aero. Quart. The Aeronautical Quarterly. (London)

AIAA J. AIAA Journal. (Easton, Pa.)

Akad. Nauk Armjan. SSR Dokl. Akademija Nauk Armjanskoi SSR. Doklady. (Erevan)

Akad. Nauk Azerbaĭdžan. SSR Dokl. Akademija Nauk Azerbaĭdžanskoĭ SSR. Doklady. (Baku)

Akad. Nauk Azerbaĭdžan. SSR Trudy Inst. Mat. Meh. Akademija Nauk Azerbaĭdžanskoĭ SSR. Trudy Instituta Matematiki i Mehaniki. (Baku)

Akad. Nauk Gruzin. SSR Trudy Tbiliss. Mat. Inst. Razmadze. Akademija Nauk Gruzinskoĭ SSR. Trudy Tbilisskogo Matematičeskogo Instituta im. A. M. Razmadze. (Tiflis)

Akad. Nauk Kazah. SSR Trudy Astrofiz. Inst. Akademija Nauk Kazahskoĭ SSR. Trudy Astrofizičeskogo Instituta. (Alma-Ata)

Akad. Nauk Kazah. SSR Trudy Sekt. Mat. Meh. Akademija Nauk Kazahskoĭ SSR. Trudy Sektora Matematiki i Mehaniki. (Alma-Ata)

Akad. Nauk Latv. SSR Trudy Inst. Fiz. Akademija Nauk Latviĭskoĭ SSR. Trudy Instituta Fiziki. (Riga)

Akad. Nauk SSSR Izv. Sibirsk. Otd. Akademija Nauk SSSR. Izvestija Sibirskogo Otdelenija Akademii Nauk SSSR. (Novosibirsk) (Continued as: *Izv. Sibirsk. Otd. Akad. Nauk SSSR Ser. Tehn. Nauk*)

Akad. Nauk Uzbek. SSR Trudy Inst. Mat. Akademija Nauk Uzbekskoĭ SSR. Trudy Instituta Matematiki im. V. I. Romanovskogo. (Tashkent)

Akad. Wiss. Lit. Mainz Abh. Math.-Natur. Kl. Akademie der Wissenschaften und der Literatur in Mainz. Abhandlungen der Mathematisch-Naturwissenschaftlichen Klasse. (Wiesbaden)

Akust. Beihefte. Akustische Beihefte. (Stuttgart)

Akust. Ž. Akademija Nauk SSSR. Akustičeskiĭ Žurnal. (Moscow) (Translated as: *Soviet Physics Acoust.*)

Algebra i Logika Sem. Akademija Nauk SSSR. Sibirskoe Otdelenie. Institut Matematiki. Algebra i Logika. Seminar. (Novosibirsk)

Algorytmy. Algorytmy. Prace. Instytutu Maszyn Matematycznych. Polskiej Akademii Nauk. (Warsaw)

Allgemein. Statist. Arch. Allgemeines statistisches Archiv. Organ der Deutschen Statistischen Gesellschaft. (Munich)

Amer. J. Math. American Journal of Mathematics. (Baltimore, Md.)

Amer. J. Phys. American Journal of Physics. (New York)

Amer. Math. Monthly. The American Mathematical Monthly. (Buffalo, N.Y.)

Amer. Math. Soc. Notices. American Mathematical Society Notices. (Providence, R.I.) (Continued as: *Notices Amer. Math. Soc.*)

Amer. Math. Soc. Transl. American Mathematical Society Translations. (Providence, R.I.)

AMR. Applied Mechanics Reviews. (New York)

An. Acad. Brasil. Ci. Anais da Academia Brasileira de Ciencias. (Rio de Janeiro)

An. Inst. Mat. Univ. Nac. Autónoma México. Anales del Instituto de Matemáticas. Universidad Nacional Autónoma de México. (Mexico City)

An. Romîno-Soviet. Mat.-Fiz. Analele Romîno-Sovietice. Matematică-Fizică. Rumyno-Sovetskie Zapiski. Matematika-Fizika. (Bucharest) (Formerly: *Acad. R. P. Romîne An. Romîno-Soviet. Ser. Mat.-Fiz.*)

An. Soc. Ci. Argentina. Anales de la Sociedad Científica Argentina. (Buenos Aires)

An. Şti. Univ. "Al. I. Cuza" Iaşi Secţ. I. Analele Ştiinţifice ale Universităţii "Al. I. Cuza" din Iaşi. (Seria Nouă). Secţiunea I. (Matematică, Fizică, Chimie). (Iaşi) (Continued as: *An. Şti. Univ. "Al. I. Cuza" Iaşi Secţ. I a Mat.*)

An. Şti. Univ. "Al. I. Cuza" Iaşi Secţ. I a Mat. Analele Ştiinţifice ale Universităţii "Al. I. Cuza" din Iaşi. (Seria Nouă). Secţiunea I. a. Matematică. (Iaşi) (Formerly: *An. Şti. Univ. "Al. I. Cuza" Iaşi Secţ. I*)

An. Univ. Bucureşti Ser. Acta Logica. Analele Universităţii Bucureşti. Seria Acta Logica. (Bucharest)

An. Univ. Bucureşti Ser. Şti. Natur. Mat.-Fiz. Analele Universităţii Bucureşti. Seria Ştiinţele Naturii. Matematică-Fizică. (Bucharest) (Formerly: *Ann. Univ. C. I. Parhon Ser. Şti. Natur. Mat.-Fiz.*)

An. Univ. C. I. Parhon Ser. Şti. Natur. Mat.-Fiz. Analele Universităţii C. I. Parhon. Seria Ştiinţele Naturii. Matematică-Fizică. (Bucharest) (Continued as: *An. Univ. Bucureşti Ser. Şti. Natur. Mat.-Fiz.*)

An. Univ. Timişoara Ser. Şti. Mat.-Fiz. Analele Universităţii din Timişoara. Seria Ştiinţe Matematice-Fizice. (Timişoara) (Formerly: *Lucrăr. Şti. Inst. Ped. Timişoara Mat.-Fiz.*)

Ann. Acad. Sci. Fenn. Ser. A I. Suomalaisen Tiedeakatemian Toimituksia. Sarja A. Annales Academiae Scientiarum Fennicae. Series A. I. Mathematica. (Helsinki)

Ann. Acad. Sci. Fenn. Ser. A VI. Suomalaisen Tiedeakatemian Toimituksia. Sarja A. Annales Academiae Scientiarum Fennicae. Series A. VI. Physica. (Helsinki)

Ann. Assoc. Internat. Calcul Anal. Annales de l'Association Internationale pour le Calcul Analogique. Proceedings of the International Association for Analog Computation. (Brussels)

Ann. Astrophys. Annales d'Astrophysique. (Paris)

Ann. Fac. Sci. Univ. Toulouse. Annales de la Faculté des Sciences de l'Université de Toulouse pour les Sciences Mathématiques et les Sciences Physiques. (Toulouse)

Ann. Geofis. Annali di Geofisica. Rivista dell'Istituto Nazionale di Geofisica. (Rome)

Ann. Inst. Fourier (Grenoble). Université de Grenoble. Annales de l'Institut Fourier. (Grenoble)

Ann. Inst. H. Poincaré. Annales de l'Institut Henri Poincaré. (Paris)

Ann. Inst. Statist. Math. Annals of the Institute of Statistical Mathematics. (Tokyo)

Ann. Japan Assoc. Philos. Sci. Annals of the Japan Association for Philosophy of Science. (Tokyo)

Ann. Mat. Pura Appl. Annali di Matematica Pura ed Applicata. (Bologna)

Ann. Math. Statist. The Annals of Mathematical Statistics. (Baltimore, Md.)

Ann. New York Acad. Sci. Annals of the New York Academy of Sciences. (New York)

Ann. of Math. Annals of Mathematics. (Princeton, N.J.)

Ann. of Sci. Annals of Science. (London)

Ann. Physics. Annals of Physics. (New York)

Ann. Physik. Annalen der Physik. (Leipzig)

Ann. Physique. Annales de Physique. (Paris)

Ann. Polon. Math. Polska Akademia Nauk. Annales Polonici Mathematici. (Warsaw)

Ann. Ponts Chaussées. Annales des Ponts et Chaussées. (Paris)

Ann. Radioélec. Annales de Radioélectricité. (Paris)

Ann. Sci. École Norm. Sup. Annales Scientifiques de l'École Normale Supérieure. (Paris)

Ann. Scuola Norm. Sup. Pisa. Annali della Scuola Normale Superiore di Pisa. Scienze Fisiche e Matematiche. (Pisa)

Ann. Soc. Sci. Bruxelles Sér. I. Annales de la Société Scientifique de Bruxelles. Série I. Sciences Mathématiques, Astronomiques et Physiques. (Brussels)

Ann. Télécommun. Annales des Télécommunications. (Paris)

Ann. Univ. Lyon Sect. A. Annales de l'Université de Lyon. Section A. Sciences Mathématiques et Astronomie. (Lyon)

Ann. Univ. Lyon Sect. B. Annales de l'Université de Lyon. Section B. Sciences Physiques et Chimiques. (Lyon)

Ann. Univ. Mariae Curie-Skłodowska Sect. A. Annales Universitatis Mariae Curie-Skłodowska. Sectio A. Mathematica. (Lublin)

Ann. Univ. Mariae Curie-Skłodowska Sect. AA. Annales Universitatis Mariae Curie-Skłodowska. Sectio AA. Physica and Chemistry. (Lublin)

Ann. Univ. Sarav. Annales Universitatis Saraviensis. (Saarbrücken) (Continued as: *Ann. Univ. Sarav. Math.-Natur. Fak.*)

Ann. Univ. Sarav. Math.-Natur. Fak. Annales Universitatis Saraviensis. Mathematisch-Naturwissenschaftliche Fakultät. (Berlin) (Formerly: *Ann. Univ. Sarav.*)

Ann. Univ. Sci. Budapest. Eötvös Sect. Math. Annales Universitatis Scientiarum Budapestinensis de Rolando Eötvös Nominatae. Sectio Mathematica. (Budapest)

Ann. Univ. Turku. Ser. A I. Annales Universitatis Turkuensis. Series A. I. Astronomica-Chemica-Physica-Mathematica. (Turku)

Annuaire Univ. Sofia Fac. Math. Godišnik na Sofiĭskija Universitet Matematičeski Fakultet. Annuaire de l'Université de Sofia. Faculté de Mathématiques. (Sofia) (Formerly: *Annuaire Univ. Sofia Fac. Sci. Phys. Math. Livre 1 Math.*)

Annuaire Univ. Sofia Fac. Phys. Godišnik na Sofiĭskija Universitet Fizičeski Fakultet. Annuaire de l'Université de Sofia. Faculté de Physique. (Sofia) (Formerly: *Annuaire Univ. Sofia Fac. Sci. Phys. Math. Livre 2 Phys.*)

Annuaire Univ. Sofia Fac. Sci. Phys. Math. Livre 1 Math. Godišnik na Sofijskija Universitet Fiziko-Matematičeski Fakultet. Kniga 1—Matematika. Annuaire de l'Université de Sofia. Faculté des Sciences Physiques et Mathématiques. Livre 1—Mathématiques. (Sofia) (Continued as: *Annuaire Univ. Sofia Fac. Math.*)

Annuaire Univ. Sofia Fac. Sci. Phys. Math. Livre 2 Phys. Godišnik na Sofijskija Universitet Fiziko-Matematičeski Fakultet. Kniga 2—Fizika. Annuaire de l'Université de Sofia. Faculté des Sciences Physiques et Mathématiques. Livre 2—Physique. (Sofia) (Continued as: *Annuaire Univ. Sofia Fac. Phys.*)

Anuário Obs. São Paulo. Universidade de São Paulo. Instituto Astronômico e Geofísico. Anuário do Observatório de S. Paulo. (São Paulo)

Apl. Mat. Československá Akademie Věd. Aplikace Matematiky. (Prague)

Appl. Mech. Rev. Applied Mechanics Reviews. (New York) (Sometimes abbreviated as *AMR* in the text of *Math. Reviews*)

Appl. Statist. Applied Statistics. A Journal of the Royal Statistical Society. (London)

Årbok Univ. Bergen Mat.-Natur. Ser. Årbok for Universitetet i Bergen. Mat.-Naturv. Serie. (Bergen)

Arch. Elektrotech. Archiv für Elektrotechnik. (Berlin-Göttingen-Heidelberg)

Arch. History Exact Sci. Archive for History of Exact Sciences. (Berlin)

Arch. Math. Archiv der Mathematik. Archives of Mathematics. Archives Mathématiques. (Basel-Stuttgart)

Arch. Math. Logik Grundlagenforsch. Archiv für mathematische Logik und Grundlagenforschung. (Stuttgart)

Arch. Mech. Stos. Polska Akademia Nauk. Instytut Podstawowych Problemów Techniki. Archiwum Mechaniki Stosowanej. (Warsaw)

Arch. Rational Mech. Anal. Archive for Rational Mechanics and Analysis. (Berlin)

Arch. Sci. Soc. Phys. Histoire Natur. Genève. Archives des Sciences. Société de Physique et d'Histoire Naturelle de Genève. (Geneva)

Archimede. Archimede. Rivista per gli Insegnanti e i Cultori di Matematiche Pure e Applicate. (Florence)

Ark. Astronom. Arkiv för Astronomi. (Stockholm)

Ark. Fys. Arkiv för Fysik. (Stockholm)

Ark. Mat. Arkiv för Matematik. (Stockholm)

Arkhimedes. Arkhimedes. Suomen Fyysikkoseura—Finlands Fysikerförening r.y. Suomen Matemaattinen Yhdistys—Finlands Matematiska Förening r.y. (Helsinki)

Arquivo Inst. Gulbenkian Ci. A Estud. Mat. Fís.-Mat. Arquivo do Instituto Gulbenkian de Ciência. A. Estudos Matemáticos e Físico-Matemáticos. (Lisbon) (Formerly: *Arquivo Inst. Gulbenkian Ci. Sec. A Estud. Mat. Fís.-Mat.*)

Arquivo Inst. Gulbenkian Ci. Sec. A Estud. Mat. Fís.-Mat. Arquivo do Instituto Gulbenkian de Ciência. Secção A. Estudos Matemáticos e Físico-Matemáticos. (Lisbon) (Continued as: *Arquivo Inst. Gulbenkian Ci. A Estud. Mat. Fís.-Mat.*)

Assoc. Roy. Actuair. Belges Bull. Association Royale des Actuaires Belges. Bulletin. (Brussels)

Astronaut. Acta. Astronautica Acta. (Vienna)

Astronom. J. The Astronomical Journal. (New Haven, Conn.)

Astronom. Jber. Astronomischer Jahresbericht. (Berlin)

Astronom. Nachr. Astronomische Nachrichten. (Berlin)

Astronom. Ž. Akademija Nauk Sojuza SSR. Astronomičeskiĭ Žurnal. (Moscow) (Translated as: *Soviet Astronom. AJ*)

Astrophys. J. The Astrophysical Journal. (Chicago, Ill.)

Astrophys. Norvegica. Det Norske Videnskaps-Akademi i Oslo. Astrophysica Norvegica. (Oslo)

Atatürk Univ. Yayınları Ser.-Mat. Atatürk Üniversitesi Yayınları. Araştırmalar Serisi—Matematik. Publications of Atatürk University. Research Series—Mathematics. (Erzurum)

Atti Accad. Gioenia Catania. Atti della Accademia Gioenia di Scienze Naturali in Catania. (Catania) See *Atti Accad. Gioenia Sci. Natur. Catania.*

Atti Accad. Gioenia Sci. Natur. Catania. Atti della Accademia Gioenia di Scienze Naturali in Catania. (Catania) (Formerly listed as: *Atti Accad. Gioenia Catania*)

Atti Accad. Ligure. Atti della Accademia Ligure di Scienze e Lettere. (Genoa) See *Atti Accad. Ligure Sci. Lett.*

Atti Accad. Ligure Sci. Lett. Atti della Accademia Ligure di Scienze e Lettere. (Genoa) (Formerly listed as: *Atti Accad. Ligure*)

Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez. I. Atti della Accademia Nazionale dei Lincei. Memorie. Classe di Scienze Fisiche, Matematiche e Naturali. Sezione Iª. (Matematica, Meccanica, Astronomia, Geodesia e Geofisica). (Rome)

Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez. II. Atti della Accademia Nazionale dei Lincei. Memorie. Classe di Scienze Fisiche, Matematiche e Naturali. Sezione IIª. (Fisica, Chimica, Geologia, Paleontologia e Mineralogia). (Rome)

Atti Accad. Naz. Lincei Rend. Adunanze Solenni. Atti della Accademia Nazionale dei Lincei. Rendiconti delle Adunanze Solenni. (Rome)

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. Atti della Accademia Nazionale dei Lincei. Rendiconti. Classe di Scienze Fisiche, Matematiche e Naturali. (Rome)

Atti Accad. Sci. Fis. Mat. Napoli. Atti dell'Accademia delle Scienze Fisiche e Matematiche di Napoli. (Naples)

Atti Accad. Sci. Ist. Bologna Cl. Sci. Fis. Mem. Atti della Accademia delle Scienze dell'Istituto di Bologna. Classe di Scienze Fisiche. Memorie. (Bologna)

Atti Accad. Sci. Ist. Bologna Cl. Sci. Fis. Rend. Atti della Accademia delle Scienze dell'Istituto di Bologna. Classe di Scienze Fisiche. Rendiconti. (Bologna)

Atti Accad. Sci. Lett. Arti Palermo Parte I. Atti della Accademia di Scienze Lettere e Arti di Palermo. Parte Prima: Scienze. (Palermo)

Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. Atti della Accademia delle Scienze di Torino. Classe di Scienze Fisiche, Matematiche e Naturali. (Turin)

Atti Sem. Mat. Fis. Univ. Modena. Atti del Seminario Matematico e Fisico dell'Università di Modena. (Modena)

Austral. J. Appl. Sci. Australian Journal of Applied Science. (Melbourne)

Austral. J. Phys. Australian Journal of Physics. (Melbourne)

Austral. J. Sci. The Australian Journal of Science. (Sydney)

Austral. J. Statist. The Australian Journal of Statistics. (Sydney)

Automat. Remote Control. Automation and Remote Control. (A translation of *Avtomatika i Telemehanika*, a publication of the Academy of Sciences of the USSR). (Pittsburgh, Pa.) (Translation of *Avtomat. i Telemeh.*)

Avh. Norske Vid.-Akad. Oslo I. Avhandlinger Utgitt av det Norske Videnskaps-Akademi i Oslo. I. Mat.-Naturv. Klasse. (Oslo)

Avtomat. i Telemeh. Akademija Nauk Sojuza SSR. Avtomatika i Telemehanika. (Moscow) (Translated as: *Automat. Remote Control*)

Avtomatika. Akademija Nauk Ukrainskoĭ RSR. Institut Elektrotehniki. Avtomatika. (Kiev)

Azerbaĭdžan. Gos. Univ. Učen. Zap. Ser. Fiz.-Mat. i Him. Nauk. Azerbaĭdžanskiĭ Gosudarstvennyĭ Universitet im. S. M. Kirova. Učenye Zapiski. Serija Fiziko-Matematičeskih i Himičeskih Nauk. (Baku)

Bayer. Akad. Wiss. Math.-Natur. Kl. Abh. Bayerische Akademie der Wissenschaften. Mathematisch-Naturwissenschaftliche Klasse. Abhandlungen. (Munich)

Bayer. Akad. Wiss. Math.-Natur. Kl. S.-B. Bayerische Akademie der Wissenschaften. Mathematisch-Naturwissenschaftliche Klasse. Sitzungsberichte. (Munich)

Behavioral Sci. Behavioral Science. (Baltimore, Md.-Ann Arbor, Mich.)

Bell System Tech. J. The Bell System Technical Journal. (New York)

Biometrics. Biometrics. Journal of the Biometric Society. (Blacksburg, Va.)

Biometrika. Biometrika. (London)

BIT = *Nordisk Tidskr. Informations-Behandling.*

Bjull. Inst. Teoret. Astronom. Akademija Nauk Sojuza Sovetskih Socialističeskih Respublik. Bjulleten' Instituta Teoretičeskoi Astronomii. (Moscow)

Bol. Fac. Ingen. Agrimens. Montevideo. Boletin de la Facultad de Ingenieria y Agrimensura de Montevideo. (Montevideo)

Bol. Soc. Mat. Mexicana. Boletin de la Sociedad Matemática Mexicana. (Mexico, D.F.)

Bol. Soc. Mat. São Paulo. Boletim da Sociedade de Matemática de São Paulo. (São Paulo)

Bol. Univ. Paraná Fís. Teórica. Boletim da Universidade do Paraná. Conselho de Pesquisas. Instituto de Física. Física Teórica. (Curitiba)

Boll. Un. Mat. Ital. Bolletino della Unione Matematica Italiana. (Bologna)

Bonn. Math. Schr. Bonner Mathematische Schriften. (Bonn)

British J. Philos. Sci. The British Journal for the Philosophy of Science. (Edinburgh)

Bul. Inst. Politehn. Bucureşti. Buletinul Institutului Politehnic Bucureşti. (Bucharest)

Bul. Inst. Politehn. Iaşi. Buletinul Institutului Politehnic din Iaşi. Serie Nouă. (Iaşi)

Bul. Şti. Inst. Politehn. Cluj. Buletinul Ştiinţific al Institutului Politehnic din Cluj. (Cluj)

Bŭlgar. Akad. Nauk. Izv. Mat. Inst. Bŭlgarska Akademija na Naukite. Otdelenie za Fiziko-Matematičeski i Tehničeski Nauki. Izvestija na Matematičeskija Institut. (Sofia) See *Bŭlgar. Akad. Nauk. Otd. Fiz.-Mat. Tehn. Nauk. Izv. Mat. Inst.*

Bŭlgar. Akad. Nauk. Otd. Fiz.-Mat. Nauk. Izv. Fiz. Inst. s Aneb. Bŭlgarska Akademija na Naukite. Otdelenie za Fiziko-Matematičeski Nauki. Izvestija na Fizičeskija Institut s Aneb. (Sofia) (Continued as: *Bŭlgar. Akad. Nauk. Otd. Mat. Fiz. Nauk. Izv. Fiz. Inst. s Aneb*)

Bŭlgar. Akad. Nauk. Otd. Fiz.-Mat. Tehn. Nauk. Izv. Mat. Inst. Bŭlgarska Akademija na Naukite. Otdelenie za Fiziko-Matematičeski i Tehničeski Nauki. Izvestija na Matematičeskija Institut. (Sofia) (Formerly listed as: *Bŭlgar. Akad. Nauk. Izv. Mat. Inst.*)

Bŭlgar. Akad. Nauk. Otd. Mat. Fiz. Nauk. Izv. Fiz. Inst. s Aneb. Bŭlgarska Akademija na Naukite. Otdelenie za Matematičeski i Fizičeski Nauki. Izvestija na Fizičeskija Institut s Aneb. (Sofia) (Formerly: *Bŭlgar. Akad. Nauk. Otd. Fiz.-Mat. Nauk. Izv. Fiz. Inst. s Aneb*)

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. Bulletin de l'Académie Polonaise des Sciences. Série des Sciences Mathématiques, Astronomiques et Physiques. (Warsaw)

Bull. Acad. Serbe Sci. Arts. Cl. Sci. Math. Natur. Sci. Math. Bulletin de l'Académie Serbe des Sciences et des Arts. Classe des Sciences Mathématiques et Naturelles Sciences. Mathématiques. (Belgrade)

Bull. Amer. Math. Soc. Bulletin of the American Mathematical Society. (Providence, R.I.)

Bull. Astronom. Bulletin Astronomique. (Paris)

Bull. Astronom. Inst. Netherlands. Bulletin of the Astronomical Institutes of the Netherlands. (Haarlem)

Bull. Calcutta Math. Soc. Bulletin of the Calcutta Mathematical Society. (Calcutta)

Bull. College Sci. (Baghdad). Bulletin of the College of Science. (Baghdad)

Bull. Earthquake Res. Inst. Univ. Tokyo. Bulletin of the Earthquake Research Institute, University of Tokyo. (Tokyo)

Bull. Fukuoka Gakugei Univ. III. Bulletin of Fukuoka Gakugei University. III. Natural Sciences. (Fukuoka)

Bull. Géodésique. Bulletin Géodésique. (Paris)

Bull. Inst. Internat. Statist. Bulletin de l'Institut International de Statistique. (The Hague)

Bull. (Izv.) Acad. Sci. USSR Geophys. Ser. Bulletin (Izvestiya) Academy of Sciences, USSR. Geophysics Series. (Washington, D.C.) (Translation of *Izv. Akad. Nauk SSSR Ser. Geofiz.*)

Bull. JSME. Bulletin of JSME. (Japan Society of Mechanical Engineers). (Tokyo)

Bull. Kyoto Gakugei Univ. Ser. B. Bulletin of the Kyoto Gakugei University. Ser. B. Mathematics and Natural Science. (Kyoto)

Bull. Kyushu Inst. Tech. Math. Natur. Sci. Bulletin of the Kyushu Institute of Technology. Mathematics, Natural Science. (Tobata)

Bull. Math. Biophys. The Bulletin of Mathematical Biophysics. (Chicago, Ill.)

Bull. Math. Soc. Nanyang Univ. Majallah Tahunan 'Ilmu Pasti. Bulletin of Mathematical Society. Nanyang University. (Singapore)

Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine. Bulletin Mathématique de la Société des Sciences Mathématiques et Physiques de la République Populaire Roumaine. (Bucharest)

Bull. Math. Statist. Bulletin of Mathematical Statistics. (Fukuoka)

Bull. Sci. Conseil Acad. RSF Yougoslav. Bulletin Scientifique. Conseil des Académies de la RSF de Yougoslavie. (Zagreb)

Bull. Sci. Math. Bulletin des Sciences Mathématiques. (Paris)

Bull. Signal. I Math. Pures Appl. Ministère de l'Éducation Nationale. Centre National de la Recherche Scientifique. Bulletin Signalétique. I. Mathématiques Pures et Appliquées. (Paris)

Bull. Soc. Amis Sci. Lettres Poznań Sér. B. Bulletin de la Société des Amis des Sciences et des Lettres de Poznań. Série B: Sciences Mathématiques et Naturelles. (Poznań)

Bull. Soc. Math. Belg. Bulletin de la Société Mathématique de Belgique. (Brussels)

Bull. Soc. Math. France. Bulletin de la Société Mathématique de France. (Paris)

Bull. Soc. Math. Grèce. Bulletin de la Société Mathématique de Grèce. (Athens)

Bull. Soc. Math. Phys. Macédoine. Bulletin de la Société des Mathématiciens et des Physiciens de la République Populaire de Macédoine. Bilten na Društvoto na Matematičarite i Fizičarite od Narodna Republika Makedonija. (Skopje)

Bull. Soc. Math. Phys. Serbie. Bulletin de la Société des Mathématiciens et Physiciens de la R. P. de Serbie. Vesnik Društva Matematičara i Fizičara Narodne Republike Srbije. (Belgrade) (Continued as: *Mat. Vesnik*)

Bull. Soc. Roy. Sci. Liège. Bulletin de la Société Royale des Sciences de Liège. (Liège)

Bull. Soc. Sci. Lettres Łódź. Bulletin de la Société des Sciences et des Lettres de Łódź. (Łódź)

Bull. Tokyo Gakugei Univ. Bulletin of Tokyo Gakugei University. (Tokyo)

Bull. Trimest. Inst. Actuaires Franç. Bulletin Trimestriel de l'Institut des Actuaires Français. (Paris)

Bull. Univ. Osaka Prefecture Ser. A. Bulletin of University of Osaka Prefecture. Series A. Engineering and Natural Sciences. (Osaka)

C. R. Acad. Bulgare Sci. Doklady Bolgarskoi Akademii Nauk. Comptes Rendus de l'Académie Bulgare des Sciences. (Sofia)

C. R. Acad. Sci. Paris. Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences. (Paris)

Cahiers Centre Études Rech. Opér. Cahiers du Centre d'Études de Recherche Opérationnelle. (Brussels) See *Cahiers Centre Études Recherche Opér.*

Cahiers Centre Études Recherche Opér. Cahiers du Centre d'Études de Recherche Opérationnelle. (Brussels) (Formerly listed as: *Cahiers Centre Études Rech. Opér.*)

Cahiers Centre Math. Statist. Appl. Sci. Social. Cahiers du Centre de Mathématique et de Statistique Appliquées aux Sciences Sociales de l'Institut de Sociologie Solvay. (Brussels)

Cahiers de Phys. Cahiers de Physique. (Paris)

Cahiers Rhodan. Cahiers Rhodaniens. (Lyon)

Calc. Automat. y Cibernet. Calculo Automatico y Cibernetica. (Madrid)

Calcolo. Calcolo. (Rome)

Calcutta Statist. Assoc. Bull. Calcutta Statistical Association Bulletin. (Calcutta)

Canad. J. Math. Canadian Journal of Mathematics. Journal Canadien de Mathématiques. (Toronto, Ont.)

Canad. J. Phys. Canadian Journal of Physics. (Ottawa, Ont.)

Canad. Math. Bull. Canadian Mathematical Bulletin. Bulletin Canadien de Mathématiques. (Toronto, Ont.)

Časopis Pěst. Mat. Československá Akademie Věd. Časopis pro Pěstování Matematiky. (Prague)

Centaurus. Centaurus. (Copenhagen)

Chalmers Tekn. Högsk. Handl. Chalmers Tekniska Högskolans Handlingar. Transactions of Chalmers University of Technology, Gothenburg, Sweden. (Gothenburg)

Chinese Math. Chinese Mathematics. Translation of Acta Mathematica Sinica. (Providence, R.I.) (Translation of *Acta Math. Sinica*) See *Chinese Math.—Acta.*

Chinese Math.—Acta. Chinese Mathematics. Translation of Acta Mathematica Sinica. (Providence, R.I.) (Translation of *Acta Math. Sinica*) (Formerly listed as: *Chinese Math.*)

Ciência (Lisboa). Ciência. Revista de Cultura Científica. (Lisbon)

Ciencias (Madrid). Las Ciencias. (Madrid)

Collect. Math. Consejo Superior de Investigaciones Científicas. Universidad de Barcelona. Collectanea Mathematica. (Barcelona)

Colloq. Math. Colloquium Mathematicum. (Warsaw)

Com. Acad. R. P. Romîne. Comunicările Academiei Republicii Populare Romîne. (Bucharest)

Comm. ACM. Communications of the Association for Computing Machinery. (New York)

Comm. Dublin Inst. Adv. Studies Ser. A. Communications of the Dublin Institute for Advanced Studies. Series A. (Dublin)

Comm. Fac. Sci. Univ. Ankara Sér. A. Communications de la Faculté des Sciences de l'Université d'Ankara. Série A. Mathématiques-Physique-Astronomie. (Ankara)

Comm. Pure Appl. Math. Communications on Pure and Applied Mathematics. (New York)

Comment. Math. Helv. Commentarii Mathematici Helvetici. (Zürich)

Comment. Math. Univ. Carolinae. Commentationes Mathematicae Universitatis Carolinae. (Prague)

Comment. Math. Univ. St. Paul. Commentarii Mathematici Universitatis Sancti Pauli. Rikkyô Daigaku Sûgaku Zasshi. (Tokyo)

Compositio Math. Compositio Mathematica. (Groningen)

Comput. Bull. The Computer Bulletin. (London)

Comput. J. The Computer Journal. (London)

Comput. Rev. Computing Reviews. (New York) (Usually abbreviated *CR* in the text of *Math. Reviews*)

Confer. Sem. Mat. Univ. Bari. Conferenze del Seminario di Matematica dell'Università di Bari. (Bari)

Contributions to Differential Equations. Contributions to Differential Equations. (New York)

CORS J. CORS J. The Canadian Operation Research Society Journal. (Toronto, Ont.)

CR. Computing Reviews. (New York)

Čuvaš. Gos. Ped. Inst. Učen. Zap. Ministerstvo Prosveščenija RSFSR. Čuvaškiĭ Gosudarstvennyĭ Pedagogičeskiĭ Institut im. I. Ja. Jakovleva. Učenye Zapiski. (Cheboksary)

Cybernetica. Cybernetica. (Namur)

Czechoslovak J. Phys. Čehoslovackaja Akademija Nauk. Čehoslovackiĭ Fizičeskiĭ Žurnal. Czechoslovak Academy of Sciences. Czechoslovak Journal of Physics. (Prague)

Czechoslovak Math. J. Čehoslovackaja Akademija Nauk. Čehoslovackiĭ Matematičeskiĭ Žurnal. Czechoslovak Mathematical Journal. (Prague)

Deutsche Geodät. Kommiss. Bayer. Akad. Wiss. Reihe B Angew. Geodäsie. Deutsche Geodätische Kommission bei der Bayerischen Akademie der Wissenschaften. Reihe B. Angewandte Geodäsie. (Munich)

Dialectica. Dialectica. (Neuchâtel)

Discussions Faraday Soc. Discussions of the Faraday Society. (London)

Diskret. Analiz. Akademija Nauk SSSR. Sibirskoe Otdelenie. Institut Matematiki. Diskretnyĭ Analiz. Sbornik Trudov. (Novosibirsk)

Dokl. Akad. Nauk BSSR. Doklady Akademii Nauk BSSR. (Minsk)

Dokl. Akad. Nauk SSSR. Doklady Akademii Nauk SSSR. (Moscow) (Mathematics section translated as: *Soviet Math. Dokl.*; Physics section translated as: *Soviet Physics Dokl.*)

Dokl. Akad. Nauk Tadžik. SSR. Doklady Akademii Nauk Tadžikskoĭ SSR. (Dushanbe)

Dopovidi Akad. Nauk Ukraïn. RSR. Dopovidi Akademiï Nauk Ukraïns'koï Radjans'koï Socialistične Respubliki. (Kiev)

Duke Math. J. Duke Mathematical Journal. (Durham, N.C.)

Econometrica. Econometrica. Journal of the Econometric Society. (Chicago, Ill.)

Eesti NSV Tead. Akad. Toimetised Füüs.-Mat. Tehn. Tead. Seer. Eesti NSV Teaduste Akadeemia Toimetised. Füüsikalis-Matemaatiliste ja Tehniliste Teaduste Seeria. Izvestija Akademii Nauk Èstonskoĭ SSR. Serija Fiziko-Matematičeskih i Tehničeskih Nauk. (Tallin) (Continued as: *Eesti NSV Tead. Akad. Toimetised Füüs.-Mat. Tehn.-tead. Seer.*)

Eesti NSV Tead. Akad. Toimetised Füüs.-Mat. Tehn.-tead. Seer. Eesti NSV Teaduste Akadeemia Toimetised. Füüsika-Matemaatika-ja Tehnikateaduste Seeria. Izvestija Akademii Nauk Èstonskoĭ SSR. Serija Fiziko-Matematičeskih i Tehničeskih Nauk. (Tallin) (Formerly: *Eesti NSV Tead. Akad. Toimetised Füüs-Mat. Tehn. Tead. Seer.*)

Elektron. Datenverarbeitung. Elektronische Datenverarbeitung. Fachberichte über programmgesteuerte Maschinen und ihre Anwendung. (Braunschweig)

Elem. Math. Elemente der Mathematik. Revue de Mathématiques Élémentaires. Rivista de Matematica Elementare. (Basel)

Enseignement Math. L'Enseignement Mathématique. (Geneva)

Ericsson Technics. Ericsson Technics. (Stockholm)

Estadíst. Española. Estadística Española. (Madrid)

Estadística. Estadística. (Washington, D.C.)

Euclides (Groningen). Euclides. Tijdschrift voor de Didactiek der Exacte Vakken. (Groningen)

Euclides (Madrid). Euclides. Revista Mensual de Ciencias Exactas, Físicas, Químicas, Naturales y Applicaciones Técnicas. (Madrid)

Fac. Ingen. Agrimens. Montevideo Publ. Didact. Inst. Mat. Estadíst. Facultad de Ingeniería y Agrimensura. Montevideo. Publicaciones Didacticas del Instituto de Matemática y Estadística. (Montevideo)

Fac. Sci. Univ. Skopje Annuaire. Faculté des Sciences de l'Université de Skopje. Annuaire. (Skopje)

Fibonacci Quart. The Fibonacci Quarterly. Official Organ of The Fibonacci Association. A journal devoted to the study of integers with special properties. (St. Mary's College, Calif.)

Fiz.-Mat. Spis. Bŭlgar. Akad. Nauk. Bŭlgarska Akademija na Naukite Fizičeski Institut. Matematičeski Institut. Fiziko-Matematičesko Spisanie. (Sofia)

Fiz. Tverd. Tela. Fizika Tverdogo Tela. (Moscow-Leningrad) (Translated as: *Soviet Physics Solid State*)

Forsch. Gebiete Ingenieurwesens. Forschung auf dem Gebiete des Ingenieurwesens. (Berlin) (Continued as: *Forsch. Ingenieurwesen*)

Forsch. Ingenieurwesen. Forschung im Ingenieurwesen. (Düsseldorf) (Formerly: *Forsch. Gebiete Ingenieurwesens*)

Fortschr. Physik. Fortschritte der Physik. (Berlin)

Fund. Math. Polska Akademia Nauk. Fundamenta Mathematicae. (Warsaw)

Funkcial. Ekvac. Fako de l'Funkcialaj Ekvacioj Japana Matematika Societo. Funkcialaj Ekvacioj. (Serio Internacia). (Kobe)

Gac. Mat. (Madrid). Gaceta Matemática. (Madrid)

Gaṇita. Gaṇita. (Lucknow)

Gaz. Mat. Fiz. Ser. A. Societatea de Ştiinţe Matematice şi Fizice din R. P. R. Gazeta Matematică şi Fizică. Publicaţie Pentru Studiul şi Răspîndirea Ştiinţelor Matematice şi Fizice. Seria A. (Bucharest) (Continued as: *Gaz. Mat. Ser. A*)

Gaz. Mat. (Lisboa). Gazeta de Matemática. (Lisbon)

Gaz. Mat. Ser. A. Societatea de Ştiinţe Matematice din R. P. R. Gazeta Matematică. Publicaţie Pentru Studiul şi Răspîndirea Ştiinţelor Matematice. Seria A. (Bucharest) (Formerly: *Gaz. Mat. Fiz. Ser. A*)

Genet. Res. Genetical Research. (New York)

Geodaet. Inst. Medd. Geodaetisk Institut. Meddelelse. (Copenhagen)

Geofys. Publ. Norske Vid.-Akad. Oslo. Geofysiske Publikasjoner Utgitt av det Norske Videnskaps-Akademi i Oslo. (Oslo)

Giorn. Ist. Ital. Attuari. Giornale dell'Istituto Italiano degli Attuari. (Rome)

Giorn. Mat. Battaglini. Giornale di Matematiche di Battaglini. (Naples)

Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hrvatske Ser. II. Glasnik Matematičko-Fizički i Astronomski. Periodicum Mathematico-Physicum et Astronomicum. Serija II. Društvo Matematičara i Fizičara Hrvatske. Societas Mathematicorum et Physicorum Croatiae. (Zagreb) See *Glasnik Mat.-Fiz. Astronom. Ser. II Društvo Mat. Fiz. Hrvatske.*

Glasnik Mat.-Fiz. Astronom. Ser. II Društvo Mat. Fiz. Hrvatske. Glasnik Matematičko-Fizički i Astronomski. Periodicum Mathematico-Physicum et Astronomicum. Serija II. Društvo Matematičara i Fizičara Hrvatske. Societas Mathematicorum et Physicorum Croatiae. (Zagreb) (Formerly listed as: *Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hrvatske Ser. II*)

Gruzin. Politehn. Inst. Trudy. Gruzinskiĭ Politehničeskiĭ Institut im. V. I. Lenina. Trudy. (Tiflis)

Helv. Phys. Acta. Helvetica Physica Acta. Societatis Physicae Helveticae Commentaria Publica. (Basel)

Hochfrequenztech. Elektroakust. Hochfrequenztechnik und Elektroakustik. (Leipzig)

Houille Blanche. La Houille Blanche. Revue de l'Ingénieur Hydraulicien. (Grenoble)

IBM J. Res. Develop. IBM Journal of Research and Development. (New York)

Icarus. Icarus. International Journal of the Solar System. (New York)

ICC Bull. ICC Bulletin. (Rome)

ICSU Rev. World Sci. ICSU Review of World Science. (Amsterdam)

IEEE Trans. Antennas and Propagation. IEEE Transactions on Antennas and Propagation. (New York)

IEEE Trans. Automatic Control. IEEE Transactions on Automatic Control. (New York)

IEEE Trans. Circuit Theory. IEEE Transactions on Circuit Theory. (New York)

IEEE Trans. Comm. and Electronics. IEEE Transactions on Communication and Electronics. (New York)

IEEE Trans. Comm. Systems. IEEE Transactions on Communication Systems. (New York) (Continued as: *IEEE Trans. Comm. Tech.*)

IEEE Trans. Comm. Tech. IEEE Transactions on Communication Technology. (New York) (Formerly: *IEEE Trans. Comm. Systems*)

IEEE Trans. Electronic Computers. IEEE Transactions on Electronic Computers. (New York)

IEEE Trans. Information Theory. IEEE Transactions on Information Theory. (New York)

IEEE Trans. Nuclear Science. IEEE Transactions on Nuclear Science. (New York)

IEEE Trans. Reliability. IEEE Transactions on Reliability. (New York)

IEEE Trans. Space Electronics and Telemetry. IEEE Transactions on Space Electronics and Telemetry. (New York)

Illinois J. Math. Illinois Journal of Mathematics. (Urbana, Ill.)

Indian J. Math. Indian Journal of Mathematics. (Allahabad)

Indian J. Mech. Math. Indian Journal of Mechanics and Mathematics. (Calcutta)

Indian J. Phys. Indian Journal of Physics and Proceedings of the Indian Association for the Cultivation of Science. (Calcutta)

Indian J. Theoret. Phys. Indian Journal of Theoretical Physics. (Calcutta)

Information and Control. Information and Control. (New York)

Information Processing in Japan. Information Processing in Japan. (Tokyo)

Information Storage and Retrieval. Information Storage and Retrieval. (Oxford-New York)

Ing.-Arch. Ingenieur-Archiv. (Berlin-Göttingen-Heidelberg)

Inst. Hautes Études Sci. Publ. Math. Institut des Hautes Études Scientifiques. Publications Mathématiques. (Paris)

Internat. J. Abstracts Statist. Theory Method. International Journal of Abstracts. Statistical Theory and Method. (London) (Continued as: *Statist. Theory Method Abstracts*)

Internat. J. Comput. Math. International Journal of Computer Mathematics. (New York)

Internat. J. Engrg. Sci. International Journal of Engineering Science. (Oxford-New York)

Internat. J. Mech. Sci. International Journal of Mechanical Sciences. (Oxford)

Inž. Ž. Inženernyĭ Žurnal. Organ Otdelenija Tehničeskih Nauk i Instituta Mehaniki Akademii Nauk SSSR. (Moscow)

Isis. Isis. An International Review Devoted to the History of Science and Civilization. (Cambridge, Mass.)

Israel J. Math. Israel Journal of Mathematics. (Jerusalem)

Ist. Lombardo Accad. Sci. Lett. Rend. A. Istituto Lombardo. Accademia di Scienze e Lettere. Rendiconti. Scienze Matematiche, Fisiche, Chimiche e Geologiche. A. (Milan)

Ist. Lombardo Accad. Sci. Lett. Rend. Parte Gen. e Atti Uff. Istituto Lombardo Accademia di Scienze e Lettere. Rendiconti. Parte Generale e Atti Ufficiali. (Milan) (Formerly: *Ist. Lombardo Sci. Lett. Rend. Parte Gen. e Atti Uff.*)

Ist. Lombardo Sci. Lett. Rend. Parte Gen. e Atti Uff. Istituto Lombardo di Scienze e Lettere. Rendiconti. Parte Generale e Atti Ufficiali. (Milan) (Continued as: *Ist. Lombardo Accad. Sci. Lett. Rend. Parte Gen. e Atti Uff.*)

Ist. Veneto Sci. Lett. Arti Atti Cl. Sci. Mat. Natur. Istituto Veneto di Scienze, Lettere ed Arti. Venezia. Atti. Classe di Scienze Matematiche e Naturali. (Venice)

İstanbul Tek. Üniv. Bül. İstanbul Teknik Üniversitesi Bülteni. Bulletin of the Technical University of Istanbul. (Istanbul)

İstanbul Üniv. Fen Fak. Mec. Ser. A. İstanbul Üniversitesi Fen Fakültesi Mecmuasi. Seri A: Sirf ve Tatbikî Matematik. Revue de la Faculté des Sciences de l'Université d'Istanbul. Série A: Mathématiques Pures et Appliquées. (Istanbul)

İstanbul Üniv. Fen Fak. Mec. Ser. C. İstanbul Üniversitesi Fen Fakültesi Mecmuasi. Seri C: Astronomi-Fizik-Kimya. Revue de la Faculté des Sciences de l'Université d'Istanbul. Série C: Astronomie-Physique-Chimie. (Istanbul)

Istor.-Astronom. Issled. Istoriko-Astronomičeskie Issledovanija. (Moscow)

Istor.-Mat. Issled. Istoriko-Matematičeskie Issledovanija. (Moscow-Leningrad)

Istoriko-Mat. Zbirnik. Akademija Nauk Ukraïns'koï RSR. Institut Matematiki. Istoriko-Matematičnii Zbirnik. (Kiev)

Izv. Akad. Nauk Armjan. SSR Ser. Fiz.-Mat. Nauk. Izvestija Akademii Nauk Armjanskoĭ SSR. Serija Fiziko-Matematičeskih Nauk. (Erevan)

Izv. Akad. Nauk Armjan. SSR Ser. Tehn. Nauk. Izvestija Akademii Nauk Armjanskoĭ SSR. Serija Tehničeskih Nauk. (Erevan)

Izv. Akad. Nauk Azerbaĭdžan. SSR Ser. Fiz.-Mat. Tehn. Nauk. Izvestija Akademii Nauk Azerbaĭdžanskoĭ SSR. Serija Fiziko-Matematičeskih i Tehničeskih Nauk. (Baku) (Continued as: *Izv. Akad. Nauk Azerbaĭdžan. SSR Ser. Fiz.-Tehn. Mat. Nauk*)

Izv. Akad. Nauk Azerbaĭdžan. SSR Ser. Fiz.-Tehn. Mat. Nauk. Izvestija Akademii Nauk Azerbaĭdžanskoĭ SSR. Serija Fiziko-Tehničeskih i Matematičeskih Nauk. (Baku) (Formerly: *Izv. Akad. Nauk Azerbaĭdžan. SSR Ser. Fiz.-Mat. Tehn. Nauk*)

Izv. Akad. Nauk Kazah. SSR Ser. Fiz.-Mat. Nauk. Izvestija Akademii Nauk Kazahskoĭ SSR. Serija Fiziko-Matematičeskih Nauk. (Alma-Ata)

Izv. Akad. Nauk SSSR Ser. Fiz. Izvestija Akademii Nauk SSSR. Serija Fizičeskaja. (Moscow)

Izv. Akad. Nauk SSSR Ser. Geofiz. Izvestija Akademii Nauk SSSR. Serija Geofizičeskaja. (Moscow) (Translated as: *Bull.* (*Izv.*) *Acad. Sci. USSR Geophys. Ser.*)

Izv. Akad. Nauk SSSR Ser. Mat. Izvestija Akademii Nauk SSSR. Serija Matematičeskaja. (Moscow)

Izv. Akad. Nauk SSSR Tehn. Kibernet. Izvestija Akademii Nauk SSSR. Tehničeskaja Kibernetika. (Moscow)

Izv. Akad. Nauk Turkmen. SSR Ser. Fiz.-Tehn. Him. Geol. Nauk. Izvestija Akademii Nauk Turkmenskoĭ SSR. Serija Fiziko-Tehničeskih, Himičeskih i Geologičeskih Nauk. (Ashkhabad)

Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk. Izvestija Akademii Nauk UzSSR. Serija Fiziko-Matematičeskih Nauk. UzSSR Fanlar Akademijasining Ahboroti. Fizika-Matematika Fanlari Serijasi. (Tashkent)

Izv. Kazan. Fil. Akad. Nauk SSSR Ser. Fiz.-Mat. i Tehn. Nauk. Izvestija Kazanskogo Filiala Akademii Nauk SSSR. Serija Fiziko-Matematičeskih i Tehničeskih Nauk. (Kazan)

Izv. Krymsk. Ped. Inst. Izvestija Krymskogo Pedagogičeskogo Instituta im. M. V. Frunze. (Simferopol)

Izv. Sibirsk. Otd. Akad. Nauk SSSR Ser. Tehn. Nauk. Izvestija Sibirskogo Otdelenija. Akademija Nauk SSSR. Serija Tehničeskih Nauk. (Novosibirsk)

Izv. Vuzov = Izv. Vysš. Učebn. Zaved. Matematika.

Izv. Vysš. Učebn. Zaved. Matematika. Ministerstvo Vysšego Obrazovanija SSSR. Izvestija Vysših Učebnyh Zavedenii. Matematika. (Kazan)

J. Acoust. Soc. Amer. The Journal of the Acoustical Society of America. (New York)

J. Algebra. Journal of Algebra. (New York)

J. Amer. Statist. Assoc. Journal of the American Statistical Association. (Washington, D.C.)

J. Analyse Math. Journal d'Analyse Mathématique. (Jerusalem)

J. Annamalai Univ. Part B Sci. Journal of the Annamalai University. Part B. Sciences. (Annamalainagar)

J. Appl. Math. Mech. Journal of Applied Mathematics and Mechanics. (Translation of the Soviet journal *Prikladnaja Matematika i Mehanika*). (New York) (Translation of *Prikl. Mat. Meh.*)

J. Appl. Phys. Journal of Applied Physics. (New York)

J. Appl. Probability. Journal of Applied Probability. (East Lansing, Mich.)

J. Assoc. Comput. Mach. Journal of the Association for Computing Machinery. (New York)

J. Atmospheric Sci. Journal of the Atmospheric Sciences. (Boston, Mass.)

J. Austral. Math. Soc. The Journal of the Australian Mathematical Society. (Sydney)

J. Chem. Phys. The Journal of Chemical Physics. (New York)

J. College Arts Sci. Chiba Univ. Journal of the College of Arts and Sciences, Chiba University. (Chiba)

J. Differential Equations. Journal of Differential Equations. (New York)

J. Electronics Control. Journal of Electronics and Control. (London)

J. Fac. Sci. Hokkaido Univ. Ser. I. Journal of the Faculty of Science. Hokkaido University. Series I. Mathematics. (Sapporo)

J. Fac. Sci. Niigata Univ. Ser. I. Journal of the Faculty of Science. Niigata University. Series I. Mathematics, Physics and Chemistry. (Niigata) (Continued as: *Sci. Rep. Niigata Univ. Ser. A*)

J. Fac. Sci. Univ. Tokyo Sect. I. Journal of the Faculty of Science. University of Tokyo. Section I. Mathematics, Astronomy, Physics, Chemistry. (Tokyo)

J. Fluid Mech. Journal of Fluid Mechanics. (London)

J. Franklin Inst. Journal of the Franklin Institute. (Philadelphia, Pa.)

J. Gakugei Tokushima Univ. Journal of Gakugei. Tokushima University. (Tokushima)

J. Geophys. Res. Journal of Geophysical Research. (Washington, D.C.)

J. Indian Inst. Sci. Journal of the Indian Institute of Science. (Bangalore)

J. Indian Math. Soc. The Journal of the Indian Mathematical Society. (Madras)

J. Indian Soc. Agric. Statist. Journal of the Indian Society of Agricultural Statistics. (New Delhi)

J. Indian Statist. Assoc. Journal of the Indian Statistical Association. (Bombay)

J. Inst. Actuar. Journal of the Institute of Actuaries. (London)

J. Karnatak Univ. Sci. Journal of the Karnatak University. Science. (Dharwar)

J. London Math. Soc. The Journal of the London Mathematical Society. (London)

J. Madras Univ. B. Journal of the Madras University. B. Contributions in Mathematics, Physical and Biological Sciences. (Madras) (Formerly: *J. Madras Univ. Sect. B*)

J. Madras Univ. Sect. B. Journal of the Madras University. Section B. (Madras) (Continued as: *J. Madras Univ. B*)

J. Math. Anal. Appl. Journal of Mathematical Analysis and Applications. (New York)

J. Math. and Phys. Journal of Mathematics and Physics. (Cambridge, Mass.)

J. Math. Kyoto Univ. Journal of Mathematics of Kyoto University. (Kyoto)

J. Math. Mech. Journal of Mathematics and Mechanics. (Bloomington, Ind.)

J. Math. Osaka City Univ. Journal of Mathematics. Osaka City University. (Osaka) (Continued as: *Osaka J. Math.*)

J. Math. Pures Appl. Journal de Mathématiques Pures et Appliquées. (Paris)

J. Math. Soc. Japan. Journal of the Mathematical Society of Japan. (Tokyo)

J. Mathematical Phys. Journal of Mathematical Physics. (New York)

J. Mathematical Psychology. Journal of Mathematical Psychology. (New York)

J. Méc. Phys. Atmos. Journal de Mécanique et de Physique de l'Atmosphère. (Paris)

J. Mécanique. Journal de Mécanique. (Paris)

J. Mech. Phys. Solids. Journal of the Mechanics and Physics of Solids. (London)

J. Nara Gakugei Univ. Natur. Sci. The Journal of Nara Gakugei University. Natural Science. (Nara)

J. Natur. Sci. and Math. The Journal of Natural Sciences and Mathematics. (Lahore)

J. Operations Res. Soc. Japan. Journal of the Operations Research Society of Japan. (Tokyo)

J. Opt. Soc. Amer. Journal of the Optical Society of America. (New York)

J. Osaka Inst. Sci. Tech. Part I. Journal of the Osaka Institute of Science and Technology. (Kinki University). Part I. Mathematics and Physics. (Osaka)

J. Phys. Soc. Japan. Journal of the Physical Society of Japan. (Tokyo)

J. Proc. Roy. Soc. New South Wales. Journal and Proceedings of the Royal Society of New South Wales. (Sydney)

J. Reine Angew. Math. Journal für die reine und angewandte Mathematik. (Berlin)

J. Res. Nat. Bur. Standards Sect. B. Journal of Research of the National Bureau of Standards. B. Mathematics and Mathematical Physics. (Washington, D.C.)

J. Res. Nat. Bur. Standards Sect. D. Journal of Research of the National Bureau of Standards. Section D. Radio Science. (Washington, D.C.)

J. Roy. Aero. Soc. The Journal of the Royal Aeronautical Society. (London) See *J. Roy. Aeronaut. Soc.*

J. Roy. Aeronaut. Soc. The Journal of the Royal Aeronautical Society. (London) (Formerly listed as: *J. Roy. Aero. Soc.*)

J. Roy. Astronom. Soc. Canada. The Journal of the Royal Astronomical Society of Canada. (Toronto, Ont.)

J. Roy. Statist. Soc. Ser. A. Journal of the Royal Statistical Society. Series A. (General). (London)

J. Roy. Statist. Soc. Ser. B. Journal of the Royal Statistical Society. Series B. (Methodological). (London)

J. Sci. Engrg. Res. Journal of Science and Engineering Research. (Kharagpur)

J. Sci. Hiroshima Univ. Ser. A-I Math. Journal of Science of the Hiroshima University. Series A-I (Mathematics). (Hiroshima)

J. Sci. Res. Banaras Hindu Univ. The Journal of Scientific Research of the Banaras Hindu University. (Banaras)

J. Ship Res. Journal of Ship Research. (New York)

J. SIAM Control Ser. A. Journal of the Society for Industrial and Applied Mathematics on Control. Series A. (Philadelphia, Pa.) See *J. Soc. Indust. Appl. Math. Ser. A Control.*

J. Soc. Indust. Appl. Math. Journal of the Society for Industrial and Applied Mathematics. (Philadelphia, Pa.)

J. Soc. Indust. Appl. Math. Ser. A Control. Journal of the Society for Industrial and Applied Mathematics. Series A: Control. (Philadelphia, Pa.) (Formerly listed as: *J. SIAM Control Ser. A*)

J. Symbolic Logic. The Journal of Symbolic Logic. (New Brunswick, N.J.)

J. Univ. Bombay. Journal of the University of Bombay. (Bombay)

J. Washington Acad. Sci. Journal of the Washington Academy of Sciences. (Washington, D.C.)

J. Zosen Kiokai. Journal of Zosen Kiokai. (Tokyo)

Janus. Janus. Revue Internationale de l'Histoire des Sciences, de la Médecine, de la Pharmacie et de la Technique. (Leiden)

Japan. J. Math. Science Council of Japan. Japanese Journal of Mathematics. (Tokyo)

Jber. Deutsch. Math.-Verein. Jahresbericht der Deutschen Mathematiker-Vereinigung. (Stuttgart)

Kabardino-Balkarsk. Gos. Univ. Učen. Zap. Ser. Fiz.-Mat. Kabardino-Balkarskiĭ Gosudarstvennyĭ Universitet. Učenye Zapiski. Serija Fiziko-Matematičeskaja. (Nalchik)

Kalinin. Gos. Ped. Inst. Učen. Zap. Kalininskiĭ Gosudarstvennyĭ Pedagogičeskiĭ Institut im. M. I. Kalinina. Učenye Zapiski. (Kalinin)

Kazan. Gos. Univ. Učen. Zap. Kazanskiĭ Ordena Trudovogo Krasnogo Znameni Gosudarstvennyĭ Universitet im. V. I. Ul'janova-Lenina. Učenye Zapiski. (Kazan)

Kišinev. Gos. Univ. Učen. Zap. Komitet Vysšego i Srednego Special'nogo Obrazovanija Soveta Ministrov Moldavskoĭ SSR. Kišinevskiĭ Gosudarstvennyĭ Universitet. Učenye Zapiski. (Kishinev)

Kōdai Math. Sem. Rep. Kōdai Mathematical Seminar Reports. (Tokyo)

Kolomen. Ped. Inst. Učen. Zap. Ser. Fiz.-Mat. Ministerstvo Prosveščenija RSFSR. Kolomenskiĭ Pedagogičeskiĭ Institut. Učenye Zapiski. Serija Fiziko-Matematičeskaja. (Kolomna)

Kös. Mat. Lapok. Középiskolai Matematikai Lapok. (Budapest)

Kristallografija. Akademija Nauk SSSR. Kristallografija. (Moscow) (Translated as: *Soviet Physics Cryst.*)

Kumamoto J. Sci. Ser. A. Kumamoto Journal of Science. Series A. (Mathematics, Physics and Chemistry). (Kumamoto)

Kungl. Svenska Vetenskapsakad. Handl. Kungl. Svenska Vetenskapsakademiens Handlingar. (Stockholm)

Kungl. Tekn. Högsk. Handl. Stockholm. Kungl. Tekniska Högskolans Handlingar. Transactions of the Royal Institute of Technology, Stockholm, Sweden. (Stockholm)

Kybernetik. Kybernetik. (Berlin)

Kyungpook Math. J. Kyungpook Mathematical Journal. (Taegu)

Latvijas PSR Zinātņu Akad. Vēstis. Latvijas PSR Zinātņu Akadēmijas Vēstis. Izvestija Akademii Nauk Latviĭskoĭ SSR. (Riga)

Latvijas PSR Zinātņu Akad. Vēstis Fiz. Tehn. Zinātņu Sēr. Latvijas PSR Zinātņu Akadēmijas Vēstis. Fizikas un Tehnisko Zinātņu Sērija. Izvestija Akademii Nauk Latviĭskoĭ SSR. Serija Fizičeskih i Tehničeskih Nauk. (Riga)

Latvijas Valsts Univ. Zinātn. Raksti. PSRS Augstākās Izglītības Ministrija. Pētera Stučkas Latvijas Valsts Universitate. Zinātniskie Raksti. Ministerstvo Vysšego Obrazovanija. Latviĭskiĭ Gosudarstvennyĭ Universitet im. Petra Stučki. Učenye Zapiski. (Riga)

Leningrad. Gos. Ped. Inst. Učen. Zap. Leningradskiĭ Gosudarstvennyĭ Pedagogičeskiĭ Institut im. A. I. Gercena. Učenye Zapiski. (Leningrad)

Leningrad. Gos. Univ. Učen. Zap. Ser. Mat. Nauk. Leningradskiĭ Gosudarstvennyĭ Ordena Lenina Universitet im. A. A. Ždanova. Učenye Zapiski. Serija Matematičeskih Nauk. (Leningrad)

Litovsk. Mat. Sb. Vysšie Učebnye Zavedenija Litovskoĭ SSR. Litovskiĭ Matematičeskiĭ Sbornik. (Vilna)

Logique et Analyse. Logique et Analyse. (Louvain)

Lucrăr. Şti. Inst. Ped. Timişoara Mat.-Fiz. Ministerul Învăţămîntului şi Culturii. Lucrările Ştiinţifice ale Institutului Pedagogic Timişoara. Matematică-Fizică. (Timişoara) (Continued as: *An. Univ. Timişoara Ser. Şti. Mat.-Fiz.*)

Lunds Univ. Årsskr. Avd. 2. Acta Universitatis Lundensis. Lunds Universitets Årsskrift. Andra Avdelningen. (Lund) (Continued as: *Acta Univ. Lund. Sect. II*)

L'vov. Politehn. Inst. Naučn. Zap. Ser. Fiz.-Mat. MVO USSR. L'vovskiĭ Politehničeskiĭ Institut. Naučnye Zapiski. Serija Fiziko-Matematičeskaja. (Lvov)

Madhya Bharati II A. Madhya Bharati. Journal of the University of Saugar. Part II, Sec. A. (Saugor)

Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. A Magyar Tudományos Akadémia Matematikai és Fizikai Tudományok Osztályának Közleményei. (Budapest)

Magyar Tud. Akad. Mat. Kutató Int. Közl. A Magyar Tudományos Akadémia Matematikai Kutató Intézetének Közleményei. (Budapest)

Management Sci. Management Science. Journal of the Institute of Management Science. (Philadelphia, Pa.)

Manchester Lit. Philos. Soc. Mem. Proc. Manchester Literary and Philosophical Society. Memoirs and Proceedings. (Manchester) (Formerly: *Mem. Proc. Manchester Lit. Philos. Soc.*)

Mat.-Fys. Medd. Danske Vid. Selsk. Matematisk-fysiske Meddelelser udgivet af Det Kongelige Danske Videnskabernes Selskab. (Copenhagen)

Mat.-Fys. Skr. Danske Vid. Selsk. Matematisk-fysiske Skrifter udgivet af Det Kongelige Danske Videnskabernes Selskab. (Copenhagen)

Mat.-Fyz. Časopis Sloven. Akad. Vied. Matematicko-Fyzikálny Časopis. Slovenská Akadémia Vied. (Bratislava)

Mat. Lapok. Matematikai Lapok. Bolyai János Matematikai Társulat. (Budapest)

Mat. Sb. Matematičeskiĭ Sbornik. Novaja Serija. (Moscow)

Mat. Vesnik. Matematički Vesnik. Nova Serija. (Belgrade) (Formerly: *Bull. Soc. Math. Phys. Serbie*)

Matematiche (Catania). Le Matematiche. (Catania)

Math. Ann. Mathematische Annalen. (Berlin-Göttingen-Heidelberg)

Math. Centrum Amsterdam Afd. Zuivere Wisk. Mathematisch Centrum Amsterdam. Afdeling Zuivere Wiskunde. (Amsterdam) (Formerly: *Math. Centrum Amsterdam Rap.*)

Math. Centrum Amsterdam Rap. Mathematisch Centrum Amsterdam. Rapport. (Amsterdam) (Continued as: *Math. Centrum Amsterdam Afd. Zuivere Wisk.*)

Math. Centrum Amsterdam Rekenafdeling. Mathematisch Centrum Amsterdam. Rekenafdeling. (Amsterdam)

Math. Comp. Mathematics of Computation. (Washington, D.C.)

Math. Gaz. The Mathematical Gazette. (London)

Math. J. Okayama Univ. Mathematical Journal of Okayama University. (Okayama)

Math. Japon. Mathematica Japonicae. (Kobe)

Math. Mag. Mathematics Magazine. (Buffalo, N.Y.)

Math. Nachr. Mathematische Nachrichten. (Berlin)

Math. Naturwiss. Unterricht. Der mathematische und naturwissenschaftliche Unterricht. (Bonn)

Math. Notae. Mathematicae Notae. Boletin del Instituto de Matemática. (Rosario)

Math.-Phys. Semesterber. Mathematisch-Physikalische Semesterberichte. (Göttingen)

Math. Reviews. Mathematical Reviews. (Providence, R.I.) (Usually abbreviated as *MR* in the text of *Math. Reviews*)

Math. Scand. Mathematica Scandinavica. (Copenhagen)

Math. Student. The Mathematics Student. (Madras)

Math.-Tech.-Wirtschaft. Mathematik-Technik-Wirtschaft. Zeitschrift für moderne Rechentechnik und Automation. (Vienna)

Math. und Wirtschaft. Mathematik und Wirtschaft. (Berlin)

Math. Z. Mathematische Zeitschrift. (Berlin-Göttingen-Heidelberg)

Mathematica (Cluj). Societatea de Ştiinţe Matematice şi Fizice din R.P.R. Filiala Cluj. Mathematica. (Cluj)

Mathematika. Mathematika. A Journal of Pure and Applied Mathematics. (London)

Mathematikunterricht. Der Mathematikunterricht. Beiträge zu seiner wissenschaftlichen und methodischen Gestaltung. (Stuttgart)

Mathesis. Mathesis. Recueil Mathématique à l'Usage des Écoles Spéciales et des Établissements d'Instruction Moyenne. (Mons)

Matrix Tensor Quart. The Tensor Club of Great Britain. The Matrix and Tensor Quarterly. (London)

Medd. Lunds Astronom. Obs. Ser. I. Meddelanden från Lunds Astronomiska Observatorium. Series I. (Lund)

Medd. Lunds Astronom. Obs. Ser. II. Meddelanden från Lunds Astronomiska Observatorium. Series II. (Lund)

Medd. Lunds Univ. Mat. Sem. Meddelanden från Lunds Universitets Matematiska Seminarium. Communications du Séminaire Mathématique de l'Université de Lund. (Lund)

Mem. Acad. Ci. Madrid. Memorias de la Real Academia de Ciencias Exactas, Fisicas y Naturales de Madrid. Serie de Ciencias Exactas. (Madrid) See *Mem. Real Acad. Ci. Exact. Fis. Natur. Madrid.*

Mem. Amer. Math. Soc. Memoirs of the American Mathematical Society. (Providence, R.I.)

Mem. Defense Acad. Memoirs of the Defense Academy (Mathematics, Physics, Chemistry and Engineering) (Yokosuka)

Mem. Fac. Ci. Univ. Habana Ser. Mat. Memorias de la Facultad de Ciencias. Universidad de la Habana. Serie Matemática. (Havana)

Mem. Fac. Ed. Kumamoto Univ. Sect. 1. Memoirs of the Faculty of Education. Kumamoto University. Section 1. (Natural Science). (Kumamoto)

Mem. Fac. Engrg. Hiroshima Univ. Memoirs of the Faculty of Engineering. Hiroshima University. (Hiroshima)

Mem. Fac. Engrg. Miyazaki Univ. Memoirs of the Faculty of Engineering, Miyazaki University. (Miyazaki)

Mem. Fac. Sci. Kyushu Univ. Ser. A. Memoirs of the Faculty of Science. Kyushu University. Series A. Mathematics. (Fukuoka)

Mem. Ist. Lombardo Accad. Sci. Lett. Cl. Sci. Mat. Natur. Memorie dell'Istituto Lombardo. Accademia di Scienze e Lettere. Classe di Scienze Matematiche e Naturali. (Milan)

Mem. Kitami College Tech. Memoirs of the Kitami College of Technology. (Kitami)

Mem. Mat. Inst. "Jorge Juan". Consejo Superior de Investigaciones Cientificas. Memorias de Matemática del Instituto "Jorge Juan". (Madrid)

Mem. Muroran Inst. Tech. Memoirs of the Muroran Institute of Technology. (Muroran)

Mem. Osaka Univ. Lib. Arts Ed. Ser. B. Memoirs of the Osaka University of the Liberal Arts and Education. Series B. Natural Science. (Osaka)

Mem. Proc. Manchester Lit. Philos. Soc. Memoirs and Proceedings of the Manchester Literary & Philosophical Society. (Manchester) (Continued as: *Manchester Lit. Philos. Soc. Mem. Proc.*)

Mem. Real Acad. Ci. Art. Barcelona. Memorias de la Real Academia de Ciencias y Artes de Barcelona. (Barcelona)

Mem. Real Acad. Ci. Exact. Fis. Natur. Madrid. Memorias de la Real Academia de Ciencias Exactas, Fisicas y Naturales de Madrid. Serie de Ciencias Exactas. (Madrid) (Formerly listed as: *Mem. Acad. Ci. Madrid*)

Mem. Roy. Astronom. Soc. Memoirs of the Royal Astronomical Society. (London)

Mem. School Sci. Engrg. Waseda Univ. Tokyo. Memoirs of the School of Science and Engineering. Waseda University, Tokyo. (Tokyo)

Mem. Soc. Astronom. Ital. Memorie della Società Astronomica Italiana. (Milan)

Mém. Soc. Roy. Sci. Liège Coll. in-4°. Mémoires de la Société Royale des Sciences de Liège. Collection in-4°. (Liège)

Mém. Soc. Roy. Sci. Liège Coll. in-8°. Mémoires de la Société Royale des Sciences de Liège. Collection in-8°. (Liège)

Mémor. Sci. Math. Mémorial des Sciences Mathématiques. (Paris)

Mémor. Sci. Phys. Mémorial des Sciences Physiques. (Paris)

Météorol. La Météorologie. (Paris)

Metrika. Metrika. (Würzburg)

Metroecon. Metroeconomica. Rivista Internazionale di Economica. (Bologna)

Metron. Metron. (Rome)

Michigan Math. J. The Michigan Mathematical Journal. (Ann Arbor, Mich.)

Min. Proc. Roy. Irish Acad. Minutes of Proceedings. Royal Irish Academy. (Dublin)

Mind. Mind. A Quarterly Review of Psychology and Philosophy. (London)

Mitt. Inst. Angew. Math. Zürich. Mitteilungen aus dem Institut für angewandte Mathematik an der Eidgenössischen Technischen Hochschule in Zürich. (Zürich)

Mitt. Inst. Theoret. Phys. Geophys. Bergakad. Freiberg. Mitteilungen aus dem Institut für Theoretische Physik und Geophysik der Bergakademie Freiberg. (Freiberg)

Mitt. Math. Ges. Hamburg. Mitteilungen der mathematischen Gesellschaft in Hamburg. (Hamburg) See *Mitt. Math. Gesellsch. Hamburg.*

Mitt. Math. Gesellsch. Hamburg. Mitteilungen der mathematischen Gesellschaft in Hamburg. (Hamburg) (Formerly listed as: *Mitt. Math. Ges. Hamburg*)

Mitt. Math. Sem. Giessen. Mitteilungen aus dem Mathematischen Seminar Giessen. (Giessen)

Mitt. Max-Planck-Inst. Strömungsforsch. Mitteilungen aus dem Max-Planck-Institut für Strömungsforschung. (Göttingen)

Mitt. Naturforsch. Ges. Bern. Mitteilungen der Naturforschenden Gesellschaft Bern. (Bern)

Mitt. Verein. Schweiz. Versich.-Math. Mitteilungen der Vereinigung Schweizerischer Versicherungsmathematiker. Bulletin de l'Association des Actuaires Suisses. (Bern)

Molecular Phys. Molecular Physics. (London)

Monatsb. Deutsch. Akad. Wiss. Berlin. Monatsberichte der Deutschen Akademie der Wissenschaften zu Berlin. (Berlin)

Monatsh. Math. Monatshefte für Mathematik. (Vienna)

Monthly Notices Roy. Astronom. Soc. Monthly Notices of the Royal Astronomical Society. (London)

Moskov. Gos. Ped. Inst. Učen. Zap. Moskovskiĭ Gosudarstvennyĭ Pedagogičeskiĭ Institut im. V. I. Lenina. Učenye Zapiski. (Moscow)

Moskov. Gos. Univ. Soobšč. Gos. Astronom. Inst. Šternberg. Moskovskiĭ Gosudarstvennyĭ Universitet im. M. V. Lomonosova. Soobščenija Gosudarstvennogo Astronomičeskogo Instituta im. P. K. Šternberga. (Moscow)

Moskov. Gos. Univ. Trudy Gos. Astronom. Inst. Šternberg. Moskovskiĭ Gosudarstvennyĭ Universitet im. M. V. Lomonosova. Trudy Gosudarstvennogo Astronomičeskogo Instituta im. P. K. Šternberga. (Moscow)

Moskov. Oblast. Ped. Inst. Učen. Zap. Moskovskiĭ Oblastnoĭ Pedagogičeskiĭ Institut. Učenye Zapiski. (Moscow)

MR. Mathematical Reviews. (Providence, R.I.)

Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II. Nachrichten der Akademie der Wissenschaften in Göttingen. II. Mathematisch-Physikalische Klasse. (Göttingen)

Nagoya Math. J. Nagoya Mathematical Journal. (Nagoya)

Nat. Bur. Standards Appl. Math. Ser. National Bureau of Standards. Applied Mathematics Series. (Washington, D.C.)

Nat. Lucht- en Ruimtevaartlab. Amsterdam Rep. Nationaal Lucht- en Ruimtevaartlaboratorium. National Aero- and Astronautical Research Institute Amsterdam. Report. (Amsterdam) (Formerly: *Nat. Luchtvaartlab. Amsterdam Rep.*)

Nat. Luchtvaartlab. Amsterdam Rep. Nationaal Luchtvaartlaboratorium. National Aeronautical Research Institute, Amsterdam. Report. (Amsterdam) (Continued as: *Nat. Lucht- en Ruimtevaartlab. Amsterdam Rep.*)

Natur. Sci. Rep. Ochanomizu Univ. Natural Science Report of the Ochanomizu University. (Tokyo)

Naturwissenschaften. Die Naturwissenschaften. (Berlin-Göttingen-Heidelberg)

Naval Res. Logist. Quart. Naval Research Logistics Quarterly. (Washington, D.C.)

Navigation. Navigation. Journal of the Institute of Navigation. (Los Angeles, Calif.)

Nederl. Akad. Wetensch. Indag. Math. Koninklijke Nederlandse Akademie van Wetenschappen. Indagationes Mathematicae ex Actis Quibus Titulus. Proceedings of the Section of Sciences. (Amsterdam) (Same as: *Nederl. Akad. Wetensch. Proc. Ser. A*)

Nederl. Akad. Wetensch. Proc. Ser. A. Koninklijke Nederlandse Akademie van Wetenschappen. Proceedings. Series A. Mathematical Sciences. (Amsterdam) (Same as: *Nederl. Akad. Wetensch. Indag. Math.*)

Nederl. Akad. Wetensch. Proc. Ser. B. Koninklijke Nederlandse Akademie van Wetenschappen. Proceedings. Series B. Physical Sciences. (Amsterdam)

Nederl. Akad. Wetensch. Verslag Afd. Natuurk. Koninklijke Nederlandse Akademie van Wetenschappen. Verslag van de Gewone Vergadering van de Afdeling Natuurkunde. (Amsterdam)

Nederl. Tijdschr. Natuurk. Nederlandse Tijdschrift voor Natuurkunde. (Utrecht)

New Zealand J. Sci. New Zealand Journal of Science. (Wellington)

Nieuw Arch. Wisk. Nieuw Archief voor Wiskunde. (Groningen)

Nieuw Tijdschr. Wisk. Nieuw Tijdschrift voor Wiskunde. (Groningen-Djakarta)

Nordisk Mat. Tidskr. Nordisk Matematisk Tidskrift. (Oslo)

Nordisk Tidskr. Informations-Behandling. Nordisk Tidskrift for Informations-Behandling. (Copenhagen)

Norske Vid. Selsk. Forh. (Trondheim). Det Kongelige Norske Videnskabers Selskabs Forhandlinger. (Trondheim)

Norske Vid. Selsk. Skr. (Trondheim). Det Kongelige Norske Videnskabers Selskabs Skrifter. (Trondheim)

Notas Mat. Notas de Matemática. (Rio de Janeiro)

Notices Amer. Math. Soc. Notices of the American Mathematical Society. (Providence, R.I.) (Formerly: *Amer. Math. Soc. Notices*)

Notre Dame J. Formal Logic. Notre Dame Journal of Formal Logic. (Notre Dame, Ind.)

Nova Acta Soc. Sci. Upsal. Nova Acta Regiae Societatis Scientiarum Upsaliensis. (Uppsala)

Nuclear Fusion. Nuclear Fusion. (Vienna)

Nuclear Phys. Nuclear Physics. (Amsterdam)

Numer. Math. Numerische Mathematik. (Berlin-Göttingen-Heidelberg)

Nuovo Cimento. Il Nuovo Cimento. (Bologna)

Nuovo Cimento Suppl. Supplemento al Nuovo Cimento. (Bologna)

O.N.E.R.A. Publ. Office National d'Études et de Recherches Aéronautiques. Publication. (Paris)

Observatory. The Observatory. A Review of Astronomy. (Hailsham)

Operations Res. Operations Research. The Journal of the Operations Research Society of America. (Baltimore, Md.) (Sometimes abbreviated as *OR* in *Math. Reviews*)

Optica Acta. Optica Acta. (London)

Optik. Optik. Zeitschrift für das gesamte Gebiet der Licht- und Elektronenoptik. (Stuttgart)

Osaka J. Math. Osaka Journal of Mathematics. (Osaka) (Formerly: *J. Math. Osaka City Univ.* and *Osaka Math. J.*)

Osaka Math. J. Osaka Mathematical Journal. (Osaka) (Continued as: *Osaka J. Math.*)

Osiris. Osiris. Commentationes de scientiarum et eruditionis historia rationeque. (Bruges)

Österreich. Akad. Wiss. Math.-Natur. Kl. S.-B. II. Österreichische Akademie der Wissenschaften. Mathematisch-Naturwissenschaftliche Klasse. Sitzungsberichte. Abteilung II. Mathematik, Astronomie, Physik, Meteorologie und Technik. (Vienna)

Österreich. Ing.-Arch. Österreichisches Ingenieur-Archiv. (Vienna)

Pacific J. Math. Pacific Journal of Mathematics. (Berkeley, Calif.)

Papers and Proc. Roy. Soc. Tasmania. Papers and Proceedings of the Royal Society of Tasmania. (Hobart)

Period. Mat. Periodico di Matematiche. (Bologna)

Perm. Gos. Univ. Učen. Zap. Mat. Ministerstvo Vysšego i Srednego Special'nogo Obrazovanija RSFSR. Permskiĭ Gosudarstvennyĭ Universitet im. A. M. Gor'kogo. Učenye Zapiski. Matematika. (Perm)

Philos. Mag. The Philosophical Magazine. A Journal of Theoretical, Experimental and Applied Science. (London)

Philos. Rev. The Philosophical Review. (Ithaca, N.Y.)

Philos. Sci. Philosophy of Science. (Baltimore, Md.)

Philos. Trans. Roy. Soc. London Ser. A. Philosophical Transactions of the Royal Society of London. Series A. Mathematical and Physical Sciences. (London)

Phys. Fluids. The Physics of Fluids. (New York)

Phys. Lett. Physics Letters. (Amsterdam)

Phys. Norveg. Physica Norvegica. (Oslo)

Phys. Rev. The Physical Review. (New York)

Phys. Rev. Lett. Physical Review Letters. (New York)

Physica. Physica. (Amsterdam)

PMTF Ž. Prikl. Meh. i Tehn. Fiz. PMTF. Žurnal Prikladnoĭ Mehaniki i Tehničeskoĭ Fiziki. (Moscow)

Polska Bibljo. Analit. Mech. Polska Akademia Nauk. Instytut Podstawowych Problemów Techniki. Polska Bibliografia Analityczna. Mechanika. Polish Scientific Abstracts. Mechanics. (Warsaw)

Portugal. Math. Portugaliae Mathematica. (Lisbon)

Portugal. Phys. Portugaliae Physica. (Lisbon)

Prace Mat. Roczniki Polskiego Towarzystwa Matematycznego. Seria I. Prace Matematyczne. (Warsaw)

Prakt. Akad. Athēnōn. *Πρακτικὰ τῆς Ἀκαδημίας Ἀθηνῶν.* (Athens)

Prikl. Mat. Meh. Akademija Nauk SSSR. Otdelenie Tehničeskih Nauk. Institut Mehaniki. Prikladnaja Matematika i Mehanika. (Moscow) (Translated as: *J. Appl. Math. Mech.*)

Prikladna Meh. Akademija Nauk Ukraïns'koï RSR. Institut Mehaniki. Prikladna Mehanika. (Kiev)

Problemy Kibernet. Problemy Kibernetiki. (Moscow)

Problemy Peredači Informacii. Akademija Nauk SSSR. Laboratorija Sistem Peredači Informacii. Problemy Peredači Informacii. (Moscow)

Proc. Amer. Math. Soc. Proceedings of the American Mathematical Society. (Providence, R.I.)

Proc. Cambridge Philos. Soc. Proceedings of the Cambridge Philosophical Society. (Cambridge, England)

Proc. Edinburgh Math. Soc. Proceedings of the Edinburgh Mathematical Society. (Edinburgh)

Proc. Edinburgh Math. Soc. Edinburgh Math. Notes. Proceedings of the Edinburgh Mathematical Society. Edinburgh Mathematical Notes. (Edinburgh)

Proc. Egyptian Acad. Sci. Proceedings of the Egyptian Academy of Sciences. (Cairo)

Proc. Fac. Engrg. Keio Univ. Proceedings of the Fujihara Memorial Faculty of Engineering. Keio University. (Tokyo)

Proc. Glasgow Math. Assoc. Proceedings of the Glasgow Mathematical Association. (Glasgow)

Proc. IEEE. Proceedings of the IEEE. (New York)

Proc. Indian Acad. Sci. Sect. A. Proceedings of the Indian Academy of Sciences. Section A. (Bangalore)

Proc. Indiana Acad. Sci. Proceedings of the Indiana Academy of Science. (Indianapolis, Ind.)

Proc. Inst. Elec. Engrs. Proceedings of the Institution of Electrical Engineers. Electronics. Power. Science & General. (London) (Formerly: *Proc. Inst. Elec. Engrs. B* and *Proc. Inst. Elec. Engrs. C*)

Proc. Inst. Elec. Engrs. B. The Proceedings of the Institution of Electrical Engineers. Part B. Radio and Electronic Engineering (including Communication Engineering). (London) (Continued as: *Proc. Inst. Elec. Engrs.*)

Proc. Inst. Elec. Engrs. C. The Proceedings of the Institution of Electrical Engineers. Part C. Monographs. (London) (Continued as: *Proc. Inst. Elec. Engrs.*)

Proc. Inst. Mech. Engrs. Proceedings of the Institution of Mechanical Engineers. (London) (Continued as: *Proc. Inst. Mech. Engrs. Part I*)

Proc. Inst. Mech. Engrs. Part I. Proceedings of the Institution of Mechanical Engineers. Part I. (London) (Formerly: *Proc. Inst. Mech. Engrs.*)

Proc. Inst. Statist. Math. The Proceedings of the Institute of Statistical Mathematics. (Tokyo)

Proc. Iowa Acad. Sci. Proceedings of the Iowa Academy of Science. (Des Moines, Iowa)

Proc. Iraqi Sci. Soc. Proceedings of the Iraqi Scientific Societies. (Baghdad)

Proc. Japan Acad. Proceedings of the Japan Academy. (Tokyo)

Proc. London Math. Soc. Proceedings of the London Mathematical Society. (London)

Proc. Math. Phys. Soc. U.A.R. Proceedings of the Mathematical and Physical Society of U.A.R. (Cairo) (Continued as: *Proc. Math. Phys. Soc. U.A.R. (Egypt)*)

Proc. Math. Phys. Soc. U.A.R. (Egypt). Proceedings of the Mathematical and Physical Society of U.A.R. (Egypt). (Cairo) (Formerly: *Proc. Math. Phys. Soc. U.A.R.*)

Proc. Nat. Acad. Sci. India Sect. A. Proceedings of the National Academy of Sciences, India. Section—A. (Allahabad)

Proc. Nat. Acad. Sci. U.S.A. Proceedings of the National Academy of Sciences of the United States of America. (Washington, D.C.)

Proc. Nat. Inst. Sci. India Part A. Proceedings of the National Institute of Sciences of India. Part A. Physical Sciences. (New Delhi)

Proc. Pakistan Statist. Assoc. Proceedings of the Pakistan Statistical Association. (Lahore)

Proc. Phys. Soc. Proceedings of the Physical Society. (London)

Proc. Rajasthan Acad. Sci. The Proceedings of The Rajasthan Academy of Sciences. (Pilani)

Proc. Roy. Irish Acad. Sect. A. Proceedings of the Royal Irish Academy. Section A. (Dublin)

Proc. Roy. Soc. Edinburgh Sect. A. Proceedings of the Royal Society of Edinburgh. Section A. (Mathematical and Physical Sciences). (Edinburgh)

Proc. Roy. Soc. New Zealand. Proceedings of the Royal Society of New Zealand. (Wellington)

Proc. Roy. Soc. Ser. A. Proceedings of the Royal Society. Series A. Mathematical and Physical Sciences. (London)

Proc. Roy. Soc. Victoria. Proceedings of the Royal Society of Victoria. New Series. (Melbourne)

Proc. Vibration Problems. Polish Academy of Sciences. Institute of Basic Technical Problems. Proceedings of Vibration Problems. (Warsaw)

Progr. Theoret. Phys. Progress of Theoretical Physics. (Kyoto)

Psychometrika. Psychometrika. A Journal Devoted to the Development of Psychology as a Quantitative Rational Science. (Chapel Hill, N.C.)

Publ. Astronom. Soc. Pacific. Publications of the Astronomical Society of the Pacific. (San Francisco, Calif.)

Publ. Inst. Math. (Beograd). Institut Mathématique. Publications de l'Institut Mathématique. Nouvelle Série. (Belgrade)

Publ. Inst. Statist. Univ. Paris. Publications de l'Institut de Statistique de l'Université de Paris. (Paris)

Publ. Math. Debrecen. Publicationes Mathematicae. (Debrecen)

Publ. Sci. Tech. Ministère de l'Air (Paris). Publications Scientifiques et Techniques du Ministère de l'Air. (Paris)

Publ. Sci. Tech. Ministère de l'Air (Paris) Notes Tech. Publications Scientifiques et Techniques du Ministère de l'Air. Notes Techniques. (Paris)

Publ. Sci. Univ. Alger Sér. A. Publications Scientifiques de l'Université d'Alger. Série A. Sciences Mathématiques. (Algiers)

Publ. Sém. Géom. Univ. Neuchâtel. Publications du Seminaire de Géométrie de l'Université de Neuchâtel. (Neuchâtel)

Quart. Appl. Math. Quarterly of Applied Mathematics. (Providence, R.I.)

Quart. J. Math. Oxford Ser. The Quarterly Journal of Mathematics, Oxford Second Series. (Oxford)

Quart. J. Mech. Appl. Math. The Quarterly Journal of Mechanics and Applied Mathematics. (Oxford)

Quart. J. Roy. Astronom. Soc. The Quarterly Journal of the Royal Astronomical Society. (London)

Quart. Rev. Sci. Publ. Polish Academy of Sciences. Distribution Centre for Scientific Publications. Quarterly Review of Scientific Publications. (Warsaw)

RAAG Newsletter. RAAG Newsletter. (Tokyo)

RAAG Res. Notes. RAAG Research Notes. (Tokyo)

Rad Jugoslav. Akad. Znan. Umjet. Odjel Mat. Fiz. Tehn. Nauke. Rad Jugoslavenske Akademije Znanosti i Umjetnosti. Odjel za Matematičke, Fizičke i Tehničke Nauke. (Zagreb)

Radio Engrg. and Electronics. Radio Engineering and Electronics. (Translation of *Radiotehnika i Èlektronika*, a U.S.S.R. Academy of Sciences Publication). (New York) (Translation of *Radiotehn. i Èlektron.*)

Radiotehn. i Èlektron. Akademija Nauk SSSR. Radiotehnika i Èlektronika. (Moscow) (Translated as: *Radio Engrg. and Electronics*)

Ratio. Ratio. (Oxford)

RCA Rev. RCA Review. (New York)

Recherche Aérospat. La Recherche Aérospatiale. (Paris)

Rend. Accad. Naz. dei XL. Rendiconti, Accademia Nazionale dei XL. (Rome)

Rend. Accad. Sci. Fis. Mat. Napoli. Società Nazionale di Napoli. Rendiconto dell'Accademia delle Scienze Fisiche e Matematiche. (Naples)

Rend. Circ. Mat. Palermo. Rendiconti del Circolo Matematico di Palermo. (Palermo)

Rend. Mat. e Appl. Università di Roma. Istituto Nazionale di Alta Matematica. Rendiconti di Matematica e delle sue Applicazioni. (Rome)

Rend. Sem. Fac. Sci. Univ. Cagliari. Rendiconti del Seminario della Facoltà di Scienze della Università di Cagliari. (Cagliari)

Rend. Sem. Mat. Fis. Milano. Rendiconti del Seminario Matematico e Fisico di Milano. (Milan)

Rend. Sem. Mat. Univ. Padova. Rendiconti del Seminario Matematico dell'Università di Padova. (Padova)

Rep. Inst. Sci. Tech. Univ. Tokyo. The Reports of the Institute of Science and Technology, University of Tokyo. (Tokyo)

Rep. Lib. Arts Sci. Fac. Shizuoka Univ. Sect. Natur. Sci. Reports of Liberal Arts and Science Faculty. Shizuoka University. Section Natural Science. (Shizuoka)

Rep. Res. Inst. Appl. Mech. Kyushu Univ. Reports of Research Institute for Applied Mechanics. Kyushu University. (Fukuoka)

Rep. Statist. Appl. Res. Un. Japan. Sci. Engrs. Reports of Statistical Application Research, Union of Japanese Scientists and Engineers. (Tokyo)

Repúb. Venezuela Bol. Acad. Ci. Fís. Mat. Natur. República de Venezuela. Boletín de la Academia de Ciencias Físicas, Matemáticas y Naturales. (Caracas)

Res. Bull. Panjab Univ. Research Bulletin of the Panjab University. (Hoshiarpur)

Res. Rep. Nagaoka Tech. College. Research Reports of the Nagaoka Technical College. (Nagaoka)

Rev. Acad. Ci. Madrid. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales de Madrid. (Madrid)

Rev. Acad. Ci. Zaragoza. Revista de la Academia de Ciencias Exactas, Físico-Químicas y Naturales de Zaragoza. (Zaragoza)

Rev. Acad. Colombiana Ci. Exact. Fís. Natur. Revista de la Academia Colombiana de Ciencias Exactas, Físicas y Naturales. (Bogotá)

Rev. Ci. (Lima). Revista de Ciencias. (Lima)

Rev. Fac. Ci. Univ. Coimbra. Revista da Faculdade de Ciências da Universidade de Coimbra. (Coimbra)

Rev. Française Traitement Information [Chiffres]. Revue Française de Traitement de l'Information. Chiffres. (Paris)

Rev. Gén. Sci. Pures Appl. Revue Générale des Sciences Pures et Appliquées et Bulletin de l'Association Française pour l'Avancement des Sciences. (Paris)

Rev. Histoire Sci. Appl. Revue d'Histoire des Sciences et de leurs Applications. (Paris)

Rev. Inst. Internat. Statist. Revue de l'Institut International de Statistique. Review of the International Statistical Institute. (The Hague)

Rev. Mat. Elem. Revista de Matemáticas Elementales. (Bogotá)

Rev. Mat. Hisp.-Amer. Revista Matemática Hispano-Americana. (Madrid)

Rev. Math. Pures Appl. Académie de la République Populaire Roumaine. Revue de Mathématiques Pures et Appliquées. (Bucharest) (Continued as: *Rev. Roumaine Math. Pures Appl.*)

Rev. Méc. Appl. Académie de la République Populaire Roumaine. Revue de Mécanique Appliquée. (Bucharest)

Rev. Mexicana Fís. Revista Mexicana de Física. (Mexico, D.F.)

Rev. Modern Phys. Reviews of Modern Physics. (New York)

Rev. Optique. Revue d'Optique Théorique et Instrumentale. (Paris)

Rev. Roumaine Math. Pures Appl. Académie de la République Populaire Roumaine. Revue Roumaine de Mathématiques Pures et Appliquées. (Bucharest) (Formerly: *Rev. Math. Pures Appl.*)

Rev. Un. Mat. Argentina. Revista de la Unión Matemática Argentina. (Buenos Aires)

Rice Univ. Studies. Rice University Studies. (Houston, Tex.)

Ricerca (Napoli). La Ricerca. Rivista di Matematiche Pure ed Applicate. (Naples)

Ricerche Mat. Ricerche di Matematica. (Naples)

Riv. Mat. Univ. Parma. Rivista di Matematica della Università di Parma. (Parma)

Rjazansk. Gos. Ped. Inst. Učen. Zap. Rjazanskiĭ Gosudarstvennyĭ Pedagogičeskiĭ Institut. Učenye Zapiski. (Ryazan)

Rostov.-na-Donu Gos. Ped. Inst. Fiz.-Mat. Fak. Učen. Zap. Rostovskiĭ-na-Donu Gosudarstvennyĭ Pedagogičeskiĭ Institut. Fiziko-Matematičeskiĭ Fakul'tet. Učenye Zapiski. (Rostov-on-Don)

Rozpravy Československé Akad. Věd. Rozpravy Československé Akademie Věd. (Prague)

Rozprawy Inż. Polska Akademia Nauk. Instytut Podstawowych Problemów Techniki. Rozprawy Inżynierskie. (Warsaw)

Rozprawy Mat. Polska Akademia Nauk. Instytut Matematyczny. Rozprawy Matematyczne. (Warsaw)

Russian Math. Surveys. Russian Mathematical Surveys. (A translation of the survey articles and of selected biographical articles in *Uspehi Matematičeskih Nauk*). (London) (Translation of *Uspehi Mat. Nauk*)

RŽ Astronom. Akademija Nauk SSSR. Institut Naučnoĭ Informacii. Referativnyĭ Žurnal. Otdel'nyĭ Vypusk 51. Astronomija. (Moscow)

RŽ Geodez. Akademija Nauk SSSR. Institut Naučnoĭ Informacii. Referativnyĭ Žurnal. Otdel'nyĭ Vypusk 52. Geodezija. (Moscow)

RŽMat. Akademija Nauk SSSR. Institut Naučnoĭ Informacii. Referativnyĭ Žurnal. Matematika Referaty. (Moscow)

RŽMeh. Akademija Nauk SSSR. Institut Naučnoĭ Informacii. Referativnyĭ Žurnal. Mehanika Referaty. (Moscow)

SA. Science Abstracts. Section A. Physics Abstracts. (London)

Saab Tech. Notes. Saab Technical Notes. Svenska Aeroplan Aktiebolaget. (Linköping) (Formerly: *Svenska Aeroplan A. B. Tech. Notes*)

Sankhyā Ser. A. Sankhyā. The Indian Journal of Statistics. Series A. (Calcutta)

Sankhyā Ser. B. Sankhyā. The Indian Journal of Statistics. Series B. (Calcutta)

S.-B. Berlin. Math. Ges. Sitzungsberichte der Berliner Mathematischen Gesellschaft. (Berlin)

S.-B. Deutsch. Akad. Wiss. Berlin Kl. Math. Phys. Tech. Sitzungsberichte der Deutschen Akademie der Wissenschaften zu Berlin. Klasse für Mathematik, Physik und Technik. (Berlin)

S.-B. Finn. Akad. Wiss. Sitzungsberichte der Finnischen Akademie der Wissenschaften. Proceedings of the Finnish Academy of Science and Letters. (Helsinki)

S.-B. Heidelberger Akad. Wiss. Math.-Natur. Kl. Sitzungsberichte der Heidelberger Akademie der Wissenschaften. Mathematisch-Naturwissenschaftliche Klasse. (Heidelberg)

S.-B. Phys.-Med. Soz. Erlangen. Sitzungsberichte der physikalisch-medizinischen Sozietät zu Erlangen. (Erlangen)

S.-B. Sächs. Akad. Wiss. Leipzig Math.-Natur. Kl. Sitzungsberichte der Sächsischen Akademie der Wissenschaften zu Leipzig. Mathematisch-naturwissenschaftliche Klasse. (Berlin)

Sb. Vysoké. Učeni Tech. Brno. Sborník Vysokého Učení Technického v Brně. (Brno)

Schr. Math. Inst. Univ. Münster. Schriftenreihe des Mathematischen Instituts der Universität Münster. (Münster)

Schweiz. Z. Vermessg. Kulturtech. Photogr. Schweizerische Zeitschrift für Vermessung, Kulturtechnik und Photogrammetrie. Revue Technique Suisse des Mensurations, du Génie Rural et de Photogrammétrie. (Winterthur)

Sci. Abstracts Sect. A. Science Abstracts. Section A. Physics Abstracts. (London) (Usually abbreviated *SA* in the text of *Math. Reviews*)

Sci. Abstracts Sect. B. Science Abstracts. Section B. Electrical Engineering Abstracts. (London)

Sci. Amer. Scientific American. (New York)

Sci. Information Notes. Scientific Information Notes. (Washington, D.C.)

Sci. Papers College Gen. Ed. Univ. Tokyo. Scientific Papers of the College of General Education. University of Tokyo. (Tokyo)

Sci. Papers Inst. Phys. Chem. Res. Scientific Papers of the Institute of Physical and Chemical Research. (Tokyo)

Sci. Proc. Roy. Dublin Soc. Ser. A. The Scientific Proceedings of the Royal Dublin Society. Series A. (Dublin)

Sci. Proc. Roy. Dublin Soc. Ser. B. The Scientific Proceedings of the Royal Dublin Society. Series B. (Dublin)

Sci. Rep. Fac. Lib. Arts Ed. Gifu Univ. Natur. Sci. Science Report of the Faculty of Liberal Arts and Education, Gifu University. (Natural Science). (Gifu)

Sci. Rep. Fac. Lit. Sci. Hirosaki Univ. Science Reports of the Faculty of Literature and Science. Hirosaki University. (Hirosaki)

Sci. Rep. Kagoshima Univ. Science Reports of the Kagoshima University. (Kagoshima)

Sci. Rep. Kanazawa Univ. The Science Reports of the Kanazawa University. (Kanazawa)

Sci. Rep. Niigata Univ. Ser. A. Science Reports of Niigata University. Series A (Mathematics). (Niigata) (Formerly: *J. Fac. Sci. Niigata Univ. Ser. I*)

Sci. Rep. Res. Inst. Theoret. Phys. Hiroshima Univ. Scientific Reports of the Research Institute for Theoretical Physics, Hiroshima University. (Hiroshima)

Sci. Rep. Res. Inst. Tôhoku Univ. Ser. A. The Science Reports of the Research Institutes. Tôhoku University. Series A. (Physics, Chemistry and Metallurgy). (Sendai)

Sci. Rep. Saitama Univ. Ser. A. The Science Reports of the Saitama University. Series A. Mathematics, Physics and Chemistry. (Urawa)

Sci. Rep. Tôhoku Univ. Ser. I. The Science Reports of the Tôhoku University. First Series. (Physics, Chemistry, Astronomy). (Sendai)

Sci. Rep. Tokyo Kyoiku Daigaku Sect. A. Science Reports of the Tokyo Kyoiku Daigaku. Section A. (Tokyo)

Sci. Rep. Yokohama Nat. Univ. Sect. I. Science Reports of the Yokohama National University. Section I. Mathematics, Physics. (Yokohama)

Sci. Sinica. Scientia Sinica. (Peking)

Scripta Math. Scripta Mathematica. A Quarterly Journal Devoted to the Philosophy, History, and Expository Treatment of Mathematics. (New York)

Shuxue Jinzhan. Shuxue Jinzhan. (Shanghai)

SIAM Rev. SIAM Review. A Publication of the Society for Industrial and Applied Mathematics. (Philadelphia, Pa.)

Sibirsk. Mat. Ž. Sibirskiĭ Matematičeskiĭ Žurnal. (Moscow)

Simon Stevin. Simon Stevin. Wis- en Natuurkundig Tijdschrift. (Groningen-Djakarta)

Skand. Aktuarietidskr. Skandinavisk Aktuarietidskrift. (Uppsala)

Skr. Norske Vid.-Akad. Oslo I. Skrifter Utgitt av det Norske Videnskaps-Akademi i Oslo. I. Mat.-Naturv. Klasse. (Oslo)

Smithsonian Contrib. to Astrophys. Smithsonian Contributions to Astrophysics. (Washington, D.C.)

Soc. Actuar. Trans. Society of Actuaries. Transactions. (Chicago, Ill.)

Soc. Parana. Mat. Anuário. Anuário da Sociedade Paranaense de Matemática. (Curitiba)

Soc. Sci. Fenn. Comment. Phys.-Math. Societas Scientiarum Fennica. Commentationes Physico-Mathematicae. (Helsinki)

Soobšč. Akad. Nauk Gruzin. SSR. Soobščenija Akademii Nauk Gruzinskoĭ SSR. (Tiflis)

Soviet Astronom. AJ. Soviet Astronomy. AJ. (A translation of *Astronomičeskiĭ Žurnal* of the Academy of Sciences of the USSR). (New York) (Translation of *Astronom. Ž.*)

Soviet Math. Dokl. Soviet Mathematics. Doklady. (A translation of the mathematics section of *Doklady Akademii Nauk SSSR*). (Providence, R.I.) (Translation of mathematics section of *Dokl. Akad. Nauk SSSR*)

Soviet Physics Acoust. Soviet Physics. Acoustics. (A translation of *Akustičeskiĭ Žurnal* of the Academy of Sciences of the USSR) (New York) (Translation of *Akust. Ž.*)

Soviet Physics Cryst. Soviet Physics. Crystallography. (A translation of the journal *Kristallografija* of the Academy of Sciences of the USSR). (New York) (Translation of *Kristallografija*)

Soviet Physics Dokl. Soviet Physics. Doklady. (A translation of the physics sections of *Doklady Akademii Nauk SSSR*). (New York) (Translation of physics sections of *Dokl. Akad. Nauk SSSR*)

Soviet Physics JETP. Soviet Physics. JETP. (A translation of *Ž. Èksperimental'noĭ i Teoretičeskoĭ Fiziki* of the USSR). (New York) (Translation of *Ž. Èksper. Teoret. Fiz.*)

Soviet Physics Solid State. Soviet Physics. Solid State. (A translation of the journal *Fizika Tverdogo Tela* of the Academy of Sciences of the USSR). (New York) (Translation of *Fiz. Tverd. Tela*)

Soviet Physics Tech. Phys. Soviet Physics. Technical Physics. (A translation of *Žurnal Tehničeskoĭ Fiziki* of the Academy of Sciences of the USSR). (New York) (Translation of *Ž. Tehn. Fiz.*)

Soviet Physics Uspekhi. Soviet Physics. Uspekhi. (A translation of *Uspehi Fizičeskih Nauk* (Advances in the Physical Sciences) of the Academy of Sciences, U.S.S.R.). (New York) (Translation of *Uspehi Fiz. Nauk*)

Spisy Přírod. Fak. Univ. Brno. Spisy Přírodovědecké Fakulty University v Brně. Trudy Estestvenno-Istoričeskogo Fakul'teta Universiteta v g. Brno. Publications de la Faculté des Sciences de l'Université à Brno. (Brno)

SRI J. SRI Journal. (Menlo Park, Calif.) (Continued as: *Stanford Res. Inst. J.*)

Stanford Res. Inst. J. Stanford Research Institute Journal. (Menlo Park, Calif.) (Formerly: *SRI J.*)

Statist. Theory Method Abstracts. Statistical Theory and Method Abstracts. (London) (Formerly: *Internat. J. Abstracts Statist. Theory Method*)

Statistica (Bologna). Statistica. (Bologna)

Statistica Neerlandica. Statistica Neerlandica. Orgaan van de Vereniging voor Statistiek. (The Hague)

Stroje na Zpracování Informací. Stroje na Zpracování Informací. (Prague)

Stud. Cerc. Mat. Studii şi Cercetări Matematice. (Bucharest) (Formerly: *Acad. R. P. Romîne Stud. Cerc. Mat.*)

Studia Logica. Polska Akademia Nauk. Komitet Filozoficzny. Studia Logica. (Warsaw)

Studia Math. Polska Akademia Nauk. Studia Mathematica. (Warsaw)

Studia Univ. Babeş-Bolyai Ser. I Math. Phys. Studia Universitatis Babeş-Bolyai. Series I. Mathematica Physica. (Cluj) (Continued as: *Studia Univ. Babeş-Bolyai Ser. Math.-Phys.*)

Studia Univ. Babeş-Bolyai Ser. Math.-Phys. Studia Universitatis Babeş-Bolyai. Series Mathematica-Physica. (Cluj) (Formerly: *Studia Univ. Babeş-Bolyai Ser. I Math. Phys.*)

Sudhoffs Arch. Sudhoffs Archiv für Geschichte der Medizin und der Naturwissenschaften. (Wiesbaden)

Sûgaku. Sûgaku. (Tokyo)

Summa Brasil. Math. Summa Brasiliensis Mathematicae. (Rio de Janeiro)

Svenska Aeroplan A. B. Tech. Notes. SAAB Aircraft Company. Svenska Aeroplan Aktiebolaget. Technical Notes. (Linköping) (Continued as: *Saab Tech. Notes*)

Tartu Riikl. Ül. Toimetised. Tartu Riikliku Ülikooli Toimetised. Učenye Zapiski Tartuskogo Gosudarstvennogo Universiteta. Matemaatika-Loodusteaduskonna Tööd. Trudy Estestvenno-Matematičeskogo Fakul'teta. (Tallin)

Taškent. Gos. Ped. Inst. Učen. Zap. Ministerstvo Prosveščenija Uzbekskoĭ SSR. Taškentskiĭ Gosudarstvennyĭ Pedagogičeskiĭ Institut im. Nizami. Učenye Zapiski. Fiziko-Matematičeskiĭ. (Tashkent)

Tbilis. Gos. Univ. Trudy Ser. Meh.-Mat. Nauk. Tbilisskiĭ Gosudarstvennyĭ Universitet. Trudy. Serija Mehaniko-Matematičeskih Nauk. (Tiflis)

Tech. Moderne. La Technique Moderne. (Paris)

Tech. Rep. Engrg. Res. Inst. Kyoto Univ. Technical Reports of the Engineering Research Institute. Kyoto University. (Kyoto)

Tech. Transl. Technical Translations. (Washington, D.C.)

Technometrics. Technometrics. A Journal of Statistics for the Physical, Chemical and Engineering Sciences. (Princeton, N.J.)

Tensor. Tensor. New Series. (Sapporo)

Teor. Verojatnost. i Primenen. Akademija Nauk SSSR. Teorija Verojatnosteĭ i ee Primenenija. (Moscow) (Translated as: *Theor. Probability Appl.*)

Thalès. Thalès. Recueil Annuel des Travaux de l'Institut d'Histoire des Sciences et des Techniques de l'Université de Paris. (Paris)

Theor. Probability Appl. Theory of Probability and its Applications. (An English translation of the Soviet journal *Teorija Verojatnosteĭ i ee Primenenija*). (Philadelphia, Pa.) (Translation of *Teor. Verojatnost. i Primenen.*)

Theoria (Lund). Theoria. A Swedish Journal of Philosophy and Psychology. (Lund)

Tôhoku Math. J. The Tôhoku Mathematical Journal. (Sendai)

Tomsk. Gos. Univ. Učen. Zap. Tomskiĭ Gosudarstvennyĭ Universitet im. V. V. Kuĭbyševa. Učenye Zapiski. (Tomsk)

Topology. Topology. An International Journal of Mathematics. (Oxford)

Trabajos Estadíst. Trabajos de Estadística. (Madrid)

Trans. Amer. Math. Soc. Transactions of the American Mathematical Society. (Providence, R.I.)

Trans. Amer. Philos. Soc. Transactions of the American Philosophical Society Held at Philadelphia for Promoting Useful Knowledge. (Philadelphia, Pa.)

Trans. ASME Ser. B. J. Engrg. Indust. Transactions of the ASME. Series B. Journal of Engineering for Industry. (New York)

Trans. ASME Ser. E. J. Appl. Mech. Transactions of the ASME. Series E. Journal of Applied Mechanics. (New York)

Trans. Faraday Soc. Transactions of the Faraday Society. (London)

Trans. New York Acad. Sci. Transactions of the New York Academy of Sciences. (New York)

Trans. Roy. Soc. Edinburgh. Transactions of the Royal Society of Edinburgh. (Edinburgh)

Trans. Roy. Soc. New Zealand General. Transactions of the Royal Society of New Zealand. General. (Wellington)

Trans. Roy. Soc. South Africa. Transactions of the Royal Society of South Africa. (Cape Town)

Trav. Soc. Sci. Lett. Wrocław Ser. B. Travaux de la Société des Sciences et des Lettres de Wrocław. Seria B. (Wrocław)

Trudy Akad. Nauk Litov. SSR Ser. B. Lietuvos TSR Mokslų Akademijos. Darbai. Serija B. Akademii Nauk Litovskoĭ SSR. Trudy. Serija B. (Vilna)

Trudy Inst. Istor. Estest. Tehn. Akademija Nauk SSSR. Trudy Instituta Istorii Estestvoznanija i Tehniki. (Moscow)

Trudy Inst. Teoret. Astronom. Akademija Nauk SSSR. Trudy Instituta Teoretičeskoĭ Astronomii. (Moscow)

Trudy Leningrad. Tehn. Inst. Trudy Kafedr. Meh. Fak. Ministerstvo Vysšego Obrazovanija SSSR. Trudy Leningradskogo Tehnologičeskogo Instituta im. Lensoveta. Trudy Kafedr Mehaničeskogo Fakul'teta. (Leningrad)

Trudy Mat. Inst. Steklov. Akademija Nauk Sojuza Sovetskih Socialističeskih Respublik. Trudy Matematičeskogo Instituta im. V. A. Steklova. (Moscow-Leningrad)

Trudy Moskov. Mat. Obšč. Trudy Moskovskogo Matematičeskogo Obščestva. (Moscow)

Trudy Samarkand. Gos. Univ. Mat. Ministerstvo Vysšego i Srednego Special'nogo Obrazovanija UzSSR. Trudy Samarkandskogo Gosudarstvennogo Universiteta im. A. Navoi. Matematika. (Samarkand)

Trudy Sem. Teor. Differencial. Uravnenii s Otklon. Argumentom Univ. Družby Narodov Patrisa Lumumby. Trudy Seminara po Teorii Differencial'nyh Uravnenii s Otklonjajuščimsja Argumentom. Universitet Družby Narodov imeni Patrisa Lumumby. (Moscow)

Trudy Sem. Vektor. Tenzor. Anal. Trudy Seminara po Vektornomu i Tenzornomu Analizu s ih Priloženijami k Geometrii, Mehanike i Fizike. (Moscow)

Trudy Taškent. Gos. Univ. Trudy Taškentskogo Gosudarstvennogo Universiteta im. V. I. Lenina. Matematika. (Tashkent)

Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. Trudy Tomskogo Gosudarstvennogo Universiteta im. V. V. Kuĭbyševa. Serija Mehaniko-Matematičeskaja. (Tomsk) (Formerly: *Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. Geom. Sb.*)

Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. Geom. Sb. Trudy Tomskogo Gosudarstvennogo Universiteta im. V. V. Kuĭbyševa. Serija Mehaniko-Matematičeskaja. Geometričeskiĭ Sbornik. (Tomsk) (Continued as: *Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat.*)

Trudy Vyčisl. Centra Akad. Nauk Gruzin. SSR. Akademija Nauk Gruzinskoĭ SSR. Trudy Vyčislitel'nogo Centra. (Tiflis)

Tul'sk. Gos. Ped. Inst. Učen. Zap. Fiz.-Mat. Nauk. Tul'skiĭ Gosudarstvennyĭ Pedagogičeskiĭ Institut im. L. N. Tolstogo. Učenye Zapiski. Fiziko-Matematičeskie Nauki. (Tula)

U.S.S.R. Comput. Math. and Math. Phys. U.S.S.R. Computational Mathematics and Mathematical Physics. (Oxford) (Translation of *Ž. Vyčisl. Mat. i Mat. Fiz.*)

Učen. Zap. Borisoglebsk. Gos. Ped. Inst. Ministerstvo Prosveščenija RSFSR. Učenye Zapiski. Borisoglebskogo Gosudarstvennogo Pedagogičeskogo Instituta. (Borisoglebsk)

Učen. Zap. Karel. Ped. Inst. Ser. Fiz.-Mat. Nauk. Ministerstvo Prosveščenija RSFSR. Karel'skiĭ Pedagogičeskiĭ Institut. Učenye Zapiski. Serija Fiziko-Matematičeskih Nauk. (Petrozavodsk)

Učen. Zap. Ural. Gos. Univ. Učenye Zapiski Ural'skogo Gosudarstvennogo Universiteta im. A. M. Gor'kogo. (Pri Učastii Ural'skogo Matematičeskogo Obščestva.) (Sverdlovsk)

Ukrain. Mat. Ž. Akademija Nauk Ukrainskoĭ SSR. Institut Matematiki. Ukrainskiĭ Matematičeskiĭ Žurnal. (Kiev)

Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. Univerzitet u Beogradu. Publikacije Elektrotehničkog Fakulteta. Serija Matematika i Fizika. (Belgrade)

Univ. Buenos Aires Contrib. Ci. Ser. Fis. Universidad de Buenos Aires. Facultad de Ciencias Exactas, Fisicas y Naturales. Contribuciones Cientificas. Serie Fisica. (Buenos Aires)

Univ. Buenos Aires Contrib. Ci. Ser. Mat. Universidad de Buenos Aires. Facultad de Ciencias Exactas y Naturales. Contribuciones Cientificas. Serie Matematica. (Buenos Aires)

Univ. Central Venezuela Bol. Fac. Ing. Universidad Central de Venezuela. Boletin de la Facultad de Ingenieria. (Caracas)

Univ. e Politec. Torino Rend. Sem. Mat. Università e Politecnico di Torino. Rendiconti del Seminario Matematico (già "Conferenze di Fisica e di Matematica"). (Turin)

Univ. Illinois Bull. Engrg. Exper. Station Bull. University of Illinois. Bulletin. Engineering Experiment Station Bulletin. (Urbana, Ill.)

Univ. Lisboa Revista Fac. Ci. A. Universidade de Lisboa. Revista da Faculdade de Ciências. 2.ª Série. A. Ciências Matemáticas. (Lisbon)

Univ. Madrid Publ. Sec. Mat. Fac. Ci. Universidad de Madrid. Publicaciones de la Sección de Matemática de la Facultad de Ciencias. (Madrid)

Univ. Nac. Ingen. Inst. Mat. Puras Apl. Notas Mat. Universidad Nacional de Ingenieria. Instituto de Matemáticas Puras y Aplicadas. Notas de Matemáticas. (Lima)

Univ. Nac. La Plata Publ. Fac. Ci. Fisicomat. Serie Segunda Rev. (or *Contrib.*) Universidad Nacional de La Plata. Publicaciones de la Facultad de Ciencias Fisicomatemáticas. Serie Segunda. Revista (or Contribuciones). (La Plata)

Univ. Nac. La Plata Publ. Fac. Ci. Fisicomat. Serie Tercera Publ. Esp. Universidad Nacional de La Plata. Publicaciones de la Facultad de Ciencias Fisicomatemáticas. Serie Tercera. Publicaciones Especiales. (La Plata)

Univ. Nac. Tucumán Rev. Ser. A. Universidad Nacional de Tucumán. Facultad de Ciencias Exactas y Tecnologia. Revista. Serie A. Matemáticas y Fisica Teórica. (Tucumán)

Univ. Repúb. Fac. Ingen. Agrimens. Montevideo Publ. Inst. Mat. Estadíst. Universidad de la República. Facultad de Ingeniería y Agrimensura. Montevideo-Uruguay. Publicaciones del Instituto de Matemática y Estadistica. (Montevideo)

Univ. Adama Mickiewicza w Poznan. Prace Wydz. Mat. Fiz. Chem. Ser. Mat. Uniwersytet im. Adama Mickiewicza w Poznaniu. Prace Wydziału Matematyki, Fiziki i Chemii. Seria Matematyka. (Poznań)

Ural. Gos. Univ. Mat. Zap. Ministerstvo Vysšego i Srednego Special'nogo Obrazovanija RSFSR. Ural'skiĭ Gosudarstvennyĭ Universitet im. A. M. Gor'kogo. Ural'skoe Matematičeskoe Obščestvo. Matematičeskie Zapiski. (Sverdlovsk)

Ural. Politehn. Inst. Ural'skiĭ Politehničeskiĭ Institut im. S. M. Kirova. (Sverdlovsk)

Uspehi Fiz. Nauk. Akademija Nauk SSSR. Uspehi Fizičeskih Nauk. (Moscow-Leningrad) (Translated as: *Soviet Physics Uspekhi*)

Uspehi Mat. Nauk. Akademija Nauk SSSR i Moskovskoe Matematičeskoe Obščestvo. Uspehi Matematičeskih Nauk. (Moscow-Leningrad) (Selected articles translated as: *Russian Math. Surveys*)

Užgorod. Gos. Univ. Naučn. Zap. MVO USSR. Užgorodskiĭ Gosudarstvennyĭ Universitet. Naučnye Zapiski. (Lvov)

Verh. Nederl. Akad. Wetensch. Afd. Natuurk. Sect. I. Verhandelingen der Koninklijke Nederlandse Akademie van Wetenschappen, Afdeling Natuurkunde. Eerste Sectie. (Amsterdam)

Vesci Akad. Navuk BSSR Ser. Fiz.-Tèhn. Navuk. Vesci Akadèmii Navuk BSSR. Serija Fizika-Tèhničnyh Navuk. (Minsk)

Vestnik Akad. Nauk Kazah. SSR. Vestnik Akademii Nauk Kazahskoi SSR. (Alma-Ata)

Vestnik Leningrad. Univ. Ser. Fiz. Him. Vestnik Leningradskogo Universiteta. Serija Fiziki, Himii. (Leningrad)

Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. Vestnik Leningradskogo Universiteta. Serija Matematiki, Mehaniki i Astronomii. (Leningrad)

Vestnik Moskov. Univ. Ser. I Mat. Meh. Vestnik Moskovskogo Universiteta. Serija I. Matematika, Mehanika. (Moscow)

Vestnik Moskov. Univ. Ser. III Fiz. Astronom. Vestnik Moskovskogo Universiteta. Serija III. Fizika, Astronomija. (Moscow)

Vierteljschr. Naturforsch. Ges. Zürich. Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich. (Zürich)

Vijnana Parishad Anusandhan Patrika. Vijnana Parishad Anusandhan Patrika [The Research Journal of the Hindi Science Academy]. (Allahabad)

Vilniaus Valst. Univ. Mokslo Darbai Mat. Fiz. Chem. Mokslų Ser. TSRS Aukštoje Mokslo Ministerija. Vilniaus Valstybinis Universitetas. Mokslo Darbai. Matematikos, Fizikos ir Chemijos Mokslų Serija. Ministerstvo Vysšego Obrazovanija SSSR. Vil'njusskiĭ Gosudarstvennyĭ Universitet. Učenye Trudy. Serija Matematičeskih, Fizičeskih i Himičeskih Nauk. (Vilna)

Virginia J. Sci. The Virginia Journal of Science. A Journal issued Quarterly by the Virginia Academy of Science. (Blacksburg, Va.)

Voprosy Kosmog. Akademija Nauk SSSR. Voprosy Kosmogonii. (Moscow)

Vyčisl. Sistemy. Akademija Nauk SSSR. Sibirskoe Otdelenie. Institut Matematiki. Vyčislitel'nye Sistemy. Sbornik Trudov. (Novosibirsk)

Wiadom. Mat. Roczniki Polskiego Towarzystwa Matematycznego. Ser. II. Wiadomości Matematyczne. (Warsaw)

Wiss. Z. Ernst-Moritz-Arndt-Univ. Greifswald Math.-Natur. Reihe. Wissenschaftliche Zeitschrift der Ernst-Moritz-Arndt-Universität Greifswald. Mathematisch-Naturwissenschaftliche Reihe. (Greifswald)

Wiss. Z. Friedrich-Schiller-Univ. Jena/Thüringen. Wissenschaftliche Zeitschrift der Friedrich-Schiller-Universität Jena/Thüringen. (Jena/Thüringen)

Wiss. Z. Hochsch. Elektrotech. Ilmenau. Wissenschaftliche Zeitschrift der Hochschule für Elektrotechnik. Ilmenau. (Ilmenau) (Continued as: *Wiss. Z. Techn. Hochsch. Ilmenau*)

Wiss. Z. Hochsch. Verkehrswesen "Friedrich List" Dresden. Wissenschaftliche Zeitschrift der Hochschule für Verkehrswesen "Friedrich List" in Dresden. Die Anwendung mathematischer Methoden im Transport- und Nachrichtenwesen. (Dresden)

Wiss. Z. Humboldt-Univ. Berlin Math.-Natur. Reihe. Wissenschaftliche Zeitschrift der Humboldt-Universität Berlin. Mathematisch-Naturwissenschaftliche Reihe. (Berlin)

Wiss. Z. Karl-Marx-Univ. Leipzig Math.-Natur. Reihe. Wissenschaftliche Zeitschrift der Karl-Marx-Universität Leipzig. Mathematisch-Naturwissenschaftliche Reihe. (Leipzig)

Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe. Wissenschaftliche Zeitschrift der Martin-Luther-Universität Halle-Wittenberg. Mathematisch-Naturwissenschaftliche Reihe. (Halle-Wittenberg)

Wiss. Z. Pädagog. Hochsch. Potsdam Math.-Natur. Reihe. Wissenschaftliche Zeitschrift der Pädagogischen Hochschule Potsdam. Mathematisch-Naturwissenschaftliche Reihe. (Potsdam)

Wiss. Z. Techn. Hochsch. Chem. Leuna-Merseburg. Wissenschaftliche Zeitschrift der Technischen Hochschule für Chemie Leuna-Merseburg. (Leuna-Merseburg)

Wiss. Z. Techn. Hochsch. Ilmenau. Wissenschaftliche Zeitschrift der Technischen Hochschule Ilmenau. (Ilmenau) (Formerly: *Wiss. Z. Hochsch. Elektrotech. Ilmenau*)

Wiss. Z. Techn. Univ. Dresden. Wissenschaftliche Zeitschrift der Technischen Universität Dresden. (Dresden)

Yokohama *Math. J.* Yokohama Mathematical Journal. (Yokohama)

Z. Angew. Math. Mech. Zeitschrift für Angewandte Mathematik und Mechanik. Ingenieurwissenschaftliche Forschungsarbeiten. (Berlin)

Z. Angew. Math. Phys. Zeitschrift für Angewandte Mathematik und Physik. ZAMP. Journal of Applied Mathematics and Physics. Journal de Mathématiques et de Physiques Appliquées. (Basel)

Z. Angew. Phys. Zeitschrift für Angewandte Physik. (Berlin)

Z. Astrophys. Zeitschrift für Astrophysik. (Berlin-Göttingen-Heidelberg)

Ž. Èksper. Teoret. Fiz. Akademija Nauk SSSR. Žurnal Èksperimental'noĭ i Teoretičeskoĭ Fiziki. (Moscow) (Translated as: *Soviet Physics JETP*)

Z. Flugwiss. Zeitschrift für Flugwissenschaften. (Braunschweig)

Z. Math. Logik Grundlagen Math. Zeitschrift für Mathematische Logik und Grundlagen der Mathematik. (Berlin)

Z. Meteorol. Zeitschrift für Meteorologie. (Berlin)

Z. Naturforsch. Zeitschrift für Naturforschung. (Tübingen)

Z. Phonetik Sprachwiss. Kommunikat. Zeitschrift für Phonetik Sprachwissenschaft und Kommunikationsforschung. (Berlin)

Z. Physik. Zeitschrift für Physik. (Berlin-Göttingen-Heidelberg)

Ž. Tehn. Fiz. Akademija Nauk SSSR. Žurnal Tehničeskoĭ Fiziki. (Moscow-Leningrad) (Translated as: *Soviet Physics Tech. Phys.*)

Ž. Vyčisl. Mat. i Mat. Fiz. Akademija Nauk SSSR. Žurnal Vyčislitel'noĭ Matematiki i Matematičeskoĭ Fiziki. (Moscow) (Translated as: *U.S.S.R. Comput. Math. and Math. Phys.*)

Z. Wahrscheinlichkeitstheorie und Verw. Gebiete. Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete. (Berlin)

Zastos. Mat. Polska Akademia Nauk. Instytut Matematyczny Zastosowania Matematyki. (Warsaw)

Zbirnik Prac' z Občisl. Mat. i Tehn. Akademija Nauk Ukraïns'koï RSR. Občisljuval'niĭ Centr. Zbirnik Prac' z Občisljuval'noï Matematiki i Tehniki. (Kiev)

Zbl. Zentralblatt für Mathematik und ihre Grenzgebiete. (Berlin-Göttingen-Heidelberg)

Zbl. Math. Zentralblatt für Mathematik und ihre Grenzgebiete. (Berlin-Göttingen-Heidelberg) (Usually abbreviated *Zbl* in the text of *Math. Reviews*)

Zeszyty Nauk. Uniw. Jagiello. Prace Mat. Zeszyty Naukowe Uniwersytetu Jagiellońskiego. Prace Matematyczne. (Matematyka, Fizyka, Chemia) (Kraków)

Zeszyty Nauk. Uniw. Łódz. Nauki Mat. Przyrod. Ser. II. Zeszyty Naukowe Uniwersytetu Łódzkiego. Nauki Matematyczno Przyrodnicze. Seria II. (Łódź)

JOURNALS IN TRANSLATION

Automat. Remote Control. Automation and Remote Control. (A translation of *Automatika i Telemehanika*, a publication of the Academy of Sciences of the USSR). (Pittsburgh, Pa.) (Translation of *Avtomat. i Telemeh.*)

Bull. (Izv.) Acad. Sci. USSR Geophys. Ser. Bulletin (Izvestiya) Academy of Sciences, USSR. Geophysics Series. (Washington, D.C.) (Translation of *Izv. Akad. Nauk SSSR Ser. Geofiz.*)

Chinese Math.—Acta. Chinese Mathematics. Translation of Acta Mathematica Sinica. (Providence, R.I.) (Translation of *Acta Math. Sinica*) (Formerly listed as: *Chinese Math.*)

J. Appl. Math. Mech. Journal of Applied Mathematics and Mechanics. (Translation of the Soviet journal *Prikladnaja Matematika i Mehanika*). (New York) (Translation of *Prikl. Mat. Meh.*)

Probleme der Kybernetik. Probleme der Kybernetik. (Berlin) (Translation of *Problemy Kibernet.*)

Radio Engrg. and Electronics. Radio Engineering and Electronics. (Translation of *Radiotehnika i Élektronika*, a U.S.S.R. Academy of Sciences Publication). (New York) (Translation of *Radiotehn. i Élektron.*)

Russian Math. Surveys. Russian Mathematical Surveys. (A translation of the survey articles and of selected biographical articles in *Uspehi Matematičeskih Nauk*). (London) (Translation of *Uspehi Mat. Nauk*)

Soviet Astronom. AJ. Soviet Astronomy. AJ. (A translation of *Astronomičeskiĭ Žurnal* of the Academy of Sciences of the USSR). (New York) (Translation of *Astronom. Ž.*)

Soviet Math. Dokl. Soviet Mathematics. Doklady. (A translation of the mathematics section of *Doklady Akademii Nauk SSSR*). (Providence, R.I.) (Translation of mathematics section of *Dokl. Akad. Nauk SSSR*)

Soviet Physics Acoust. Soviet Physics. Acoustics. (A translation of *Akustičeskiĭ Žurnal* of the Academy of Sciences of the USSR). (New York) (Translation of *Akust. Ž.*)

Soviet Physics Cryst. Soviet Physics. Crystallography. (A translation of the journal *Kristallografija* of the Academy of Sciences of the USSR). (New York) (Translation of *Kristallografija*)

Soviet Physics Dokl. Soviet Physics. Doklady. (A translation of the physics sections of *Doklady Akademii Nauk SSSR*). (New York) (Translation of physics sections of *Dokl. Akad. Nauk SSSR*)

Soviet Physics JETP. Soviet Physics. JETP. (A translation of *Ž. Èksperimental'noĭ i Teoretičeskoĭ Fiziki* of the USSR). (New York) (Translation of *Ž. Èksper. Teoret. Fiz.*)

Soviet Physics Solid State. Soviet Physics. Solid State. (A translation of the journal *Fizika Tverdogo Tela* of the Academy of Sciences of the USSR). (New York) (Translation of *Fiz. Tverd. Tela*)

Soviet Physics Tech. Phys. Soviet Physics. Technical Physics. (A translation of *Žurnal Tehničeskoĭ Fiziki* of the Academy of Sciences of the USSR). (New York) (Translation of *Ž. Tehn. Fiz.*)

Soviet Physics Uspekhi. Soviet Physics. Uspekhi. (A translation of *Uspehi Fizičeskih Nauk* (Advances in the Physical Sciences) of the Academy of Sciences, U.S.S.R.). (New York) (Translation of *Uspehi Fiz. Nauk*)

Theor. Probability Appl. Theory of Probability and its Applications. (An English translation of the Soviet journal *Teorija Verojatnosteĭ i ee Primenenija*). (Philadelphia, Pa.) (Translation of *Teor. Verojatnost. i Primenen.*)

U.S.S.R. Comput. Math. and Math. Phys. U.S.S.R. Computational Mathematics and Mathematical Physics. (Oxford) (Translation of *Ž. Vyčisl. Mat. i Mat. Fiz.*)

ERRATA AND ADDENDA

VOLUME 22

#2976: Landsberg, Max

In lines 1 and 2, the name "Condry" should be replaced by "Cauchy".

VOLUME 26

#2918: Zylbertrest, S.

In connection with the last sentence of the first paragraph of the review, the reviewer calls attention to a paper of R. Raghavendran [Math. Student **28** (1960), 65–70; MR **26** #2917].

Wightman, A. S.

On p. 1514 of the Index issue the review numbered 1339 should be 7339.

VOLUME 27

#5867: Teleman, Silviu

The reviewer adds the following correction. The algebra $C^*(\Sigma_d)$ is isomorphic (as a Banach algebra) to $C_{rs}(G)$ as discussed, for example, in (19.23.b) of Hewitt and Ross, *Abstract harmonic analysis*, Vol. I: *Structure of topological groups. Integration theory, group representations* [Academic Press, New York, 1963; MR **28** #158]. Thus, contrary to a statement in the review, $C^*(\Sigma_d)$ need not be commutative even if G is [op. cit., (19.24.b)].

Newman, Morris

On p. 1338 of the Index issue, review #2551 should be attributed to Michael F. Newman.

VOLUME 28

#2545: Gray, John W.

The correct bibliographical reference in the heading should be *Topology* **3** (1965), 1–18.

#2554: Raymond, Frank

The correct bibliographical reference in the heading should be *Topology* **3** (1965), 43–57.

#2937: Barbashin, E. A.; Tabueva, V. A.

The reviewer wishes to replace the first sentence of the review by the following. "The system studied in the authors' earlier paper [Avtomat. i Telemeh. **23** (1962), 1290–1297; MR **27** #6649] is again considered; the notation used below will be that of the review [MR **27** #6649] of the earlier paper. Assume that $A = B^2 - aB + b > 0$, $B > 0$."

#4713: Tetere, G. A.

The title of the journal should be completed by *Fiz. Tehn. Zinātņu Sēr.*

#4974: Mucenieks, V. A.; Rastrigin, L. A.

The title of the journal should be completed by *Fiz. Tehn. Zinātņu Sēr.*

#5063: Kolchin, E. R.

The correct bibliographical reference in the heading should be *Topology* **3** (1965), *suppl.* **2**, 309–318.

#5402: Tabata, Morio

The author's name should read Obata, Morio.

Tabata, Morio

On p. 1210 of the Index issue, the author's name should read Obata, Morio.

Volume 29

#196: Kobayashi, Shoshichi; Nagano, Tadashi

In line 3 of the review, replace (1906) by (1909).

#618: Hilton, P. J.

The correct bibliographical reference in the heading should be *Topology* **3** (1965), *suppl.* 2, 161–176.

#656: Georgiou, P.

The reviewer wishes to change "... of B) ..." in line 8 from the bottom of the review to "... of the boundary of B) ...".

#827: Green, A. E.

The reviewer wishes his comment at the end of the review to read as follows. "In the reviewer's opinion, equation (2) above is not at least unjustified. The utility of the author's continuum theory of anisotropic fluids will better wait till some specific problems are solved using that theory. It is too early to decide."

#1229: Taft, Earl J.

The author communicates that, although the hypotheses for the existence result are stated correctly, the hypotheses for the conjugacy result are too stringent; all that one needs is that G be completely reducible and that the characteristic not be 2.

#1248: Mendelsohn, N. S.

The author points out that the proof of Lemma 1 on p. 512 is incorrect, but that it can be rectified. A corrected version will appear in the same journal.

#1581: Coxeter, H. S. M.

In line 4 of the review, the expression $2f_{n-1}(x)/f_n(x)$ should read $2f_{n-1}(n)/f_n(n)$, according to the author.

#2198: Kislicyn, S. S.

The reviewer wishes to replace the last line of the review by the following. "This conjecture was disproved by L. R. Ford, Jr., and S. M. Johnson [Amer. Math. Monthly **66** (1959), 387–389; MR **21** #1942]."

#2463: Kukles, I. S.; Nurov, T.

The second displayed formula should be $\omega_2(x, y) = \frac{1}{2}(x^2+y^2)$ and not $\omega_2(x, y) = \frac{1}{2}(x^2-y^2)$. In addition, a more accurate reference to Ljapunov is the following: *General problem of the stability of motion* (Russian), pp. 206–207, GITTL, Moscow, 1950 [MR **12**, 612].

#2780: Edelstein, Michael

In line 2, replace "$f^m(E^n)$" by "$f^m(A)$, for any bounded $A \subset E^n$,".

TRANSLITERATION OF RUSSIAN

Russian Letter Cap. Ital.	Mathematical Reviews	Zentralblatt für Mathematik	Bulletin Signalétique	Applied Mechanics Reviews	Science Abstracts	U.S. Library of Congress	Amer. Slavic & E. European Review	Journal of Symbolic Logic
А *а*	a	a	a	a	a	a	a	a
Б *б*	b	b	b	b	b	b	b	b
В *в*	v	v	v	v	v	v	v	v
Г *г*	g	g	g	g	g	g	g	g
Д *д*	d	d	d	d	d	d	d	d
Е *е*	e	e	e	e	e	e	e	é
Ё *ё*	e	e	ë	ë	ë	ë	ë	ë
Ж *ж*	ž	ž	zh	zh	zh	z͡h	ž	ž
З *з*	z	z	z	z	z	z	z	z
И *и*	i	i	i	i	i	i	i	i
Й *й*	ĭ	j	j	ĭ	ĭ	ĭ	j	j
К *к*	k	k	k	k	k	k	k	k
Л *л*	l	l	l	l	l	l	l	l
М *м*	m	m	m	m	m	m	m	m
Н *н*	n	n	n	n	n	n	n	n
О *о*	o	o	o	o	o	o	o	o
П *п*	p	p	p	p	p	p	p	p
Р *р*	r	r	r	r	r	r	r	r
С *с*	s	s	s	s	s	s	s	s
Т *т*	t	t	t	t	t	t	t	t
У *у*	u	u	u	u	u	u	u	u
Ф *ф*	f	f	f	f	f	f	f	f
Х *х*	h	ch	kh	kh	kh	k͡h	kh	h
Ц *ц*	c	c	c	ts	ts	t͡s	c	c
Ч *ч*	č	č	ch	ch	ch	c͡h	č	č
Ш *ш*	š	š	sh	sh	sh	s͡h	š	š
Щ *щ*	šč	šč	shch	shch	shch	s͡hch	šč	šč
Ъ *ъ*	″			″	'	″	'	
Ы *ы*	y	y	y	ī	ȳ	y	y	y
Ь *ь*	′	′	'	′	'	'	'	′
Э *э*	è	ė	e	é	é	ė	è	e
Ю *ю*	ju*	ju	ju	yu	yu	i͡u	ju	ú
Я *я*	ja†	ja	ja	ya	ya	i͡a	ja	á

* formerly yu
† formerly ya